A Simple Sub-Polynomial Degree Coboundary Expander

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Abstract

High dimensional expanders simultaneously satisfying spectral and combinatorial (coboundary) expansion have recently played a major role in breakthroughs in PCP and coding theory, but the only known construction of such complexes is extremely involved, requiring deep algebraic number theory. In this work, we give an extremely simple combinatorial construction of a sub-polynomial degree complex based on projections of the flags complex (subspace chains) that is (i) a local spectral expander, (ii) a coboundary expander, and (iii) a swap coboundary expander. As a corollary, we also give the first near-linear size combinatorial hypergraphs with good agreement tests in the '1%' regime, and a simple PCP construction with near-linear size.

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1 Introduction

High dimensional expanders, sparse but robustly connected hypergraphs, have seen a surge of interest and application in theoretical computer science, driving breakthrough progress in coding theory [40, 25, 2, 57, 73, 42, 31, 62, 51], approximate sampling [8, 9, 6, 7, 64], property testing [35, 21, 59, 47, 22, 13, 26], hardness of approximation [32, 53, 14], and pseudorandomness [54, 28]. Unlike their widely-studied graph analogs, there are very few known constructions of high dimensional expanders, and all known methods are algebraic. In this work, we give a simple construction of sub-polynomial degree hypergraphs based on subspaces that are both spectral and combinatorial high dimensional expanders, and a corresponding simple near-linear size PCP construction.

We consider three core notions of expansion: local spectral expansion (see Definition 16), a generalization of spectral expansion in graphs, coboundary expansion (see Definition 20), a generalization of edge expansion, and swap-coboundary expansion, a key notion related to the construction of PCPs measuring the coboundary expansion of higher level disjoint faces (see Definition 25). Unlike the graph setting, these generalizations behave quite differently in high dimensions, e.g., there are random two-dimensional simplicial complexes that are good local spectral expanders but not good coboundary expanders with high probability [49]. Constructing sparse hypergraphs satisfying both spectral and coboundary expansion has been a major challenge since the notions were introduced [65, 48, 58], and has received renewed attention after it was shown such hypergraphs admit low soundness "agreement tests" [47, 22, 13] and efficient PCPs [15].

The first sparse hypergraphs satisfying all three notions of expansion were constructed recently in [17, 26, 12], based on deep algebraic number theory. This was followed very recently (and independently to our work) by [71], who gave a simpler algebraic construction based on variants of the elementary coset complexes of [60]. In this work, we show that if one is willing to sacrifice somewhat on sparsity, it is possible to give completely elementary combinatorial constructions of local-spectral and (swap)-coboundary HDX of sub-polynomial degree. Following [14], we show these give rise to a simpler PCP construction roughly matching the parameters of the breakthrough algebraic PCP of [69] in the small constant soundness regime.

Our complexes are based upon projections of the *flags complex*, a well-known dense hypergraph more typically studied as the local structure of [67]'s quotients of the affine building, and shown to be a swap-coboundary expander in [24]. The vertices of flags complex are the subspaces of $\mathbb{F}_q^d$ (excepting 0 and $\mathbb{F}_q^d$). Its hyperedges are given by inclusion chains ("flags") of these subspaces.

▶ **Definition 1** (Flags Complex). *Given $d \in \mathbb{N}$ and prime power q, the flags complex is the $(d-1)$-partite complex with vertices given by all subspaces of $\mathbb{F}_q^d$, and top-level faces given by inclusion chains (flags) of subspaces:*

$$\mathfrak{G}_q^d(d-2) = \{(W_1 \subseteq W_2 \subseteq \ldots \subseteq W_{d-1}) : \dim(W_i) = i\}\,.$$

While the flags complex is known to be a high-dimensional expander under all three definitions [24], it has two fatal flaws: it is *dense*, with roughly $q^{d^2/2}$ top faces over only $q^{d^2/4}$ vertices (a quadratic blow-up), and has *extremely high max-degree*. In fact, both flaws stem from the same source: the number of i-dimensional subspaces of $\mathbb{F}_q^d$ varies wildly with i. In particular, since there are roughly $q^{i(d-i)}$ subspaces of dimension i, the levels are sharply imbalanced: each of the q^{d-1} lines, for instance, show up in far more hyperedges than any of the $q^{d^2/4}$ subspaces of dimension $d/2$. Measured by size and max-degree (the key parameters used in applications), the flags complex is thus, asymptotically, essentially no better than simply taking an imbalanced product $[q^{d/4}]^{d-1} \times [q^{d^2/4}]$. As such, despite its fantastic expansion, the flags complex has seen little use outside its role as the local structure of [67]'s Ramanujan complexes in the literature.

The above, however, suggests a simple fix. The imbalance in the flags complex comes from *extremal* dimensions, those near 1 and d, whereas *central* dimensions, i.e., those near $d/2$, are much closer to balanced, each containing $q^{d^2/4\pm\Theta(d)}$ subspaces. As such, we should simply discard the problematic levels and only consider inclusion chains through the well-behaved ones.

▶ **Definition 2** (Projected Flags Complex). *Given $d \in \mathbb{N}$, a prime power q, and a subset of dimensions $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [d-1]$, the* projected flags complex *$\mathfrak{G}_q^{d,S}$ is the k-partite simplicial complex with vertices given by subspaces $W \subset \mathbb{F}_q^d$ such that $\dim(W) \in S$ (partitioned into parts by dimension), and top-level faces given by inclusion chains using one subspace of each dimension in S:*

$$\mathfrak{G}_q^{d,S}(k-1) = \{W_1 \subset W_2 \subset \cdots \subset W_k \;:\; \dim(W_i) = s_i\}\,.$$

Equivalently, $\mathfrak{G}_q^{d,S}$ *is the complex whose top faces are obtained by projecting each flag of* $\mathfrak{G}_q^d$ *onto its components of dimension in* S*. We drop* d *and* q *from the notation when clear from context.*

Taking S to be the $2k+1$ central dimensions of the flags complex over $\mathbb{F}_q^{2d}$,

$$S_k = \{d-k, \ldots, d-1, d, d+1, \ldots, d+k\},$$

every level has $q^{d^2/4 \pm \Theta(kd)}$ of subspaces, eliminating the imbalance up to lower order factors. Fixing k and letting $d \to \infty$, we get a strongly explicit infinite family of complexes with $q^{\Theta(d^2)}$ vertices and max-degree $q^{O(kd)} \approx 2^{O(k\sqrt{\log N})}$ – sub-polynomial in the number of vertices N. The main technical content of this work is to show this projection retains the spectral and (swap)-coboundary expansion of the full complex, yielding simple near-linear-size low-soundness PCPs.

1.1 Our Results

Local Spectral Expansion. A k-dimensional simplicial complex $X = \{\emptyset\} \sqcup X(0) \sqcup X(1) \sqcup \cdots \sqcup X(k)$, where the set of i-dimensional faces $X(i)$ contains faces with cardinality $i+1$, is a (two-sided) λ-local spectral expander if (a) the underlying graph of X (i.e. the graph with vertices $X(0)$ and edges $X(1)$) is a (two-sided) spectral expander, and (b) for any face $s \in X$ with dimension at most $k-2$, the underlying graph of the s-link $X_s = \{\tau \setminus s | \tau \in X,\ \tau \supseteq s\}$ is also a (two-sided) spectral expander.

Our first result states any projection of the flags complex is a local spectral expander.[1]

▶ **Theorem 3** (Spectral Expansion of the Projected Flags Complex)**.** *Fix any* $d \geqslant 3$*, any prime power* q*, and a set* $S \subseteq [d-1]$ *of size at least 2. Let* $s \in \mathfrak{G}_q^{d,S}$ *be any face with co-dimension at least 2. Then, the second eigenvalue of the underlying graph of the* s*-link of* $\mathfrak{G}_q^{d,S}$ *is* $\frac{2}{(|S|-|s|-1)(\sqrt{q}-1)}$*. In particular, the complex* $\mathfrak{G}_q^{d,S}$ *is a* $\frac{2}{\sqrt{q}-1}$*-local-spectral expander for any prime power* q*. Moreover, the complex* $\mathfrak{G}_q^{d,S}$ *is also a* $\frac{1}{\sqrt{q}}$*-product for any prime power* q*.*

We remark that while the local-spectral expansion of the projected flags complex has not been studied explicitly to our knowledge, similar bounds (up to a worse constant in the denominator) can be derived from combining improved "Trickling-Down" Theorems [1, 64] with well known bounds on the spectrum of subspace inclusion (Grassmann) graphs [20]. We give a simple and direct proof that does not require the former machinery.

Coboundary Expansion. One way to view combinatorial (edge) expansion of a (connected) graph is that if a graph has expansion η, then for any function $f : V \to \mathbb{F}_2$ there exists a constant function $\delta g : V \to \mathbb{F}_2$ such that

$$\eta \cdot \mathsf{dist}(f, \delta g) \leqslant \mathsf{wt}(\delta f)$$

where $\delta f(u,v) = f(u) - f(v)$ is the "coboundary" of f (similarly δg is the "coboundary" of some constant $g : \{\emptyset\} \to \mathbb{F}_2$). In other words, δf is the size of the cut formed by f, and any f whose cut has few edges must be close to 0 (a very small set) or 1 (a very large set).

[1] The below also references the notion of a λ-product, a slight variant of local-spectral expansion in the partite case which promises every bi-partite component of every link has expansion $\lambda_2 \leqslant \lambda$. See Definition 19.

For general groups, this definition can be extended to anti-symmetric[2] functions taking values from any group Γ instead of $\mathbb{F}_2$ by taking $\delta f(u,v) = f(u)f(v)^{-1}$. For a simplicial complex, it is possible to generalize this notion to the 1-coboundary expansion with respect to a group Γ that compares weights of the coboundary of 1-dimensional faces to their distance from 0-dimensional coboundary, as opposed to comparing weights of coboundary functions on 0-dimensional faces to their distance from the corresponding coboundary. In particular, a simplicial complex has 1-coboundary expansion $h^1(X,\Gamma)$ if for any anti-symmetric function $f : \overrightarrow{X}(1) \to \Gamma$

$$h^1(X,\Gamma) \cdot \mathsf{dist}(f, \delta g) \leqslant \mathsf{wt}(\delta f)$$

for some anti-symmetric function $g : X(0) \to \Gamma$ on the directed edges $\overrightarrow{X}(1)$ and vertices $X(0)$, where $\delta f(u,v,w) = f(u,v)f(v,w)f(w,u)$ is the "coboundary" of f. Our second result states that the projection of the flags complex onto its central dimensions is a coboundary expander.

▶ **Theorem 4** (Coboundary Expansion of Projected Flags Complex). *There exists a constant q_0 such that the following holds. Let Γ be any group. Let $d > k \geqslant 2$ be two integers and $q \geqslant q_0$ be a prime power. Let*

$$S_k = \{d-k, \dots, d-1, d, d+1, \dots, d+k\}.$$

Then, for any face $s \in \mathfrak{G}_q^{2d,S_k}$ with codimension at least 3, $\mathfrak{G}_{q,s}^{2d,S_k}$ is 1-coboundary expander with

$$h^1(\mathfrak{G}_{q,s}^{2d,S_k}, \Gamma) = \Omega(1).$$

As a corollary, we get a completely elementary construction of local-spectral coboundary expanders of sub-polynomial degree:

▶ **Corollary 5** (Elementary Sub-polynomial Degree HDX). *There exists a constant q_0 such that for any $d \geqslant k+1 \geqslant 3$, prime power $q \geqslant q_0$, and group Γ, $X = \mathfrak{G}_q^{2d,S_k}$ has $O(q^{d^2})$ vertices and*

(i) ***Local-spectral Expansion:*** *$\frac{2}{(j-1)(\sqrt{q}-1)}$-spectral expansion at co-dimension j,*
(ii) ***Coboundary Expansion:*** *$h^1(X,\Gamma) \geqslant \Omega(1)$,*
(iii) ***Maximum Degree:*** *$\Delta_{max}(X) \leqslant q^{O(kd)}$.*

Moreover, the complex is strongly explicit.

In particular, for any $\lambda > 0$ we get a family of λ-local-spectral and coboundary expanders with degree $\exp(k\sqrt{\log(N)\log(1/\lambda)})$. To the best of our knowledge, these are the first sub-polynomial degree high dimensional expanders for which the cohomology is known to vanish for *every group*, rather than for all sufficiently small copies of Sym_m. As we discuss below, this is an important point toward the construction of better soundness agreement tests and PCPs.

We briefly comment on the proof before moving on, which is divided into two main steps. First, we need to argue that the projected flags complex (and all its links) have *vanishing cohomology* (Lemma 32), meaning that every $f : E \to \Gamma$ with $\delta f = 0$ is a coboundary δg for some $g : V \to \Gamma$. This is a well-known fact for the full flags-complex, and can be derived for projections of the flags complex via fairly standard tools in algebraic topology, e.g., from [16, 74]. We give a simplified, self-contained proof via [16]'s theory of shellability.[3]

[2] We discuss this standard mild restriction on the family of functions in Section 2.3.
[3] We thank Roy Meshulam for pointing us to this approach.

Second, we show that the central projections of the flags complex also satisfy "cocycle" expansion, which states that if $|\delta f|$ is small, then f must be close to a function in $\mathrm{Ker}(\delta)$ (which due to vanishing cohomology, is exactly the set of coboundaries). The proof relies on several techniques from [39, 23] to reduce the analysis to simple local components of the complex. In particular, we first use a local-to-global argument (Lemma 44) from [23] to move to links of the complex, and a "color restriction lemma" (Lemma 45) to reduce to showing most projections of the links to three colors are coboundary expanders. Finally, we use the cones method [61] (see Section 2.5) to argue directly that most projections to three colors indeed have constant coboundary expansion.

Swap Coboundary Expansion. Towards construction of agreement testers, [24, 13] define a third notion of expansion called swap coboundary expansion. Swap coboundary expansion essentially says the faces complexes (defined below) of the simplicial complex under consideration is a 1-coboundary expander.

▶ **Definition 6.** *Let X be a d-dimensional simplicial complex. Let $r \leqslant d$. We use $\mathrm{F}^r X$ to denote the simplicial complex whose vertices are $\mathrm{F}^r X(0) = X(r)$ and whose faces are all $\{s_0, s_1, ..., s_j\}$ such that $s_0 \sqcup s_1 \sqcup \cdots \sqcup s_j \in X((j+1)(r+1)-1)$. We call this the r-*faces complex *of X.*

Now, we can define swap coboundary expansion.

▶ **Definition 7.** *A simplicial complex X is said to have (β, r)-*swap coboundary expansion* if $\mathrm{F}^r X$ has 1-coboundary expansion at least β with respect to every group.*

Finally, we show swap coboundary expansion of the projections of the flags complex.

▶ **Theorem 8** (Swap coboundary expansion of the projected flags complex)**.** *Let d, k, r be integers such that $d > k \geqslant r^5$. There is some $q_0 = q_0(k, r)$ such that the following holds. Let $q \geqslant q_0$ be any prime power, $\mathfrak{G}$ the $\mathfrak{G}_q^{2d}$-flags complex, and*

$$S_k = \{d-k, \ldots, d-1, d, d+1, \ldots, d+k\}.$$

Then $\mathfrak{G}^{S_k}$ is a $(\exp(-O(\sqrt{r}\log^2 r)), r)$-swap coboundary expander.

Theorem 8 is an adaptation of the strategy of [24] for the full un-projected flags complex. We break the faces complex $\mathrm{F}^r(\mathfrak{G}^{S_k})$ into progressively simpler complexes (projections and deep links) until we can bound the coboundary expansion directly via the cones method. At a high level, the critical observation is simply that once one has moved to the link of a vertex, the projected flags complex behaves very similarly to a product (join) of full flags complexes, allowing us to apply [24]'s machinery without substantial modification.

Applications. Following [22, 13, 14], we give two main applications of our swap coboundary expansion result: (i) an agreement test in the low-soundness regime, and (ii) a corresponding simple, near-linear size PCP construction.

Agreement testing. An agreement test is a randomized property tester that, given an ensemble of local functions $\{f_s : s \to \Sigma | s \in \mathcal{F}\}$ on $\mathcal{F} \subseteq 2^{[n]}$, tries to determine whether there is a global function $G : [n] \to \Sigma$ such that $f_s = G|_s$ for all $s \in \mathcal{F}$. Many recent works have examined the case where the local functions are defined on some level of a high-dimensional expander. In this work, we consider the standard two-query agreement test (the "V test") where we pick two functions $f_s, f_{s'}$ on sets of size ℓ where $|s \cap s'| = \sqrt{\ell}$, and the test accepts if $f_s|_{s\cap s'} = f_{s\cap s'}$, i.e., the two functions agree on all the common points in their domain. [22, 13, 14] recently showed that if a complex has good local-spectral and swap coboundary expansion, then it supports this two-query test. Thus, as a corollary of our results above, we immediately get that the projected flags complex gives rise to a simple near-linear size low soundness agreement tester.

