Beyond Bilinear Complexity: What Works and What Breaks with Many Modes?

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Abstract

The complexity of bilinear maps (equivalently, of 3-mode tensors) has been studied extensively, most notably in the context of matrix multiplication. While circuit complexity and tensor rank coincide asymptotically for 3-mode tensors, this correspondence breaks down for $d \geq 4$ modes. As a result, the complexity of d-mode tensors for larger fixed d remains poorly understood, despite its relevance, e.g., in fine-grained complexity. Our paper explores this intermediate regime.

First, we give a "graph-theoretic" proof of Strassen's $2\omega/3$ bound on the asymptotic rank exponent of 3-mode tensors. Our proof directly generalizes to an upper bound of $(d-1)\omega/3$ for d-mode tensors. Using refined techniques available only for $d \geq 4$ modes, we improve this bound beyond the current state of the art for ω. We also obtain a bound of $d/2+1$ on the asymptotic exponent of *circuit complexity* of generic d-mode tensors and optimized bounds for $d \in \{4, 5\}$.

To the best of our knowledge, asymptotic circuit complexity (rather than rank) of tensors has not been studied before. To obtain a robust theory, we first ask whether low complexity of T and U imply low complexity of their Kronecker product $T \otimes U$. While this crucially holds for rank (and thus for circuit complexity in 3 modes), we show that assumptions from fine-grained complexity rule out such a *submultiplicativity* for the circuit complexity of tensors with many modes. In particular, assuming the Hyperclique Conjecture, this failure occurs already for $d = 8$ modes. Nevertheless, we can salvage a restricted notion of submultiplicativity.

From a technical perspective, our proofs heavily make use of the *graph tensors* T_H, as employed by Christandl and Zuiddam (*Comput. Complexity* 28 (2019) 27–56) and Christandl, Vrana and Zuiddam (*Comput. Complexity* 28 (2019) 57–111), whose modes correspond to the vertices of undirected graphs H. We make the simple but conceptually crucial observation that Kronecker products $T_G \otimes T_H$ are isomorphic to T_{G+H}, and that G and H may also be *fractional* graphs. By asymptotically converting generic tensors to specific graph tensors, we can use nontrivial results from algorithmic graph theory to study the rank and complexity of d-mode tensors for fixed d.

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1 Introduction

In several fundamental computational problems, the input data is represented by two vectors u, v and the output is a vector $f(u, v)$ that captures a meaningful combination of the input data. Prominent examples are *matrix multiplication* (when the input matrices are flattened to vectors), *polynomial multiplication* (when the input polynomials are given as coefficient vectors), and more general *convolution* problems such as the *subset convolution* in parameterized and exact algorithms, which is essentially multiplication in the algebra of square-free multivariate polynomials.

Bilinear Maps and Rank. The above problems all ask to evaluate maps $f : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ that are *bilinear* over the field $\mathbb{F}$; they satisfy $f(\alpha x + \beta x', y) = \alpha f(x, y) + \beta f(x', y)$, with an analogous rule for the second argument. Bilinear maps enjoy a strong connection between computational complexity and *rank*, an algebraic complexity measure. The rank $\mathrm{R}(f)$ is the minimum number of terms required to express f as a linear combination of rank-1 maps, where a rank-1 map is a product of linear functions in disjoint variable sets.

Rank and Kronecker Products. Several extremely well-studied bilinear maps, such as matrix multiplication and subset convolution, have a very useful structure: They can be expressed as *Kronecker products* of lower-dimensional bilinear maps. To make this precise, given vectors $u, u' \in \mathbb{F}^n$, define $u \otimes u' \in \mathbb{F}^{n^2}$ by setting $(u \otimes u')_{i,j} = u_i u'_j$ and viewing the index set $[n]^2$ as $[n^2]$ under some bijection. For bilinear maps $f, f' : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$, the *Kronecker product* $f \otimes f' : \mathbb{F}^{n^2} \times \mathbb{F}^{n^2} \to \mathbb{F}^{n^2}$ is defined by mapping

$$(u \otimes u', v \otimes v') \mapsto f(u, v) \otimes f'(u', v'),$$

and extending this rule bilinearly. For example, the Kronecker product of the bilinear map g_n for multiplication of two $n \times n$ matrices and the map g_m for multiplication of two $m \times m$ matrices is the map $g_n \otimes g_m \equiv g_{nm}$ for multiplication of two $nm \times nm$ matrices. Likewise, subset convolution over a universe of size n (i.e., square-free products of n-variate multilinear polynomials) is the Kronecker power of subset convolution over a universe of size 1.

Rank and Kronecker products interact very favorably: If f_1 and f_2 have rank $r_1 = \mathrm{R}(f_1)$ and $r_2 = \mathrm{R}(f_2)$, then $\mathrm{R}(f_1 \otimes f_2) \leq r_1 r_2$, since the rank decompositions for f_1 and f_2 can be combined to a rank decomposition for $f_1 \otimes f_2$. Unlike for linear maps, this combined rank decomposition of bilinear maps may however not be optimal, and we may have $\mathrm{R}(f_1 \otimes f_2) < r_1 r_2$. Consequently, we say that rank is *submultiplicative* under Kronecker products.

Asymptotic Rank and Circuit Complexity. The rank of bilinear maps like matrix multiplication and subset convolution of increasing dimensions can be understood by fixing a constant-dimension bilinear map $g : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ of rank $r = \mathrm{R}(g)$ and considering its *Kronecker power* $g^{\otimes k} = g \otimes \ldots \otimes g$, a bilinear map between spaces of dimension n^k. As discussed above, we then have $\mathrm{R}(g^{\otimes k}) \leq r^k$, but this bound may be exponentially loose: One of the simplest examples of this phenomenon is the subset convolution map $g : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}^2$

on 1-element universes, or what is essentially the same, the three-qubit W-state in quantum information theory. We have $\mathrm{R}(g) = 3$ and thus $\mathrm{R}(g^{\otimes k}) \leq 3^k$, while it is known that $\mathrm{R}(g^{\otimes k}) \leq O(2^k k)$. Combined with algorithmic ideas dating back to Yates [50], this nontrivial rank bound for $g^{\otimes k}$ translates into nontrivial $2^{k+o(k)}$ size circuits for subset convolution on k-element universes. Similar observations are crucial in the study of matrix multiplication.

This algorithmic connection between rank and algorithmic complexity of Kronecker powers motivates defining the *asymptotic rank exponent* $\omega(g)$ of a bilinear map $g : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ as the infimum over all $\beta > 0$ such that $\mathrm{R}(g^{\otimes k}) \leq O(n^{\beta k})$. Likewise, we define the *asymptotic circuit complexity exponent* $\eta(g)$ as the infimum over all $\beta > 0$ such that $g^{\otimes k}$ has arithmetic circuits of size $O(n^{\beta k})$. For bilinear maps, these two exponents are known to be *equal* (cf. the discussion following Theorem 8), so asymptotic circuit complexity can be understood entirely in terms of asymptotic rank, which in turn is amenable to techniques from pure mathematics.

Trivially, the asymptotic rank exponent of every bilinear map is bounded by 2, as $\mathrm{R}(g) \leq n^2$. However, Strassen [45] showed a surprising bound of $2\omega/3 < 2$, where $\omega < 3$ is the asymptotic rank exponent of matrix multiplication. This translates directly to improved algorithms, e.g., for convolution problems [7]. The *asymptotic rank conjecture* [46] postulates that this upper bound can be pushed further down to 1, i.e., that every bilinear map g among n-dimensional vector spaces satisfies $\mathrm{R}(g^{\otimes k}) \leq O(n^k)$. This would lead to near-linear time algorithms for evaluating any map of the form $g^{\otimes k}$, which directly implies that $n \times n$ matrices can be multiplied with $n^{2+o(1)}$ operations. Moreover, it was recently shown that the asymptotic rank conjecture implies breakthrough exponential-time algorithms, e.g., for the permanent [5], for the chromatic number of graphs [3], and for the set cover problem [4, 40].

More Modes. So far, we described *bi*-linear maps, which take *two* input vectors and produce an output vector. As a natural generalization, a *multi*-linear map takes d input vectors, called *modes*, to a single output vector, and is linear in each argument. Multilinear maps are mentioned as an open research direction, e.g., in the survey by Wigderson and Zuiddam [49, Sect. 13]. Beyond their intrinsic algebraic appeal, they capture multi-party states in quantum information theory and problems in fine-grained complexity.

A prominent example for a multilinear map with d modes is the *iterated matrix multiplication*, which takes as input d matrices $A_1, \ldots, A_d$ (flattened to vectors) and outputs their product $A_1 \ldots A_d$. Other examples include the *determinant* and *permanent* of an $n \times n$ matrix A, which are multilinear *forms* (rather than *maps*, i.e., they output scalars) with n modes corresponding to the columns of A. In fine-grained complexity, the k-hyperclique problem admits a natural formulation as a multilinear form with k modes [35]. The complexity of this problem is interesting even for small, fixed values of k, starting with $k = 4$, and has attracted significant attention in recent years, especially as a source of conditional hardness [32, 8, 24].

▶ Remark. We will only consider multilinear *forms* in the following, since maps with $d-1$ modes correspond to forms with d modes by taking duals. After fixing coordinates, multilinear forms g with d modes can be viewed as set-multilinear polynomials: Denoting the entries of the input vector $\mathbf{x^{(i)}}$ at mode $i \in [d]$ by $x_1^{(i)}, \ldots, x_n^{(i)}$, we have

$$g(\mathbf{x^{(1)}}, \ldots, \mathbf{x^{(d)}}) = \sum_{i_1,\ldots,i_d} t_{i_1,\ldots,i_d}\, x_{i_1}^{(1)} \ldots x_{i_d}^{(d)}$$

with coefficients $t_{i_1,\ldots,i_d} \in \mathbb{F}$ that fully specify g. Arranging these coefficients in an array with d modes, we call this array the *tensor* of g, and overloading terminology further, we call g *itself* a tensor with d modes. After tacitly performing all required identifications, we can view bilinear maps as 3-mode tensors.

