Rank Bounds and Polynomial-Time PIT for $\Sigma^k\Pi\Sigma\Pi^2$ Circuits

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Abstract

A depth-4 algebraic circuit with top fan-in k and bottom fan-in 2 is a circuit Φ of the form $\Phi = \sum_{i=1}^{k} \prod_{j=1}^{m_i} Q_{ij}$, where the polynomials $Q_{ij} \in \mathbb{K}[x_1, \ldots, x_n]$ have degree at most 2. The class of all such circuits is denoted by $\Sigma^k\Pi\Sigma\Pi^2$. We say that the circuit Φ is an identity if it formally computes the zero polynomial. An important parameter of $\Sigma^k\Pi\Sigma\Pi^2$ circuits Φ is their (linear) rank, which is defined as the vector space dimension of the polynomials $\{Q_{ij}\}_{i\in[k],j\in[m_i]}$.

We prove that, when the base field $\mathbb{K}$ is of characteristic zero, the rank of any (simple and minimal) $\Sigma^k\Pi\Sigma\Pi^2$ identity is upper bounded by a function which depends only on the top fan-in k. This result makes progress on [11, Conjecture 28], being the first work to establish a bound on the rank of such identities that depends only on the top fan-in. Moreover, when combined with [11, Theorem 2], our main result yields the first deterministic, *polynomial time* PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^2$ circuits.

One of the key components of our proof of the rank bounds is the derivation of an approximate Hansen-type result, which is interesting in its own right. This result can be seen as an algebraic and higher-dimensional analogue of the approximate Sylvester-Gallai result of [5], and a distinct approximate fractional Sylvester-Gallai result than the one from [27]. Additionally, we prove a robust version of it, in the spirit of the generalization of Hansen's theorem by [9].

This paper is an extended abstract of the full version of the paper, which can be found at [28].

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1 Introduction

Polynomial Identity Testing (PIT) is the problem of checking whether a given algebraic circuit over a field $\mathbb{K}$ computes the zero polynomial. When the base field $\mathbb{K}$ is sufficiently large, the PIT problem is equivalent to asking whether the given algebraic circuit evaluates to zero on all inputs. This equivalent formulation suggests a very simple randomized algorithm: simply evaluate the circuit at a randomly chosen point. The correctness of this algorithm is given by the polynomial identity lemma.

Derandomizing PIT is a foundational problem in theoretical computer science: it is intrinsically related to proving lower bounds for algebraic circuits [39, 41, 2, 49, 50] and the derandomization of PIT even for restricted classes of circuits has resulted in the derandomization of important problems in mathematics and computer science [3, 24, 53, 22, 37, 68]. For an overview on PIT and its connections, we refer the reader to [61, 67, 62, 16].

Given the difficulty of derandomizing the (full) PIT problem, much of the focus has shifted to natural subclasses of circuits, such as sparse polynomials (equivalently, depth-2 circuits) [12, 32, 47] and depth-3 circuits with bounded top fan-in [19, 45, 44, 42, 63, 64]. Recent advances in depth reduction [1, 48, 70, 36] have shown that a deterministic PIT algorithm for unrestricted depth-3 circuits or homogeneous depth-4 circuits would yield a quasipolynomial-time deterministic PIT algorithm for general circuits. These insights have renewed interest in these classes and spurred substantial research activity on PIT within these settings, as seen in [8, 43, 60, 11, 56, 23, 4, 33, 15, 58, 31, 34] and references therein.

The work [19], which first studied the PIT problem for the class of depth-3 circuits of bounded top fan-in (denoted $\Sigma^k\Pi\Sigma$), defined the *rank* of a depth-3 circuit as the dimension of the vector space of the linear forms appearing in the bottom layer of the circuit. Their high-level strategy for obtaining a deterministic, polynomial-time PIT algorithm for this class was as follows. Suppose there exists a function $R : \mathbb{N} \to \mathbb{N}$ such that any "non-trivial" (simple and minimal) $\Sigma^k\Pi\Sigma$ identity has rank less than $R(k)$. If one can devise a variable reduction procedure that maps the variables into an $R(k)$-dimensional space while preserving the rank of any $R(k)$ linear forms, then composing this map with the circuit yields a nonzero circuit in only $R(k)$ variables. A simple interpolation on this reduced circuit suffices to verify nonzeroness.

While [19] was not able to prove the existence of such a function R, they managed to prove a rank bound which depended both on the top fan-in k and on the logarithm of the degree of the circuit. This bound, combined with the variable reduction procedure was enough to yield a quasi-polynomial time PIT algorithm for $\Sigma^k\Pi\Sigma$ circuits. In [19] the above variable reduction strategy was achieved in the *white-box* setting, where the circuit is given explicitly as input and one can inspect the linear functions appearing in it. Later, building on [25], Karnin and Shpilka obtained a corresponding variable reduction procedure in the *black-box* setting, where the algorithm only has query access to the circuit [42].

The work [19] also highlighted a connection between rank bounds for depth-3 identities and discrete geometry, specifically to colored versions of the classical Sylvester-Gallai theorem. In particular, [19] suggested that insights from discrete geometry could lead to improved rank bounds. This direction was effectively pursued by Kayal and Saraf [44], who observed that one of the conjectures posed in [19] corresponds to the Edelstein-Kelly theorem [20]. By applying higher dimensional incidence theorems, [44] obtained the first rank bounds which depended only on the top fan-in k. Subsequent research further improved the upper bound on $R(k)$ culminating in $R(k) = 3k^2$ over fields of characteristic zero [64].

Motivated by the success of rank-bounds for depth-3 circuits, Beecken, Mittmann and Saxena [11] conjectured that if one defines an analogous notion of rank for depth-4 circuits, replacing linear dimension with algebraic rank, then "nontrivial" depth-4 identities should have small rank. Motivated by the connection between depth-3 identities and questions in incidence geometry, Gupta suggested far-reaching generalizations of the Sylvester-Gallai theorem and its colored and high-dimensional variants [35]. These conjectures suggest a program of generalizing incidence geometry from studying intersections of lines, hyperplanes or curves, to studying intersections between zero sets of low-degree polynomials (or more generally sparse polynomials).

The first progress on these conjectures was made by Shpilka [66], who established an analogue of the Sylvester-Gallai theorem for quadratic polynomials. Subsequent work gradually confirmed several other conjectures of Gupta [57, 58, 59, 54, 29, 27, 55, 31, 30]. The results of [58, 31], combined with [11, Theorem 2], immediately yield black-box PIT algorithms for $\Sigma^3\Pi\Sigma\Pi^d$ circuits, for $d = 2$ and more generally for any $d = O(1)$, respectively.