▶ **Theorem 9** (Agreement testing over the projected flags complex)**.** *Fix a $\delta > 0$. There is some q_0 such that the following holds.*

Let $q \geqslant q_0$ be any prime power. Let d, k, ℓ be such that $d > k \geqslant \mathsf{poly}(\ell)$ and $\ell \geqslant \exp(\mathsf{poly}(1/\delta))$. Let $\mathfrak{G}$ denote the $\mathfrak{G}_q^{2d}$-flags complex and

$$S_k = \{d-k, \dots, d-1, d, d+1, \dots, d+k\}.$$

Then, the $(\ell, \sqrt{\ell})$-agreement test over $\mathfrak{G}^{S_k}(\ell)$ with respect to every alphabet Σ has soundness δ, i.e., if $F : \mathfrak{G}^{S_k}(\ell-1) \to \Sigma^\ell$ passes the $(\ell, \sqrt{\ell})$-agreement test with respect to $\mathfrak{G}^{S_k}$ with probability at least δ, then there is $f : \mathfrak{G}^{S_k}(0) \to \Sigma$ such that

$$\mathop{\mathbb{P}}_{A\sim\mu_\ell}[\mathsf{dist}(F[A], f|A) \leqslant \delta] \geqslant \mathsf{poly}(\delta).$$

We remark that on the complete complex, it is known that the V test has soundness $\delta = 1/\mathsf{poly}(\ell)$, rather then $1/\mathsf{poly}\log(\ell)$ as given above. In prior HDX constructions, there is a barrier toward improving the soundness that comes from the cohomology of the complex becoming non-trivial for larger $\mathrm{Sym}_m = \Gamma$. Since our complexes are coboundary expanders over *every group*, this obstruction vanishes and raises the interesting possibility of showing agreement tests with optimal soundness (an important factor in the eventual goal of constructing subconstant soundness PCPs from HDX). Indeed, it is plausible that 3-query tests like the Z-test [56, 36] may have soundness $2^{-\Omega(\ell)}$.

Probabilistically Checkable Proofs (PCPs). The PCP theorem [41, 11, 10, 30] is one of the most celebrated results in theoretical computer science. Here, we will take the classic view of PCPs through hardness of approximation for the Label Cover problem.

▶ **Definition 10.** *An instance of* Label Cover $\Psi = (G = (L \cup R, E), \Sigma_\ell, \Sigma_r, \Phi = \{\phi_e\}_{e\in E})$ *consists of a bipartite graph G, alphabets Σ_ℓ, Σ_r and constraints $\phi_e : \Sigma_\ell \to \Sigma_r$, for each edge $e \in E$.*

Given a label cover instance Φ, the goal is to find assignments $A_\ell : L \to \Sigma_\ell$ and $A_r : R \to \Sigma_r$ that satisfy as many of the constraints as possible, namely that maximize the quantity

$$\mathsf{val}_\Phi(A_\ell, A_r) = \frac{1}{|E|}\left|\{e = (u,v) \in E \mid \phi_e(A_\ell(u)) = A_r(v)\}\right|.$$

This maximum value is referred to as the value of the instance Φ and denoted by

$$\mathsf{val}(\Phi) = \max_{A_\ell, A_r} \mathsf{val}_\Phi(A_\ell, A_r).$$

When viewed through the label cover problem, the PCP theorem takes the form of NP-hardness of distinguishing between satisfiable label cover instances and instances with value less than δ provided the alphabet size is large enough. Proofs of the PCP theorem proceed by reducing the 3SAT problem to a label cover instance. A longstanding goal in such reductions is to achieve small δ (soundness) without dramatically blowing up the size of the resulting label-cover instance.

[14] give the first construction of quasi-linear sized PCP with low constant soundness. They do this by showing how to use a local-spectral expander with few additional structural properties and that supports the aforementioned agreement test to construct such instances. Finally, they show that variants of the complexes from [17] have the required properties. Unfortunately, these complexes also rely on extremely involved algebra. In contrast, our construction is elementary and combinatorial, and as a corollary to our agreement testing result (Theorem 9) we obtain the following elementary low-soundness PCP construction.

▶ **Theorem 11.** *For all $\delta \in (0,1)$, there exist a constant $C > 0$ such that for all sufficiently large integers d and n there is a polynomial time reduction mapping a 3SAT instance ϕ of size n to a label cover instance Ψ over a bipartite graph G of size n' for some $n' \leqslant n \exp\left(C\sqrt{\log n \log\log n}\right)$, such that*

1. *If ϕ is satisfiable, then Ψ is satisfiable.*
2. *If ϕ is unsatisfiable, then $\mathsf{val}(\Psi) \leqslant \delta$.*
3. *The left alphabet of Ψ is $\Sigma^{\exp(C/\delta)}$ and right alphabet is $\Sigma^{\sqrt{\exp(C/\delta)}}$ for some alphabet Σ with $\log|\Sigma| \leqslant \exp\left(C\sqrt{\log n \log\log n}\right)$.*

We remark that instantiating the base PCP with [68], one can make the entire construction combinatorial with the exception of the use of a multiplication code, of which only algebraic constructions are known. Finally, we note the alphabet size in the above statement is considerably larger compared to [14]. There are standard methods to reduce this [34], but they also rely on algebraic techniques. It would be interesting to give a purely combinatorial method, and we leave this as a direction for future work.

1.2 Further Related Works

Simple High Dimensional Expanders. Beyond the celebrated construction of Kaufman and Oppenheim [60] and its variants [70, 19], a number of recent works have considered simplified constructions of local-spectral expanders. [66, 45] construct bounded degree local-spectral HDX via a product of expanders and the complete complex, but cannot achieve strong enough expansion for most applications [27]. [46] constructs local-spectral HDX of any expansion from low-rank matrices, achieving roughly the same degree as our construction. This was improved in [29] via a recursive procedure to degree $2^{\log(N)^{\varepsilon}}$ for any constant $\varepsilon > 0$. Neither construction has vanishing cohomology, and therefore cannot be used in the construction of PCPs.

Comparison with [71]. In recent independent work, O'Donnell and Singer [71], building on [63], showed that [14]'s PCPs may be instantiated on a variant of the coset-complex HDX construction of Kaufman and Oppenheim [60]. These complexes are remarkably simpler than those originally used in [14], relying only elementary matrix groups, and achieve parameters matching [14]'s construction while our flags-complex based PCP is closer to the original construction of Moshkovitz-Raz [69]. Nevertheless, [71] still requires a fairly substantial amount of algebraic manipulation, and while the complexes are undeniably simpler than [17], they are certainly more involved than the completely elementary subspace-based construction suggested in this work. Finally, as discussed our complexes have vanishing cohomology over *all* groups, rather than just Sym_m for small m, removing a major barrier toward better agreement tests and PCPs with strong sub-constant soundness.

2 Preliminaries

General Notations. We will use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. We will use Sym_A to denote the group of permutations on a set A. We will simply use Sym_n for the group $\mathrm{Sym}_{[n]}$. For a tuple t, we will write $\mathsf{set}(t)$ to denote the set of elements that occur in the tuple. For a set of sets J, we write $\cup J = \bigcup_{s \in J} s$. We abuse the symbol 0 to mean the subspace $\{0\}$ as well; such usage should be clear from context. We denote by $\lambda(\mathsf{A})$ the second largest eigenvalue of an operator A. We denote by $|\lambda|(\mathsf{A})$ the second largest eigenvalue of an operator A in absolute value. We will use $\mathbb{F}_q$ to denote the field of order q.

Let A, B be two set and $\mathcal{D}$ and distribution on A. Then, for any two functions $f : A \to B$ and $g : A \to B$, we define

$$\mathsf{dist}_{\mathcal{D}}(f, g) = \mathop{\mathbb{P}}_{x \sim \mathcal{D}} [f(x) \neq g(x)] .$$

When the distribution is not specified the uniform distribution is assumed.

2.1 Graphs and Expansion

We will use $V(G)$ and $E(G)$ to denote the vertices and the edges of a graph G, respectively. We will drop the parenthetical and simply write V and E when G is clear from the context. We will use $\Delta(G)$ to denote the max degree of the graph G. We will use $\mathsf{diam}(G)$ denote the diameter of the graph G, i.e., the maximum (over all pairs of vertices) length of shortest paths.

Probability density over Graphs. We can define an arbitrary probability density π_1 over the edges of a graph. This induces a distribution π_0 over the vertices in which we first sample an edge $\{u, v\} \sim \pi_1$ and then uniformly pick u or v. Note that this is the stationary distribution for the random walk on the graph.

Partite Graphs. A *k-partite* graph is a graph such that we can decompose $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ such that for every edge $\{u, v\} \in E$ we have $u \in V_i$ and $v \in V_j$ for $i \neq j$. If $v \in V_i$, we define the color $\mathtt{col}(v) = i$. Let $c \subseteq [k]$ be a set of colors. We define the *projection* of a graph G onto c as the graph G^c induced by the vertex set $V(G^c) = \{v \in V | \mathtt{col}(v) \in c\}$. We will often shorten $V(G^c)$ as V^c and $E(G^c)$ as E^c.

Expansion. Let $G = (V, E)$ be a graph and let $\pi_1 : E \to (0, 1]$ be a probability distribution. Let A_G be the *normalized adjacency operator*. This operator takes as input $f : V \to \mathbb{R}$ and outputs $\mathsf{A}_G f : V \to \mathbb{R}$, with

$$\mathsf{A}_G f(v) = \frac{1}{2\pi_0(v)} \sum_{u \sim v} \pi_1(uv) f(u).$$

The (normalized) adjacency operator is self adjoint with respect to the inner product on $\ell_2(V) = \{f : V \to \mathbb{R}\}$ given by

$$\langle f, g \rangle = \sum_{v \in V} \pi_0(v) f(v) g(v).$$

We write $\lambda(G) := \lambda(\mathsf{A}_G)$ and $|\lambda|(G) := |\lambda|(\mathsf{A}_G)$ for the the second largest eigenvalue and the second largest absolute value of the eigenvalue of the (normalized) adjacency operator, respectively. We say that G is a λ-*one sided spectral expander* if for every $\lambda(G) \leqslant \lambda$ and say that G is a λ-*two sided spectral expander* if $|\lambda|(G) \leqslant \lambda$. We say that G is an η-edge expander if for every subset $S \subseteq V$, $S \neq \emptyset$, it holds that

$$\mathop{\mathbb{P}}_{uv \sim \mu_1}[u \in S, v \in V \setminus S] \geqslant \eta \mathop{\mathbb{P}}_{\mu_0}[S] \mathop{\mathbb{P}}_{\mu_0}[V \setminus S].$$

Spectral and edge expansion are classically related to each other via Cheeger's Inequality [4, 5]. We will only use the following "easy direction" of Cheeger.

▶ **Lemma 12** (Folklore). *Let G be a λ-one sided spectral expander, then G is a $(1-\lambda)$-edge expander.*

We will also need the standard expander-mixing lemma, which states that the weight of edges between any two sets A, B is roughly the expected weight based on the sizes of A and B.

▶ **Lemma 13** (Expander Mixing Lemma). *Let $G = (U, V, E)$ be a bipartite graph in which the $\lambda(G) \leqslant \lambda$, and let μ be the stationary distribution over G. Then for all $A \subseteq U$ and $B \subseteq V$ we have that*

$$\left| \mathop{\mathbb{P}}_{(u,v) \in E}[u \in A, v \in B] - \mu(A)\mu(B) \right| \leqslant \lambda \sqrt{\mu(A)(1-\mu(A))\mu(B)(1-\mu(B))}.$$

We also note a well-known sampling property of the bipartite expanders.

▶ **Lemma 14.** *Let $G = (U, V, E)$ be a weighted bipartite graph with $\lambda(G) \leqslant \lambda$. Let $B \subseteq U$ be a subset with $\mu(B) = \delta$ and set*

$$T = \left\{ v \in V \,\middle|\, \mathop{\mathbb{P}}_{u \text{ neighbour of } v}[u \in B] - \delta > \varepsilon \right\}.$$

Then $\mathbb{P}[T] \leqslant \lambda^2 \delta / \varepsilon^2$.

Majority and expansion. A second very useful fact about expander graphs is that local agreement on the graph implies global agreement with majority. Let $G = (V, E)$ be a graph and $g : V \to \{1, 2, ..., n\}$ be some function. Let $S_i = \{v \in V | g(v) = i\}$. The majority assignment $\mathsf{maj}(g) \in \{1, 2, ..., n\}$ is the i such that $\mathbb{P}[S_i]$ is largest (ties broken arbitrarily). Observe that if $\mathbb{P}_v[g(v) = \mathsf{maj}(g)] \approx 1$ then for most edges $uv \in E$, it holds that $g(v) = g(u)$ (since with high probability they are both equal to $\mathsf{maj}(g)$). In expander graphs a converse to this statement also holds. That is, if for most edges $g(v) = g(u)$, then $\mathbb{P}_v[g(v) = \mathsf{maj}(g)] \approx 1$.

▶ **Lemma 15.** *Let G be an η-edge expander. Let $S_1, S_2, \ldots, S_m$ be a partition of the vertices of V. Assume that*

$$\mathop{\mathbb{P}}_{uv \sim \mu_1}[\exists i, u \in S_i \text{ and } v \notin S_i] \leqslant \varepsilon.$$

Then, there exists i such that $\mathbb{P}[S_i] \geqslant 1 - \frac{\varepsilon}{\eta}$. Stated differently, for every $g : V \to \{1, 2, ..., n\}$, we have

$$\mathop{\mathbb{P}}_{v}[g(v) \neq \mathsf{maj}(g)] \leqslant \mathop{\mathbb{P}}_{uv \sim \mu_1}[g(u) \neq g(v)].$$

2.2 Basics of Simplicial Complexes

We will now describe simplicial complexes and related notions. Our exposition and notation in this section closely follow [23, 24].

Simplicial Complexes. A *simplicial complex* X is a downward closed set system, i.e., if $s \in X$ and $t \subseteq s$, then $t \in X$.

- The sets in a simplicial complex are called *faces*. The *dimension* of a face s is $|s| - 1$.
- The dimension of a complex is the dimension of the largest face. In a *pure* simplicial complex, all maximal faces have the same dimension.
- $X(i)$ is the set of all the faces of dimension i of X. In particular, we have $X(-1) = \{\emptyset\}$. We will sometimes refer to $X(i)$ as the level i of the complex.
- The set of *oriented* k-faces is $\overrightarrow{X}(k) = \{(v_0, v_1, \dots, v_k) | \{v_0, v_1, \dots, v_k\} \in X(k)\}$.
- Let $s = (v_0, v_1, \dots, v_k)$ be an oriented face and $i \in \{0, 1, \dots, k\}$ be an index. Then, we get the face s_i by removing v_i from s.
- For convenience, typically we drop the set brackets and the tuple brackets for faces.

We will use Δ_n to denote the *complete complex* on n vertices, i.e., the $n-1$-dimensional complex with $[n]$ as the top face.

Partite Complexes. A $(d+1)$-partite simplicial complex is a d-dimensional simplicial complex such that one can decompose $X(0) = A_0 \sqcup A_1 \sqcup \dots \sqcup A_d$ such that for every $s \in X(d)$ and $i \in \{0, 1, \dots, d\}$ it holds that $|s \cap A_i| = 1$. The color of a vertex $\texttt{col}(v) = i$ such that $v \in A_i$. More generally, the color of a face s is $c = \texttt{col}(s) = \{\texttt{col}(v) | v \in s\}$. We denote by X^c, the *projection* of X to c, the set of faces $s \in X$ with the color $\texttt{col}(s) \subseteq c$, and for a singleton $\{i\}$ we sometimes write X^i instead of $X^{\{i\}}$. Note X^c is a $|c|$-partite complex.

Links, Underlying Graphs and Skeletons. Let X be a d-dimensional simplicial complex. For $k \leqslant d$, the k-*skeleton* of X is the complex $X^{\leqslant k} = \bigcup_{j=-1}^{k} X(j)$. When $k = 1$, we call this complex the *underlying graph* of X, as it consists of the vertices and edges in X (as well as the empty face). A d-dimensional complex is called a *clique complex* if every $(d+1)$-clique in its underlying graph is a face in X. The *diameter* $\mathsf{diam}(X)$, and the *max degree* $\Delta(X)$ are the diameter $\mathsf{diam}(G = (X(0), X(1)))$, and the max degree $\Delta(G)$ of the underlying graph of X. The *link* of a face $s \in X$ is the $d' = (d - |s|)$-dimensional complex $X_s = \{t \setminus s | t \in X, t \supseteq s\}$.

Joins. Given two simplicial complexes X_1 and X_2, we define the *join* $X_1 * X_2$ to be the simplicial complex defined on the vertices $X_1(0) \sqcup X_2(0)$ with

$$X_1 * X_2 = \{\sigma \sqcup \tau | \sigma \in X_1, \tau \in X_2\}.$$

Extend this definition inductively to define joins of multiple complexes.

Probability density over a simplicial complex. Let X be a d-dimensional (pure) simplicial complex. The top faces can be assigned an arbitrary probability density π_d. The probability density π_d induces a probability density π_k on the k-dimensional faces in which we first sample $t \sim \pi_d$ and then uniformly sample a subface $s \subseteq t$. We observe that

$$\pi_k(s) = \frac{1}{\binom{d+1}{k+1}} \sum_{t \supseteq s} \pi_d(t).$$

This also defines a probability density on oriented faces where all permutations of a face have the same probability. So, we have $\pi_k(s) = \frac{1}{(k+1)!} \pi_k(\mathsf{set}(s))$ for any face $s \in \overrightarrow{X}(k)$. For a simplicial complex X with a measure $\pi_d : X(d) \to (0, 1]$, the induced measure $\pi_{d',X_s} : X_s(d - |s|) \to (0, 1]$ on X_s, for $t \in X_s(d - |s|)$, i.e., the probability $t \sim \pi_{d',X_s}$ is the probability of sampling $t \sqcup s$ from π_d conditioned on containing s. Observe that we have

$$\pi_{d',X_s}(t \setminus s) = \frac{\pi_d(t)}{\sum_{t' \supseteq s} \pi_d(t')}.$$

Whenever the complex X is clear we shall write $\pi_{d',s}$ instead of π_{d',X_s}.