Rank versus Circuit Complexity for More Modes. Rank and circuit complexity can be generalized directly to tensors with an arbitrary number of modes. For $d \geq 4$ modes however, the correspondence between asymptotic circuit complexity and asymptotic rank breaks. As a folklore example of this, consider the tensor P_n with four n-dimensional modes defined by

$$P_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = \left(\sum_{j=1}^{n} x_j y_j\right)\left(\sum_{j=1}^{n} w_j z_j\right). \tag{1}$$

In algebraic complexity, P_n is known as a product of inner products (PIP). Since $P_n \otimes P_m \equiv P_{nm}$, the asymptotic rank exponent of every single P_n equals the exponent of the increasing sequence $\mathrm{R}(P_n)$ as $n \to \infty$. The defining formula for P_n of size $O(n)$ implies an asymptotic circuit complexity exponent of 1, while the asymptotic rank exponent of P_n is 2: Multiplying out (1) gives an upper bound of n^2, matched by a simple flattening lower bound.

Thus, unlike for $d = 3$ modes, asymptotic rank and asymptotic circuit complexity of 4-mode tensors for $d \geq 4$ need not agree. The separation can be exacerbated by generalizing the PIP tensor to a fixed number of $2d$ modes, which increases the asymptotic rank exponent to d, while the asymptotic circuit complexity exponent stays 1. As the example shows, the separation holds even for $\Pi\Sigma\Pi$-formula complexity, which morally still resembles rank in that rank decompositions can be viewed as $\Sigma\Pi\Sigma$-formulæ.

Full Version. Before explaining our results, we wish to highlight that an extended discussion as well as all omitted proofs can be found in the full version of the present paper [6].

Our Results: Phenomena of Tensors with More Modes

Starting from the observation that rank and circuit complexity need not agree for relevant tensors with $d \geq 4$ modes, we are led to systematically study asymptotic rank and asymptotic circuit complexity for d-mode tensors when $d \geq 3$ is fixed and moderately large or may tend to infinity. As part of this study, we obtain better upper bounds on asymptotic rank and asymptotic circuit complexity for generic d-mode tensors. Moreover, we investigate to which extent desirable properties like submultiplicativity fail for the circuit complexity of d-mode tensors with $d \geq 4$. The foundation to our results are *graph tensors* in the form used by Christandl and Zuiddam [15] and Christandl, Vrana and Zuiddam [14], which allow us to import techniques from graph theory.

We elaborate on these tensors later; for now, it suffices to note that the modes of the graph tensor T_H correspond to the vertices of a graph H. Graph tensors can be viewed as a special class of tensors that admit representation as a *tensor network*. Tensor networks have received extensive attention across a range of disciplines, each pursuing its own distinctive terminology and notation, from pure mathematics and physics [33, 37, 39] to the study of probabilistic graphical models in artificial intelligence and machine learning [30, 31, 38], and to the study of restricted circuit models in algebraic computation, where tensor networks give rise to a class of set-multilinear circuits for evaluating multilinear maps [1].

Generic Asymptotic Rank. To our surprise and best of knowledge, an analogue of Strassen's $2\omega/3$ bound on the asymptotic exponent of 3-mode tensors for d-tensors with $d \geq 4$ is not given in the literature. We give a simple proof for an upper bound of $(d-1)\omega/3$ on the asymptotic exponent of all d-mode tensors for $d \geq 3$ using the framework of graph tensors.

Our proof admits some freedom in the choice of low-rank tensor L to reduce to: While Strassen's original proof requires L to be the matrix multiplication tensor, which will turn out to be the graph tensor of a triangle K_3, our proof allows L to be the graph tensor of the complete graph K_s for $s \leq d$. In the case $d = 3$, we recover the aforementioned bound of

$(d-1)\omega/3$, but for $d \geq 4$, better upper bounds on the exponent of the graph tensor of K_d are known. Moreover, with a nontrivial application of Strassen's *laser method*, we improve the best known exponent on the graph tensor for K_4 and obtain:

▶ **Theorem 1** (Upper bound on asymptotic rank). *For every d-mode tensor with $d \geq 4$, the asymptotic rank exponent of T is at most* $0.772318(d-1)$.

For comparison, the PIP tensor presented above (1) establishes a lower bound of $\lfloor d/2 \rfloor$ on the rank exponent of d-mode tensors. Moreover, $\omega = 2$ would imply an upper bound of $2/3 \cdot (d-1)$. This is elaborated more formally in Remark 21.

Submultiplicativity. While the PIP tensor rules out a functional dependence between rank and circuit complexity for $d \geq 4$ modes, we ask whether circuit complexity is at least *submultiplicative*: From *rank* decompositions for tensors T and U, we can explicitly construct a *rank* decomposition for $T \otimes U$ of rank $\mathrm{R}(T) \cdot \mathrm{R}(U)$. If T and U are given by circuits of small size, can we analogously expect a circuit of small size for $T \otimes U$? Restricted circuit models are known to have submultiplicativity under Kronecker powering; in particular, tensor networks admit submultiplicativity of amortized cost via contraction trees [1]. But does *unrestricted* circuit complexity have submultiplicativity?

Our results strongly suggest a negative answer, but we obtain such statements only *conditionally*, as an unconditional result would imply strong circuit lower bounds. In the following, we write $\mathrm{C}(T)$ for the size of a smallest arithmetic circuit computing the tensor T.

▶ **Theorem 2** (Submultiplicativity of C implies VP = VNP). *There are explicit tensors $T_1, T_2, \ldots$ and $U_1, U_2, \ldots$, where T_d, U_d for $d \in \mathbb{N}$ are d-mode 0-1-tensors, such that the following holds: If $\mathrm{C}(T_d \otimes U_d) \leq \mathrm{poly}(d, \mathrm{C}(T_d), \mathrm{C}(U_d))$ for all $d \in \mathbb{N}$, then* VP = VNP.

Thus, the assumption VP $\neq$ VNP rules out submultiplicativity on d-mode tensors for all $d \in \mathbb{N}$, even when a polynomial slack on the complexity bound of the product tensor is allowed. Note that VP $\neq$ VNP implies super-polynomial circuit lower bounds for permanents [47]. If we assume even further that permanents require exponential-size circuits, then we can rule out submultiplicativity on d-mode tensors for concrete values of d.

▶ **Theorem 3** (Submultiplicativity of C implies faster permanents). *For all $0 < c < 1$, there are explicit $\lceil 4/c \rceil^2$-mode 0-1-tensors $T_1, T_2, \ldots$ and $U_1, U_2, \ldots$ such that the following holds: If $\mathrm{C}(T_n \otimes U_n) \leq O(\mathrm{C}(T_n) \cdot \mathrm{C}(U_n))$, then permanents have circuits of size $2^{cn} n^{O(1)}$.*

In particular, unless the permanent has circuits of size $2^{o(n)}$, there is a constant $D \in \mathbb{N}$ such that C is not submultiplicative for D-mode tensors. This assumption on the permanent is implied by a suitable non-uniform variant of the *exponential-time hypothesis*. For a more fine-grained approach, we can consider concrete values of $c > 0$ in Theorem 3. For example, if the permanent does not have circuits of size $2^{0.1n}$, then submultiplicativity fails for tensors with $d = \lceil 4/0.1 \rceil^2 = 40^2 = 1600$ modes. If even circuits of size $2^{0.8n}$ can be ruled out for the permanent, then submultiplicativity fails already on $d = 25$ modes, the smallest d for which Theorem 3 still yields a statement.

As for the plausibility of this assumption, Knuth asks in *The Art of Computer Programming* [29, Volume 2, §4.6.4, Problem 11] whether there is any way to evaluate the permanent of a general $n \times n$ matrix using fewer than 2^n arithmetic operations. This is still an open problem. Note that there is no asymptotic notation around 2^n. [1]

[1] Knuth's question is currently known to admit an affirmative answer only under the asymptotic rank conjecture [5]; more specifically, the asymptotic rank conjecture implies uniform arithmetic circuits of size 2^{cn} for the permanent where $c \approx 0.9183$.

Using another conjecture from fine-grained complexity theory, we approach $d = 4$, the smallest number of modes where rank and circuit complexity differ: The (h, k)-*hyperclique conjecture* [35] rules out $O(n^{k-\varepsilon})$-time algorithms for detecting k-hypercliques in h-uniform hypergraphs (see the full version for details).

▶ **Theorem 4** (Submultiplicativity of C implies faster hypercliques)**.** *For every even $d \geq 8$, there are explicit d-mode 0-1-tensors $T_1, T_2, \ldots$ and $U_1, U_2, \ldots$ such that: If $\mathrm{C}(T_n \otimes U_n) \leq O(\mathrm{C}(T_n) \cdot \mathrm{C}(U_n))$, then the non-uniform $(\frac{d}{2} - 1, \frac{d}{2})$-hyperclique conjecture fails.*

While our results suggest that circuit complexity is not submultiplicative, we can efficiently construct a circuit for $T \otimes U$ from a rank decomposition of T and a circuit for U.

▶ **Theorem 5** (Mixed asymptotic rank and circuit complexity)**.** *Given d-mode tensors T and U, where T has asymptotic rank exponent $\psi \geq 1$ and U has asymptotic circuit complexity exponent $\eta \geq 1$, the asymptotic circuit complexity exponent of $T \otimes U$ is at most $\eta + \psi$.*

Asymptotic Circuit Complexity. With tools like Theorem 5 and insights from graph theory, we adapt the graph-theoretic proof of our asymptotic rank bound in Theorem 1 to obtain stronger upper bounds for asymptotic circuit complexity rather than rank.

