Plethysm is in #BQP

Matthias Christandl
University of Copenhagen, Denmark

Aram W. Harrow
Massachusetts Institute of Technology, Cambridge, MA, USA

Greta Panova
University of Southern California, Los Angeles, USA

Pietro M. Posta
University of Copenhagen, Denmark

Michael Walter
Ludwig-Maximilians-Universität München, Germany
Munich Center for Quantum Science and Technology (MCQST), Germany
Korteweg-de Vries Institute for Mathematics, Amsterdam, The Netherlands
QuSoft, University of Amsterdam, The Netherlands

Abstract

Some representation-theoretic multiplicities, such as the Kostka and the Littlewood-Richardson coefficients, admit a combinatorial interpretation that places their computation in the complexity class #P. Whether this holds more generally is considered an important open problem in mathematics and computer science, with relevance for geometric complexity theory and quantum information. Recent work has investigated the quantum complexity of particular multiplicities, such as the Kronecker coefficients and certain special cases of the plethysm coefficients.

Here, we show that a broad class of representation-theoretic multiplicities is in #BQP. This includes the result that plethysm coefficients are in #BQP, which was only known in certain cases. It also implies all known results on the quantum complexity of previously studied coefficients as special cases, thus unifying, simplifying, and extending prior work. We obtain our result by multiple applications of the Schur transform; recent work has improved its dependence on the local dimension, which is crucial for our work. We further describe a general approach for showing that representation-theoretic multiplicities are in #BQP that captures the approaches of our and previous work. We complement the above by showing that the same multiplicities are also naturally in GapP and obtain polynomial-time classical algorithms when certain parameters are fixed.

2012 ACM Subject Classification Mathematics of computing → Combinatorics; Theory of computation → Quantum complexity theory

Keywords and phrases Quantum witness counting, #BQP, algebraic combinatorics, representation theory, representation-theoretic multiplicities

Digital Object Identifier 10.4230/LIPIcs.CCC.2026.28

Related Version *Full Version*: https://arxiv.org/abs/2602.08441 [1]

Funding *Matthias Christandl*: MC acknowledges support from the European Research Council (ERC Grant Agreement No. 818761), VILLUM FONDEN via the QMATH Centre of Excellence (Grant No. 10059) and the Novo Nordisk Foundation (grant NNF20OC0059939 "Quantum for Life").
Aram W. Harrow: AWH acknowledges support from a grant from the Simons Foundation (MP-SIP-00001553, AWH) and from a Fulbright Scholar Grant.
Greta Panova: GP acknowledges partial support from NSF grant CCF:2302174 and from an AMS Birman fellowship, and a residence at ICERM (NSF grant DMS-1929284) in Fall 2025.
Pietro M. Posta: PMP acknowledges support from the European Research Council (ERC Grant Agreement No. 818761), VILLUM FONDEN via the QMATH Centre of Excellence (Grant No. 10059), the Novo Nordisk Foundation (grant NNF20OC0059939 "Quantum for Life") and from a PhD scholarship on Quantum Algorithms from the Danish e-infrastructure Consortium (DeiC), and partial support from Istituto Nazionale di Alta Matematica "Francesco Severi"(INdAM).

41st Computational Complexity Conference (CCC 2026).
Editor: Dana Moshkovitz; Article No. 28; pp. 28:1–28:11

Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Michael Walter: MW acknowledges support from the European Research Council (ERC Grant SYMOPTIC, 101040907) by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation, 556164098), from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy– EXC-2111– 390814868, and from the German Federal Ministry of Research, Technology and Space (QuSol, 13N17173). Part of this work was conducted while MW was visiting Q-FARM and the Leinweber Institute for Theoretical Physics at Stanford University and the Simons Institute for the Theory of Computing at UC Berkeley.

Acknowledgements We are grateful to Christian Ikenmeyer, Vojtech Havlicek and Martin Larocca for fruitful discussions.