2.3 Expansion on a Simplicial Complex

Expansion on a simplicial complex can be defined in numerous ways. We discuss *local spectral expansion* and *coboundary expansion* here.

Local Spectral Expansion. The notion of local spectral expansion requires that underlying graph of each of its links is a spectral expander. Before formally defining local spectral expansion, let us setup some notation. We use $\lambda(X_s)$ to denote the second largest eigenvalue of the (normalized) adjacency operator of the underlying graph of X_s and we use $|\lambda|(X_s)$ to denote $|\lambda|(G_{X_s})$, where G_{X_s} is the underlying graph of X_s.

▶ **Definition 16** (Local Spectral Expansion [58, 72])**.** *Let X be a d-dimensional simplicial complex and let $\lambda \in (0,1)$. X is a λ-one sided local spectral expander if for every $s \in X^{\leqslant d-2}$ it holds that $\lambda(X_s) \leqslant \lambda$. Similarly, X is a λ-two sided local spectral expander if for every $s \in X^{\leqslant d-2}$ it holds that $|\lambda|(X_s) \leqslant \lambda$.*

▶ Remark 17. In the literature, the term *high-dimensional expander* is also sometimes used to mean local spectral expanders.

The following useful result relates the expansion of the links of faces with codimension 2 with local spectral expansion of the entire complex.

▶ **Theorem 18** (Trickle Down Theorem [72])**.** *Let X be a d-dimensional simplicial complex such that the 1-skeleton of every link is connected and for every $s \in X(d-2)$, the link X_s is a one-sided λ-expander. Then X is a μ-local spectral expander, where $\mu = \frac{\lambda}{1-(d-1)\lambda}$.*

We will also use the following variant of local-spectral expansion for partite complexes which asks that the projection of any link onto two colors is a good expander.

▶ **Definition 19** (λ-product [50])**.** *A d-partite complex X is called a λ-product if for every link X_τ of co-dimension at least 2 and colors $i, j \notin \mathtt{col}(\tau)$, we have:*

$$\lambda(X_\tau^{ij}) \leqslant \lambda.$$

Coboundary Expansion. Another useful notion of expansion in a simplicial complex is coboundary expansion. Let X be a d-dimensional simplicial complex for $d \geqslant 1$ and let Γ be any group. First, we define the notion of cochains. For $i = -1, 0$, the set of *i-cochains* is $C^i(X,\Gamma) = \{f : X(i) \to \Gamma\}$. We sometimes identify $C^{-1}(X,\Gamma) \cong \Gamma$. For $i = 1, 2$, the i-cochains are defined as follows. The 1-cochains

$$C^1(X,\Gamma) = \left\{ f : \overrightarrow{X}(1) \to \Gamma \;\middle|\; f(u,v) = f(v,u)^{-1} \right\}$$

and the 2-cochains

$$C^2(X,\Gamma) = \left\{ f : \overrightarrow{X}(i) \to \Gamma \;\middle|\; \begin{array}{l} \forall \pi \in \mathrm{Sym}_3, (v_0, v_1, v_2) \in X(2) \\ f(v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)}) = f(v_0, v_1, v_2)^{\mathsf{sign}(\pi)} \end{array} \right\}$$

are the spaces of the so-called anti-symmetric functions on edges and triangles, respectively. For $i = -1, 0, 1$, we define the "coboundary operators" as $\delta_i : C^i(X,\Gamma) \to C^{i+1}(X,\Gamma)$ by

(i) $\delta_{-1} : C^{-1}(X,\Gamma) \to C^0(X,\Gamma)$ is $\delta_{-1}h(v) = h(\emptyset)$,
(ii) $\delta_0 : C^0(X,\Gamma) \to C^1(X,\Gamma)$ is $\delta_0 h(v,u) = h(v)h(u)^{-1}$,
(iii) $\delta_1 : C^1(X,\Gamma) \to C^2(X,\Gamma)$ is $\delta_1 h(v,u,w) = h(v,u)h(u,w)h(w,v)$.

We write just δf for $\delta_i f$ when the dimension i is clear from context.

Although we want results on a general group Γ, it is instructive to consider the case where $\Gamma = \mathbb{F}_2$. Specifically, in this case, each i-cochain f is an indicator for some subset $S_f \subseteq X(i)$ of i-faces. Furthermore, the coboundary of an 0-cochain f corresponds to the indicator edges crossing the set S_f.

Let $Id = Id_i \in C^i(X, \Gamma)$ be the function that always outputs the identity element. It is easy to check that $\delta_{i+1} \circ \delta_i h = Id_{i+2}$ for all $i = -1, 0$ and $h \in C^i(X, \Gamma)$. Thus, if we write

$$Z^i(X, \Gamma) = \ker(\delta_i) \subseteq C^i,$$

and

$$B^i(X, \Gamma) = \mathrm{im}(\delta_{i-1}) \subseteq C^i,$$

for the sets of the so-called *cocycles* and *coboundaries*, respectively, we see that $B^i \subseteq Z^i$. We say i-th cohomology is *trivial* whenever $B^i = Z^i$.

▶ **Definition 20** (Cocycle and coboundary expansion [65, 48]). *Let X be a d-dimensional simplicial complex for $d \geqslant 2$. Let $-1 \leqslant k \leqslant d-1$. The k-cocycle expansion $h^k(X, \Gamma)$ of X with respect to a group Γ is given by*

$$h^k(X, \Gamma) = \min_{f \in C^k \setminus Z^k} \frac{\mathsf{wt}(\delta f)}{\mathsf{dist}(f, Z^k)}.$$

Additionally, if the k-th cohomology is trivial, i.e., we have $B^k(X, \Gamma) = Z^k(X, \Gamma)$, then we say $h^k(X, \Gamma)$ is the k-coboundary expansion of X with respect to a group Γ.

We remark the above is often called "cosystolic" rather than "cocycle" expansion in the literature. However the former originally referred to complexes with other additional properties [58], so there has been a recent shift in the it the literature to using "cocycle expansion" for this notion instead.

2.4 Walks on Simplicial Complexes

Simplicial complexes come equipped with several natural families of random walks generalizing the standard random walk on a graph.

Up-down and Down-up walks. Let $\ell \leqslant k \leqslant d$. The (k, ℓ)-*containment graph* $G_{k,\ell} = G_{k,\ell}(X)$ is the bipartite graph whose vertices are $L = X(k)$, $R = X(\ell)$ and whose edges are all (t, s) such that $t \supseteq s$. The distribution we associate to the edges is the natural distribution induced by the complex X, i.e., sampling $t \sim \pi_k$ and then uniformly sampling $s \subseteq t$ of size $|s| = \ell + 1$.

The *down operator* $\mathsf{D}_{k,\ell} : \ell_2(X(k)) \to \ell_2(X(\ell))$ is the bipartite operator of the containment graph $G_{k,l}$, i.e.,

$$\mathsf{D}_{k,\ell} f(s) = \mathop{\mathbb{E}}_{t \supseteq s} [f(t)].$$

The adjoint of $\mathsf{D}_{k,\ell}$ is the *up operator*, $\mathsf{U}_{\ell,k} : \ell_2(X(\ell)) \to \ell_2(X(k))$, given by:

$$\mathsf{U}_{\ell,k} g(t) = \mathop{\mathbb{E}}_{s \subseteq t} [g(s)].$$

Then the composition of the up and down operators $\mathsf{U}_{\ell,k}\mathsf{D}_{k,\ell}$ is called the "down-up" walk, and the "down-up" walk is similarly defined.

Swap Walks. Let X be a d-dimensional simplicial complex. Let i, j be so that $i + j + 1 \leqslant d$. The swap walk $\mathsf{S}_{i,j} = \mathsf{S}_{i,j}(X)$ is the bipartite adjacency operator of the graph $(X(i), X(j), E)$, where an edge $\{s_i, s_j\}$ is chosen in this graph by first selecting a face $t \sim \pi_{i+j+1}$, and then partitioning it to $t = s_i \sqcup s_j$ where $s_i \in X(i)$ and $s_j \in X(j)$. This walk was defined and studied independently by [3] and [21].

▶ **Lemma 21** ([50]). *Let X be a λ-two-sided local spectral expander. Then the second largest eigenvalue of $\mathsf{S}_{k,\ell}(X)$ is upper bounded by $\sqrt{(k+1)(\ell+1)}\lambda$.*

Colored Swap Walks. The standard swap walks are not well behaved on partite complexes, but there is a useful analog for this setting defined in [21]. Let X be a d-partite simplicial complex, and $F_1, F_2 \subseteq [d]$ be two disjoint subsets. The colored swap walk $\mathsf{S}_{F_1,F_2} = \mathsf{S}_{F_1,F_2}(X)$ is the bipartite adjacency operator of the graph (X^{F_1}, X^{F_2}, E), where an edge $\{s_1, s_2\}$ is chosen in this graph by first selecting a face $t \in X^{F_1 \sqcup F_2}$, and then partitioning it to $t = s_1 \sqcup s_2$ according to its colors.

▶ **Lemma 22** ([50]). *Let X be a d-partite λ-product, and $F_1, F_2 \subseteq [d]$ disjoint subsets. Then*

$$\lambda(\mathsf{S}_{F_1,F_2}) \leqslant \sqrt{|F_1||F_2|}\lambda.$$

2.5 Cones

Cones are the main machinery used in most proofs of coboundary expansion. We define (non-abelian) cones. First, we define the composition of two paths $P_0 = (v_0, v_1, \ldots, v_m)$ and $P_1 = (v_m, v_{m+1}, \ldots, v_n)$ by $P_0 \circ P_1 = (v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_n)$. Note that composition is only defined when the end point of the first path is equal to the starting point of the second.

Now, fix a simplicial complex X and some $v_0 \in X(0)$. We define two symmetric relations on loops around v_0:

(BT) We say that $P_0 \overset{BT}{\sim} P_1$ if $P_i = Q_0 \circ (u, v, u) \circ Q_1$ and $P_{1-i} = Q_0 \circ (u) \circ Q_1$ for $i = 0, 1$ (i.e. going from u to v and then backtracking is trivial).

(TR) We say that $P_0 \overset{TR}{\sim} P_1$ if $P_i = Q_0 \circ (u, v) \circ Q_1$ and $P_{1-i} = Q_0 \circ (u, w, v) \circ Q_1$ for some triangle $uvw \in X(2)$ and $i = 0, 1$.

We denote by $P \sim_1 P'$ if there is a sequence of loops $(P_0 = P, P_1, \ldots, P_m = P')$ and $j \in [m-1]$ such that:

1. $P_j \overset{TR}{\sim} P_{j+1}$ and
2. For every $j' \neq j$, $P_{j'} \overset{BT}{\sim} P_{j'+1}$.

In other words, We can get from P to P' by a sequence of "equivalences", where exactly one equivalence is by (TR).

Finally, for a path $P = (u_0, u_1, \ldots, u_m)$ on the graph underlying X, we denote by P^{-1} the inverse path $(u_m, \ldots, u_1, u_0)$.

▶ **Definition 23** (Cone). *A cone is a triple $C = (v_0, \{P_u\}_{u \in X(0)}, \{T_{uw}\}_{uw \in X(1)})$ such that*

1. *$v_0 \in X(0)$,*
2. *for every $v_0 u \in X(0)$, we have a walk P_u from v_0 to u,*
3. *for $u = v_0$, we take P_{v_0} to be the loop with no edges from v_0,*
4. *and for every $uw \in X(1)$, T_{uw} is a sequence of walks $(P_0, P_1, \ldots, P_m)$ such that*
   - **a.** *$P_0 = P_u \circ (u, w) \circ P_w^{-1}$,*
   - **b.** *for every $i = 0, 1, \ldots, m-1$, $P_i \sim_1 P_{i+1}$ and*
   - **c.** *P_m is equivalent to the trivial loop by a sequence of (BT) relations.*

We call T_{uw} a contraction, and we write $|T_{uw}| = m$. The diameter of a cone is defined as follows.

$$\operatorname{diam}(C) = \max_{uw \in X(1)} |T_{uw}|.$$

▶ **Lemma 24** ([61, 24]). *Let X be a simplicial complex such that $\operatorname{Aut}(X)$ is transitive on k-faces. Suppose that there exists a cone C with diameter R. Then the 1-coboundary expansion of X at least $\frac{1}{\binom{k+1}{3} \cdot R}$.*

2.6 Faces Complexes

▶ **Definition 25.** *Let X be a d-dimensional simplicial complex. Let $r \leqslant d$. We use $\mathrm{F}^r X$ to denote the simplicial complex whose vertices are $\mathrm{F}^r X(0) = X(r)$ and whose faces are all $\{s_0, s_1, ..., s_j\}$ such that $s_0 \sqcup s_1 \sqcup \cdots \sqcup s_j \in X((j+1)(r+1)-1)$. We call this the r-*faces complex *of X.*

Observe that this is a $\left(\lfloor \frac{d+1}{r+1} \rfloor - 1\right)$-dimensional complex.

Probability density over a faces complex. Let X be a d-dimensional simplicial complex, and let $r < d$. The distribution on the top-level faces of $\mathrm{F}^r X$ is given by the following. Let $m = \lfloor \frac{d+1}{r+1} \rfloor - 1$.

1. Sample a d-face $t = \{v_0, v_1, \ldots, v_d\} \in X(d)$.
2. Sample $s_0, s_1, \ldots, s_m \subseteq t$ such that $|s_i| = r+1$, $s_i \cap s_j = \emptyset$ and output $\{s_0, s_1, \ldots, s_m\}$.

A faces complex can be viewed as a subcomplex of the following (non-pure) simplicial complex.

▶ **Definition 26.** *Let X be a d-dimensional simplicial complex. We denote by $\mathrm{F}X$ the simplicial complex whose vertices are $\mathrm{F}X(0) = X \setminus \{\emptyset\}$ and whose faces are all $\{s_0, s_1, ..., s_j\}$ such that $s_0 \sqcup s_1 \sqcup \cdots \sqcup s_j \in X$.*

This complex is not pure in general, hence we do not define a distribution on it.

Links of a faces complex. The links of a faces complex correspond to a faces complexes of the links of the original complex. More precisely, we have the following statement.

▶ **Observation 27.** *Let $s \in \mathrm{F}X$. Then $\mathrm{F}X_s = \mathrm{F}(X_{\cup s})$, where $\cup s = \bigcup_{t \in s} t$. Furthermore, if $s \in \mathrm{F}^r X$, then we have $\mathrm{F}^r X_s = \mathrm{F}^r(X_{\cup s})$.*

We introduce the notion of *generalized links* to simplify notation for links of faces complex.

▶ **Definition 28** (Generalized links). *Let $w \in X$. We write $\mathrm{F}X_w$ for $\mathrm{F}(X_w)$. We also write $\mathrm{F}^r X_w$ for $\mathrm{F}^r X \cap \mathrm{F}X_w$.*

Note that $\mathrm{F}^r X_w$ is not necessarily a proper link of $\mathrm{F}^r X$. However, using this notation we can write the conclusions of Observation 27 more compactly as $\mathrm{F}X_s = \mathrm{F}X_{\cup s}$ and $\mathrm{F}^r X_s = \mathrm{F}^r X_{\cup s}$ for a face $s \in \mathrm{F}^r X$.

Colors on a faces complex. Let X be a k-partite complex. We denote the set of colors of $\mathrm{F}^r X$ by $\mathcal{C} = \binom{[k]}{r+1}$. Fix a set $J \subset \Delta_k$, denoted $J = \{c_1, \dots, c_m\}$, such that $c_j \subset [k]$ are pairwise disjoint. Let $\mathrm{F}^J X = \{s \in \mathrm{F}X | \mathtt{col}(s) \subseteq J\}$ be the sub-complex of $\mathrm{F}X$ whose vertex colors are in J, so $\mathrm{F}^J X(0) = \bigcup_{j=1}^m X^{c_j}$. We call such a complex a J-faces complex and also a *colored* faces complex. We will be particularly interested in the case where $J \in \mathrm{F}^r\Delta$, namely, J consists of pairwise disjoint subsets of cardinality $r+1$. In this case, $\mathrm{F}^J X$ is $|J|$-partite. We abuse notation in this section allowing multiple c_j's to be empty sets. In this case X^{c_j} are copies of $\emptyset$, and every empty set is in all top level faces of $\mathrm{F}^J X$.

The measure induced on the top level faces of $\mathrm{F}^J X$ is obtained by sampling $t \in X^{\cup J}$ and partitioning it into $t = s_1 \sqcup s_2 \sqcup \cdots \sqcup s_m$ such that $s_i \in X^{c_i}$. Finally, throughout the paper, we use the following notation. Let $J', J \subseteq \mathrm{F}\Delta$. We write $J' \leqslant J$, if $J = \{c_1, c_2, \dots, c_m\}$ and $J' = \{c'_1, \dots, c'_m\}$ where $c'_j \subseteq c_j$.

Swap Coboundary Expansion. We end this section by defining swap coboundary expansion.

▶ **Definition 29.** *A simplicial complex X is said to have (β, r)-*swap coboundary (cocycle) expansion *if $\mathrm{F}^r X$ has 1-coboundary (cocycle) expansion (with respect to every group) at least β.*

In general, our goal is to prove swap co-boundary expansion for $\beta \geqslant 2^{-o(r)}$, which is the pre-condition to a 1% agreement test and resulting PCP construction [13, 22, 14].