First, using a known nontrivial upper bound on the treewidth of line graphs of complete graphs [25], we obtain an upper bound on the circuit complexity of generic d-mode tensors:

▶ **Theorem 6** (Upper bound on asymptotic circuit complexity)**.** *The asymptotic circuit complexity exponent of every d-mode tensor T is less than $d/2 + 1$.*

While a lower bound of $d/2$ on the asymptotic *rank* exponent of d-mode tensors is immediate from (1), an unbounded *circuit complexity* exponent in d would constitute a strong circuit lower bound and hence seems exceedingly unlikely to be obtainable.

Next, we observe that Theorem 6 does not yield optimal upper bounds for small fixed d. For example, the theorem gives a circuit complexity exponent bound of $5/2$ for $d = 4$, but $5/2$ is even above the asymptotic *rank* upper bound of 2.317. To obtain meaningful upper bounds for small fixed d, we combine Theorem 5 with fractional decompositions of specific small graphs into low-rank and low-treewidth parts. These graph decompositions are found through computer-aided optimization. Our technique is applicable for arbitrary fixed d, albeit with higher computational load, and we obtain exemplary results for $d = 4, 5$:

▶ **Theorem 7** (Upper bound on asymptotic circuit complexity for small orders)**.** *For every* 4*-mode tensor T, the asymptotic circuit complexity exponent of T is at most* 2.2928*. For every* 5*-mode tensor, it is at most* 2.8738*.*

For comparison, the upper bounds on the asymptotic *rank* exponents from Theorem 1 for $d = 4, 5$ are about 2.317 and 3.0893, respectively. Table 1 provides a comprehensive overview of the kinds of results we obtain.

Our Techniques: New Insights into Graph Tensors

The protagonists in our proofs are so-called *graph tensors*, i.e., tensors with many modes that are composed from 2-mode tensors by using graphs as "composition blueprints." Such tensors appear in quantum information theory as *matrix-product states* and *projected entangled pairs* [16], in quantum machine learning as (restricted) *tensor networks* [26], and in counting complexity (under the right abstraction) as *Holant problems* [48, 12, 13]. For us, graph tensors act as a bridge between graph theory and multilinear algebra that allows us to import

Table 1 Bounds on asymptotic rank and circuit complexity exponents of generic d-mode tensors. The first row states rank upper bounds obtained via Theorem 1. The next two rows state upper bounds on asymptotic circuit exponents. For comparison, we include the known case $d = 3$ and the strongest known lower bounds.

Method	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 10$
$\underset{\sim}{\mathrm{R}}(T)$, asymptotic submult. (Theorem 1)	1.59	2.32	3.09	3.87	6.96
$\underset{\sim}{\mathrm{C}}(T)$, treewidth-based (Theorem 30)	2.00	2.50	3.20	3.67	5.80
$\underset{\sim}{\mathrm{C}}(T)$, specialized decompositions (Theorem 7)	–	2.30	2.88	–	–
Flattening lower bound on $\underset{\sim}{\mathrm{R}}(T)$ from (1)	1.00	2.00	2.00	3.00	5.00

nontrivial graph-theoretic concepts (e.g., graph decompositions and treewidth) and results (e.g., bounds on treewidth for specific graph classes) to the study of tensors with many modes. Moreover, graph tensors are closed under Kronecker products, which makes them particularly well-suited for studying the asymptotic behavior of rank and circuit complexity. They can be generalized further to tensors admitting representation by tensor networks or factor-graph models (e.g. [1, 31]), but graph tensors will be sufficient for our present purposes.

As a simple example, the outer tensor product of k tensors on b modes gives a tensor T with kb modes. In the extreme case of $b = 1$, the resulting tensor T has rank 1, but already for $b = 2$, examples of large rank such as the PIP tensor (1) emerge. Graph tensors are obtained by additionally allowing identification (with flattening) of modes. Such tensors admit a natural interpretation as generic instances of Holant problems; we discuss the former in the following. The interpretation as quantum states is discussed in the full version.

Graph Tensors in Terms of Holant Problems. To analyze the complexity of tensors $T_{H,n}$ for graphs H, we observe that they can be interpreted as *Holant problems*, which are very well-studied in counting complexity [48, 12, 10, 11, 44]: On input a graph $G = (V, E)$, a Holant problem asks to compute a weighted count of edge-assignments $s\colon E \to [r]$, with weights determined locally at vertices; the vast majority of the literature focuses on the case $r = 2$. As a concrete example, the number of *perfect matchings* in G is the number of edge-assignments $s\colon E(G) \to \{0, 1\}$ such that every vertex $v \in V$ has exactly one 1-labeled edge in the local restriction $s|_{I(v)}$ of the global assignment s.

In general Holant problems, the weights are determined by *signatures* $f_v\colon [r]^{I(v)} \to \mathbb{C}$ at the vertices $v \in V$. On input G and signatures $\{f_v\}_{v\in V}$, we then wish to determine

$$\mathrm{Holant}(G) = \sum_{s\colon E\to[r]} \prod_{v\in V} f_v(s|_{I(v)}). \tag{2}$$

Holant problems have been studied from a very different perspective in the literature: Usually, a set of possible signatures $\mathcal{F}$ is fixed, and the input is a graph G with signatures $f_v \in \mathcal{F}$. We consider the converse setting: The graph G is fixed, but the signatures can vary freely in that they are provided as input vectors to the $|V(G)|$ modes of the multilinear form $T_{G,n}$.

As an example of particular relevance for us, fix G as the 4-regular toroidal grid on $t \times t$ vertices. Then $T_{G,2}$ is a t^2-mode tensor with a copy of $(\mathbb{C}^2)^{\otimes 4}$ at each mode. By specifying a vertex signature $f_v : \{0, 1\}^4 \to \mathbb{C}$ for each grid vertex and inputting it as a vector $u_v \in \mathbb{C}^{16}$ into the multilinear form $T_{G,2}$, we can count perfect matchings, Eulerian subgraphs, or evaluate any Holant problem definable on the fixed grid G. In particular, for growing grid sizes, this includes VNP-hard problems: By appropriately engineering the signatures, we express the permanent of arbitrary $n \times n$ matrices as a particular Holant problem on the fixed $n \times n$ *grid*, and we obtain hardness of the grid graph tensor. On the algorithmic side, known fixed-parameter tractable algorithms over low-treewidth graphs establish a nontrivial complexity upper bound of $n^{O(t)}$ on the evaluation of $T_{G,n}$, where our generic results would only yield an $n^{O(t^2)}$ bound. Such upper and lower complexity bounds informed by (parameterized) algorithmic graph theory are ubiquitous in our arguments.

2 Preliminaries

For a nonnegative integer n we write $[n] = \{1, 2, \ldots, n\}$. We write $\mathbb{N} = \{1, 2, \ldots\}$. The Kronecker delta δ_{ij} is 1 if $i = j$ and 0 else.

Algebraic Complexity

Tensors. For convenience, we work with tensors in coordinates and view them as set-multilinear polynomials with an underlying tuple of modes. Let $\mathbb{F}$ be a field, and $\mathbb{F}[X]$ the ring of polynomials over $\mathbb{F}$ in a tuple X of one or more indeterminates. We assume X is partitioned into d nonempty sets $X_1, X_2, \ldots, X_d$ called *modes*, each of *dimension* $|X_i|$. A monomial is *set-multilinear* if each mode has exactly one indeterminate of degree one in the monomial, and all others have degree zero. A polynomial is *set-multilinear* if all of its monomials are. A *d-mode tensor* (or a *d-tensor*) T is a set-multilinear polynomial together with a d-tuple $(X_1, X_2, \ldots, X_d)$ of modes. We call d the *degree* or *order* of the tensor. We suppress d and the modes when they are clear from context. Suppressing also the variable names, we write $(\mathbb{F}^n)^{\otimes d}$ for the set of d-tensors over $\mathbb{F}$ with each mode of dimension n.

Addition and scalar multiplication of tensors with identical tuples of modes are inherited from polynomial arithmetic. The *Kronecker product* of d-tensors $S \in \mathbb{F}[X]$ and $T \in \mathbb{F}[Y]$ with X, Y disjoint is the d-tensor $S \otimes T$ obtained from the polynomial product ST by viewing the Cartesian products $X_i \times Y_i$ for $i \in [d]$ as the d modes. When the sets X, Y are not disjoint, such as when taking a Kronecker power of a tensor, we tacitly assume a disjoint copy of one of the sets of indeterminates is formed before taking the Kronecker product.

Projection, Equivalence, and Restriction. A *simple substitution* for sets of indeterminates X, Y is a map $\sigma : Y \to \mathbb{F} \cup X$. For tensors $S \in \mathbb{F}[X]$ and $T \in \mathbb{F}[Y]$ we say that S is a *projection* of T and write $S \leq_{\mathrm{s}} T$ if there exists a simple substitution σ such that the polynomial identity $S = T^\sigma$ holds and the image of each mode Y_j of T under σ intersects at most one mode X_i of S. A simple substitution is *nonscalar* if it does not assume values in $\mathbb{F}$. We say that S and T are *equivalent* and write $S \equiv T$ if the polynomial identity $S = T^\sigma$ holds for a bijective nonscalar σ that is bijective on modes. For d-tensors S and T we say that T *restricts to* S and write $S \leq T$ if the polynomial identity $S = T^\mu$ holds for a substitution $\mu : Y \to \mathbb{F}[X]$ such that for all for all modes Y_j and all indeterminates $y \in Y_j$ the polynomial $\mu(y)$ is a linear polynomial in the indeterminates X_j.