1 Introduction

Computing multiplicities is a central problem in representation theory and algebraic combinatorics, and is directly relevant to quantum information theory and geometric complexity theory. In most cases, no closed-form formulas are known. A natural and fruitful question is to ask whether these multiplicities count combinatorial objects. If so, then one obtains a *positive combinatorial interpretation*. Stanley stated the problem of finding such interpretations for important representation-theoretic multiplicities in his influential paper [48]. Specific multiplicities of interest include: the *Littlewood-Richardson (LR) coefficients*, discussed in 1934 [36], giving the multiplicity of an irreducible representation of the general linear group $\mathrm{GL}(n)$ in the tensor product of two others; the *Kronecker coefficients* of the symmetric group, defined in 1938 [38], giving the multiplicity of an irreducible representation of the symmetric group S_n in the tensor product of two others; and the *plethysm coefficients*, first mentioned by Littlewood in 1936 [34] and defined formally in Ref. [35], which gives the multiplicity of an irreducible $\mathrm{GL}(n)$-representation in the composition of two irreducible GL-representations (see Section 2). By Schur-Weyl duality, the Kronecker coefficients can also be defined in terms of the general linear group, and the LR and plethysm coefficients can also be defined in terms of the symmetric group.

Multiplicities arise naturally across a broad range of areas and applications in mathematics, physics, and computer science, and accordingly they have a long history of study using different methods, from representation theory and invariant theory, to geometry and algebraic combinatorics. For the plethysm coefficients in particular, initial properties were understood via geometric means, notably their stability [6]. The combinatorial investigations started with the unfulfilled hope of finding an interpretation as counting integer points in polytopes (as is known for the LR coefficients, see below), see e.g. the works of Kahle and Michalek [29]. Approaches via tableaux combinatorics were developed by many groups, see e.g. Refs. [21, 20, 39]. The general goal is to give an interpretation as counts of some simple-to-describe but exponentially large set. Ultimately, however, combinatorial formulas are only known in very special situations, see Refs. [22, 40, 43] for some of the known cases.

1.1 The complexity of computing multiplicities

While the notion of a "positive combinatorial interpretation" is not a priori formally defined, the modern interpretation formalizes it as "proving the computational problem is in #P, and deciding positivity is in NP" [41, 45]. Indeed, the complexity class #P captures the problem of counting gadgets (witnesses) which can be recognized efficiently (in polynomial time). This perspective uncovered surprising connections between mathematics and computer science and led to new insights in both fields. The LR coefficients turn out to be in #P, as they admit an interpretation as certain tableaux and integer points in polytopes, whereas their

positivity is even decidable in P [37, 12] as a consequence of the Knutson-Tao proof of the saturation conjecture [32]. In contrast, the problems for Kronecker and plethysm coefficients are still wide open [46]. We know that both coefficients are #P-hard to compute and NP-hard to decide positivity of [26, 23]. However, it is not known whether they are in #P, that is, whether they admit a positive combinatorial interpretation, or whether their positivity is in NP. [1]

As representation-theoretic multiplicities are linear-algebraic quantities defined in high dimensions, it is natural to study their complexity from the perspective of quantum computation. In particular, their very definition as the dimension of the space of multiplicities of some irreducible representation, gives the hope that one could place them in the quantum complexity class #BQP, the quantum analogue of #P, by the intuition that these multiplicities might count the dimension of easy-to-recognize subspaces in some bigger vector space. Indeed, in Ref. [5] it was shown that deciding positivity of Kronecker coefficients is in QMA, the quantum analogue of NP, and computing a certain multiple of these coefficients is in #BQP. Shortly after, it was shown in Ref. [28] that computing the Kronecker coefficients, as well as certain *special* plethysm coefficients (see below), is in #BQP. See also Ref. [17]. In certain parameter regimes, efficient quantum algorithms for computing these multiplicities were proposed in Ref. [33], however many of those cases were later shown to be polynomial-time computable by classical algorithms as well [47], see also Ref. [15]. Thus the quantum and classical complexity of representation-theoretic multiplicities is subtle even in special cases. Here we consider this problem from a general perspective, motivated by the outstanding problem of characterizing the complexity of the plethysm coefficients.