3 Expansion of the Projected Flags Complex

3.1 Local Spectral Expansion Bound

In this section, we show Theorem 3, that any projection of the flags complex is a good local-spectral-expander.

▶ **Theorem 3** (Spectral Expansion of the Projected Flags Complex)**.** *Fix any $d \geqslant 3$, any prime power q, and a set $S \subseteq [d-1]$ of size at least 2. Let $s \in \mathfrak{G}_q^{d,S}$ be any face with co-dimension at least 2. Then, the second eigenvalue of the underlying graph of the s-link of $\mathfrak{G}_q^{d,S}$ is $\frac{2}{(|S|-|s|-1)(\sqrt{q}-1)}$. In particular, the complex $\mathfrak{G}_q^{d,S}$ is a $\frac{2}{\sqrt{q}-1}$-local-spectral expander for any prime power q. Moreover, the complex $\mathfrak{G}_q^{d,S}$ is also a $\frac{1}{\sqrt{q}}$-product for any prime power q.*

Towards proving Theorem 3, we first show Proposition 30 which relates the spectral expansion of a k-partite graph to its bipartite components.

▶ **Proposition 30.** *Let G be a k-partite graph with the vertex-set $V = \bigsqcup_{i\in[k]} P_i$ with (normalized) edge weights given by $\pi_2 : E \to [0,1]$ and stationary distribution given by $\pi_1 : V \to [0,1]$. Suppose $\lambda_2(G^{\{i,j\}}) \leqslant \lambda_{ij}$ for all $i \neq j \in [k]$. Let K be the complete graph on k vertices with edge weight $\kappa_2(i,j) = \pi_2(E^{\{i,j\}})$. Then, the second eigenvalue of G is at most*

$$\lambda_2(G) \leqslant \lambda_2(K) + \max_{i\in[k]} \mathop{\mathbb{E}}_{j\sim\kappa_2[i]} [\lambda_{ij}] \,.$$

where $\kappa_2[i]$ denotes the distribution on the neighbors of i induced by κ_2.

The proof is fairly elementary linear algebra, and defer it to the full version. We will also need the following bound on the expansion of the Grassmann graphs.

▶ **Fact 31** ([20, 43])**.** *Fix any $d \geqslant 3$. For any $i < j \leqslant d-1$, the second eigenvalue of the underlying graph of $\mathfrak{G}_d^q[i,j]$ is at most $q^{-(j-i)/2}$.*

Proof of Theorem 3. The underlying graph of the complex is a k-partite graph where $k = |S|$ with vertex-set $V = \bigsqcup_{i \in S} P_i$ for P_i subspaces of dimension i with edges given by subspace inclusion and weight given by the number of chains containing end points of the edge with appropriate normalization. Moreover, the weights $\pi_2(E^{\{i,j\}})$ are all equal. Thus, using Proposition 30 we can bound the second eigenvalue of this graph by

$$\begin{aligned} 0 + \max_{i \in S} \mathop{\mathbb{E}}_{j \in S \setminus \{i\}} [\lambda_{ij}] &\leqslant \max_{i \in [d-1]} \mathop{\mathbb{E}}_{j \in S \setminus \{i\}} [\lambda_{ij}] \\ &\leqslant \frac{1}{(|S|-1)} \max_{i \in [d-1]} \sum_{j \in S \setminus \{i\}} \lambda_{ij} \\ &\leqslant \frac{1}{(|S|-1)} \max_{i \in [d-1]} \sum_{j \in [d-1] \setminus \{i\}} \lambda_{ij} \end{aligned}$$

Using the bounds on random walks on the projections of flags complex stated in Fact 31 we get the final bound of

$$\begin{aligned} \frac{1}{(|S|-1)} \max_{i \in [d-1]} \sum_{j \in [d-1] \setminus \{i\}} \lambda_{ij} &\leqslant \frac{1}{(|S|-1)} \max_{i \in [d-1]} \left(\sum_{j=1}^{i-1} q^{-(i-j)/2} + \sum_{j=i+1}^{d-1} q^{-(j-i)/2} \right) \\ &\leqslant \frac{1}{(|S|-1)} \max_{i \in [d-1]} \frac{(1 - q^{-(i-1)/2}) + (1 - q^{-(d-i-1)/2})}{\sqrt{q} - 1} \\ &\leqslant \frac{2}{(|S|-1)(\sqrt{q}-1)}. \end{aligned}$$

Now, consider a face $(W_{i_1} \subseteq W_{i_2} \subseteq \ldots \subseteq W_{i_\ell})$ of $\mathfrak{G}^S(\ell - 1)$. Notice that the link of this face is isomorphic to the join of projected complexes of form $\mathfrak{G}^{i_j - i_{j-1}, S_j}$ where $S_j \subseteq [i_j - i_{j-1}]$ with $i_0 = 0$. As before, the underlying graph is some $(k - \ell)$-partite graph G with each $G^{\{p,q\}}$ being either the underlying graph of $\mathfrak{G}^{i_j - i_{j-1}, S_j}$ or a complete bipartite graph. Again, similarly applying Proposition 30, Fact 31, and the fact that the second eigenvalue of a complete bipartite graph is 0 gives us the required bound of $\frac{2}{(|S|-\ell-1)(\sqrt{q}-1)}$. This observation about the structure of the links, Fact 31, and the fact that the second eigenvalue of a complete bipartite graph is 0 also lets us conclude that the complex is a $\frac{1}{\sqrt{q}}$-product. ◀

3.2 Coboundary Expansion

In this section we prove Theorem 4, that central projections of the Flags complex and their links have good 1-coboundary expansion over any group.

▶ **Theorem 4** (Coboundary Expansion of Projected Flags Complex)**.** *There exists a constant q_0 such that the following holds. Let Γ be any group. Let $d > k \geqslant 2$ be two integers and $q \geqslant q_0$ be a prime power. Let*

$$S_k = \{d - k, \ldots, d - 1, d, d + 1, \ldots, d + k\}.$$

Then, for any face $s \in \mathfrak{G}_q^{2d, S_k}$ with codimension at least 3, $\mathfrak{G}_{q,s}^{2d, S_k}$ is 1-coboundary expander with

$$h^1(\mathfrak{G}_{q,s}^{2d, S_k}, \Gamma) = \Omega(1).$$

Broadly speaking, the proof of Theorem 4 has two main steps, vanishing cohomology (Lemma 32) and co-cycle expansion (Lemma 44 + Lemma 47). We start with the former, which we prove via the theory of shellability.

3.2.1 Shellability of the Flags Complex

In this section, we prove projections of the flags complex have trivial cohomology over any group Γ.

▶ **Lemma 32.** *Fix any $d \geqslant 3$, any prime power q, any group Γ, and a set $S \subseteq [d-1]$ of size at least 2. Let $s \in \mathfrak{G}_q^{d,S}$ be any face of codimension at least 3. Then, the 1-cohomology of $\mathfrak{G}_{q,s}^{d,S}$ with respect to Γ is trivial.*

Shellability and Vanishing Cohomology. To prove Lemma 32, it is enough to show projections of the Flags complex are *shellable*, a condition roughly stating that the complex can be built up one (top-level) face at a time in such a way that each new face is glued to the prior faces along co-dimension 1 facets. It is a standard fact that shellable complexes have vanishing 1-cohomology, since they are homotopy equivalent to a wedge of spheres [16]. Here, we give a simple, self-contained proof of vanishing cohomology that does not rely on the latter machinery.

First, we formalize the standard notion of shellability.

▶ **Definition 33** (Topological Shellability)**.** *Suppose that X is a simplicial complex. A total ordering $\sigma_1, \sigma_2, \ldots, \sigma_k$ of maximal faces of X is a* topological shelling *of X if*

$$\dim(\sigma_1) \geqslant \dim(\sigma_2) \geqslant \ldots \geqslant \dim(\sigma_k),$$

and for any $j > 1$,

$$\left(\bigcup_{i<j} \sigma_i\right) \cap \sigma_j$$

is a pure $(\dim(\sigma_j) - 1)$-dimensional complex.[4] *A complex X is called* shellable *whenever a topological shelling exists.*

In other words, a complex is shellable if it may be constructed by sequentially gluing top level faces along a (union of) co-dimension 1 facets. As discussed, it is a standard fact that shellable complexes have vanishing 1-cohomology over every group (see [18, Proposition B.44], [52, Corollary 2.11], and [18, Proposition B.42]). We give an elementary proof of this fact by inductively constructing a contraction for any loop along the shelling order, where the inductive step is essentially a simple special case of Van Kampen's Theorem we handle directly.

▶ **Theorem 34.** *If X is a shellable complex of dimension at least 2, its 1-cohomology vanishes with respect to every group Γ.*

Proof. It is enough to prove any finite loop in the 1-skeleton of X can be contracted via a series of triangle and backtracking relations. We show this inductively. Given a shelling order $\sigma_1 \ldots \sigma_k$, define

$$X_j \coloneqq \bigcup_{i \leqslant j} \sigma_j$$

[4] Note the union denotes the simplicial complex given by taking all the σ_i and their downward closures.

Observe $X_1 = \sigma_1$, and $X_k = X$. We will prove by induction that every loop in X_i can be contracted via a series of triangle and backtracking relations. The base case is immediate from the fact that X_1 is a simplex (complete complex), for which it is a standard exercise to construct such a contraction.

Assume the inductive hypothesis holds for X_{j-1}, and let γ be a loop in X_j. Assume further that the loop γ does not lie entirely inside σ_j, else a contraction exists within the full simplex σ_j itself. We will prove in this case it is always possible to contract the loop γ to a loop entirely contained within X_{j-1}, at which point we may apply the inductive hypothesis.

Assume γ is not already entirely inside X_{j-1} (else we are done). Let $v \in \gamma$ be an offending vertex in $\sigma_j \setminus X_{j-1}$, and let $(v_i, \dots, v_j)$ be the largest arc of γ containing v such that the entire arc lies inside σ_j. We will rely on the following two useful facts:

1. v_i and v_j are in $X_{j-1} \cap \sigma_j$, and
2. The 1-skeleton of $X_{j-1} \cap \sigma_j$ is connected.

Let's complete the proof assuming these facts. Since $X_{j-1} \cap \sigma_j$ is connected, we may find a path between v_i and v_j entirely inside $X_{j-1} \cap \sigma_j$. Concatenating this path with the arc (v_i, v_j) gives a loop entirely inside σ_j, so in particular, the arc can be contracted to this path by triangle and backtracking relations. Repeating this procedure until there are no further loop elements outside X_{j-1} completes the proof.[5]

It is left to prove the two facts. The latter is immediate from shellability, which promises $X_{j-1} \cap \sigma_j$ is a pure $(d-1)$-dimensional complex on at most $(d+1)$ vertices. Such complexes are automatically connected (indeed have diameter at most 2), since for any vertices w, w' in the complex, either (w, w') are already in a shared face, or the top-level faces σ_w and $\sigma_{w'}$ containing them respectively share a joint vertex v, so the path (w, v, w') exists on the 1-skeleton as desired.

Toward the former, observe that since we may assume $(v_i, \dots, v_j)$ is not the entire loop (else γ is fully contained in σ_j), the arc has *boundary point(s)* v_{i-1} and v_{i+1} which are in $X_{j-1} \setminus \sigma_j$. Further, since (v_{i-1}, v_i) and (v_j, v_{j+1}) are edges in X_j, observe by definition they must come from some face σ_i for $i \leqslant j$ in the shelling order. Since v_{i-1} and v_{j+1} are not in σ_j, the edges cannot come from σ_j, so in particular must come from a prior σ_i included in X_{j-1}, so v_i and v_j are also in X_{j-1} as desired. ◀

Posets and Shellability of Projected Flags. It is left to prove that projections of the flags complex are shellable. To do so, we will rely on the fact that the flags complex arises from taking maximal chains of a graded poset, i.e., it is the so-called "order complex" of the Grassmann poset. Such complexes have a simple combinatorial characterization of shellability. Moreover, if the poset is ranked, the notion is closed under projecting to a subset of ranks, exactly our projecting operation.

We need a few background definitions concerning posets. A poset P is a set P endowed with a relation $\preceq_P$ that is (i) reflexive, (ii) anti-symmetric, and (iii) transitive. Whenever it is clear we drop the subscript P and use $\preceq$. Also, if $a \neq b$ and $a \preceq b$, we write $a \prec b$.

Graded Poset. A graded poset additionally has a rank function $\rho : P \to \mathbb{N}$ that has the following properties.

(i) If $x \prec y$, then $\rho(x) < \rho(y)$.
(ii) If $x \prec y$ and there is no element z such that $x \prec z \prec y$, then $\rho(y) = \rho(x) + 1$.

---

[5] Note the process indeed terminates by an elementary potential argument measuring the number of loop vertices in $\sigma_j \setminus X_{j-1}$ since 1) the loop is finite and 2) in every round we strictly decrease the number of elements in $\sigma_j \setminus X_{j-1}$.

We stress that such a rank function may not exist for every poset. We assume all posets we work with this in this section are graded.

Joins. Given two posets P_1 and P_2, the *join* $P_1 * P_2$ is the poset on the set $P_1 \sqcup P_2$ with $\leqslant_{P_1 * P_2}$ being such that if $a \leqslant_{P_1 * P_2} b$, then either (i) $a \preceq_{P_1} b$, or (ii) $a \preceq_{P_2} b$, or (iii)$a \in P_1$ and $b \in P_2$. Extend this definition inductively to define joins of multiple posets.

▶ **Definition 35** (Order Complex). *Let P be a poset. The* order complex X_P *of the poset P is a simplicial complex consisting of chains of the poset. In other words, whenever $a_1 \preceq_P a_2 \preceq_P \cdots \preceq_P a_k$ we have $\{a_1, a_2, \ldots, a_k\} \in X_P$.*

▶ **Definition 36** (Grassmann Poset). *Given $d \in \mathbb{N}$ and prime power q, the* Grassmann poset GP_q^d *is the poset whose elements are the subspaces of $\mathbb{F}_q^d$ and $V_1 \preceq_{\mathrm{GP}_q^d} V_2$ whenever $V_1 \subseteq V_2$.*

Thus the flags complex is exactly the order complex of the Grassmann poset. We now move to defining shellability of posets.

▶ **Definition 37** (Shellability of Posets). *Suppose $(P, \preceq)$ is a poset. Moreover, assume that P is* pure*, i.e., all its maximal chains are of the same length. If its maximal chains can be ordered as $m_1, m_2, \ldots, m_t$ so that for every $1 \leqslant i < j \leqslant t$, there exists a $1 \leqslant k < j$ and $x \in m_j$ such that*

$$m_i \cap m_j \subseteq m_k \cap m_j = m_j \setminus \{x\},$$

then we say P is a (pure) shellable poset*. Moreover, such an ordering is called a* shelling order*.*

The following two lemmas, which state respectively that 1) projections of shellable posets are shellable, and 2) the order complex of a shellable poset is topologically shellable, are simple exercises and can be found in [18].

▶ **Lemma 38** ([18, Theorem C.9]). *Suppose $(P, \preceq, \rho)$ is a pure ranked poset and r is the rank of the maximal elements. If P is shellable, then for any $S \subseteq [r]$, $P_S = \{x \in P | \rho(x) \in S\}$ is a shellable poset.*

▶ **Lemma 39** ([18, Proposition C.6]). *Let P be a shellable poset. Then, the order complex X_P is pure and shellable.*

Thus it suffices for us to prove the Grassmann poset is shellable. This fact is fairly standard, and can be considered a special case of [16, Theorem 3.7]. We give a self-contained direct proof here adapted from the proof of shellability of the Boolean poset in [18, Proposition C.8].

▶ **Lemma 40.** *For d any positive integer and q a prime power, the Grassmann poset GP_q^d is shellable.*

Proof. The idea is to label each maximal chain by an ordered basis $(v_1, \ldots, v_d)$ of $\mathbb{F}_q^d$ with $W_i = \mathrm{span}\{v_1, \ldots, v_i\}$ for all i, taking each v_i to be the smallest vector in $W_i \setminus W_{i-1}$ under a fixed total order on $\mathbb{F}_q^d$. We show lexicographically ordering chains by these labels gives a shelling. We formalize the argument below.

Labelling maximal chains. Fix an arbitrary total order on $\mathbb{F}_q^d$. To each maximal chain

$$m\colon \quad 0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_d = \mathbb{F}_q^d,$$

associate the label-vector $\lambda(m) = (v_1, \dots, v_d)$, where v_i is the smallest vector in $W_i \setminus W_{i-1}$ under our ordering. Note by definition we have $W_i = \text{span}\{v_1, \dots, v_i\}$.

We also extend the labeling to maximal chains lying *between* two subspaces $M \subseteq N$.[6] In particular, given $M \subseteq N$ and a maximal chain $c\colon M = W_0 \subseteq \cdots \subseteq W_k = N$, let $\lambda(c) = (v_1, \dots, v_k)$ where v_i is the smallest vector in $W_i \setminus W_{i-1}$, so that $W_i = W_0 \oplus \text{span}\{v_1, \dots, v_i\}$.

The proposed shelling order. Order all maximal chains of GP_q^d as $m_1, m_2, \dots$ in lexicographic order of their labels:

$$\lambda(m_1) < \lambda(m_2) < \lambda(m_3) < \cdots .$$

We claim this is a shelling. Fix $i < j$; we must produce some $k < j$ together with a vertex $V_t \in m_j$ such that

$$m_j \cap m_k \;=\; m_j \setminus \{V_t\} \quad \text{and} \quad m_j \cap m_i \;\subseteq\; m_j \cap m_k.$$

Setup. Write the two chains as

$$m_i\colon \quad 0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d = \mathbb{F}_q^d,$$
$$m_j\colon \quad 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_d = \mathbb{F}_q^d,$$

and let $r \geqslant 0$ be the largest index such that $V_i = U_i$ for all $i \leqslant r$, and $s > r$ be the smallest index such that we again have $V_s = U_s$. Abusing notation, we write $\lambda(V_a)$ for the a-th entry of $\lambda(m_j)$.