Tensor Rank. A nonzero d-tensor T has *rank one* if there exist linear polynomials $\ell_j \in \mathbb{F}[X_j]$ for $j \in [d]$ such that the polynomial identity $T = \ell_1 \ell_2 \cdots \ell_d$ holds. The *rank* $\mathrm{R}(T)$ of a d-tensor T is the minimum number of rank one tensors whose sum is T. (Note that $d = 2$ recovers matrix rank.) For two d-tensors S and T, we have $\mathrm{R}(S \otimes T) \leq \mathrm{R}(S) \cdot \mathrm{R}(T)$, with equality for $d = 2$ but strict inequality may hold for $d \geq 3$. This property is commonly referred to as the *submultiplicativity* of tensor rank. Moreover, the rank of any tensor $T \in (\mathbb{F}^n)^{\otimes d}$ is obviously bounded by n^d, the number of set-multilinear monomials (and the stronger bound n^{d-1} can easily be shown). Consequently, $T^{\otimes k}$ has rank at most $n^{(d-1)k}$. The *asymptotic rank* of $T \in (\mathbb{F}^n)^{\otimes d}$ is defined as the limit $\underset{\sim}{\mathrm{R}}(T) = \lim_{k\to\infty} \mathrm{R}(T^{\otimes k})^{1/k}$, which exists by submultiplicativity of rank and Fekete's Lemma (the result is folklore, but see [49, Lemma 2.10] and the references given there). Like rank, also the asymptotic rank satisfies submultiplicativity with respect to Kronecker products. We say that a d-tensor is *concise* if all of its d flattenings to a matrix with one mode forming the rows and all the other modes forming the columns have full rank. A concise tensor $T \in (\mathbb{F}^n)^{\otimes d}$ in particular satisfies both $\mathrm{R}(T) \geq n$ and $\underset{\sim}{\mathrm{R}}(T) \geq n$. The *exponent* of a concise tensor $T \in (\mathbb{F}^n)^{\otimes d}$ is defined by

$$\omega(T) = \inf\{\beta \geq 0 : \mathrm{R}(T^{\otimes k}) = O(n^{\beta k})\}$$

and we have

$$\underset{\sim}{\mathrm{R}}(T) = n^{\omega(T)} .$$

For example, the tensor $\mathrm{MM}_m \in \mathbb{F}^{m^2} \otimes \mathbb{F}^{m^2} \otimes \mathbb{F}^{m^2}$ that represents the bilinear map that multiplies two $m \times m$ matrices for any constant $m \geq 2$ is concise and recovers the exponent ω of square matrix multiplication either via the asymptotic rank identity $\underset{\sim}{\mathrm{R}}(\mathrm{MM}_m) = m^{\omega}$ or, equivalently, via the tensor exponent identity $\omega(\mathrm{MM}_m) = \omega/2$.

Arithmetic Circuits. Tensor rank fails to capture the arithmetic complexity of tensors for $d \geq 4$, see Example 11. In contrast, circuits model arithmetic complexity more faithfully. An *arithmetic circuit over* $\mathbb{F}[X]$ is a directed acyclic graph with sinks called *outputs*, sources called *inputs*, vertices called *gates*, and arcs called *wires*. Non-input gates are labeled with $+$ and $\times$, while inputs are labeled with elements from $\mathbb{F} \cup X$, and wires are labeled with elements of $\mathbb{F}$. A gate *computes* a polynomial in the obvious inductive manner, and the *size* of a circuit is the number of wires (see e.g. Kayal's [28] lecture notes for background on this particular implementation of the formalism). For a tensor $T \in (\mathbb{F}^n)^{\otimes d}$, we write $\mathrm{C}(T)$ for the size of a smallest circuit computing T, called the *arithmetic circuit complexity* of T.

A rank-r decomposition of $T \in (\mathbb{F}^n)^{\otimes d}$ yields an arithmetic circuit of size $O(d \cdot n \cdot r)$. This trivial bound $\mathrm{C}(T^{\otimes k}) \leq O(d \cdot n^{(d+1)k})$ allows us to define, in analogy with the tensor case, the *circuit exponent* $\eta(T)$ and the *asymptotic circuit complexity* $\underset{\sim}{\mathrm{C}}(T)$ of T via

$$\eta(T) = \inf\{\beta \geq 0 : \mathrm{C}(T^{\otimes k}) \leq O(n^{k\beta})\}, \quad \underset{\sim}{\mathrm{C}}(T) = n^{\eta(T)} .$$

Again, $\underset{\sim}{\mathrm{C}}(T) = \limsup_{k\to\infty} \mathrm{C}(T^{\otimes k})^{1/k}$ can be shown, but it is not clear if the limit exists. The bound $\mathrm{C}(T) = O(d \cdot n \cdot \mathrm{R}(T))$ shows $\eta(T) \leq \omega(T) + 1$, and a classic result of Yates [50] (cf. Knuth [29, §4.6.4]) shows that for $d \geq 2$, concise $T \in (\mathbb{F}^n)^{\otimes d}$, and all $k \geq 1$,

$$\mathrm{C}(T^{\otimes k}) = O(dk \cdot \mathrm{R}(T)^{k+1}). \tag{3}$$

The treatments in the literature we are aware of focus on the cases $d = 2, 3$ of Kronecker powers of matrices and 3-tensors, but the result easily generalizes to $d \geq 4$.

▶ **Theorem 8** (Yates's algorithm)**.** *Let $d \geq 2$ and let $T \in (\mathbb{F}^n)^{\otimes d}$ be concise. Then,*

$$\eta(T) \leq \omega(T), \text{ and therefore } \underset{\sim}{\mathrm{C}}(T) \leq \underset{\sim}{\mathrm{R}}(T).$$

For $d = 3$, it is also true that $\underset{\sim}{\mathrm{C}}(T) \geq \underset{\sim}{\mathrm{R}}(T)$ holds, justifying tensor rank as a measure of arithmetic complexity. As mentioned, this behavior does not extend to $d \geq 4$.

Closure Properties. Since restrictions of rank-1 tensors are themselves rank-1 (or identically zero), $\mathrm{R}(S) \leq \mathrm{R}(T)$ whenever $S \leq T$, and $\mathrm{R}(S) = \mathrm{R}(T)$ if $S \equiv T$. The same statements hold for asymptotic rank. For circuits, it is only true that $\mathrm{C}(S) \leq \mathrm{C}(T)$ when $S \leq_{\mathrm{s}} T$ for $S, T \in (\mathbb{F}^n)^{\otimes d}$, but $S \leq T$ only implies $\mathrm{C}(S) \leq \mathrm{C}(T) + n^{d+1}$. Still, asymptotic circuit complexity admits $\underset{\sim}{\mathrm{C}}(S) \leq \underset{\sim}{\mathrm{C}}(T)$ if $S \leq T$ as a consequence of Yates's algorithm (3). Moreover, we do have that $\mathrm{C}(U) \leq \mathrm{C}(U \otimes V)$ provided $V \neq 0$. The questions of whether and when $\underset{\sim}{\mathrm{C}}$ and C are robust with respect to Kronecker products will be a main theme of the article.

A very general description of these properties in relation to tensor rank (and far beyond) is laid out in the work of Wigderson and Zuiddam [49]. For further background on algebraic complexity we refer to the book of Bürgisser, Clausen and Shokrollahi [9].

Graph Theory

A *graph* G consists of a finite set $V(G)$ of *vertices* and a finite set $E(G)$ of *edges* such that each edge is associated with a set of two vertices called the *end-vertices* or *ends* of the edge. We stress that multiple edges may have the same ends; such edges are called *parallel*. For a vertex $v \in V(G)$ we write $I_G(v) \subseteq E(G)$ for the set of all edges that have v as an end, and call $|I_G(v)|$ the *degree* of v. The maximum degree of a vertex in G is denoted by $\Delta(G)$.

For graphs G and H with disjoint edge sets, the *sum* $G + H$ is the graph defined by $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H)$. If the edge sets are not disjoint, such as for $G + G$, we assume a disjoint copy of one of the edge sets is taken beforehand.

We call two graphs G and H *isomorphic*, written $G \equiv H$, if there are bijections $\phi\colon V(G) \to V(H)$, $\psi\colon E(G) \to E(H)$ such that every $e \in E(G)$ has the ends $\{v_1, v_2\} \subseteq V(G)$ if and only if $\psi(e)$ has the ends $\{\phi(v_1), \phi(v_2)\} \subseteq V(H)$. Equivalently, $\psi(I_G(v)) = I_H(\phi(v))$ should hold.

Subgraphs. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and common edges of H and G have the same ends. A *subdivision* of a graph G is a graph obtained by applying the following operation zero or more times: select an edge e with end u and v, insert a new vertex w, delete the edge e, insert a new edge with ends u and w, and insert a new edge with ends v and w. A graph H is a *topological subgraph* of a graph G if there exists a subdivision of H that is isomorphic to a subgraph of G.

Treewidth. A *tree decomposition* of a graph G is a pair (T, β) where T is a tree and β assigns to each vertex $a \in V(T)$ a *bag* $\beta(a) \subseteq V(G)$ such that (i) the union of all bags is $V(G)$; (ii) both ends of each edge of G are contained in at least one bag; and (iii) for all $a, c \in V(T)$ and all $b \in V(T)$ on the path from a to c in T, $\beta(a) \cap \beta(c) \subseteq \beta(b)$ holds. The *width* of (T, β) is $\max_{a \in V(T)} |\beta(a)| - 1$. The *treewidth* $\mathrm{tw}(G)$ is the minimum width of a tree decomposition of G. More background on algorithmic graph theory, in particular algorithms using tree decompositions of low width, can be found in the textbook of Cygan et al. [19].

3 Graph Tensors and the Asymptotic Rank

This section develops our conventions for graph tensors and their basic properties in relation to the underlying graphs. We then prove our main theorem (Theorem 1) on bounding the asymptotic rank for d-mode tensors for $d \geq 4$, extending Strassen's result for $d = 3$.