In this work we obtain the first polynomial time black-box PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^2$ circuits, for any constant top fan-in k. We achieve this by combining the conceptual approach of [44] with the techniques developed in [55]. This allows us to prove an upper bound on the rank of $\Sigma^k\Pi\Sigma\Pi^2$ identities. Before stating our main results we discuss the relation between PIT and (non-linear) incidence geometry.

1.1 Edelstein-Kelly Theorem and Small Depth Identities

The Sylvester-Gallai theorem, posed independently by Sylvester [69] and Erdös [21], and solved independently by Melchior [52] and Gallai [26], asserts that if a finite set of points in $\mathbb{R}^n$ has the property that every line passing through any two points in the set also contains a third point from the set, then all the points in the set are collinear. In [46], Kelly observed that a result of Hirzebruch [40] proves that over $\mathbb{C}^n$ the points must lie on a 2-dimensional affine space, thereby resolving a question of Serre [65]. Many variants of this theorem were studied: extensions to higher dimensions, colored versions, robust versions and many more. For a survey on the Sylvester-Gallai theorem and its variants see [13]. Of particular relevance to PIT is the Edelstein-Kelly theorem [20], a colored variant of the Sylvester-Gallai theorem. We begin by defining the notion of an Edgelstein-Kelly configuration.

▶ **Definition 1** (Edelstein-Kelly configurations). *Let $\mathbb{K}$ be a field, $\mathcal{A} \coloneqq \{u_1, \ldots, u_a\}$, $\mathcal{B} \coloneqq \{v_1, \ldots, v_b\}$ and $\mathcal{C} \coloneqq \{w_1, \ldots, w_c\}$ be pairwise disjoint subsets of $\mathbb{P}(\mathbb{K}^N)$. We say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ forms a* linear Edelstein-Kelly configuration *if any pair of points taken from distinct sets is collinear with a point from the third set. The* rank *of an Edelstein-Kelly configuration $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is given by* $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}\}$.

The Edelstein-Kelly theorem asserts that, for $\mathbb{K} = \mathbb{R}$, the rank of any linear Edelstein-Kelly configuration is at most 3. Over $\mathbb{C}$, the fractional Sylvester-Gallai theorems of [10, 18, 17] imply that the dimension is bounded by some absolute constant.

Consider now the simplest nontrivial $\Sigma^3\Pi\Sigma$ identity over $S \coloneqq \mathbb{K}[\mathbf{x}]$:

$$\prod_i \ell_{1,i}(\mathbf{x}) + \prod_j \ell_{2,j}(\mathbf{x}) + \prod_k \ell_{3,k}(\mathbf{x}) \equiv 0\,, \tag{1}$$

where the $\ell_{a,b}$ are linear forms. By factoring out the common gcd we may assume that the products in (1) are pairwise prime. Indeed, if a form divides two terms then it must divide the third term as well. The *rank* of the identity is defined as $\dim \operatorname{span}_{\mathbb{K}}\{\ell_{a,b}\}_{a,b}$. Since

the three terms sum to zero, if we pick any two linear forms from distinct terms, say $\ell_{1,i}$ and $\ell_{2,j}$, and restrict to the subspace $U = \{\mathbf{a} : \ell_{1,i}(\mathbf{a}) = \ell_{2,j}(\mathbf{a}) = 0\}$, we obtain that also $(\Pi_k \ell_{3,k}(\mathbf{x}))\,|_U \equiv 0$. In other words, if we consider the corresponding zero sets,[1]

$$V(\ell_{1,i}, \ell_{2,j}) \subseteq V\left(\prod_k \ell_{3,k}(\mathbf{x})\right) = \bigcup_k V(\ell_{3,k}(\mathbf{x})). \tag{2}$$

Since $V(\ell_{1,i}, \ell_{2,j})$ is irreducible[2] there exists some $k = k(i,j)$ such that $\ell_{3,k} \in \mathrm{span}_{\mathbb{K}}\{\ell_{1,i}, \ell_{2,j}\}$. Thus, any two linear forms from distinct terms span a linear form from the third term.

To fit this into the setting of Definition 1, identify each linear form $\ell_{a,b}$ with the point $p_{a,b} \in \mathbb{P}(\mathbb{K}^N)$ spanned by its coefficient vector $\mathbf{a}_{a,b}$. Then, $\ell_{3,k} \in \mathrm{span}\{\ell_{1,i}, \ell_{2,j}\}$ if and only if $p_{3,k}$ is collinear with $p_{1,i}$ and $p_{2,j}$. Hence, the sets $\mathcal{A} = \{p_{1,i}\}$, $\mathcal{B} = \{p_{2,j}\}$ and $\mathcal{C} = \{p_{3,k}\}$ satisfy the conditions of Definition 1. The Edelstein-Kelly theorem then implies that, for $\mathbb{K}$ of characteristic zero, all the points $p_{a,b}$ lie on an $O(1)$-dimensional space, and consequently, $\dim_{\mathbb{K}} \mathrm{span}\{\ell_{a,b}\} = O(1)$. In conclusion, this argument proves that any nontrivial (i.e., gcd-free) depth-3 identity depends essentially on only $O(1)$ variables, thereby enabling an efficient PIT algorithm.

Consider now a nontrivial $\Sigma^4\Pi\Sigma$ identity. Henceforth, by nontrivial we mean gcd-free and that any 3 terms are linearly independent (i.e., there is no shorter nontrivial identity):

$$\prod_{i=1}^{d} \ell_{1,i}(\mathbf{x}) + \prod_{i=1}^{d} \ell_{2,i}(\mathbf{x}) + \prod_{i=1}^{d} \ell_{3,i}(\mathbf{x}) + \prod_{i=1}^{d} \ell_{4,i}(\mathbf{x}) \equiv 0\,. \tag{3}$$

The natural attempt to bound the rank is to reduce to the case of three terms as in (1). For this we can consider the restriction to $U = \{\mathbf{a} : \ell_{1,1}(\mathbf{a}) = 0\}$. This restriction cancels out the first term. However, it may now be the case that after restricting to U, the resulting identity

$$\prod_{i=1}^{d} \ell_{2,i}(\mathbf{x})|_U + \prod_{i=1}^{d} \ell_{3,i}(\mathbf{x})|_U + \prod_{i=1}^{d} \ell_{4,i}(\mathbf{x})|_U \equiv 0$$

is no longer nontrivial. This complication becomes more evident as we increase the number of terms, and we will discuss it more in Section 1.3. Thus, moving from 3 terms to 4 already poses nontrivial challenges.