Reduction to finding a descent. We claim it suffices to produce an index $r < t < s$ at which the labels of m_j *descend*, i.e., $\lambda(V_t) > \lambda(V_{t+1})$. Suppose we have such a t, and write $\beta = \lambda(V_{t+1})$. Set

$$V \;=\; V_{t-1} + \text{span}\{\beta\}.$$

Since $\beta \in V_{t+1} \setminus V_{t-1}$, V is a subspace strictly between V_{t-1} and V_{t+1} and distinct from V_t, so replacing V_t with V in m_j produces a maximal chain

$$m_k\colon \quad V_0 \subseteq \cdots \subseteq V_{t-1} \subseteq V \subseteq V_{t+1} \subseteq \cdots \subseteq V_d.$$

We now argue that
1. m_k comes before m_j in the order, and
2. $m_j \cap m_i \subseteq m_j \setminus \{V_t\} = m_j \cap m_k$.

We start with the former. Because $\beta \in V \setminus V_{t-1}$, the t-th entry of $\lambda(m_k)$ is at most β. On the other hand, $\beta = \lambda(V_{t+1}) < \lambda(V_t)$ since t is our descent. Since $\lambda(m_k)$ and $\lambda(m_j)$ agree in positions $1, \dots, t-1$, we therefore have $\lambda(m_k) < \lambda(m_j)$ lexicographically as desired.

Towards the latter, first observe that by construction $m_j \cap m_k = m_j \setminus \{V_t\}$. As $r < t < s$, the chains m_i and m_j disagree on V_t. Thus, we have $V_t \notin m_i$ and $m_j \cap m_i \subseteq m_j \setminus \{V_t\} = m_j \cap m_k$ as desired.

[6] A chain from A to B is *maximal between them* if no element $A \preceq C \preceq B$ can be added.

Existence of a descent. It remains to produce such a t. This is a consequence of the following fact.

▷ Claim 41. Let $M \subseteq N$ be two subspaces in this poset. Then, there is a unique maximal chain c starting at M and ending at N with increasing label-vector $\lambda(c)$. Moreover, $\lambda(c)$ is lexicogaphically minimal among labelings of maximal chains between M and N.

Namely, consider the sub-chains of m_i and m_j between the matching endpoints $V_r = U_r$ and $V_s = U_s$. Because $\lambda(m_i) < \lambda(m_j)$ and the two chains first disagree at step $r+1$, the labels of the m_i-sub-chain are lexicographically smaller than those of the m_j-sub-chain. In particular, the m_j-sub-chain is *not* the lexicographical minimum, and so its label-vector is not strictly increasing and there must exist a descent. It is left to prove the claim.

Proof of Claim 41. We start with uniqueness. First note that a maximal chain $M = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k = N$ starting at M and ending at N with increasing label-vector can be constructed by taking $W_0 = M$ and then at each step i adding the smallest vector $w_i \in N \setminus W_{i-1}$ to W_{i-1} to get $W_i = W_{i-1} \oplus \mathsf{span}\,\{w_i\}$. This implies $(w_1 < w_2 < \cdots < w_k)$ is the label-vector for the constructed chain.

Towards a contradiction, assume that there are two (distinct) maximal chains c, c' starting at M and ending at N both with increasing label vector. Suppose the chains c, c' are given by

$$M = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k = N,$$
$$M = W'_0 \subseteq W'_1 \subseteq \cdots \subseteq W'_k = N,$$

respectively, and the label-vectors are given by $(w_1, w_2, \ldots, w_k)$ and $(w'_1, w'_2, \ldots, w'_k)$, respectively. Suppose i is the first index such that $w_i \neq w'_i$ and without loss of generality assume that $w_i < w'_i$. We must have $w_i \notin W'_i$; otherwise, $W_i = W'_i$ which implies w_i is the correct vector to associate with $W'_i \setminus W'_{i-1}$. This also means that the w_i must be contained in some W'_j with $k \geqslant j > i$. So, the corresponding label $w'_j \leqslant w_i$. On the other hand, since $(w'_1, w'_2, \ldots, w'_k)$ is increasing we must have $w'_j > w'_i > w_i$, which gives us a contradiction.

To see $\lambda(c)$ is minimal, consider another chain $\tilde{c}$ given by

$$M = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = N$$

with label-vector $(s_1, \ldots, s_k)$. If $s_1 > w_1$, then we are done. Otherwise, let i be the smallest index such that $s_i \neq w_i$ but $s_{i-1} = w_{i-1}$. Then, as w_i is the smallest vector in $N \setminus W_{i-1} = S_{i-1}$, we must have $s_i > w_i$. ◁

◀

We already have everything we need to conclude that the cohomology of the projected flags complex is trivial. Furthermore, we can use the following observations to conclude that every link of the the projected flags complex also has trivial cohomology.

The following observation trivially follows from the definitions of join of simplicial complexes and join of posets.

▶ **Observation 42.** *Let P_1, P_2 be two posets. Then, we have $X_{P_1 * P_2} = X_{P_1} * X_{P_2}$.*

▶ **Observation 43.** *Let P_1 and P_2 be shellable posets. Then, the join $P_1 * P_2$ is also shellable.*

Proof. Let $m_1, m_2, \ldots, m_t$ be the maximal chains of P_1 in a shelling order. Let $m'_1, m'_2, \ldots, m'_{t'}$ be the maximal chains of P_2 in a shelling order. Then, it is easy to verify that the following is the sequence of maximal chains of $P_1 * P_2$ in a shelling order:

$$m_1 \circ m'_1, m_2 \circ m'_1, \ldots, m_t \circ m'_1, m_1 \circ m'_2, m_2 \circ m'_2, \ldots, m_t \circ m'_2, \ldots\ldots, m_1 \circ m'_{t'}, m_2 \circ m'_{t'}, \ldots, m_t \circ m'_{t'},$$

where $\circ$ is used to denote concatenation. ◀

Since links of the flags complex are joins of flags complexes in lower dimension, Lemma 32 immediately follows.

3.2.2 Cocycle Expansion of the Projected Flags Complex

With vanishing co-homology out of the way, it is enough to prove the complexes are also good 1-cocycle expanders. Toward this end, we'll rely on some useful techniques from [23] for coboundary and cocycle expansion of a complex with respect to a general (possibly non-abelian) group. The first allows us to derive cocycle expansion from spectral and coboundary expansion of the links of X.

▶ **Lemma 44** ([23]). *Let $\beta, \lambda > 0$ and let $k > 0$ be an integer. Let X be a d-dimensional simplicial complex for $d \geqslant k+2$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every non-empty $r \in X$, X_r is a coboundary expander and that $h^{k+1-|r|}(X_r, \Gamma) \geqslant \beta$. Then*

$$h^k(X, \Gamma) \geqslant \frac{\beta^{k+1}}{(k+1)! \cdot 4} - e\lambda.$$

Here, $e \approx 2.71$ is the Euler's number.

The second tool is the "color restriction lemma" which states that if most projections of a complex are coboundary expanders, then the complex itself is a coboundary expander.

▶ **Lemma 45** ([23]). *Let ℓ, d be integers so that $3 \leqslant \ell \leqslant d$ and let $\beta, p \in (0,1]$. Let Γ be some group. Let X be a $(d+1)$-partite simplicial complex so that*

$$\mathop{\mathbb{P}}_{I \in \binom{[d+1]}{\ell}} \left[X^I \text{ has 1-coboundary expansion } \beta \text{ and } \forall s \in X(0), X^I_s \text{ is a } \frac{1}{2}\text{-local spectral expander} \right] \geqslant p.$$

Then X is a coboundary expander with $h^1(X) \geqslant \frac{p\beta}{36}$.

To take advantage of Lemma 45, we start by showing any projection of the flags complex onto a "well-separated" set of three colors has constant coboundary expansion by directly constructing a family of cones.

▶ **Lemma 46.** *Fix any integer $d \geqslant 3$, any prime power q, and Γ any group. Let $I = \{i_0 < i_1 < i_2\} \subseteq [d-1]$. Let $s \in \mathfrak{G}_q^{d,[d-1]\setminus I}$. Assume that*

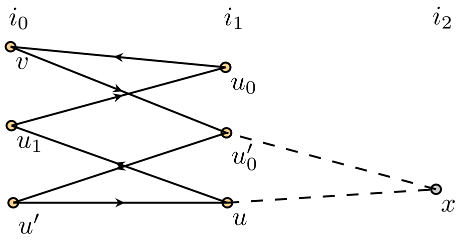
1. *$2i_0 \leqslant i_1$ or there is a subspace $V \in s$ with $i_0 < \mathtt{col}(V) < i_1$, and*
2. *either $2i_1 - i_0 \leqslant i_2$, or $2i_1 - \dim(V) \leqslant i_2$ where $V \in s$ is a subspace with $i_0 < \mathtt{col}(V) < i_1$, or there is a subspace $W \in s$ with $i_1 < \mathtt{col}(W) < i_2$.*

Then, the complex $\mathfrak{G}_{q,s}^{d,I}$ has 1-coboundary expansion at least

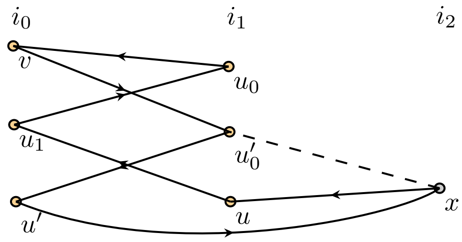
$$h^1\left(\mathfrak{G}_{q,s}^{d,I}, \Gamma\right) \geqslant \frac{1}{13}. \tag{1}$$

Proof. Let $\mathfrak{G} = \mathfrak{G}_q^d$. It is elementary to see the group of automorphisms $\mathrm{Aut}(\mathfrak{G}_s^I)$ acts transitively on the triangles of $\mathfrak{G}_s^I$ so by Lemma 24 it is sufficient to construct a cone of diameter 13.[7]

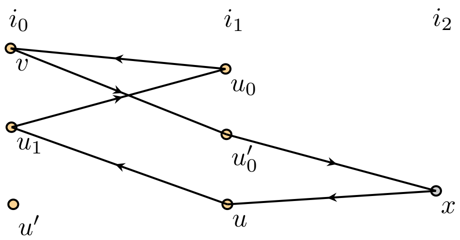
[7] Namely taking an appropriate basis for the vector spaces in any two such flags, we can transform one to the other with the corresponding basis exchange matrix.



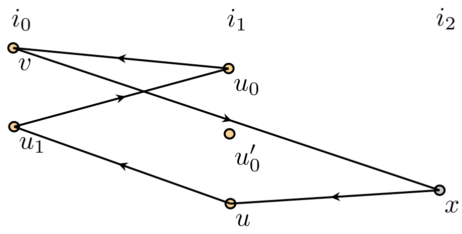
(a) $P_0 = P_u \circ (u', u) \circ P_{u'}^{-1} = vu_0'u'uu_1u_0v$.



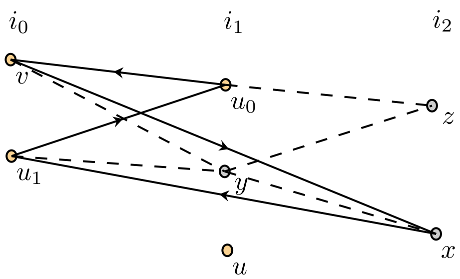
(b) $P_1 = vu_0'u'xuu_1u_0v$.



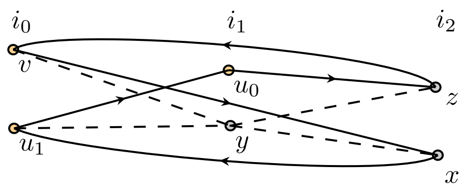
(c) $P_2 = vu_0'xuu_1u_0v$.



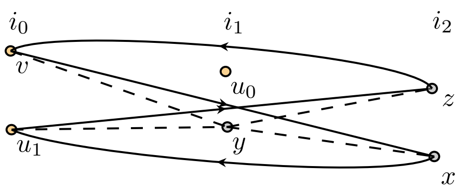
(d) $P_3 = vxuu_1u_0v$.



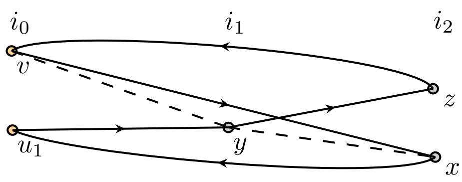
(e) $P_4 = vxu_1u_0v$.



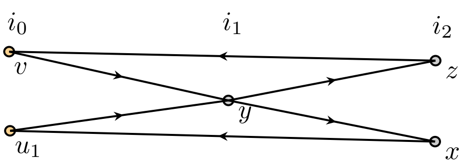
(f) $P_5 = vxu_1u_0zv$.



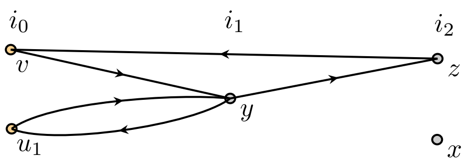
(g) $P_6 = vxu_1zv$.



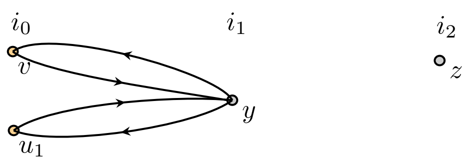
(h) $P_7 = vxu_1yzv$.



(i) $P_8 = vyxu_1yzv$.

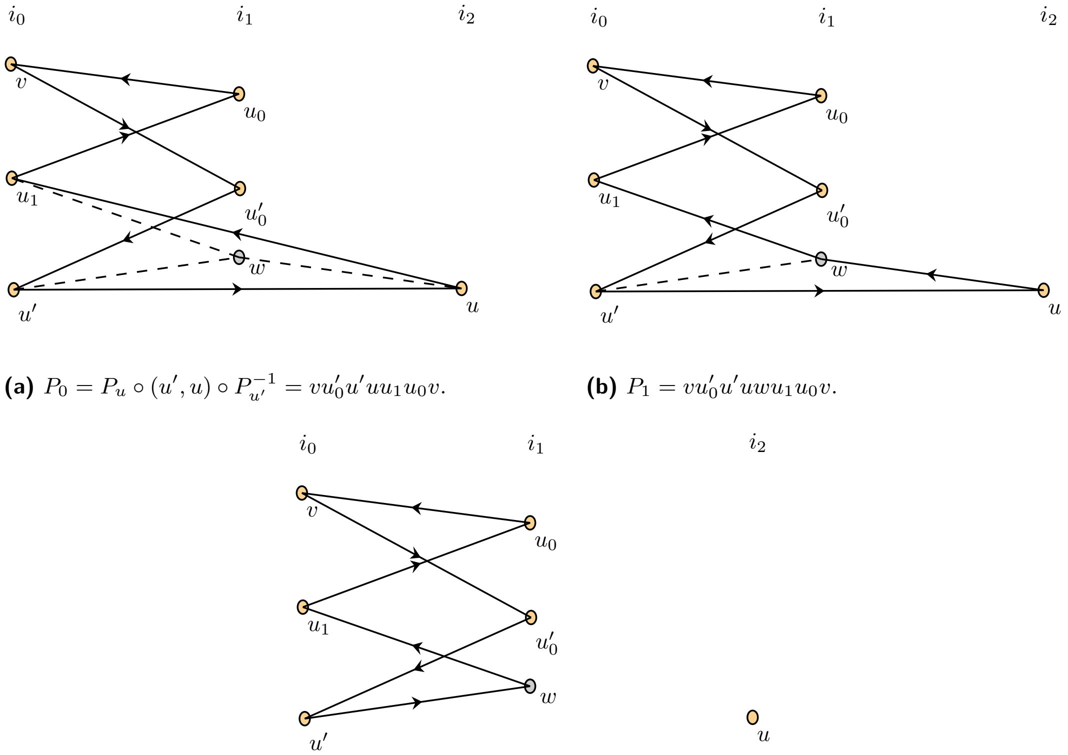


(j) $P_9 = vyu_1yzv$.



(k) $P_{10} = vyu_1yv$.

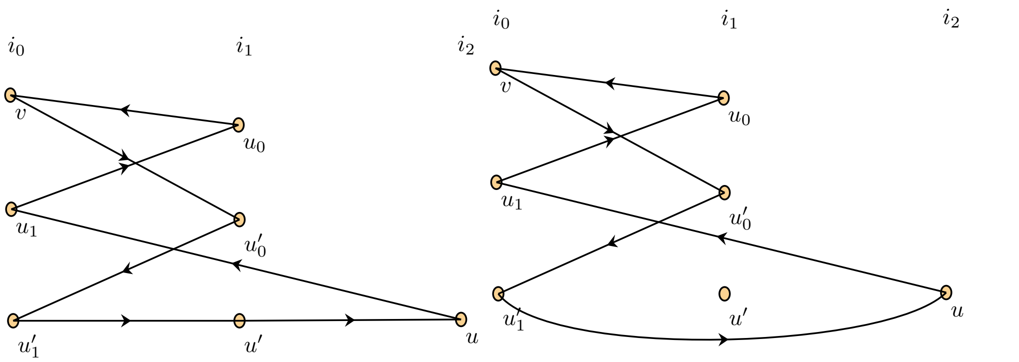
Figure 1 contraction $T_{u'u}$ when $u \in \mathfrak{G}^{\{i_1\}}$ and $u' \in \mathfrak{G}_s^{\{i_0\}}$.