3.1 Graph Tensors

Our graphs are undirected, loopless, and may have parallel edges. Let G be a graph and let $n \in \mathbb{N}$. For a mapping $f \colon E(G) \to [n]$ and a vertex $v \in V(G)$, let us write $f|_{I(v)}$ for the restriction of f to the set $I(v)$ of edges incident to v in G. For each vertex $v \in V(G)$, introduce a mode $X^{(v)} = \{x_g^{(v)} \mid g \colon I(v) \to [n]\}$.

▶ **Definition 9** (Graph tensor). *The* graph tensor $T_{G,n}$ *of the graph G with* length parameter *n has the modes $(X^{(v)} : v \in V(G))$ and is defined by the polynomial identity*

$$T_{G,n} = \sum_{f \colon E(G) \to [n]} \prod_{v \in V} x^{(v)}_{f|_{I(v)}}. \tag{4}$$

Note that $T_{G,n}$ has $|V(G)|$ modes and the length *of the mode $X^{(v)}$ for $v \in V(G)$ is $|X^{(v)}| = n^{|I(v)|}$. Graph tensors are concise.*

▶ Remark 10 (The Holant view). As laid out in the introduction, graph tensors and Holant problems are intimately connected. Namely, the tensor $T_{G,n}$ essentially captures all Holant problems definable on the fixed graph G by choosing concrete signatures $s_v : [n]^{I(v)} \to \mathbb{C}$ for the vertices $v \in V(G)$ and substituting $x_a^{(v)} \leftarrow s_v(a)$ for all $a \in [n]^{I(v)}$.

▶ **Example 11** (The k-matching tensor). Let us write H_k for the *k-matching graph*, i.e., the 1-regular graph consisting of vertices $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ and edges $u_i v_i$ for all $i \in [k]$. For $n \in \mathbb{N}$, the *k-matching tensor* $T_{H_k,n}$ is known as the *product of inner products* in algebraic complexity theory. For $k \geq 2$, $T_{H_k,n}$ provides an example of the separation of algebraic complexity and tensor rank for tensors with at least four modes. Namely, we observe

$$T_{H_k,n} \equiv \sum_{f \colon [k] \to [n]} \prod_{i \in [k]} x^{(u_i)}_{i \mapsto f(i)} x^{(v_i)}_{i \mapsto f(i)} = \prod_{i \in [k]} \sum_{j \in [n]} x^{(u_i)}_{i \mapsto j} x^{(v_i)}_{i \mapsto j},$$

holds as a polynomial identity, where the last formula shows $\mathrm{C}(T_{H_k,n}) \leq 2kn$, yet we have $\mathrm{R}(T_{H_k,n}) = n^k$. The first formula above gives $\mathrm{R}(T_{H_k,n}) \leq n^k$. The lower bound $\mathrm{R}(T_{H_k,n}) \geq n^k$ holds since tensor rank is lower-bounded by the matrix rank of any matrix flattening of the tensor. For instance, the $n^k \times n^k$ matrix flattening of $T_{H_k,n}$ defined by the modes $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ is a permutation of the $n^k \times n^k$ identity matrix.

Several useful connections between algebraic properties of $T_{G,n}$ and graph-theoretic properties of G can be shown. Crucially for us, the Kronecker product of graph tensors of graphs G and H is the graph tensor of the *sum graph* $G + H$. (Recall that the sum may introduce multiedges.) The elementary proof is given in the full version.

▶ **Lemma 12** (Product of graph tensors corresponds to graph sum). *Let G and H be graphs and let $n \in \mathbb{N}$. Then, $T_{G,n} \otimes T_{H,n} \equiv T_{G+H,n}$.*

▶ Remark 13 (Single-edge decomposition). Christandl and Zuiddam [15] used a special case of Lemma 12 to *define* $T_{G,n}$. Specifically, let $G = (V, E)$ with $E = \{e_1, \ldots, e_m\}$, and let $F_i = (V, \{e_i\})$ for $i \in [m]$ be the graph on vertex set V that contains only e_i as edge. Then $G = F_1 + \ldots + F_m$ and

$$T_{G,n} \equiv T_{F_1,n} \otimes \ldots \otimes T_{F_m,n}. \tag{5}$$

Lemma 12, which was not stated in [15], then follows by commutativity of $\otimes$ up to relabeling modes. The definition as a sum of single-edge graphs is more elegant, but the monomial expansion of $T_{G,n}$ in (4) directly connects to algebraic complexity, as witnessed in Example 11.

▶ Remark 14 (Length rule and sum rule). With the same proof as in Lemma 12, for $n_1, n_2 \in \mathbb{N}$ we have the length rule $T_{G,n_1} \otimes T_{G,n_2} \equiv T_{G,n_1 \cdot n_2}$. Indeed, the set of pairs of mappings (f_1, f_2) with $f_i : E(G) \to [n_i]$ is in bijective correspondence with the set of mappings $E(G) \to [n_1 \cdot n_2]$. Consequently, writing $2 \cdot G = G + G$ for any graph G, and more generally $k \cdot G = G + \cdots + G$ for the k-fold sum for $k > 2$, we have the sum rule $T_{k \cdot G,n} \equiv T_{G,n^k}$.

The following lemma is similarly elementary.

▶ **Lemma 15** (Projection under topological subgraphs). *Let G and H be graphs and let $n \in \mathbb{N}$. If H is isomorphic to a topological subgraph of G, then $T_{H,n} \leq_{\mathrm{s}} T_{G,n}$.*

Moreover, we will use *contractions* of graphs: Given a set $U \subseteq V(G)$, the graph G/U is obtained by replacing U by a single vertex w incident with all edges *leaving* U; this process can create multiedges. Contractions are efficient if they don't involve too many edges:

▶ **Lemma 16** (Complexity of contractions). *For every G and $n \in \mathbb{N}$ and $U \subseteq V(G)$, it holds that $\mathrm{C}(T_{G,n}) \leq \mathrm{C}(T_{G/U,n}) + |U| \cdot n^{a(U)}$, where $a(U) := \sum_{u \in U} \deg_G(u) - |E(G[U])|$ is the number of edges incident with vertices in U.*

A proof is provided in the full version. Contractions for graph tensors have been extensively studied for probabilistic inference in factor graphs and Bayesian networks [2, 21, 31, 34, 38, 41, 42, 43, 52]. Even a contraction-based model of computation was studied [1].

3.2 Asymptotic Rank

We now prove the main result of this section, Theorem 1. In line with Christandl, Vrana and Zuiddam [14], define the *exponent* of a graph G as

$$\omega(G) := \inf \{\beta \mid \mathrm{R}(T_{G,n}) = O(n^\beta)\}.$$

A related quantity is the *exponent per edge* of G, defined as

$$\tau(G) := \omega(G)/|E(G)|.$$

The following useful property can be shown in a standard manner, see the full version.

▶ **Lemma 17** (Sum rule for exponents). *Let G be a graph and let $k \in \mathbb{N}$. Then,*

$$\omega(k \cdot G) = k \cdot \omega(G). \tag{6}$$

To motivate the use of graph tensors in our result, we recall Strassen's upper bound on the maximal asymptotic rank of bilinear maps (i.e., 3-mode tensors) together with a high-level intuition of its proof. This property is often referred to as asymptotic submultiplicativity.

▶ **Theorem 18** (Strassen [45]; Upper bound on asymptotic rank for 3-tensors). *Let $T \in (\mathbb{F}^n)^{\otimes 3}$ be an arbitrary 3-mode tensor. Then, $\underset{\sim}{\mathrm{R}}(T) \leq n^{2\omega/3}$.*

Strassen's proof of Theorem 18 works with the tensor corresponding to the universal bilinear map $U_3 : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n \otimes \mathbb{F}^n,\ (u, v) \mapsto u \otimes v$, which in our notation can be written as

$$U_3 = \sum_{i,j \in [n]} x_i y_j z_{ij}.$$

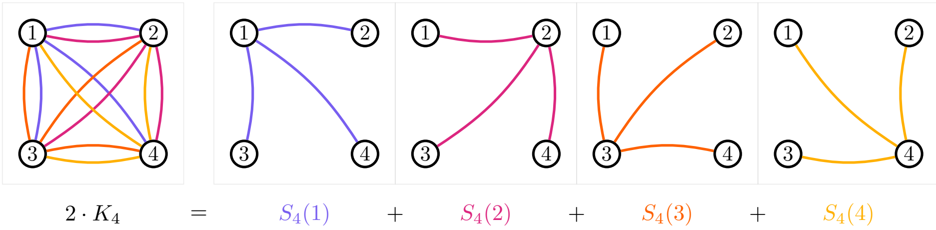


Figure 1 Decomposition of $2 \cdot K_d$ into d stars, (9), for $d = 4$.

In the language of graph tensors, U_3 can be represented as $U_3 \equiv T_{P_2,n}$, where P_2 is the path graph with two edges. This generalizes to $d > 3$ modes as follows. The universal $(d-1)$-linear map $U_d : \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to (\mathbb{F}^n)^{\otimes d-1}$ is the d-mode tensor

$$U_d = \sum_{i_1,\ldots,i_d \in [n]} x^{(1)}_{i_1} \cdots x^{(d-1)}_{i_{d-1}} \cdot x^{(d)}_{i_1,\ldots,i_{d-1}} .$$

We observe that $U_d \equiv T_{S_d,n}$, where S_d is the star graph with $d-1$ edges, and $S_3 \equiv P_2$ in particular. This enables us to consider universality under Kronecker powers combinatorially via graph sums and graph decompositions (cf. Lemma 12) and thus arrive at our main theorem; all our other upper bounds also follow from this observation. To this end, let $d \geq 3$ and write K_d for the complete graph on d vertices. We recall that $\tau(K_d) = \omega(K_d)/\binom{d}{2}$, and write $\tau(d) := \tau(K_d)$ for brevity in what follows.