Let us now consider a $\Sigma^3\Pi\Sigma\Pi^d$ identity:

$$\prod_i A_i(\mathbf{x}) + \prod_j B_j(\mathbf{x}) + \prod_k C_k(\mathbf{x}) \equiv 0\,, \tag{4}$$

where each A_i, B_j, C_k is a homogeneous polynomial of degree $\leq d$, and, as before, we assume that (4) is gcd-free. In analogy to (2) we obtain[3]

$$V(A_i, B_j) \subseteq V\left(\prod_k C_k(\mathbf{x})\right). \tag{5}$$

Trying to repeat the previous argument, we immediately encounter a new difficulty. Since A_i and B_j are forms of degree higher than 1, their variety may not be irreducible. Equivalently,

[1] By symmetry, a similar containment holds for any two forms from distinct terms and the third term.
[2] This can be shown directly by simple linear algebra, but we use the more general terminology for later purposes.
[3] By symmetry, a similar containment holds for any two forms from distinct terms and the third term.

the ideal $I = (A_i, B_j) \subset \mathbb{K}[\mathbf{x}]$ is not necessarily prime. As a result, it may no longer be true that there exists $C_k \in \text{span}\{A_i, B_j\}$. Even worse, we are not guaranteed that there is a C_k such that $V(A_i, B_j) \subseteq V(C_k)$.

When $d = 2$, i.e., the forms are quadratic or linear polynomials, Peleg and Shpilka [57, 58] proved that if the ideal generated by two quadratic polynomials is not prime, then some of its associated primes have very simple form, and with that information [58] managed to extend the Edelstein-Kelly theorem to quadratic polynomials. Oliveira and Sengupta [54] generalized this approach to the case of cubic forms by characterizing non-radical ideals generated by two cubic forms. However, when $d > 3$ the situation becomes significantly more complex and there is no such "structure theorem" for ideals generated by two higher degree forms. On the other hand, structure theorems such as the primality conditions in [30], allowed [30] to control the number and structure of "bad events". To use this result inductively, Garg, Oliveira and Sengupta [55, 31] asked more difficult questions, but which are more amenable to induction. Instead of considering dependencies over $S = \mathbb{K}[\mathbf{x}]$, they considered dependencies over quotient rings $R = S/(U)$ where U is a vector space of forms up to degree d, such that R is a unique factorization domain (UFD). In addition, and to capture (and generalize) the technique of projection introduced in [44, 66] and generalized in [55], [31] gave the following definition that greatly generalizes Definition 1

▶ **Definition 2** (High-degree EK configurations). *Let $U \subseteq S$ be a graded, finitely generated vector space such that $R := S/(U)$ is a UFD, and let $z \in R_1$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset R$ be pairwise disjoint finite sets of irreducible forms of degree at most d. We say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a* (d, z, R)–EK configuration *if the following hold:*

1. *$z \notin \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, and the union $\{z\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ consists of pairwise non-associate forms.*
2. *For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have*

$$z \cdot \prod_{C \in \mathcal{C}} C \in \text{rad}(A, B),$$

and similarly for any two forms from any two distinct sets, where $\text{rad}(A, B)$ is the radical of the ideal (A, B) in the quotient ring R.

While this definition is more intricate than Definition 1, it has the important advantage of greater flexibility. In particular, it allows one to add forms to U while remaining within the same framework. Their high level approach was to prove an upper bound on the rank using induction. The case $d = 1$ can be handled as in previous work, even though the setting is more general. Now, if all the forms in the identity Equation 4 are of degree at most $d - 1$, then we are done by induction. Otherwise, they consider two cases. The first is when there is a form $A \in \mathcal{A}$ such that for any other form $B \in \mathcal{B}$ (similarly, for $C \in \mathcal{C}$), $\text{span}\{A, B\}$ does not contain any form of small "strength". That is, no nonzero linear combination of A, B can be written as a sum of a few terms of the form $P_i P_i'$, where $\deg(P_i), \deg(P_i') < d$. [31] prove that in such a case, we can add A to U and by going to $R' = S/(U, A)$, obtain a simpler identity. In the other extreme case, assume that every form in the identity has small strength. Moreover, let us assume that it depends on a few variables. Consider now a forms $A \in \mathcal{A}$. If for many forms $B_j \in \mathcal{B}$, the ideal (A, B_j) is prime, then A spans a form C_k with each of these "good" B_j. If this holds for most $A \in \mathcal{A}$ then we can conclude by a *robust* version of the linear Edelstein-Kelly theorem. If this is not the case, then the following important structural result of [30, 31] is needed: there is a universal upper bound, depending only on the degree d, on the number of irreducible forms P such that (A, P) is not a prime ideal. Thus, modulo A, many forms in $\mathcal{B}$ become reducible, and we've made progress. Furthermore, since A depends on a few variables, we can pass to a smaller polynomial ring in which these forms factor.

While the description above greatly simplifies the approach of [31], it highlights some challenges that arise when trying to go from three terms to more terms and from linear forms to higher degree forms.

1.2 Our results

In this work, we overcome the challenges posed by depth-4 identities of constant top fan-in, at the expense of restricting the bottom fan-in to be bounded by 2.

Before we state our main result, we need to establish standard definitions of "nontrivial identities" for $\Sigma^k\Pi\Sigma\Pi^2$ circuits. As we mentioned in the previous subsection, one way in which an identity $\Phi = T_1 + \cdots + T_k$ can be trivial is if each gate is a multiple of a quadratic (or linear) polynomial. To disallow such triviality, we say that Φ is a simple circuit if $\gcd(T_1, \ldots, T_k) = 1$. Another way to obtain a trivial identity is by adding two smaller identities together. To disallow such triviality, we say that Φ is a minimal circuit if no subset of its gates sum to zero.

We also need to define the rank of a circuit. In this work, the rank of a $\Sigma^k\Pi\Sigma\Pi^2$ circuit $\Phi = T_1 + \cdots + T_k$ is the dimension of the $\mathbb{K}$-span of all the forms appearing in any of the gates T_i. While in [11] the authors use an algebraic notion of rank of a depth-4 circuit, we use a stricter (and simpler to state) notion of rank of a circuit, which in particular upper bounds the notion of rank in [11]. With these definitions in mind, we can state our main theorem.