(a) $P_0 = P_u \circ (u', u) \circ P_{u'}^{-1} = vu_0'u'uu_1u_0v$. **(b)** $P_1 = vu_0'u'uwu_1u_0v$.

(c) $P_2 = vu_0'u'wu_1u_0v$.

Figure 2 contraction $T_{u'u}$ when $u \in \mathfrak{G}^{\{i_2\}}$ and $u' \in \mathfrak{G}_s^{\{i_0\}}$ (reduction to $u \in \mathfrak{G}^{\{i_1\}}$ and $u' \in \mathfrak{G}_s^{\{i_0\}}$).



(a) $P_0 = P_u \circ (u', u) \circ P_{u'}^{-1} = vu_0'u_1'u'uu_1u_0v$. **(b)** $P_1 = vu_0'u_1'uwu_1u_0v$.

Figure 3 contraction $T_{u'u}$ when $u \in \mathfrak{G}^{\{i_2\}}$ and $u' \in \mathfrak{G}_s^{\{i_1\}}$ (reduction to $u \in \mathfrak{G}^{\{i_2\}}$ and $u' \in \mathfrak{G}_s^{\{i_0\}}$).

Fix an arbitrary $v \in \mathfrak{G}_s^{\{i_0\}}$. For any vertex $u \in \mathfrak{G}_s^{\{i_0\}} \setminus \{v\}$, we take $P_u = vu_0u$, where u_0 is an arbitrary vertex in $\mathfrak{G}_s^{\{i_1\}}$ such that $u_0 \supseteq v + u$. For any vertex $u \in \mathfrak{G}_s^{\{i_1,i_2\}}$, take $P_u = P_{u_1} \circ (u_1, u)$, where u_1 is an arbitrary vertex in $\mathfrak{G}_s^{\{i_0\}} \setminus \{v\}$ such that $u \supseteq u_1$.

We now describe the contraction $T_{u'u}$ for each edge $u'u$. First, suppose $u \in \mathfrak{G}^{\{i_1\}}$ and $u' \in \mathfrak{G}_s^{\{i_0\}}$. Then the first walk in the contraction is given by

$$P_0 = P_u \circ (u', u) \circ P_{u'}^{-1} = vu_0'u'uu_1u_0v.$$

As either $u_0' \cap u \supseteq u'$ and $i_2 \geqslant 2i_1 - i_0$, or $u_0' \cap u \supseteq V$ and $i_2 \geqslant 2i_1 - \dim(V)$, or $W \supseteq u_0', u$ there exists a vertex $x \in \mathfrak{G}_s^{\{i_2\}}$ such that $x \supseteq u + u_0'$. Now we get the following set of $\sim_1$ relations.

1. $P_0 \sim_1 P_1 = vu_0'u'xuu_1u_0v$ using the loop uxu and then the triangle $u'xu$.
2. $P_1 \sim_1 P_2 = vu_0'xuu_1u_0v$ using the triangle $u_0'u'x$.
3. $P_2 \sim_1 P_3 = vxuu_1u_0v$ using the triangle $vu_0'x$.
4. $P_3 \sim_1 P_4 = vxu_1u_0v$ using the triangle xuu_1.

Observe that there exists $y \in \mathfrak{G}_s^{\{i_1\}}$ such that $v + u_1 \subseteq y \subseteq x$ as $2i_0 \leqslant i_1$ or $v + u_1 \subseteq V$. Also there is a $z \in \mathfrak{G}_s^{\{i_2\}}$ such that $z \supseteq y + u_0$ since $y \cap u_0 \supseteq v$ or $y + u_0 \subseteq W$. This gives us another sequence of $\sim_1$ relations.

5. $P_4 \sim_1 P_5 = vxu_1u_0zv$ using the triangle u_0zv.
6. $P_5 \sim_1 P_6 = vxu_1zv$ using the triangle u_1u_0z.
7. $P_6 \sim_1 P_7 = vxu_1yzv$ using the triangle u_1yz.
8. $P_7 \sim_1 P_8 = vyxu_1yzv$ using the triangle vyx.
9. $P_8 \sim_1 P_9 = vyu_1yzv$ using the triangle yxu_1.
10. $P_9 \sim_1 P_{10} = vyu_1yv$ using the triangle yzv.

This completes the contraction $T_{u'u}$ as P_{10} is just two loops. Now, we reduce the task of finding contraction in the other two cases to the case above. Suppose $u \in \mathfrak{G}_s^{\{i_2\}}$ and $u' \in \mathfrak{G}_s^{\{i_0\}}$. Then, we start with $P_0 = vu_0'u'uu_1u_0v$. Note that there exists a $w \in \mathfrak{G}_s^{\{i_1\}}$ such that $u \supseteq w \supseteq u' + u_1$. Observe that $P_0 \sim_1 P_1 = vu_0'u'uwu_1u_0v$ using the triangle uwu_1, and $P_1 \sim_1 P_2 = vu_0'u'wu_1u_0v$ using the triangle $u'wu$ and then the loop wuw. This is precisely the walk at the start of the first case (except for the labeling of vertices). Finally, assume $u \in \mathfrak{G}^{\{i_2\}}$ and $u' \in \mathfrak{G}_s^{\{i_1\}}$. Then, we start with $P_0 = vu_0'u_1'u'uu_1u_0v$. We reduce it to the second case by observing $P_0 \sim_1 P_1 = vu_0'u_1'uu_1u_0v$ using the triangle $u_1'u'u$. ◀

While we cannot use Lemma 46 directly to show the projected flags complex is a coboundary expander (since *no* 3-color projection there-in is well-separated), we *can* use it to show the complex's *links* are good coboundary expanders. We may then combine this with the local-to-global lemma Lemma 44 to complete the proof.

▶ **Lemma 47.** *Fix integers $d > k \geqslant 2$, and a prime power $q \geqslant 25$. Let*

$$S_k = \{d-k, \dots, d-1, d, d+1, \dots, d+k\}$$

and let $s \in \mathfrak{G}_q^{2d,S_k}$ be a non-empty face in the projected building $\mathfrak{G}_q^{2d,S_k}$ with co-dimension at least 3. Then, the link $\mathfrak{G}_{q,s}^{2d,S_k}$ has 1-coboundary expansion

$$h^1(\mathfrak{G}_{q,s}^{2d,S_k}, \Gamma) = \Omega(1).$$

Proof. Let $c^* = \mathtt{col}(s) \subseteq S_k$ and write $c^* = \{j_1, \dots, j_b\}$. Let $S' = S_k \setminus \mathtt{col}(s)$. Now, we want to show that for a constant fraction of $I = \{i_0 < i_1 < i_2\} \in \binom{S'}{3}$, $\mathfrak{G}_{q,s}^{2d,I}$ has 1-coboundary expansion at least β for some constant $\beta > 0$. The proof is then complete using Lemma 45, as by Theorem 3 we already know that $\mathfrak{G}_{q,s}^{2d,I'}$ is $\frac{1}{2}$-local spectral expander for any I' for sufficiently large q. Pick a uniformly random set $I = \{i_0 < i_1 < i_2\} \in \binom{S'}{3}$.

Define $b+1$ *bins*

$$B_i = \{j \in S_k | j_i < j < j_{i+1}\}$$

for $0 < i < b$ and also $B_0 = \{j \in S_k | j < j_1\}$ and $B_b = \{j \in S_k | j > j_b\}$. We divide the proof into two cases based on the size of the s

Case 1: Small s. Assume $|s| \leqslant 2k+1-80$. In this case, we may assume there is a bin B_i with $|B_i| > \frac{2k+1-|s|}{4} \geqslant 20$, as otherwise a union bound implies the probability our three random indices lie in separate bins is at least

$$1 - \frac{(2k+1-|s|)/4}{2k+1-|s|} - \frac{2(2k+1-|s|)/4}{2k+1-|s|} \geqslant \frac{1}{4},$$

so $h^1(\mathfrak{G}^{2d,I}_{q,s}, \Gamma) \geqslant \frac{1}{13}$ by Lemma 46 and $h^1(\mathfrak{G}^{2d,S_k}_{q,s}, \Gamma) \geqslant \frac{1}{1872}$. We may also assume $j_i \neq 0$; otherwise, observe that any projected building is isomorphic to the projected building formed by replacing each subspace with its orthogonal complement, where $j_i = b$ instead. Since $|B_i| > \frac{2k+1-|s|}{4}$, with probability at least

$$\frac{|B_i|/20}{2k+1-|s|} \cdot \frac{|B_i|/5}{2k+1-|s|} \cdot \frac{|B_i|/4}{2k+1-|s|} \geqslant \frac{1}{25600}$$

i_0 is in the second twentieth of the indices B_i, i_1 is in the second fifth of the indices B_i, and i_2 is in the final quarter of the indices B_i. In this case, the link is isomorphic to $\mathfrak{G}_q^{j_{i+1}-j_i,\{i_0-j_i,i_1-j_i,i_2-j_i\}}$ where $\{i_0-j_i, i_1-j_i, i_2-j_i\}$ satisfy the spreadness conditions of Lemma 46. Thus $h^1(\mathfrak{G}^{2d,I}_{q,s}, \Gamma) \geqslant \frac{1}{13}$ with probability at least $\frac{1}{25600}$ and $h^1(\mathfrak{G}^{2d,S_k}_{q,s}, \Gamma) \geqslant 10^{-8}$ as desired.

Case 2: Large s. When $|s| > 2k+1-80$, it suffices to show there exists a single choice of I with constant coboundary expansion, since each valid I appears with probability at least $\frac{1}{\binom{2k+1-|s|}{3}} = \Omega(1)$. We now show such a choice of I always exists. We break into cases based on maximum bin size

1. Suppose we have a bin B_i with at least three colors. As before, by taking orthogonal complements if necessary, we may assume $i \neq 0$ (similarly for the cases below). We may then take $I = \{j_i+1, j_i+2, j_i+3\}$. As $\mathfrak{G}^{2d,I}_{q,s}$ isomorphic to $\mathfrak{G}_q^{j_{i+1}-j_i,\{1,2,3\}}$, Lemma 46 implies $h^1(\mathfrak{G}^{2d,I}_{q,s}, \Gamma) \geqslant \frac{1}{13}$.
2. Suppose we have a bin B_i with two colors and there is a non-empty bin $B_{i'}$ where $i' > i > 0$. Then, taking $I = \{j_i+1, j_i+2, j_{i'}+1\}$ we get that $\mathfrak{G}^{2d,I}_{q,s}$ is isomorphic to $\mathfrak{G}_{q,s'}^{j_{i'+1}-j_i,\{1,2,j_{i'}-j_i+1\}}$ for the appropriate $s' \subset s$ between these indices. In particular, s' contains a subspace of dimension $j_{i'} - j_i$, so $\mathfrak{G}^{2d,I}_{q,s}$ has constant coboundary expansion by Lemma 46. If we instead have B_i with one color and a bin $B_{i'}$ with two colors where $i' > i$, the proof is analogous.
3. Finally, we may have three non-empty bins $B_i, B_{i'}, B_{i''}$, where $i'' > i' > i$. Taking $I = \{j_i+1, j_{i'}+1, j_{i''}+1\}$ we get that $\mathfrak{G}^{2d,I}_{q,s}$ is isomorphic to $\mathfrak{G}_{q,s'}^{j_{i''+1}-j_i,\{1,j_{i'}-j_i+1,j_{i''}-j_i+1\}}$ where $s' \subset s$ contains subspaces of dimensions $j_{i'}-j_i$ and $j_{i''}-j_i$, and constant coboundary expansion again follows from Lemma 46. ◀

The proof of Theorem 4 now follows directly from Lemma 44: if all links of a local spectral expander with vanishing cohomology are coboundary expanders, then so is the complex itself.

Proof of Theorem 4. By Lemma 47, it is enough to handle just the case of the full complex $h^1(\mathfrak{G}_q^{d,S_k},\Gamma)$. Toward this end, let q_0 be large enough so that we have $\frac{2e}{\sqrt{q_0}-1} \leqslant \frac{\beta^2}{16}$. Let $V, W \in \mathfrak{G}_q^{2d,S_k}(1)$ be any edge. By Theorem 3, we get that $\mathfrak{G}_{q,\{W,V\}}^{d,S_k}$ is a $1/2$-local spectral expander and by Lemma 47 we have $h^1(\mathfrak{G}_{q,V}^{d,S_k},\Gamma) = \Omega(1)$. So, there is a constant $\beta > 0$ such that $h^0(\mathfrak{G}_{q,\{W,V\}}^{d,S_k},\Gamma) \geqslant \beta$ and $h^1(\mathfrak{G}_{q,V}^{d,S_k},\Gamma) \geqslant \beta$. As $\frac{2e}{\sqrt{q_0}-1} \leqslant \frac{\beta^2}{16}$ by Lemma 44 we get $h^1(\mathfrak{G}_q^{d,S_k},\Gamma) \geqslant \beta^2/16$. ◀

4 The Faces Complexes

In this section, we analyze the expansion properties of the faces complexes of the projected flags complex.

4.1 Local Spectral Expansion

We first bound the local spectral expansion of the faces complexes. The following lemma, which relates the local spectral expansion of the faces complex on disjoint colors to the local spectral expansion of the partite base complex, is standard but we include it for completeness.

▶ **Lemma 48.** *Let X be a k-partite λ-product. Let $2r+2 \leqslant k$. Let $J \subseteq \binom{[k]}{r+1}$ be a subset of disjoint colors. Then $\mathrm{F}^J X$ is a $\frac{(r+1)\lambda}{1-(k-r)\lambda}$-one-sided local spectral expander.*

Proof. Let $s \in \mathrm{F}^J X$ with co-dimension exactly 2. Observe that the random walk matrix of $(F^J X)_s$ is

$$\mathsf{W}_s = \begin{bmatrix} 0 & \mathsf{S}_{c_1,c_2} \\ \mathsf{S}_{c_2,c_1} & 0 \end{bmatrix},$$

where $c_1, c_2 \sqcup \texttt{col}(s) = J$. By Lemma 22, we get that $\lambda(W_s) \leqslant (r+1)\cdot\lambda$. Similarly, links of co-dimension i are i-partite graphs where each bi-partite component is a corresponding swap walk in the original complex as above, and are therefore connected. The Trickle-Down Theorem (Theorem 18) then gives the desired statement. ◀

Now, the following lemma is an easy consequence of Theorem 3 and Lemma 48.

▶ **Lemma 49.** *Fix any $d \geqslant 3$, any prime power q, and a set $S \subseteq [d-1]$ of size at least 2. Let $2r+2 \leqslant |S|$ and $J \subseteq \binom{S}{r+1}$ be disjoint color sets. Then $\mathrm{F}^J(\mathfrak{G}_q^{d,S})$ is a $\frac{2(r+1)}{\sqrt{q}-1-2(|S|-r)}$-local-spectral expander for any prime power q.*

4.2 Coboundary Expansion

We now prove the faces complexes associated with the projected flags complex is good 1-coboundary expander.

▶ **Theorem 50** (Swap coboundary expansion of the projected flags complex)**.** *Let d, k, r be integers such that $d > k \geqslant r^5$. There is some $q_0 = q_0(k,r)$ such that the following holds. Let $q \geqslant q_0$ be any prime power, $\mathfrak{G}$ be the $\mathfrak{G}_q^{2d}$-flags complex, and*

$$S_k = \{d-k, \dots, d-1, d, d+1, \dots, d+k\}.$$

Let $\mathcal{Z} = \mathrm{F}^r(\mathfrak{G}^{S_k})$ be its faces complex. Then $\mathcal{Z}$ is a coboundary expander and $h^1(\mathcal{Z}) \geqslant \exp(-O(\sqrt{r}\log^2 r))$.

As in [24, Lemma 8.2], we start by showing it is enough to argue that a constant fraction of projections of the faces complex are themselves co-boundary expanders, a variant of Lemma 45 from [23] for the faces complex. In particular, we start with the r-faces complex of a k-partite complex and consider the projections onto m disjoint r-sets of colors from $[k]$, where each such r-set acts as a color for the faces complex. If (i) all the 1-skeletons of vertex-links of all such projections are spectral expanders, and (ii) a large fraction of such projections are 1-coboundary expanders, then (abstracting [24, Lemma 8.2]) this implies coboundary expansion of the faces complex.

▶ **Lemma 51.** *Let k, r be integers such that $k \geqslant r^5$. Let X be a k-partite simplicial complex and $\mathcal{Y} = \mathrm{F}^r X$. Let $m = \sqrt{r+1}$. Assume that for each $I \in \mathrm{F}\Delta_k(m-1)$ and for each $v \in \mathcal{Y}^I$, the underlying graph of $\mathcal{Y}^I_v$ is a $\Theta(1)$-edge expander. If*

$$\mathop{\mathbb{P}}_{I\sim \mathrm{F}^r\Delta_k(m-1)}\left[\mathcal{Y}^I \text{ is a 1-coboundary expander with } h^1(\mathcal{Y}^I) \geqslant \beta\right] = p$$

where I is a uniformly chosen set of $m = \sqrt{r+1}$ pairwise disjoint colors from $\mathcal{C}$, then $\mathcal{Y}$ is a 1-coboundary expander with $h^1(\mathcal{Y}) = \Omega(\beta p^2)$.