▶ Remark 19 (The exponents $\tau(d)$ for $d \geq 3$). Recall that we write ω for the exponent of square matrix multiplication. Since $T_{K_3,n}$ is equivalent to the tensor of $n \times n$ matrix multiplication, we have $\tau(3) = \omega/3$. Moreover, $\tau(d) \leq \tau(e)$ whenever $d \geq e$ [14, Proposition 1.31].

The following lemma generalizes Strassen's result through graph tensors.

▶ **Lemma 20** (Generalization of Strassen's upper bound to d modes). *Let $T \in (\mathbb{F}^n)^{\otimes d}$ be an arbitrary d-mode tensor with $d \geq 3$. Then, $\underset{\sim}{\mathrm{R}}(T) \leq n^{(d-1)\tau(d)}$.*

Proof. For $u \in [d]$, let us write $S_d(u)$ for the d-vertex star graph with vertex set $[d]$ such that u is the unique *center* vertex of degree $d-1$. An arbitrary d-mode tensor $T \in (\mathbb{F}^n)^{\otimes d}$ admits representation in coordinates as the polynomial

$$T = \sum_{i_1,\ldots,i_d \in [n]} T_{i_1,\ldots,i_d} x^{(1)}_{i_1} \cdots x^{(d)}_{i_d} = \sum_{i_1,\ldots,i_{d-1} \in [n]} x^{(1)}_{i_1} \cdots x^{(d-1)}_{i_{d-1}} \left(\sum_{i_d \in [n]} T_{i_1,\ldots,i_d} x^{(d)}_{i_d} \right) . \tag{7}$$

Explicitly, the linear substitution

$$x_{i_1,\ldots,i_{d-1}} \mapsto \sum_{i_d \in [n]} T_{i_1,\ldots,i_d} x^{(d)}_{i_d}$$

into the d-tensor $T_{S_d(d),n}$ shows by (7) and symmetry that

$$T \leq T_{S_d(u),n} \quad \text{for all } u \in [d] . \tag{8}$$

We now observe the graph-sum identity (see Figure 1)

$$S_d(1) + \cdots + S_d(d) \equiv 2 \cdot K_d , \tag{9}$$

where we recall that K_d is the complete graph on d vertices and $2 \cdot K_d$ is the graph obtained from K_d by taking two copies of each edge. From (8), Lemma 12, (9), and Remark 14 thus

$$T^{\otimes d} \leq T_{2 \cdot K_d, n} \equiv T_{K_d, n^2}. \tag{10}$$

By definition of $\tau(d)$ and the properties of $\underset{\sim}{\mathrm{R}}$, it follows from (10) that

$$\underset{\sim}{\mathrm{R}}(T) \leq \underset{\sim}{\mathrm{R}}(T_{K_d,n})^{2/d} \leq n^{2\binom{d}{2}\tau(d)/d} = n^{(d-1)\tau(d)}. \qquad \blacktriangleleft$$

For $d = 3$, Lemma 20 replicates Strassen's upper bound from Theorem 18, and generalizes it for $d \geq 4$. Our main result of this section is the following upper bound for $d \geq 4$ that combines Lemma 20 with an improved upper bound for $\tau(d)$.

▶ **Theorem 1** (Upper bound on asymptotic rank). *For every d-mode tensor with $d \geq 4$, the asymptotic rank exponent of T is at most* $0.772318(d-1)$.

Proof. Christandl, Vrana, and Zuiddam [14] showed that $\underset{\sim}{\mathrm{R}}(T_{K_d}) \leq n^{\binom{d}{2}\log_7(9/2)}$ for $d \geq 4$, which implies $\tau(d) < 0.7729$. By a careful analysis and modification of their argument, we obtain the sharper bound $\tau(d) < 0.772318$; see the appendix of the full version. The theorem then follows by Lemma 20. ◀

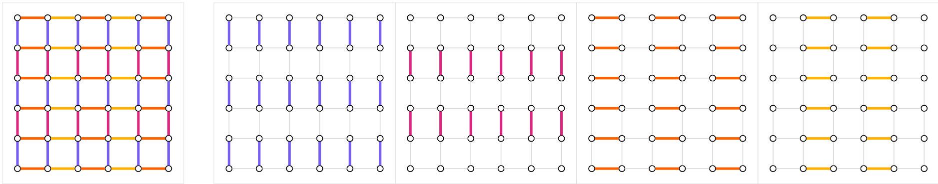
▶ Remark 21 (Bounds on $\tau(d)$ for $d \geq 4$). To obtain the constant 0.772318 in Theorem 1, an upper bound for $\tau(4)$ suffices since $\tau(d) \leq \tau(4)$ for $d \geq 4$. It is natural to ask whether better upper bounds can be obtained by finding better upper bounds on $\tau(d)$ for $d \geq 4$. The best-known lower bounds on $\tau(d)$ allow for a scenario in which $\tau(d) \leq 1/2 + o(1)$, implying $\underset{\sim}{\mathrm{R}}(T) \leq n^{(d-1)/2+o(d)}$ would follow for d-mode tensors. On the other hand, the current best-known upper bound on ω yields an exponent greater than the exponent $(d-1) \cdot 0.772318$ in Theorem 1, while $\omega = 2$ would imply $\tau(3) = \tau(4) = 2/3$. It is an open problem whether there is a d such that $\tau(d) < 2/3$. Recalling from Example 11 that the k-matching tensor $T_{H_k,n}$ has $d = 2k$ modes and its rank $\mathrm{R}(T_{H_k,n}) = n^k$ equals a matrix-flattening-rank lower bound, this lower bound applies also to asymptotic rank and thus $\underset{\sim}{\mathrm{R}}(T_{H_k,n}) = n^k$, implying $\tau(d) \geq 1/2$ by Theorem 1 for all $d \geq 4$.

4 Asymptotic Circuit Complexity

This section proceeds to study tensors with $d \geq 4$ modes from the standpoint of the asymptotic circuit complexity of a tensor viewed as a set-multilinear polynomial. Recalling Example 11 and our discussion in the introduction, for $d \geq 4$ the asymptotic circuit complexity and the asymptotic tensor rank are no longer tightly connected as is the case for $d = 3$. Yet tensor families of interest from the standpoint of algorithms and complexity (e.g. matrix permanent, hypercliques, general convolutions, iterated matrix multiplication, ...) can be captured as Kronecker powers of individual base tensors with $d \geq 4$ modes, making the asymptotic circuit complexity of these constant-size base tensors worth studying. Graph tensors provide a convenient tool for this study.

4.1 Complexity of Kronecker Powers

We ask whether tensors $U, T \in (\mathbb{F}^n)^{\otimes d}$ of low circuit complexity also have Kronecker products $U \otimes T$ of low circuit complexity. Recalling that tensor rank is submultiplicative for tensors of all orders, a natural first hope promoted by the $d = 3$ connection of circuit complexity to tensor rank would be to establish the *submultiplicativity* property $\mathrm{C}(T \otimes U) \leq \mathrm{C}(T) \cdot \mathrm{C}(U)$ of the complexity measure C for all orders $d \in \mathbb{N}$. However, already at $d = 4$ submultiplicativity would have a breakthrough consequence.



$$G = M_1 + M_2 + M_3 + M_4$$

$$T_{G,N} = T_{M_1,N} \otimes T_{M_2,N} \otimes T_{M_3,N} \otimes T_{M_4,N}$$

Figure 2 The grid G is the sum of four matchings, so Lemma 12 shows that its graph tensor $T_{G,N}$ is the Kronecker product of four graph tensors of matchings. But since grids are hard by Lemma 25 and matchings are easy by Example 11, we do not expect complexity to be submultiplicative.

The proofs in this section rely on the graph tensors $T_{M_k,n}$ of matchings M_k, for $k, n \in \mathbb{N}$, which are explicit 0-1 tensors. Recall their definition in Example 11, where it is also shown that the circuit complexity of $T_{M_k,n}$ is $O(kn)$. In a nutshell, if circuit complexity were submultiplicative, then $T_{G,n}$ would have low complexity for every graph G that can be decomposed into few matchings, which would contradict common beliefs. We make this statement precise in the remainder of the section.

▶ **Theorem 22** (Fast matrix multiplication under submultiplicativity). *Let M, M' be the matching graphs with $V(M) = V(M') = \{v_1, \ldots, v_4\}$ and $E(M) = \{v_1v_2, v_3v_4\}$ and $E(M') = \{v_1v_3, v_2v_4\}$. If*

$$\mathrm{C}(T_{M,n} \otimes T_{M',n}) \leq O(\mathrm{C}(T_{M,n}) \cdot \mathrm{C}(T_{M',n})), \tag{11}$$

then $\omega = 2$.

Proof. The cycle graph C_4 is the sum of the two matchings M and M'. By virtue of Lemma 12 and Example 11, Equation (11) implies $\mathrm{C}(T_{C_4,n}) \leq O(n^2)$. Moreover, $T_{C_3,n}$ is a projection of $T_{C_4,n} \equiv \sum_{i,j,k,\ell} x_{ij}y_{jk}z_{k\ell}w_{\ell i}$ via $z_{k\ell} \mapsto \delta_{k\ell}$, see Lemma 15. Hence, $\mathrm{C}(T_{C_3,n}) \leq O(n^2)$ and $\omega = \omega(C_3) = 2$. ◀

The remainder of this subsection strengthens the case for the absence of submultiplicativity, illustrating the serendipity of graph tensors as a tool when working with more modes.