▶ **Theorem 3.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $S := \mathbb{K}[x_1, \ldots, x_n]$. There is a function $R : \mathbb{N} \to \mathbb{N}$, independent of $\mathbb{K}, n$, such that for any simple and minimal $\Sigma^k\Pi\Sigma\Pi^2$ identity Φ over S, we have* $\operatorname{rank}(\Phi) \leq R(k)$.

The above theorem makes progress on [11, Conjecture 28], being the first work to obtain bounds for circuits with top fan-in larger than 3. Combining the above theorem with [11, Theorem 2], we obtain the following corollary.

▶ **Corollary 4.** *There is a deterministic, polynomial-time black-box PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^2$ circuits.*

Theorem 3 is a corollary of our main technical result [28, Theorem 6.1]. In [28, Section 6] we prove our main technical result and prove Theorem 3 as a corollary. An important ingredient in the proof of Theorem 3 (more accurately, of [28, Theorem 6.1]) is an extension of Hansen's theorem to quadratic polynomials [38], which we prove in [28, Section 4].

Our result both extends Hansen's theorem to the quadratic setting and relaxes its exact linear dependence condition to an approximate one. This can also be viewed as a high-dimensional quadratic analogue of the approximate collinear Sylvester–Gallai theorem of Ai, Dvir, Saraf, and Wigderson [5].

To state the theorem, we first introduce the notion of an (M, b)-strong configuration. The following simplified version captures the main idea, while the precise definition and theorem appear in [28, Section 4].

▶ **Definition 5.** *We say that a finite set of quadratic polynomials $\mathcal{F}$ is an (M, b)-strong span configuration if, for any $b-1$ polynomials $Q_1, \ldots, Q_{b-1} \in \mathcal{F}$, every other $Q \in \mathcal{F}$ satisfies one of the following:*

1. $|\operatorname{span}_{\mathbb{K}}\{Q_1, \ldots, Q_{b-1}, Q\} \cap \mathcal{F}| > b$, *or*
2. *Q is M-close to* $\operatorname{span}_{\mathbb{K}}\{Q_1, \ldots, Q_{b-1}\}$.

Here, M-closeness means that there exists $F \in \operatorname{span}_{\mathbb{K}}\{Q_1, \ldots, Q_{b-1}\}$ such that the quadratic form $Q - F$ has rank less than M.

We now present a simplified version of [28, Theorem 4.6].

▶ **Theorem 6.** *For any (M, b)-strong span configuration $\mathcal{F}$, there exists a subspace V of dimension $O_b(1)$ such that every quadratic form in $\mathcal{F}$ is $O_b(M)$-close to V. In particular, the dimension of V is independent of the number of variables, of $|\mathcal{F}|$, and of M.*

Interestingly, for the proof of Theorem 6 we actually first need to prove a *robust* version of it, in the spirit of the robust version of Hansen's theorem proved in [9]. We think that this result is also interesting on it own. We now state a simplified version of [28, Definition 4.7].

▶ **Definition 7.** *Let $0 < \nu \leq 1$ We say that a finite set of quadratic polynomials $\mathcal{F}$ is a (ν, M, b)-strong span configuration if, for any $b-1$ polynomials $Q_1, \ldots, Q_{b-1} \in \mathcal{F}$, for ν fraction of the remaining $Q \in \mathcal{F}$ the following holds:*

1. $|\mathrm{span}_{\mathbb{K}} \{Q_1, \ldots, Q_{b-1}, Q\} \cap \mathcal{F}| > b$, *or*
2. Q *is M-close to* $\mathrm{span}_{\mathbb{K}} \{Q_1, \ldots, Q_{b-1}\}$.

We now give a simplified version of [28, Theorem 4.10].

▶ **Theorem 8.** *For any (ν, M, b)-strong span configuration $\mathcal{F}$, there exists a subspace V of dimension $O_b(1/\nu)$, such that $O_b(\nu)$ fraction of the forms in $\mathcal{F}$ are M-close to V.*

We now proceed to describe an outline of our proof of Theorem 3.

1.3 Proof outline

As we previously mentioned, our approach generalizes the framework of [44] via the techniques developed in [55, 30]. In this subsection, we present our proof strategy and its implementation, emphasizing the conceptual contributions and the technical challenges that we overcome in developing the inductive framework that underlies our rank bound proof for $\Sigma^k\Pi\Sigma\Pi^2$ identities. To better understand our proof, we briefly outline the inductive approach of [44] to construct their rank upper bound function $R : \mathbb{N} \to \mathbb{N}$ for $\Sigma^k\Pi\Sigma$ circuits.

1.3.1 Overview of the framework in [44]

When $k \leq 2$, only the trivial circuit is a simple and minimal identity, by the unique factorization property in polynomial rings. This gives us a trivial starting upper bound on the rank of such identities. In particular, we can define $R(0) = R(1) = R(2) = 1$.

Now, suppose that we have constructed our function R up to $k-1$, where $k \geq 3$, and let $\Phi = T_1 + \cdots + T_k$ be our simple and minimal $\Sigma^k\Pi\Sigma$ identity, where each $T_i = \prod_{j=1}^{m} \ell_{ij}$ (note that ℓ_{ij}'s may or may not be scalar multiples of one another). We can still use unique factorization of the polynomial ring to quantify "how different two distinct gates are." This motivated the authors to define the "symmetric difference" of two gates as follows: $T_i \Delta T_j$ equals the set of linear forms (up to scalar multiples) which appear in T_i and T_j with different multiplicities (if $\ell \nmid T_i$ then we say that the multiplicity of ℓ in T_i is zero). Note that the size of the symmetric difference is not too important as a parameter, but a better choice of parameter is the dimension of the span of this set, i.e. $\dim \mathrm{span}_{\mathbb{K}} \{T_i \Delta T_j\}$, as this quantifies the "distinct number of variables" that distinguish the two gates. This leads to the definition of *pairwise rank* of Φ: it is defined as $\min_{1 \leq i < j \leq k} \dim \mathrm{span}_{\mathbb{K}} \{T_i \Delta T_j\}$.