We defer the proof of Lemma 51 to the full version of this paper as it is essentially identical to the proof of [24, Lemma 8.2] up to changes in nomenclature. We will also rely on [24, Proposition 7.1.1] to relate the coboundary expansion of the faces complex to the base complex.

▶ **Proposition 52** ([24, Proposition 7.1.1])**.** *Let X be an k-partite complex that is a λ-local spectral expander for $\lambda \leqslant \frac{1}{2r^2}$. Let $\ell \geqslant 5$ and let $J = \{c_1, c_2, \dots, c_\ell\}$ be a set of mutually disjoint colors $c_j \subseteq [k]$, with $|c_j| \leqslant r$. Denote by $R = \sum_{j=1}^{\ell} |c_j|$. Let $\beta > 0$ and assume that for every $I = \{i_1, i_2, \dots, i_\ell\}$ such that $i_j \in c_j$ and every $w \in X^{\cup J\setminus I}$, we have $h^1(X^I_w) \geqslant \beta$. Then $h^1(\mathrm{F}^J X) \geqslant \beta_1^R$ for $\beta_1 = \Omega_\ell(\beta)$.*

Before we begin proving Theorem 50, it will be helpful to establish some notation and definitions.

Notations.

- Let $\mathfrak{G} = \mathfrak{G}_q^{2d}$. Note that $\mathfrak{G}$ is $(2d-1)$-partite.
- Let $\mathcal{Z} = \mathrm{F}^r(\mathfrak{G}^{S_k})$ and $\tilde{\mathcal{Z}} = \mathrm{F}(\mathfrak{G}^{S_k})$.
- Let $\mathcal{C} = \binom{S_k}{r+1}$.
- Let $m = \sqrt{r+1}$

Given $w \in \mathfrak{G}^{S_k}$, let $c^* = \mathtt{col}(w) \subseteq S_k$ and write $c^* = \{j_1, \dots, j_b\}$. Define $b+1$ c^*-*bins*

$$B_i = \{j \in S_k | j_i < j < j_{i+1}\}$$

for $0 < i < b$ and also $B_0 = \{j \in S_k | j < j_1\}$ and $B_b = \{j \in S_k | j > j_b\}$. Note that the link $\mathfrak{G}_w^{S_k}$ is isomorphic to the join of $\left\{(\mathfrak{G}_q^{j_{i+1}-j_i})^{\tilde{B}_i}\right\}_{i=0}^{b}$ where we have taken $j_0 = 0$, $j_{b+1} = 2d$ and $\tilde{B}_i = \left\{\tilde{j} | \tilde{j} = j - j_i \text{ where } j \in B_i\right\}$ for $i = 1, 2, \dots, b$. We will refer to a c^*-bin as just a bin when the c^* is clear from context.

Now, recall that for $J', J \subseteq \binom{S_k}{\leqslant r+1}$ we write $J' \leqslant J$, if $J' = \{c'_1, \dots, c'_m\}$ where $c'_j \subseteq c_j$. Let $c^i_j = c_j \cap B_i$.

▶ **Definition 53** (Classification of bins). *Let $J = \{c_1, c_2, \ldots, c_\ell\} \subseteq \mathcal{C}$ be a set of colors of $\mathcal{Z}$. A bin B_i is* J-crowded *if $J_i = (c_1^i, \ldots, c_\ell^i)$ has more than one coordinate for which $c_j^i \neq \emptyset$. It is* J-lonely *if there is exactly one such coordinate. Otherwise, it is* J-empty. *When the set J is clear from the context we simply say crowded, lonely, and empty for J-crowded, J-lonely, and J-empty, respectively.*

▶ **Definition 54** (Well-spread colors). *Let J be a set of m colors in $\mathcal{C}$. We say that J is* well-spread *if the following properties hold.*

1. *J consists of pairwise disjoint colors.*
2. *For every $\ell_1, \ell_2 \in (\cup J) \cup \{d-k, d+k\}$ it holds that $|\ell_1 - \ell_2| \geqslant \frac{2k}{(m(r+1))^3}$.*
3. *For every $J' \subseteq J$ of size $|J'| = 5$ and $\overline{J'} = J \setminus J'$ the following hold.*
   - **a.** *For every $\ell \in \cup\overline{J'} \cup \{d-k\}$, there exists some $\ell' \in \cup\overline{J'} \cup \{d+k\}$ such that $1 < \ell' - \ell \leqslant \frac{200k\log(r+1)}{(r+1)m}$.*
   - **b.** *For every $c \in J$, the number of colors $i \in c$ that are in J'-crowded $\cup\overline{J'}$-bins is at most $\frac{100(r+1)\log(r+1)}{m\log m} = \frac{200(r+1)}{m}$.*

▶ Remark 55. Our definition of well-spread colors does not include an analogue of [24, Item 3(c)]. The bin size bounds in Item 3(c) lets them obtain a stronger bound for the flags complex. Here, we have adapted the proof used for weaker swap-expansion parameter where we don't need the bin size bounds.

Proof of Theorem 50. Our proof strategy follows the overall framework of [24] for the full flags complex. The idea is to break $\mathcal{Z}$ into progressively simpler complexes (projections, then deep links) until we have a complex whose coboundary expansion we can bound directly via our bounds for $\mathfrak{G}_w^{S_k}$.

Moving to projections of $\mathcal{Z}$ to well-spread colors. First, we move from bounding the coboundary expansion of $\mathcal{Z}$ to coboundary expansion of the projections of $\mathcal{Z}$ on well-spread color-sets J (see Definition 54). We do this in two (sub)steps: (i) first, we show most color-sets are well-spread (see Claim 56), (ii) then, we show that if most color-sets are well-spread and all well-spread color sets are good co-boundary expanders, then $\mathcal{Z}$ itself is a good co-boundary expander (see Claim 57).

More concretely, denote by $\mathcal{J}$ the set of well-spread J's as per Definition 54. We claim the measure of $\mathcal{J}$ tends to 1 as r grows large, i.e., most projections are well-spread as follows.

▷ Claim 56. Let k, r be such that $k \geqslant r^5$. Let $6 \leqslant m \leqslant (r+1)$. The probability that m colors chosen uniformly from S_k are well-spread tends to 1 as $r \to \infty$.

Claim 56 is an elementary combinatorial calculation and follows directly from an analogous computation in [24, Proposition 8.4.1] for the correct setting of parameters,[8] so we omit the proof.

By Claim 56, for large enough r at least half of the sets J are in $\mathcal{J}$. Thus, applying, Lemma 51 to $\mathfrak{G}$, we immediately get the coboundary expansion of $\mathcal{Z}$ is lower bounded by the worst-case expansion over spread projections as stated in Claim 57.

▷ Claim 57. $h^1(\mathcal{Z}, \Gamma) \geqslant \Omega(1) \cdot \min_{J \in \mathcal{J}} h^1(\mathcal{Z}^J, \Gamma)$.

---

[8] To be precise, the dimension parameter in [24, Proposition 8.4.1] should be set to $2k+1$ and the arity parameter to r. After this, observe that the bijection from $[2k+1]$ to S_k that maps $x \mapsto d-(k+1)+x$ has the property that (i) a well-spread set of color according the definition of [24] maps to a well-spread set of colors according to our definition, and (ii) a uniform distribution on $[2k+1]$ induces a uniform distribution on S_k.

▶ Remark 58. We note that

$$\mathcal{Z}^J = \mathrm{F}^r(\mathfrak{G}^{S_k})^J = \mathrm{F}^r(\mathfrak{G})^J.$$

Comparing this to the proof of flags coboundary expansion in [24] we find that they also end up with the projection of the r-faces complex of the flags complex to well-spread colors. As a result, we will be able to proceed with the proof in exactly the same manner from hereon with the changes due to changes in parameters of the well-spreadness definition being addressed when they come up.

Trickling down the links. In the previous step, we reduced the task of showing the coboundary expansion of $\mathcal{Z}$ to coboundary expansion of $\mathcal{Z}^J$ for spread color sets. Our next step will be to reduce the analysis of $h^1(\mathcal{Z}^J)$ to the coboundary expansion of its links. In particular, in Claim 59 below, we fix $J \in \mathcal{J}$ and use a "local to global" argument to move to the links of $\mathcal{Z}^J$. We note that we need the local spectral expansion to be constant for Claim 59 to hold and indeed there is a q_0 such that we have local spectral expansion $\mathcal{Z}^J$ to be at least $1/2$ for any $q \geqslant q_0$ due to Claim 59.

▷ Claim 59 ([24, Claim 8.5.1]). Let J be a set of m pairwise disjoint colors for $\mathcal{Z}$, and let $i = -1, 0, \ldots, m-6$, then

$$h^1(\mathcal{Z}^J, \Gamma) \geqslant \exp(-O(i)) \cdot \min_{s \in \mathcal{Z}^J(i)} h^1(\mathcal{Z}^J_s, \Gamma),$$

where we understand $\mathcal{Z}^J_s$ to mean $\mathcal{Z}^{J'}_s$ for $J' = J \setminus \texttt{col}(s)$.

We note that, as stated in [24, Claim 8.5.1] the set J should be well-spread, but it is easy to check the proof only relies on disjointness. Thus, by Claim 59,

$$h^1(\mathcal{Z}^J, \Gamma) \geqslant \exp(-O(m)) \cdot \min_{s \in \mathcal{Z}^J(m-6)} h^1(\mathcal{Z}^J_s, \Gamma).$$

Reducing to Crowded Colors. Fix any $s \in \mathcal{Z}^J(m-6)$. Our goal is now to reduce from analyzing coboundary expansion of $\mathcal{Z}^J_s$ to $\mathcal{Z}^{\tilde{J}}_s$ where $\tilde{J}$ are the crowded colors in the link. In particular, since by definition of spreadness there are few crowded colors, this will allow us to apply Proposition 52 with much smaller $R = \sum_{j=1}^{5} |\tilde{c}_i|$ to achieve the final desired coboundary expansion. Toward this end, we'd like to use the following result of [24].

▶ **Corollary 60** ([24, Corollary 8.7]). *Let $w \in \mathfrak{G}$. There is some constant $\beta > 0$ such that*

$$h^1(\tilde{\mathcal{Z}}^J_w, \Gamma) \geqslant \beta \cdot h^1(\tilde{\mathcal{Z}}^{\tilde{J}}_w, \Gamma),$$

where $\tilde{J} = \{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_\ell\}$ and $\tilde{c}_j = \{i \in c_j | i$ is not in a lonely or empty bin$\}$.

As stated, the only problem is the above bounds $\tilde{\mathcal{Z}}^J_w$ for a vertex $w \in \tilde{\mathcal{Z}}$ (equivalently a face in $\mathfrak{G}$), while we need to bound $\mathcal{Z}^J_s$. However, by Observation 27 we have $\mathcal{Z}^J_s \cong \mathrm{F}^J\mathfrak{G}_{\cup s}$, and since J consists entirely of $(r+1)$-sets we indeed have $\mathrm{F}^J\mathfrak{G}_{\cup s} \cong \tilde{\mathcal{Z}}^J_{\cup s}$. Thus, applying Corollary 60, we have

$$h^1(\mathcal{Z}^J_s, \Gamma) = h^1(\mathrm{F}^J\mathfrak{G}_{\cup s}, \Gamma) = h^1(\tilde{\mathcal{Z}}^J_{\cup s}, \Gamma) \geqslant \Omega(h^1(\tilde{\mathcal{Z}}^{\tilde{J}}_{\cup s}, \Gamma)).$$

Reduction to the Base Complex. Next, denote by $\beta = \min_{w,I} h^1(\mathfrak{G}^{\cup I}_{\cup s \sqcup w}, \Gamma)$ where the minimum is taken over sets I consisting of five singletons, i.e., $I = \{\{i_0\}, \ldots, \{i_4\}\}$ such that $I \cup \texttt{col}(s) \leqslant \tilde{J}$, and $w \in \mathfrak{G}_{\cup s}$ such that $\texttt{col}(w) \subseteq \bigcup \tilde{J}$ and $\texttt{col}(w) \cap I = \emptyset$. Since $\tilde{Z}^{\tilde{J}}_s \cong \mathrm{F}^{\tilde{J}} \mathfrak{G}_{\cup s}$, by Proposition 52,

$$h^1(\tilde{Z}^{\tilde{J}}_s, \Gamma) \geqslant \text{const} \cdot \beta_1^R,$$

where $\beta_1 = \Omega(\beta)$ and $R = \sum_j |\tilde{c}_j|$. By Item 3b of Definition 54, for every c_j,

$$|\tilde{c}_j| = \text{the number of indices from } c_j \text{ in crowded bins} = O\left(\frac{r \log r}{m \log m}\right),$$

so, in total $R = O\left(\frac{r \log r}{m \log m}\right) = O\left(\sqrt{r}\right)$

$$h^1(\mathcal{Z}, \Gamma) \geqslant \text{const} \cdot \exp(-O(\sqrt{r})) \cdot \beta^{\sqrt{r}}. \tag{2}$$

Using bounds for the Flags complex. It is left to bound β, the coboundary expansion of deep spread out links in the projected complex.

▷ Claim 61. Let $I = \{\{i_0\}, \ldots, \{i_4\}\}$, and let $s \in \mathcal{Z}$ be such that $|s| = m - 5$ and there is a well-spread set of colors $J \in \mathcal{J}$ such that $I \cup \texttt{col}(s) \leqslant J$. Let $w' \in \mathfrak{G}^{\cup J}_{\cup s}$ be such that $\texttt{col}(w') \cap I = \emptyset$. Then

$$h^1(\mathfrak{G}^{\cup I}_{\cup s \sqcup w'}, \Gamma) \geqslant \exp(-O(\log^2 r)).$$

Claim 61 is a variant of [24, Lemma 8.8], which makes the same statement for the full flags complex under a modified notion of spreadness dependent on the ambient dimension d (n in their definition), while our definition of spreadness critically depends only on k, as d grows with the number of vertices and none of the projections we consider would be spread under [24]'s definition. Nevertheless, because we are inside the link of s, "morally" we may treat the ambient dimension as k, in which case our definition reduces to theirs.

In somewhat more detail, the proof of Claim 61 essentially proceeds by (1) first, using Lemma 45 to move to projections to 4 indices, (2) relating the coboundary expansion of the complexes to the diameter, and (3) finally, obtaining a bound on the diameter as result of well-spreadness of J. Because we are inside a link, the diameter bounds depend on the ratio of the parameters in Item 2 and parameters in Item 3 which remains unchanged from the definition of [24]. We reproduce relevant parts of the proof of Claim 61 in the full version of the paper.

We can now complete the proof. Namely, by Claim 61, we have

$$\beta = \min_{w,I} h^1(\mathfrak{G}^{\cup I}_{\cup s \sqcup w}, \Gamma) \geqslant \exp(-O(\log^2 r)).$$

Plugging this back into Equation (2) we get

$$h^1(\mathcal{Z}, \Gamma) \geqslant \text{const} \cdot \exp(-O(\sqrt{r} \log^2 r))$$

as needed. ◀

5 Applications

5.1 Background

5.1.1 Agreement testing

Agreement Testing. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family of k element subsets of $[n]$ and let $\{f_s : s \to \Sigma | s \in \mathcal{F}\}$ be an ensemble of local functions, each defined over a subset $s \in \mathcal{F}$. An agreement test is a randomized property tester for the question:

Is there a global function $G : [n] \to \Sigma$ such that $f_s = G|_s$ for all $s \in \mathcal{F}$?

Towards answering this question, we will work with the following basic restricted type of 2-query test.

▶ **Definition 62** (2-query agreement test). *Let Σ be an arbitrary finite set which we will call the* alphabet*. Given a family $\mathcal{F}$ of k element subsets of $[n]$ and an ensemble of functions $\mathcal{E} = \{f_s : s \to \Sigma | s \in \mathcal{F}\}$ from $[n]$ to Σ which we will call* local functions*, a 2-query agreement test is a distribution $\mathcal{D}$ over a pair of subsets $s_1, s_2 \in \mathcal{F}$, with the agreement of the ensemble being*

$$\mathsf{agr}_{\mathcal{D}}(\mathcal{E}) = \mathop{\mathbb{P}}_{s_1,s_2\sim\mathcal{D}}\left[f_{s_1}|_{s_1\cap s_2} = f_{s_2}|_{s_1\cap s_2}\right].$$

▶ Remark 63. In the literature, sometimes more general agreement tests are considered where the distribution $\mathcal{D}$ is over ℓ subsets in $\mathcal{D}$.

We note that for any property testing question we would like the test to be complete, i.e., we would want an object having the property in consideration to pass the test. By design, agreement tests (as described above) have completeness as one can clearly see restriction of a global function to sets in $\mathcal{F}$ would pass any 2-query test of the above form.