We proceed to reduce matrix permanent tensors to graph tensors of square grid graphs G. Since grids have large treewidth, related results [22, 18] suggest their graph tensors have large algebraic complexity. Indeed, we show in Lemma 25 that the $d \times d$ matrix permanent is a projection of the graph tensors of $(d+2) \times (d+2)$ grids, even for mode dimension 2. On the other hand, as shown in Figure 2, every grid G can be written as the sum

$$G \equiv M_1 + M_2 + M_3 + M_4 \tag{12}$$

of edge-disjoint matching graphs M_1, M_2, M_3, M_4. Hence, Lemma 12 implies for $N \in \mathbb{N}$ that

$$T_{G,N} \equiv T_{M_1,N} \otimes T_{M_2,N} \otimes T_{M_3,N} \otimes T_{M_4,N}. \tag{13}$$

Graph tensors of matchings have low complexity by Example 11. So, if C were submultiplicative, then $T_{G,N}$ would also have low complexity. More generally, this holds for every graph G of small edge-chromatic number, i.e., whenever G can be partitioned into few matchings:

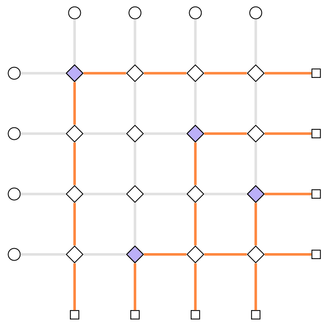


Figure 3 The reduction from the permanent to the graph tensors of grids in Lemma 25. The gray and orange edges are assigned 0 and 1, respectively. Each horizontal and each vertical path flips from 0 to 1 exactly once; whenever this happens at some vertex $v_{i,j}$ (a *flip vertex*, shown purple), we ensure that both the i-th horizontal and the j-th vertical path flip at $v_{i,j}$ (so purple vertices always lie on orange corners). Each flip vertex $v_{i,j}$ contributes $x_{i,j}$ to the weight of the assignment, others 1. The weight of the assignment shown here is $x_{11}x_{23}x_{34}x_{42} = \prod_i x_{i,\pi(i)}$ for $\pi = 1342$.

▶ **Lemma 23** (Graphs with small edge-chromatic number under submultiplicativity). *Let G be a graph such that $E(G)$ can be partitioned into t matchings $M_1, \dots, M_t$ with maximum edge-multiplicity $b \in \mathbb{N}$. Let $N \in \mathbb{N}$. If*

$$\mathrm{C}(T_{M_1,N} \otimes \ldots \otimes T_{M_t,N}) \le O(\mathrm{C}(T_{M_1,N}) \cdot \ldots \cdot \mathrm{C}(T_{M_t,N})), \tag{14}$$

then $\mathrm{C}(T_{G,N}) \le O(2^t |V(G)|^t \cdot N^{tb})$.

Proof. Example 11 and Remark 14 give $\mathrm{C}(T_{M_i,N}) \le 2|V(G)| \cdot N^b$. The assumed submultiplicativity of C, together with Lemma 12 gives $\mathrm{C}(T_{G,N}) \le \prod_{i=1}^t \mathrm{C}(T_{M_i,N}) \le 2^t |V(G)|^t \cdot N^{tb}$, thus proving the claim. ◀

Using (13), we obtain the following corollary for grids.

▶ **Corollary 24.** *Let G denote the $n \times m$ grid, for $n, m \in \mathbb{N}$, with maximum edge-multiplicity $b \in \mathbb{N}$. If Equation (14) holds for $t = 4$ matchings, each on $|V(M_1)| = \ldots |V(M_4)| = nm$ vertices, then* $\mathrm{C}(T_{G,N}) \le O(n^4 m^4 \cdot N^{4b})$.

Our results are obtained by invoking this corollary for various choices of grids G and dimensions N. We begin under the comparatively weak assumption $\mathsf{VP} \neq \mathsf{VNP}$ [47]. Towards this end, we first show how the permanent reduces to grid graph tensors.

▶ **Lemma 25** (The permanent tensor reduces to a grid-graph tensor). *The $n \times n$ permanent tensor* per_n *is a projection of $T_{G,2}$ for the simple $(n+2) \times (n+2)$ grid graph G.*

Proof sketch. The reduction is illustrated in Figure 3: G is obtained by attaching pendant vertices of degree 1 to the boundary of an $n \times n$ grid; the pendant vertices are needed to ensure boundary conditions. Each edge of the resulting graph carries a Boolean state, and we substitute values for the indeterminates of $T_{G,2}$ such that the resulting polynomial counts assignments a to the edges of the grid with the following properties:

1. We ensure that each horizontal and each vertical path P starts with a 0-edge and the edge states flip from 0 to 1 at exactly one vertex; we call this the *flip vertex* of P under a.
2. For all $i, j \in [n]$, we ensure that a vertex $v_{i,j}$ is the flip vertex of the i-th horizontal path if and only if the same vertex $v_{i,j}$ also is the flip vertex of the j-th vertical path.
3. Finally, we ensure that a flip vertex contributes a factor $x_{i,j}$ to the weight of the assignment, while non-flip vertices do not contribute (i.e., they contribute 1).

These conditions can be realized by a 4-leg graph tensor (signature): if we denote by t, r, b, ℓ the top, right, bottom and left input of the signature, then its weight is the corresponding entry in the matrix if $t = \ell = 0, r = b = 1$, is 1 if $t = b, r = \ell$, and 0 otherwise. If these conditions hold, the flip vertices under an assignment a induce an $n \times n$ permutation matrix P_π, and the weight of a is the product of the indeterminates $x_{i,\pi(i)}$ selected by P_π. Summing over valid assignments a gives per_n. ◀

▶ **Theorem 2** (Submultiplicativity of C implies VP = VNP). *There are explicit tensors $T_1, T_2, \ldots$ and $U_1, U_2, \ldots$, where T_d, U_d for $d \in \mathbb{N}$ are d-mode 0-1-tensors, such that the following holds: If $\mathrm{C}(T_d \otimes U_d) \leq \mathrm{poly}(d, \mathrm{C}(T_d), \mathrm{C}(U_d))$ for all $d \in \mathbb{N}$, then* VP = VNP*.*

Proof. Let $d \in \mathbb{N}$ be a perfect square, otherwise use padding. Let G be the $\sqrt{d} \times \sqrt{d}$ grid and $M_1, \ldots, M_4$ be its decomposition into matchings. Then

$$T_{G,2} \equiv \underbrace{T_{M_1,2} \otimes T_{M_2,2}}_{=:T_d} \otimes \underbrace{T_{M_3,2} \otimes T_{M_4,2}}_{=:U_d}.$$

Note that T_d and U_d are graph tensors, each corresponding to a disjoint union of paths.

Under the assumption of the theorem, Corollary 24 and Lemma 25 would imply $\mathrm{C}(T_{G,2}) \leq \mathrm{poly}(d)$, which would imply VP = VNP due to the VNP-hardness of the permanent. ◀

▶ Remark 26. As pointed out to us by an anonymous referee, a weaker version of the preceding Theorem 2 can be obtained immediately by considering the so-called Hadamard product $\odot$ of tensors. This is simply the point-wise product of monomial coefficients and can be obtained as a simple substitution of the Kronecker product. In particular, $\mathrm{per}_n = \det_n \odot \det_n$. We note that the appearing tensors have d modes of dimension d, whereas Theorem 2 applies already to tensors with modes of constant dimension.

If we assume that permanents require exponential-size circuits, then we can obtain stronger bounds. Towards this, we first contract large grids into smaller grids with larger edge-multiplicities. (Larger edge-multiplicities correspond to larger mode dimensions: For every graph H and $b, k, n \in \mathbb{N}$ with k dividing n, and writing $B = b^{n/k}$, we have $T_{H,B} \equiv T_{n/k \cdot H, b}$.) Namely, first partition the $n \times n$ grid into $k \times k$ pieces of size $n/k \times n/k$ each, then contract each piece to a single vertex by Lemma 16. The complexity bound in Lemma 16 requires a little care in the choice of contraction sequence to obtain the desired lower bounds later.

▶ **Lemma 27** (Grid graph tensor contraction by subgrids). *Let k divide n and let H and G be the $k \times k$ and $n \times n$ grid graphs, respectively. Let $b \in \mathbb{N}$ and $B = b^{n/k}$. Then,* $\mathrm{C}(T_{G,b}) \leq (2b^4 + b)n^2 \cdot b^{3(n/k)} + \mathrm{C}(T_{H,B})$*.*

Proof. Partition the vertex set of G into $k \times k$ square blocks $A_{i,j}$, each of size $n/k \times n/k$. For each $i, j \in [k]$, we then perform the following:

1. For $t = 1, \ldots, n/k$, contract the vertices in row t of block $A_{i,j}$ by Lemma 16. Each contraction involves $3n/k + 1$ edges and thus incurs an additive cost term of $bn/k \cdot b^{3(n/k)}$ in Lemma 16. Block $A_{i,j}$ now consists of a vertical path of length n/k, with edge-multiplicity n/k between consecutive vertices, and with two additional edges per vertex. The total cost of the contractions in this step is $b(n/k)^2 \cdot b^{3(n/k)}$.
2. For $t = 1, \ldots, n/k - 1$, we contract vertices t and $t + 1$ on the path remaining in block $A_{i,j}$. This involves $3n/k + 4$ edges per contraction and incurs a cost term of $2b^4 \cdot b^{3(n/k)}$ in Lemma 16. Block $A_{i,j}$ now consists of a single vertex incident with all edges leaving $A_{i,j}$. The total cost of the contractions in this step is $2b^4(n/k) \cdot b^{3(n/k)}$.

In total over the k^2 cells, we incur cost at most $(2b^4 + b)k^2(n/k)^2 \cdot b^{3(n/k)} = (2b^4 + b)n^2 \cdot b^{3(n/k)}$ for all contractions and obtain the $k \times k$ grid with edge-multiplicity n/k from G. ◀

A lower bound now follows easily.