If two gates T_i and T_j "are almost the same," that is, $\dim \mathrm{span}_{\mathbb{K}} \{T_i \Delta T_j\}$ is "small enough," then it is possible to "merge" these two gates via a *general quotient* by the linear forms in $T_i \Delta T_j$. This essentially preserves simplicity and minimality of the circuit, and since we are quotienting by linear forms the new circuit is still over a polynomial ring, in which

case we would get an $\Sigma^{k-1}\Pi\Sigma$ identity, for which we know rank bounds by induction. This would allow us to lift the rank bounds. If one believes that such rank bounds ought to exist, then in particular it must be the case that any two gates should "depend on almost the same variables", in which case the above scenario must always happen. The main strategy of [44] is to show that this case always happens, and it takes up most of the work in [44].

Assume we are not in the above case, that is, we have a simple and minimal circuit $\Phi = T_1 + \cdots + T_k$ in which *every* pair of gates T_i, T_j are "far enough apart." We would like to show that Φ is not an identity. Since we have an inductive proof, we assume that we know rank bounds for $\Sigma^{k-1}\Pi\Sigma$ identities. One way to show this is via a *fan-in reduction lemma*: to show the existence of a linear form ℓ which appears in one of the gates of the circuit, such that by "zeroing out" ℓ – in other words, work over the quotient ring $\mathbb{K}[\mathbf{x}]/(\ell)$, which is still a polynomial ring – the gates of our new circuit $\overline{\Phi}$ (which will now have at most $k-1$ gates) are still "far enough apart from one another", which implies that $\operatorname{rank}(\overline{\Phi})$ is too large for it to be an identity.

This linear form ℓ, if exists, is rather special, as it could happen that by taking a quotient (which is not "general"), the new circuit is no longer simple and minimal. One terrible thing that could happen because of failure of simplicity is that the simple part of the new circuit "is missing" many linear forms from the original circuit – this could happen if many forms "go into the GCD" after the quotient. For instance, if all gates but two in our circuit were divisible by ℓ and the remaining two gates are $\prod_{i=1}^m x_i - \prod_{i=1}^m (\ell + x_i)$, when we quotient by ℓ these two gates become an identity with trivial simple part, but our bounds say nothing about the rank of the GCD (which in this case is everything!).

One ingredient in the fan-in reduction lemma of [44] is Hansen's incidence theorem [38], which in essence states that if a set $\mathcal{F}$ of linear forms is such that $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\}$ is large, then it must contain a $(k+1)$-subset of forms $\ell_1, \ldots, \ell_{k+1} \in \mathcal{F}$ such that $|\operatorname{span}_{\mathbb{K}} \{\ell_1, \ldots, \ell_{k+1}\} \cap \mathcal{F}| = k+1$.[4] In other words, these forms do not span any other form in $\mathcal{F}$. With this theorem in mind, [44] proceed as follows: since the pairwise rank of Φ is large enough (as a function of k), one can construct a large enough "core" $\mathcal{C} := \bigcup_{i,j} \mathcal{C}_{ij}$ of linear forms, where each $\mathcal{C}_{ij} \subset T_i \Delta T_j$ has "enough distinct variables" appearing in the symmetric difference. The objective of having this core is that we want to prove that there is a linear form ℓ in the circuit that "does not affect any subcore $\mathcal{C}_{ij}$" if $\ell \nmid T_i \cdot T_j$. If one achieves this, we will be able to quotient by (ℓ), thereby reducing fan-in of Φ, and preserving "too many variables" in the symmetric differences, which implies that the new circuit $\overline{\Phi}$ will have too large of a rank, in which case it cannot be an identity.

The above summarizes the core ideas behind (a simplification of) the approach of [44], but several challenges present themselves in implementing such approach. In general they are not always able to preserve the core itself, but a "proxy core," which emerges from the interaction of the core with the remainder of the circuit. This will be the same for us in our generalization of their approach.

1.3.2 Generalizing the above approach

Several new challenges arise in the above approach when trying to generalize it to the setting of $\Sigma^k\Pi\Sigma\Pi^2$ circuits, as we have to deal with quadratic forms, in addition to linear forms. These challenges are both algebro-geometric and combinatorial. We now outline the main conceptual and technical challenges we face, and our approaches to overcome them.

[4] Note that Hansen's theorem is over $\mathbb{K} = \mathbb{R}$ only, but a generalization of it is proved over any characteristic zero field in [10, 18].

1. In order to reduce the top fan-in, we may have to "zero out" a set of quadratic forms, that is, work over a polynomial ring modulo an ideal with quadratic generators. This forces us to work with rings which are not polynomial rings. If one is not careful, such rings will not even be unique factorization domains (UFDs), which makes the entire approach collapse.

   We overcome this issue by building on the concept of *general quotients* developed in previous works [44, 66, 55, 30, 31]. While the general quotients developed in these previous works enable us to reduce the top fan-in in certain situations, they are not able to handle the case where we want to set a *particular* quadratic form to zero. To achieve this, we develop the notion of *targeted quotients* and establish some of its algebro-geometric properties, which will allow us to handle all the necessary quotients that "zero out" a particular quadratic form. We establish the properties that we need from our quotients in [28, Section 3.5].

   Moreover, to be able to always work within "nice enough rings", we invoke the strong algebras and vector spaces developed in [6, 55]. Analogously to previous works, we generalize the notion of $\Sigma^k\Pi\Sigma\Pi^2$ identities to rings of the form $S/(U)$ where S is a polynomial ring and U is a strong enough vector space in S. This generalized notion of identities allows our inductive approach more flexibility to handle the general and targeted quotients that we need to apply in order to reduce the top fan-in. However, unlike in previous works, it is important for us to be able to define the concept of "essential variables" inside of quotient rings, as these variables allow us to exploit the geometric techniques coming from absolute irreducibility. We define strong vector spaces and algebras in [28, Section 3], and define and establish properties of essential variables in quotient rings in [28, Section 3.4].

2. In [44], the drop in pairwise rank of the circuit can only happen due to linear relations between linear forms. This allows the authors to both argue about large pairwise rank with Hansen's theorem and to control the drop in pairwise rank by preserving non-colinearity relations. However, the presence of quadratic forms yields more intricate algebro-geometric phenomena, which makes the understanding of the behaviour of pairwise rank more complex.