We also want agreement tests to be sound, i.e., whenever the tests passes there is a global function whose restrictions agree with most functions in the ensemble. There are two regimes of soundness of agreement tests that are typically considered:

The 99% regime or the high acceptance regime. In this regime, we want to claim that there is a global function $G : [n] \to \Sigma$ such that $G|_s = f|_s$ for most $s \in \mathcal{F}$ whenever $f_s|_{s\cap s'} = f_{s'}|_{s\cap s'}$ for most $s, s' \in \mathcal{F}$. More formally, we want to show

$$\mathsf{agr}_{\mathcal{D}}(\{f_s\}_{s\in\mathcal{F}}) \geqslant 1-\varepsilon \quad\Longrightarrow\quad \exists G : [n] \to \Sigma,\ \mathbb{P}\left[G|_s = f_s\right] \geqslant 1 - O(\varepsilon).$$

The 1% regime or the low acceptance regime. In this regime, if there is at least some, say ε, fraction of $s, s' \in \mathcal{F}$ so that $f_s|_{s\cap s'} = f_{s'}|_{s\cap s'}$, we want to claim that there is a global function $G : [n] \to \Sigma$ such that $G|_s \approx f|_s$ for some fraction of $s \in \mathcal{F}$, say $\mathsf{poly}(\varepsilon)$. More formally, we want to show there there is some polynomial poly such that for any $\varepsilon \geqslant \varepsilon_0$,

$$\mathsf{agr}_{\mathcal{D}}(\{f_s\}_{s\in\mathcal{F}}) \geqslant \varepsilon \quad\Longrightarrow\quad \exists G : [n] \to \Sigma,\ \mathbb{P}\left[\mathsf{dist}(G|_s, f_s) \leqslant \eta\right] \geqslant \mathsf{poly}(\varepsilon),$$

where η is some "closeness" parameter. Assuming (for simplicity) that the polynomial is of form $\mathsf{poly}(x) = x^c$, we call (ε_0, η, c) the *soundness parameters* and call a test $\mathcal{D}$ with these parameters a (ε_0, η, c)-*sound test.* More consisely, we may leave ε_0, c unspecified say a test is η-sound, i.e., a η-sound test is (ε_0, η, c)-sound for some $\varepsilon_0, c > 0$.

In the low soundness regime, agreement testing is quite non-trivial even over $\mathcal{F} = \binom{[n]}{k}$, and there is a long line of works which settled this [44, 33, 37, 38, 55, 36].

Here, we will study agreement testing on sparser families of sets, in particular, we will take $\mathcal{F} = X(k-1)$ for some simplicial complex X of dimension $d > k-1$ and define an agreement test based on its weights. In particular, we will consider the following agreement test.

$(k, \sqrt{k})$-test. The $(k, \sqrt{k})$-test $\mathcal{D}$ is a 2-query test given by the following process.

1. Sample $t \sim \Pi_d$.
2. Sample $s \sim \Pi_{\sqrt{k}-1}$, conditioned on $s \subseteq t$.
3. Sample $s_1, s_2 \sim \Pi_{k-1}$, conditioned on $s \subseteq s_1, s_2 \subseteq t$.

Finally, we remark that for any function $G : [n] \to \Sigma$ the ensemble $\{G|_s\}_{s \in \mathcal{F}}$ can be thought of as an encoding of G that is locally testable using the agreement test. This code is often referred to as a derandomized direct product code and agreement testing is referred to as direct product testing in this context.

5.1.2 Probabilistically Checkable Proofs (PCPs)

For the most part, we will forego a discussion of the "proofs" view of probabilistically checkable proofs and take a more combinatorial view of PCPs using the language of the Label Cover problem.

▶ **Definition 64.** *An instance of* Label Cover $\Psi = (G = (L \cup R, E), \Sigma_\ell, \Sigma_r, \Phi = \{\phi_e\}_{e \in E})$ *consists of a bipartite graph G, alphabets Σ_ℓ, Σ_r and constraints $\phi_e : \Sigma_\ell \to \Sigma_r$, for each edge $e \in E$.*

Given a label cover instance Φ, the goal is to find assignments $A_\ell : L \to \Sigma_\ell$ and $A_r : R \to \Sigma_r$ that satisfy as many of the constraints as possible, namely that maximize the quantity

$$\mathsf{val}_\Phi(A_\ell, A_r) = \frac{1}{|E|} \left| \{ e = (u,v) \in E | \phi_e(A_\ell(u)) = A_r(v) \} \right|.$$

This maximum value is referred to as the value of the instance Φ and denoted by

$$\mathsf{val}(\Phi) = \max_{A_\ell, A_r} \mathsf{val}_\Phi(A_\ell, A_r).$$

The label cover problem is a special case of the 2-ary constraint satisfaction problem where each constraint is a relation on the alphabet instead of being a function.

▶ **Definition 65.** *An instance $\Psi = (G = (V, E), \Sigma, \{\Phi_e\}_{e \in E})$ of a 2-CSP consists of a weighted digraph G, alphabet Σ, and constraints $\Phi_e \subseteq \Sigma \times \Sigma$, one for each edge.*

Let $\mathsf{gapLabelCover}[\alpha, \beta]$ be the promise problem wherein the input is an instance Φ of label cover promised to either have $\mathsf{val}(\Phi) \geqslant \alpha$ or else $\mathsf{val}(\Phi) \leqslant \beta$, and the goal is to distinguish between these two cases.

▶ **Definition 66.** *An instance of* Generalized Label Cover $\Psi = (G = (L \cup R, E), \Sigma_\ell, \Sigma_r, \Phi = \{\phi_e\}_{e \in E})$ *consists of a bipartite graph G, alphabets Σ_ℓ, Σ_r, $\{\Sigma_\ell(u)\}_{u \in L}$ with $\Sigma_\ell(u) \subseteq \Sigma_\ell$, for each $u \in L$, and constraints $\phi_e : \Sigma_\ell(u) \to \Sigma_r$, for each edge $e = (u,v) \in E$.*

5.2 New Agreement Testers

[22, 13, 14] showed the following agreement testing result for complexes with strong expansion properties. We refer the reader to [13, Remark 4.2] for the specific parameters.

▶ **Theorem 67** ([22]). *There is a $c > 0$ such that for all $\delta > 0$, there is $r \in \mathbb{N}$ such that for any $\ell \geqslant \exp(\mathsf{poly}(1/\delta))$, any $k \geqslant \mathsf{poly}(\ell)$ and there is a $\gamma_0(k)$ such that for any $\gamma \leqslant \gamma_0$ the following holds.*

If a k-dimensional clique complex X is a γ-local spectral expander and $(c2^{-o(r)}, r)$-swap coboundary expander, then the $(\ell, \sqrt{\ell})$-agreement test over $X(\ell)$ with respect to every alphabet Σ has soundness δ. Namely, if $F : X(\ell - 1) \to \Sigma^\ell$ passes the $(\ell, \sqrt{\ell})$-agreement test with respect to X with probability at least δ, then there is $f : X(0) \to \Sigma$ such that

$$\mathop{\mathbb{P}}_{A\sim\mu_\ell}[\mathsf{dist}(F[A], f|A) \leqslant \delta] \geqslant \mathsf{poly}(\delta).$$ ⌟

As an immediate corollary we get that the $(\ell, \sqrt{\ell})$-test on the projected flags complex has a low soundness test.

▶ **Theorem 9** (Agreement testing over the projected flags complex). *Fix a $\delta > 0$. There is some q_0 such that the following holds.*

Let $q \geqslant q_0$ be any prime power. Let d, k, ℓ be such that $d > k \geqslant \mathsf{poly}(\ell)$ and $\ell \geqslant \exp(\mathsf{poly}(1/\delta))$. Let $\mathfrak{G}$ denote the $\mathfrak{G}_q^{2d}$-flags complex and

$$S_k = \{d - k, \dots, d - 1, d, d + 1, \dots, d + k\}.$$

Then, the $(\ell, \sqrt{\ell})$-agreement test over $\mathfrak{G}^{S_k}(\ell)$ with respect to every alphabet Σ has soundness δ, i.e., if $F : \mathfrak{G}^{S_k}(\ell - 1) \to \Sigma^\ell$ passes the $(\ell, \sqrt{\ell})$-agreement test with respect to $\mathfrak{G}^{S_k}$ with probability at least δ, then there is $f : \mathfrak{G}^{S_k}(0) \to \Sigma$ such that

$$\mathop{\mathbb{P}}_{A\sim\mu_\ell}[\mathsf{dist}(F[A], f|A) \leqslant \delta] \geqslant \mathsf{poly}(\delta).$$

Proof. Let $c > 0, r, \gamma_0(k)$ be as in Theorem 67. First note that by taking q_0 the flags complex $\mathfrak{G}_q^{2d,S_k}$ has $\gamma \leqslant \gamma_0$ by Theorem 3. Now, by Theorem 50 we have the the swap coboundary expansion required by Theorem 67. Hence, by Theorem 67, the $(\ell, \sqrt{\ell})$-agreement test over $\mathfrak{G}^{S_k}(\ell)$ with respect to every alphabet Σ has soundness δ for any $\ell \geqslant \exp(\mathsf{poly}(1/\delta))$. ◀

5.3 A Near-linear Size PCP

Finally, we show the projected flags complex can be used to construct PCPs using techniques in [14].

▶ **Theorem 11.** *For all $\delta \in (0, 1)$, there exist a constant $C > 0$ such that for all sufficiently large integers d and n there is a polynomial time reduction mapping a 3SAT instance ϕ of size n to a label cover instance Ψ over a bipartite graph G of size n' for some $n' \leqslant n \exp\left(C\sqrt{\log n \log\log n}\right)$, such that*

1. *If ϕ is satisfiable, then Ψ is satisfiable.*
2. *If ϕ is unsatisfiable, then $\mathsf{val}(\Psi) \leqslant \delta$.*
3. *The left alphabet of Ψ is $\Sigma^{\exp(C/\delta)}$ and right alphabet is $\Sigma^{\sqrt{\exp(C/\delta)}}$ for some alphabet Σ with $\log|\Sigma| \leqslant \exp\left(C\sqrt{\log n \log\log n}\right)$.*

We first record the set of properties of the projected flags complex in Theorem 69 that are required for the PCP construction. The statement of Theorem 69 is an analog of Theorem 2.13 in [14], albeit with slightly different parameters, as we replace the Chapman-Lubotzky complexes [17] with the projected buildings.

▶ **Definition 68** (Symmetrically-Connected Complex). *Let L be a k-partite complex. We call L* symmetrically-connected *if*

1. *For any $i \neq j \in [k]$, the diameter of L^{ij} is bounded by $O(\frac{k}{|i-j|})$.*
2. *Sym_L acts transitively on its top faces.*

▶ **Theorem 69.** *For all $\delta \in (0,1)$, and for all $C > 0$ the following holds. For large enough $\ell, k \in \mathbb{N}$, and for large enough $n \in \mathbb{N}$, there is a prime power $q = \Theta(\log^C n)$ and $d = \Theta\left(\sqrt{\frac{\log(n)}{\log\log(n)}}\right)$, one can construct in time $\mathsf{poly}(n)$ a $2k$-dimensional complex X for which $n \leqslant |X(0)| \leqslant nq^{O(d)}$ such that the following holds.*

1. *The complex X is $(2k+1)$-partite.*
2. *The complex X is a clique complex.*
3. *$\Delta_{max}(X) \leqslant q^{O(kd)}$.*
4. *Each vertex-link L of X is symmetrically-connected. In other words, we have the following:*
   a. *For each vertex link L of X there is a group $\mathrm{Sym}(L)$ that acts transitively on the $(k-2)$-faces $L(k-2)$.*
   b. *For each vertex link L of X and any colors $i \neq j$, the bipartite graph $(L_i(0), L_j(0))$ is uniformly weighted (over edges) and has diameter $O(\frac{\max\{i,j\}}{|i-j|})$.*
5. *X is a $\frac{1}{\sqrt{q}}$-product.*
6. *The 1-skeleton $G = (X(0), X(1))$ satisfies $\lambda(G) \leqslant \frac{1}{k(\sqrt{q}-1)}$ and $|\lambda|(G) \leqslant \frac{1}{2k}$.*
7. *The $(\ell, \sqrt{\ell})$-direct product test on X has soundness δ.*

Proof. Given $\delta > 0$, $C > 0$, and n, take $q = 2^{\lceil \log_2 \log_2^C n \rceil}$ and $d = \lceil \sqrt{\log_q n} \rceil$. If q and k are large enough, then the complex $X = \mathfrak{G}_q^{2d, S_k}$ has $(\ell, \sqrt{\ell})$-agreement test with soundness δ for large enough ℓ by Theorem 9 (this shows Item 7). We get Items 5 and 6 from Theorem 3, Fact 31, and the Trickle-Down theorem (Theorem 18). It is elementary to see the group of automorphisms $\mathrm{Aut}(L)$ acts transitively on the top faces of L where L is a vertex link (this gives us Item 4a).[9] Item 4b follows directly from Claim 8.7.2 in [24] and hence we get Item 4. Item 3 is a straight-forward counting which we omit for the sake of brevity. Items 1 and 2 also follow directly from the definition. ◀

Overview of PCP construction in [14]. The first step of [14]'s PCP involves reducing an arbitrary 2-query PCP, viewed as a 2-CSP, to a 2-CSP whose constraint graph underlies an HDX. In particular, they show how to embed an arbitrary 2-CSP on a graph G while maintaining the soundness provided that G has a fault-tolerant routing protocol, and show any sufficiently local-spectral HDX (see Items 5 and 6) with dense symmetric links (see Items 4a and 4b) indeed satisfies such a protocol. The second step uses a size-efficient direct product tester from HDX. This allows derandomized parallel repetition for 2-CSPs on an HDX, and reduce their soundness to a small constant close while incurring a poly-logarithmic blow-up in size. Combining the first and second steps they get a quasi-linear PCP with small soundness, albeit with a large alphabet.

Since the projected flags complex satisfies essentially the same properties as the Chapman-Lubotzky complex albeit with different parameters (compare Theorem 69 with [14, Theorem 2.13]), the same construction also works even if we use projected flags complex. Theorem 70 states a general version of this theorem. In the full version of this paper, we provide a sketch of the proof to highlight the changes in parameters.

[9] As in the proof of Lemma 46, taking an appropriate basis for the vector spaces in any two such flags, we can transform one to the other with the corresponding basis exchange matrix.

▶ **Theorem 70** ([14]). *There exist $\alpha, \varepsilon_0, d_0 > 0$, and $r, C \in \mathbb{N}$ such that for large enough $n \in \mathbb{N}$ the following holds. Let X be d-partite λ-product such that (i) its underlying graph G has $|\lambda|(G) \leqslant \alpha/4$; (ii) its vertex links are symmetrically-bounded; (iii) its max degree $\Delta(X)$ is bounded by Δ; (iii) the dimension $d \geqslant d_0$ and its size $|X(0)| = n$. Also, assume that the $(\ell, \sqrt{\ell})$-direct product tester on X has soundness $\delta^2 \leqslant \varepsilon_0^2/16$. Then there is a $\mathsf{poly}(n)$-time procedure mapping any 2-CSP Ψ' with n vertices on a r-regular graph H and alphabet Σ, to an instance of Label Cover Φ over the weighted inclusion graph $(X(\sqrt{\ell}), X(\ell))$, over left alphabet $\Sigma^{\mathsf{poly}(\Delta \log(n) \log|\Sigma|)\ell}$ and right alphabet $\Sigma^{\mathsf{poly}(\Delta \log(n) \log|\Sigma|)\sqrt{\ell}}$ satisfying the following properties.*

(i) ***Completeness:*** *If* $\mathsf{val}(\Psi') = 1$ *then* $\mathsf{val}(\Psi) = 1$.

(ii) ***Soundness:*** *If* $\mathsf{val}(\Psi') \leqslant 1 - \mathsf{poly}(d) r^2 \lambda^2 \log^C n - \frac{1}{\log^C n}$, *then* $\mathsf{val}(\Psi) \leqslant \delta$.

We now start with a size efficient construction of PCP due to [68] stated in a more convenient formulation in terms of hardness of 2-CSPs.[10]

▶ **Theorem 71** ([68]). *There exist constants $c > 0, \delta \in (0, 1)$ such that there is a polynomial-time reduction mapping a 3-SAT instance ϕ of size n to a 2-CSP instance Ψ over the regular graph $G = (V, E)$ and alphabet Σ where*

- $\mathsf{size}(G) \leqslant n(\log n)^{c \log\log n}$ *and* $|\Sigma| = O(1)$.
- *If ϕ is satisfiable, then* $\mathsf{val}(\Psi) = 1$.
- *If ϕ is not satisfiable, then* $\mathsf{val}(\Psi) \leqslant 1 - \delta$.

▶ Remark 72. The construction of PCP in [68] is almost combinatorial. The only algebraic component is the multiplicative codes used in the paper.

Applying this to our complex, i.e., combining Theorems 69–71, we obtain Theorem 73.

▶ **Theorem 73.** *For all $\delta \in (0, 1)$, there exist constants $C, C' > 0$ such that for all sufficiently large integers d and n, there exist k, ℓ with $k \geqslant \mathsf{poly}(\ell)$ and $\ell \geqslant \exp(\mathsf{poly}(1/\delta))$ so that the following holds. Let $\{X_{n'}\}_{n' \in \mathbb{N}}$ be the infinite sequence of complexes from Theorem 69, where every $X_{n'}$ is a $(2k+1)$-dimensional complex on n' vertices with parameters $q = \Theta(\log^{C'} n')$ and δ that is constructible in time $\mathsf{poly}(n')$. There is a polynomial time reduction mapping a 3SAT instance ϕ of size n to a label cover instance Ψ over the weighted inclusion graph $(X_{n'}(\ell), X_{n'}(\sqrt{\ell}))$ for some $n' \leqslant n \log^{C \log\log n} n$, such that*

1. *If ϕ is satisfiable, then Ψ is satisfiable.*
2. *If ϕ is unsatisfiable, then* $\mathsf{val}(\Psi) \leqslant \delta$.
3. *The left alphabet of Ψ is Σ^{ℓ} and right alphabet is $\Sigma^{\sqrt{\ell}}$ for some alphabet Σ with $\log|\Sigma| \leqslant q^{Ckd}$.*

As an immediate corollary, we get the version of the PCP theorem as stated in Theorem 11.

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