▶ **Theorem 3** (Submultiplicativity of C implies faster permanents). *For all $0 < c < 1$, there are explicit $\lceil 4/c \rceil^2$-mode 0-1-tensors $T_1, T_2, \dots$ and $U_1, U_2, \dots$ such that the following holds: If $\mathrm{C}(T_n \otimes U_n) \leq O(\mathrm{C}(T_n) \cdot \mathrm{C}(U_n))$, then permanents have circuits of size $2^{cn} n^{O(1)}$.*

Proof. Choose $k = \lceil 4/c \rceil$ so $4/k \leq c$, and let H be the $k \times k$ grid on k^2 vertices. By Remark 14 and Lemma 27 with $b = 2$, if the k^2-mode tensor $T_{n/k \cdot H,2}$ admits a circuit of size s, then the n^2-mode tensor $T_{G,2}$ for the $n \times n$ grid G admits a circuit of size $s' \leq n^{O(1)} \cdot 8^{n/k} + s$, and Lemma 25 gives a circuit of size $s' \cdot n^{O(1)}$ for per_n.

If submultiplicativity indeed held, then Corollary 24 would give a circuit of size $s \leq n^{O(1)} \cdot 2^{4n/k}$ for $T_{n/k \cdot H,2}$, so we obtain a circuit of size $s' \leq n^{O(1)} \cdot 2^{4n/k} \leq 2^{cn} n^{O(1)}$ for $T_{G,2}$ and the permanent. ◀

The above strategy for conditional arguments can be refined in different ways to rule out submultiplicativity on fewer modes, e.g., by choosing graph tensors other than grids. For instance, we show:

▶ **Theorem 4** (Submultiplicativity of C implies faster hypercliques). *For every even $d \geq 8$, there are explicit d-mode 0-1-tensors $T_1, T_2, \dots$ and $U_1, U_2, \dots$ such that: If $\mathrm{C}(T_n \otimes U_n) \leq O(\mathrm{C}(T_n) \cdot \mathrm{C}(U_n))$, then the non-uniform $(\frac{d}{2} - 1, \frac{d}{2})$-hyperclique conjecture fails.*

4.2 Upper Bounds via Line Graph Treewidth

The *line-treewidth* $\mathrm{ltw}(G)$ is the treewidth $\mathrm{tw}(L(G))$ of the *line graph* $L(G)$ of G, where the vertex set of $L(G)$ is $E(G)$ and two vertices corresponding to edges e, e' are adjacent in $L(G)$ if they share a vertex in G. Of special interest is the treewidth of the line graph of the complete graph, which has been determined exactly.

▶ **Theorem 28** (Line-treewidth of a complete graph [25, Theorem 1]). *For $d \geq 1$, we have*

$$\mathrm{ltw}(K_d) = \begin{cases} \left(\frac{d-1}{2}\right)^2 + d - 2, & \text{if } d \text{ is odd,} \\ \left(\frac{d-2}{2}\right)\left(\frac{d}{2}\right) + d - 2, & \text{if } d \text{ is even.} \end{cases}$$

Similar to standard dynamic programming arguments on tree-decompositions in, e.g., [51, Section 5.1], [20, Lemma 5.23], and [23, Example 3.4], we can show:

▶ **Lemma 29.** *For every H and $n \in \mathbb{N}$, we have $\mathrm{C}(T_{H,n}) \leq |V(H)| \cdot n^{\mathrm{ltw}(H)+1}$.*

As an immediate consequence the above lemma proves $\underset{\sim}{\mathrm{C}}(T_{H,n}) \leq n^{\mathrm{ltw}(H)+1}$ for every graph H. For general tensors, recall that we write $\eta(T)$ for the exponent of the asymptotic circuit complexity $\underset{\sim}{\mathrm{C}}(T) = n^{\eta(T)}$ of a tensor $T \in (\mathbb{F}^n)^{\otimes d}$. We then have the following implication, which gives a proof of Theorem 6 in a stronger version:

▶ **Theorem 30** (Upper bound on the asymptotic circuit complexity of a d-tensor). *Let $T \in (\mathbb{F}^n)^{\otimes d}$ be a d-mode tensor of dimension n. Then, we have*

$$\eta(T) \leq \frac{2}{d}(\mathrm{ltw}(K_d) + 1) = d/2 + 1 - \frac{7 + (-1)^d}{4d}\,.$$

The proofs of Theorem 30 and Lemma 29 are given in the full version.

▶ Remark 31. Our proof of Theorem 30 in fact constructs a monotone circuit, and so does the circuit constructed in Lemma 29, as it implements a standard dynamic programming approach through tensor contractions.

▶ Remark 32. It is natural to ask whether the exponent from Theorem 30 is asymptotically tight under standard assumptions from complexity theory. To the best of our knowledge, current techniques would only rule out exponents of the form $o(d/\log d)$.

For instance, known results on the complexity of subgraph isomorphism [36, 27, 17] imply $\eta(T_{H,n}) \in \Omega(d/\log d)$ for a sequence of d-vertex graphs H of maximum degree 3. It is also possible to adapt a recent bound by Pratt [40] to a higher number of modes, which would yield a lower bound under the set cover conjecture. Since all of the resulting lower bounds are fairly loose, we refrain from formally stating and proving them here.

5 Restricted Submultiplicativity

The results in Section 4.1 strongly suggest the absence of circuit complexity submultiplicativity for d-mode tensors with $d \geq 4$: For d-mode tensors T and U of low complexity, no useful upper bounds for the circuit complexity $\mathrm{C}(T \otimes U)$ as a function of $\mathrm{C}(T)$ and $\mathrm{C}(U)$ are known. However, we can construct small circuits for $T \otimes U$ from small circuits for T and U, provided that one of the circuits is sufficiently simple.

To this end, we will need the following property of tensor contraction. Let $\ell_1, \dots, \ell_d \colon \mathbb{F}^m \to \mathbb{F}$ and $h_1, \dots, h_d \colon \mathbb{F}^n \to \mathbb{F}$ be linear forms. We consider the Kronecker product of the corresponding rank-one tensors $L = \prod_{i=1}^d \ell_i(x^{(i)})$ and $H = \prod_{i=1}^d h_i(y^{(i)})$, where the i-th mode of $L \otimes H$ has coordinates $z_{jk}^{(i)} := x_j^{(i)} \otimes y_k^{(i)}$, with $j \in [m], k \in [n]$. For brevity, we write $z^{(i)}$ for the $m \times n$ matrix with entries $z_{jk}^{(i)}$, and identify h_i with its vector of coefficients in $\mathbb{F}^n$. Then, one can check that

$$L \otimes H = \prod_{i=1}^{d} \ell_i(h_i(z_{11}^{(i)}, \dots, z_{1n}^{(i)}), \dots, h_i(z_{m1}^{(i)}, \dots, z_{mn}^{(i)})) = L(z^{(1)} \cdot h_1, \dots, z^{(d)} \cdot h_d).$$

By linearity and the fact that every tensor U can be decomposed into rank-1 tensors,

$$U \otimes H = U(z^{(1)} \cdot h_1, \dots, z^{(d)} \cdot h_d) \tag{15}$$

holds for all tensors U with d modes in coordinates $z_{jk}^{(i)}$. We can use this to show:

▶ **Lemma 33.** *Let $T \in (\mathbb{F}^n)^{\otimes d}$ and $U \in (\mathbb{F}^m)^{\otimes d}$ with T of rank at most r, with $T = \sum_{i=1}^r \prod_{j=1}^d \ell_{ij}(x^{(j)})$. For $1 \leq j \leq d$, let $\Lambda^{(j)}$ be the $n \times r$ matrix of linear forms $(\ell_{ij})_i$ appearing in the j-th mode of the rank-decomposition of T, and we recall that $z^{(i)}$ is an $m \times n$ matrix of indeterminates $z_{j,k}^{(i)}$ for $1 \leq i \leq d$. Then,*

$$\mathrm{C}(U \otimes T) \leq r(\mathrm{C}(U) + 1) + \sum_{i=1}^{d} \mathrm{C}(z^{(i)} \cdot \Lambda^{(i)}) + r \cdot m \cdot d.$$

Proof. Distributing (15) linearly across the terms of the assumed rank-r decomposition of T, we see that

$$U \otimes T = \sum_{i=1}^{r} \left(U \otimes \prod_{j=1}^{d} \ell_{ij}(x^{(j)}) \right) = \sum_{i=1}^{r} U(z^{(1)} \cdot \ell_{i,1}, \dots, z^{(d)} \cdot \ell_{i,d}). \tag{16}$$

On the right-hand side of (16), the j-th mode of the instance of U appearing in the i-th term in the sum obtains as an input the i-th column of $z^{(j)} \cdot \Lambda^{(j)}$. This is a column vector of dimension m. To implement (16) as a circuit, place a fan-in r +-gate at the top, contributing

r wires. The i-th input of this topmost $+$-gate is a copy of the circuit for U, contributing $r \cdot \mathrm{C}(U)$ to the overall size. In addition, we need to compute the inputs to the individual r copies of U (which becomes crucial in the asymptotic regime.) As mentioned, for this it suffices to compute the d matrix products $z^{(1)} \cdot \Lambda^{(1)}, \ldots, z^{(d)} \cdot \Lambda^{(d)}$. Finally, we wire each of the entries of $z^{(j)} \cdot \Lambda^{(j)}$ to the respective input of U, amounting to a total of $r \cdot m \cdot d$ wires. ◀

For our applications, we need a variant of Lemma 33 for higher Kronecker powers of T and U. It can be proven by applying Lemma 33 and combining it with Yates's algorithm.

▶ **Theorem 5** (Mixed asymptotic rank and circuit complexity). *Given d-mode tensors T and U, where T has asymptotic rank exponent $\psi \geq 1$ and U has asymptotic circuit complexity exponent $\eta \geq 1$, the asymptotic circuit complexity exponent of $T \otimes U$ is at most $\eta + \psi$.*

The preceding Theorem 5 provides a useful tool to study more general graph decompositions to obtain asymptotic circuit complexity upper bounds. This can be done by generalizing the graph tensor framework to fractional graphs, and then performing computer-aided search for fractional decompositions under relaxed constraints. The details of this approach can be found in the full version. As a result, these techniques allow us to obtain Theorem 7.

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