   We overcome this issue by proving structural results on primality of certain special ideals generated by quadratic forms using absolute irreducibility over algebras, in a similar way as was done in [55, 30, 31]. Since we are dealing with quadratic forms only, the primality results we need are simpler and stronger, and thus we do not need to use the primality results in these works. Moreover, we use resultant-based methods to control the drop in pairwise rank under targeted quotients. Once we find a desired set of forms that we want to quotient out, both general quotients and targeted quotients will take care of pairwise rank preservation under quotients. It is important to note that, unlike for general quotients, we do not need to prove any lifting results for the targeted quotients. This phenomenon happens because of the same reasons as its linear analogue in [44]: if the pairwise rank is large enough (by assumption), then our circuit must not be an identity. Hence, to obtain the rank bounds one only need to lift the general quotients that one applies when dealing with the cases of small pairwise rank.

3. Unlike the linear case, we do not have at hand an algebro-geometric analog of Hansen's theorem. Note that the main result of [27] does not seem to be the right analog to what is needed in our case, since they only handle radical dependencies (whereas we would need product dependencies), and they do not handle the same combinatorial conditions as Hansen's theorem.

We overcome this issue in a 2-step approach: in the first step, we "control" the high strength quadratics (the ones that "depend on too many variables") in any $\Sigma^k\Pi\Sigma\Pi^2$ identity by reducing our identity to a non-trivial $\Sigma^k\Pi\Sigma\Pi^2$ identity where all the quadratics have bounded strength (i.e., "depend on few variables"). Interestingly enough, to control the high strength quadratics, we need to combine the approach of [44] to reduce the top fan-in – which is done in [28, Section 4.1] – together with a Hansen-like result, which we establish in [28, Theorem 4.6]. Note that [28, Theorem 4.6] is different from the statements of [27, Section 6.2] (where they consider configurations of strong forms), since in our case we have to handle a Hansen-like condition, whereas [27] only requires to handle a stronger condition, similar to the SG_k^* conditions from [10]. We establish our Hansen-like result by reducing our strong span configurations to fractional configurations, which then we can further reduce to several fractional Sylvester-Gallai configurations, with an approach inspired by [10]. For details, see [28, Section 4] for the entire approach and [28, Section 4.2] for our Hansen-like result.

Once we complete the strength reduction above, the second step is to prove rank bounds for $\Sigma^k\Pi\Sigma\Pi^2$ identities where all quadratics "depend on few variables." In this case we are able to combine our primality structure theorems similar to the ones from [30, 31] and the integral sequences developed in [27] with the approach of [44] described above and construct a core, along with a "proxy core," which allows us to upper bound the rank of such $\Sigma^k\Pi\Sigma\Pi^2$ identities.

While the above outlines how we overcome the challenges posed by working with quadratic and linear forms, we now elaborate on our technical contributions and how they compare with previous works.

1.3.3 Elaboration of our techniques

The two main conceptual and technical contributions of this work are the concept of *targeted quotients* and *essential variables in (strong) quotient rings*. Once we have these two concepts and their properties established, we can execute our strategy above with a substantial amount of intricate combinatorial work.

- **Targeted quotients:** Previous works, such as [44, 66, 58, 55, 31], only required working with general quotients – where forms are mapped to general multiples of (appropriate powers of) a special linear form z. These quotients have very useful properties, and allowed these works to reduce the complexity of their main objects of study (polynomial identities or Sylvester-Gallai type configurations). However, in [44] the authors need to also quotient by especially chosen linear forms. This type of quotient was not captured (and not needed) in subsequent works [66, 58, 55, 31], but it is crucial here. While the definition is clear on what quotient we will need for linear forms and for high strength quadratics, as these essentially behave "as variables in a polynomial ring," it is much less clear what to do in the case where we need to make a quadratic of low strength vanish "in a generic way" that does not substantially affect the other forms in the circuit. The latter case of a low strength quadratic F is addressed by our notion of targeted quotients, whose idea is as follows: if F only depends on the variables $x_1, \ldots, x_r$, to "make F vanish", we need to set the variables $x_1, \ldots, x_r$ to a zero of F (i.e. a point in the variety defined by F) in "a generic way." The way to obtain such a generic zero of F is to simply pick a general point $(\alpha_1, \ldots, \alpha_r)$ in the zero set of F and set $x_i \mapsto \alpha_i z$, where z is a new variable. It turns out that this choice of quotients map F to zero, and at the same time "preserves" the algebraic and geometric properties of other forms that we need. We also show that

the only instance where another form G is not preserved under such quotients is when G also depends only on the variables $x_1, \ldots, x_r$, and in this case, we also show that G will be sent to zero iff it is a multiple of F. [28, Proposition 3.42] establishes the properties we need from targeted quotients and in [28, Section 5] we show how such quotients interact with the other forms in our circuit.

- **Essential variables in quotient rings:** previous works on Sylvester-Gallai type theorems involving quadratics [66, 57, 58, 59, 29] heavily used the fact that they were working over polynomial rings, which in particular was very useful when defining the space of linear forms that defined a quadratic. In this setting, the uniqueness of representation of any polynomial in the polynomial ring was key in their definitions of the relative linear space of forms of a quadratic of low strength over an algebra. However, in our more general setting, where we must work with quotients of a polynomial ring by an ideal generated by a strong vector space, it is much less clear how to define the relative space of a quadratic element. The main difficulty stems from the fact that an element of the quotient ring has many distinct representatives in the corresponding polynomial ring, and in our setting we must make sure that any such representative must be compatible with a unique space of linear forms. To address this issue, in [28, Section 3.4], we establish some necessary results needed in order to define two key concepts: the *pseudo-distance* of a quadratic to an algebra in a quotient ring ([28, Definition 3.28]) and then after establishing some results about the pseudo-distance, we can finally define the *relative space of essential variables* ([28, Definition 3.32]), which generalize the related notions in previous works. With these notions and their properties at hand, we can analyze the case where all of our quadratics have bounded strength, which we do in [28, Section 5]. In this section, we crucially use the relative space of essential variables, combined with the geometric properties established in previous sections, to obtain our rank bounds.
- **Controlling Hansen-like configurations:** in our reduction from general $\Sigma^k\Pi\Sigma\Pi^2$ identities to the case of $\Sigma^k\Pi\Sigma\Pi^2$ identities where all quadratics are close to a small strong algebra, one important step to control the strong quadratics (i.e. the ones depending on many variables) is our Hansen-like result [28, Theorem 4.6]. This result bounds the size of a subset needed to bring a set of forms satisfying certain Hansen-like conditions (see [28, Definition 4.2]) "close to a given algebra." Although the result of [28, Theorem 4.6] is stated in a manner which seems specific to depth-4 identities, note that [28, Definition 4.2] can be stated independently of the setting of depth-4 identities. We believe that this result is also an interesting geometric-combinatorial result on its own right, as it allows one to control such sets of forms. A weaker statement was proved in [27, Section 6], where the authors handle a geometric-combinatorial condition similar to the SG_k^* configurations defined in [10, Definition 5.1]. In our case, our configurations are more related to the fractional configurations generalizing Hansen, given in [10, Definition 5.2].

1.3.4 Choice of approach, challenges of higher degree and open questions

Given the above overview and the history about rank bounds in the depth-3 case, one may wonder what makes the approach of [44] the preferred one in this paper. One of the main reasons that motivated this choice is the fact that the approach of [44] proves rank bounds by controlling pairs of gates, whereas the other approaches make simultaneous use of more gates in their analysis. The latter approaches require geometric results involving ideals generated by several forms (of potentially unbounded degree), such as the ideal chinese remaindering results of [45, 64], whereas the approach of [44] only requires the use of geometric statements

about ideals generated by forms of constant degree. Extending these other approaches is an interesting open problem, and the geometric statements needed to do so may be important in their own right.

Another question that may arise, in light of the recent rank bounds for $\Sigma^3\Pi\Sigma\Pi^d$ circuits and structural results for ideals generated by pairs of degree d forms from [30, 31], is what makes the approach above not work for bottom fan-in larger than 2. The main issue in extending the above result to bottom fan-in larger than 2 is the fact that we currently lack the correct analogue of "essential variables" when decomposing a low strength form of degree larger than 2, Note that, even for the case of degree 3, a minimum decomposition of a low strength form may have quadratic factors, which may interact non-trivially with the remainder of the forms in the circuit. On the other hand, in our case, once we establish the uniqueness of the space of linear forms that we will quotient by, their infinite strength (as they are truly variables in the polynomial ring) allows us to argue that they do not interact non-trivially with any other linear form lying outside of the given space. Generalizing our notion of essential variables to the higher degree setting is an interesting open question, as it would yield a valuable tool which can have uses not only for PIT but also for other problems in the computational complexity of other computer algebra problems.

Lastly, we conclude by emphasizing that a full resolution of [11, Conjecture 28] is the biggest problem left open by this work: proving rank bounds for $\Sigma^k\Pi\Sigma\Pi^d$ circuits which depend only on k, d.

1.3.5 Flow of argument for rank bound

Starting assumption: rank bounds for simple and minimal $\Sigma^{k-1}\Pi\Sigma\Pi^2$ identities over rings $S/(U)$, where U is a strong enough vector space (the required lower bound on the strength of U is a function of $k-1$).

Suppose we are given a simple minimal $\Sigma^k\Pi\Sigma\Pi^2$ identity $\Phi = T_1 + \cdots + T_k \equiv 0$ in the ring $R = S/(U)$, where U is a strong enough vector space.

- Step 1: Are two gates close to each other? If so then apply a general quotient (see [28, Section 3.5]) to merge two gates, and invoke inductive rank bounds for circuits with top fan-in $k-1$. Then lift the rank bound to deduce a rank bound for Φ using [28, Proposition 3.41].
  Here the measure of closeness of two gates is "pairwise rank" or the size of their symmetric difference (see [28, Lemma 4.1], item 1 for quantitative details).
- Step 2: Suppose that no two gates are close to each other, i.e. all pairwise ranks are high.
  Case 2.1. (Fanin-reduction) If there exists a strong form H in some gate T_i such that after applying a quotient by H, the resulting circuit has high pairwise rank, then we obtain a contradiction to the starting assumption.
  Case 2.2. If there is no strong form H with the property above, then we have a (M, b)-strong configuration (see [28, Section 4.1] where we prove the contrapositive: a fanin-reduction lemma). This argument follows the strategy of Kayal and Saraf [44].
- Step 3: If a set of forms is an (M, b)-configuration, then there is a vector space V such that all forms in the set are close to V (see [28, Section 4.2]). This result is itself proved by induction. Refer to [28, Lemma 4.1], item 3, [28, Theorem 4.6], [28, Theorem 4.10] for qualitative versions that prove approximate and robust Hansen-type results.
  Therefore, all the forms in the circuit Φ are close to some vector space V.
- Step 4: Now, all the forms in the circuit Φ are close to some vector space V, i.e. Φ is of bounded pseudo-distance with respect to V (see [28, Section 3.4] for the notions of essential variables and pseudo-distance in quotient rings).

In [28, Section 5] we handle such circuits and [28, Lemma 5.3] shows that we will land in one of the following cases, each handled separately. These statements are proved using a potential function (see [28, Section 5.1]) and an iterative process that decreases this potential (see [28, Section 5.3]).

Case 4.1. There exists a vector space Y such that some symmetric difference lies in the algebra generated by Y. Apply a general quotient with respect to Y and lift the inductive bounds.

Case 4.2. There exists a vector space Y such that either every quadratic form in Φ is absolutely reducible with respect to Y. Apply a general quotient with respect to Y to reduce the degree of forms in the configuration and use inductive rank bounds on $\Sigma^k\Pi\Sigma$ circuits similar to [44].

Case 4.3. (Targeted fanin-reduction). There exists a form H such that the targeted quotient with respect to H (see [28, Proposition 3.42]) results in a circuit with high pairwise rank (see [28, Section 5.4]). Then we have a contradiction to the starting assumption.

At the end of these steps, we have obtained a rank bound for $\Sigma^k\Pi\Sigma\Pi^2$ circuits.

1.4 Previous and related work

1.4.1 Polynomial identity testing

Depth-4 circuits with bounded top and bottom fan-in are the simplest known circuit class for which no deterministic polynomial-time algorithm for PIT is currently known. Consequently, this model has been studied using techniques that extend beyond the incidence geometry–based methods discussed earlier.

In [15], the authors devise and prove correctness of a quasipolynomial-time deterministic PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits, leveraging the Jacobian method introduced in [4]. Their key insight involves applying the logarithmic derivative and its associated power series expansion to transform the top summation gate of the circuit into a powering gate. While this transformation technically violates the bounded top fan-in constraint, circuits with powering gates are well studied, and efficient PIT algorithms are known for such models. This reduction enables the use of existing tools for PIT in powering-gate circuits, ultimately leading to their quasipolynomial-time result.

The breakthrough work of [51] on proving lower bounds for bounded-depth arithmetic circuits offers another route toward derandomizing PIT for this model. The *hardness-versus-randomness paradigm* has been a powerful tool for establishing connections between circuit lower bounds and derandomization [2, 41]. Notably, the results of [14] extended these tradeoffs to the bounded-depth regime, showing that analogous principles continue to hold even in this restricted setting. By leveraging these developments, one obtains a *subexponential-time* PIT algorithm for depth-4 circuits. Building further on the lower bound techniques of [51], [7] constructed another *hitting set generator* tailored to bounded-depth circuits, yielding an alternative subexponential-time PIT algorithm with improved parameters. This construction exploits the fact that such circuits are inherently unable to detect low-rank matrices, due to the algebraic hardness of computing the determinant in low-depth settings. It is worth emphasizing, however, that neither of these approaches appears capable of yielding a *fully polynomial-time* PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits.

Karnin et al. [43] and Saraf and Volkovich [60] studied *multilinear* $\Sigma^k\Pi\Sigma\Pi$ circuits and obtained quasi-polynomial- and subsequently polynomial-sized hitting sets, respectively. However, as their techniques crucially rely on multilinearity, none of these works can handle even $\Sigma^3\Pi\Sigma$ circuits.

Peleg and Shpilka [58] gave the first polynomial sized hitting sets for $\Sigma^3\Pi\Sigma\Pi^2$ circuits. As their technique relied on their structural theorem, which argues about the structure of ideals generated by two quadratic polynomials, they were not able to extend their proof beyond three terms, as more terms would involve ideals generated by three or more quadratics.

Garg et al. [27] studied high-dimensional Sylvester-Gallai configurations and obtained a generalization of both [38, 66]. This can be viewed as a problem analogous to ours, but in the *single color class* case, where each and every k-tuple of forms satisfies some nontrivial relation. In contrast, to obtain a PIT algorithm, we need to consider the case where forms belong to different sets and we can only argue about k-tuples coming from k disjoint sets.

Recently, two independent works [31, 34] have made progress in the PIT problem for $\Sigma^3\Pi\Sigma\Pi^d$ circuits. Note that progress in these works is made on the *bottom fan-in* of the circuit, while the current work focuses on the *top fanin* of depth-4 circuits. These two settings present different challenges, which must be overcome in order to fully resolve the PIT problem for $\Sigma^k\Pi\Sigma\Pi^d$ circuits.

In [34], the authors use normalization of algebraic varieties to give a deterministic, polynomial time PIT algorithm for $\Sigma^3\Pi\Sigma\Pi^d$ circuits over *any base field* (of any characteristic), but with the *additional assumption* that one summand of the top gate is *square-free* (that is, a multiplication of coprime polynomials). Their approach circumvents the need of incidence theorems and their generalizations, which provides an advantage over finite fields, as it is known that even in the case of depth-3 circuits over fields of positive characteristic, incidence-based approaches can only yield quasi-polynomial time algorithms. On the other hand, the requirement on one summand to be square-free seems to be very restrictive.

In [31, Theorem 1.4], as discussed above, the authors prove that there is a function $R_{EK} : \mathbb{N} \to \mathbb{N}$ such that any degree d EK configurations over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, where $\operatorname{char} \mathbb{K} = 0$, has rank upper bounded by $R_{EK}(d)$. As any simple and minimal $\Sigma^3\Pi\Sigma\Pi^d$ identity gives rise to a degree d EK configuration over $\mathbb{K}[x_1, \ldots, x_n]$, the EK rank bound implies a rank bound for these identities, and when one combines this result with [11, Theorem 2], one gets a deterministic, polynomial time algorithm for the PIT problem for this class.

1.4.2 High-dimensional and approximate variants of Sylvester-Gallai theorem

Let us now elaborate on the relation between Theorem 6, Hansen's theorem [38], the strong sequences of [27] and the result of Ai, Dvir, Saraf, and Wigderson [5].

Both [38, 5] consider configurations of points in Euclidean space, so in that context $\mathcal{F}$ should be viewed as a set of points rather than quadratic polynomials. Hansen requires that for every $b-1$ points $p_1, \ldots, p_{b-1}$ and any other point $p \in \mathcal{F}$, either the flat they define contains another point of $\mathcal{F}$, or that p lies exactly in the flat defined by $p_1, \ldots, p_{b-1}$. Thus, Hansen's condition enforces exact containment in the span rather than proximity to it.

Ai et al. consider an approximate version of this condition. Roughly speaking, an ϵ-approximate collinear Sylvester–Gallai configuration is a finite set of points $\mathcal{F} \subset \mathbb{R}^n$ that are neither "too close" nor "too far" from one another, such that for any two points in $\mathcal{F}$, there exists a third point in $\mathcal{F}$ that is ϵ-close to the line spanned by them. In other words, the standard Sylvester–Gallai requirement that the third point lie exactly on the line is replaced by an approximate one. The conclusion in this setting is that there exists a subspace V of dimension $O(1)$ such that all the points in $\mathcal{F}$ are $O(\epsilon)$-close to V. This is similar in spirit to our result, except that it concerns points rather than polynomials and corresponds to the case $b = 3$.

Hence, our Theorem 6 (and the formal [28, Theorem 4.6]) generalizes both Hansen's theorem and the result of Ai et al., yielding a quadratic, high-dimensional, approximate Sylvester–Gallai theorem.

Barak, Dvir, Wigderson, and Yehudayoff [9] proved a robust version of Hansen's theorem. They call a $(b-1)$-dimensional flat spanned by b points of $\mathcal{F}$ *elementary* if its intersection with $\mathcal{F}$ does not contain any other point. They showed that if, for every choice of $b-1$ points in $\mathcal{F}$, at least a ν-fraction of the remaining points are such that each of them either lies in the flat spanned by these $b-1$ points or spans with them a flat that is not elementary, then $\mathcal{F}$ is contained in an $O_b(1/\nu^2)$-dimensional flat. This was improved in [18, 17] to $O_b(1/\nu)$.

If we now replace points by quadratic polynomials, and the condition of belonging to a flat by being close to it, we obtain a setting similar to that of Theorem 8. The conclusion we derive is also analogous to that of [9]: to approximate all the quadratics, we must repeat the argument about $1/\nu$ times, resulting in an $O_b(1/\nu^2)$-dimensional subspace. The work of [27] generalizes the version of *ordinary* spaces defined by [10, Definition 5.1].

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