1.2 Relevance in algebraic complexity theory and quantum information theory

In algebraic complexity theory, geometric complexity theory (GCT) is a program that seeks to establish lower bounds, such as separating VNP from VP, by exploiting the symmetries of the universal polynomials, for example the permanent and determinant of a symbolic matrix. One approach via algebraic geometry and representation theory goes via so-called *multiplicity obstructions*. Namely, decompose the coordinate rings of the GL-orbit closures of a VNP-complete polynomial (usually the permanent) into irreducible components, and compare the multiplicities of these components to the ones corresponding to a universal polynomial for VP (usually the determinant). The plethysm coefficients (as well as the Kronecker coefficients) play a natural and significant role in such computations, for example as upper bounds or even as part of explicit formulas for such multiplicities, see Refs. [8, 11, 13, 14, 22, 27]. The GCT program can also be adapted to other situations, such as finding lower bounds for the complexity of the matrix multiplication problem [11], and representation-theoretic multiplicities again turn out to be crucial.

In quantum physics, representation-theoretic multiplicities are ubiquitous since the state of quantum systems are described by unit vectors in high-dimensional complex vector spaces, which are acted upon unitarily by symmetry groups. In some situations, such as high energy physics, the groups and representations of interest are fixed, while in quantum information and computation one naturally considers sequences of groups and representations. One important setting is the one-body quantum marginal problem. Indeed, the compatibility of the one-body

[1] However, it is known that, for a broad class multiplicities, a natural *asymptotic* version of the positivity problem is in NP ∩ coNP [9].

marginals with an overall pure state is guided by the Kronecker coefficients [16, 18, 30], while the plethysm coefficients appear in the fermionic version of the problem [2], known as the one-body N-representability problem in quantum chemistry [19], which has been studied at least since the early 1970s [4].

2 Our contributions

We now present our main results, starting with the quantum complexity of plethysm coefficients. We first give a brief definition of plethysms (see the full version of the paper [1] for more details). Let $V \cong \mathbb{C}^n$ be a finite-dimensional complex vector space. Recall that the irreducible polynomial complex representations of the general linear group $\mathrm{GL}(V)$ are labeled by integer partitions with no more than n parts. Given such a partition ν, we denote by $\rho_{\mathrm{GL}(V),\nu}\colon \mathrm{GL}(V) \to \mathrm{GL}(\{\nu\}_{\mathrm{GL}(V)})$ the corresponding irreducible representation, where $\{\nu\}_{\mathrm{GL}(V)}$ denotes the representation space or *Weyl module*. If ν has more than n parts, we define $\rho_{\mathrm{GL}(V),\nu}$ as the zero-dimensional representation. We abbreviate $H \coloneqq \mathrm{GL}(V)$, and consider the group $G \coloneqq \mathrm{GL}(\{\nu\}_H)$ along with one of its polynomial representations $\rho_{G,\mu}$, where μ is again an integer partition. Then the composition $\rho_{G,\mu} \circ \rho_{H,\nu}\colon H \to \mathrm{GL}(\{\mu\}_G)$ is a representation of $H = \mathrm{GL}(V)$. We shall denote the corresponding representation space by $\{\mu\}{\downarrow}_H^{\mathrm{GL}(\{\nu\}_H)}$ to remember that we regard it as a module of H, by *restriction* along the irreducible representation $\rho_{H,\nu}\colon H \to G = \mathrm{GL}(\{\nu\}_H)$. In general, the representation $\rho_{G,\mu} \circ \rho_{H,\nu}$ is reducible and hence it decomposes into irreducible H-representations. Thus:

$$\{\mu\}{\downarrow}_H^{\mathrm{GL}(\{\nu\}_H)} \cong \bigoplus_\lambda \underbrace{\{\lambda\}_H \oplus \cdots \oplus \{\lambda\}_H}_{a^\lambda_{\mu,\nu}} \cong \bigoplus_\lambda \{\lambda\}_H \otimes \mathbb{C}^{a^\lambda_{\mu,\nu}}. \tag{2.1}$$

The *plethysm coefficient* $a^\lambda_{\mu,\nu}$ is defined as the multiplicity of the irreducible module $\{\lambda\}_H$ in this decomposition. Here we assume that λ has no more than $n = \dim(V)$ parts, in which case the coefficient does not depend on the choice of n but only on the three partitions λ, μ, ν. An equivalent definition of the plethysm coefficient $a^\lambda_{\mu,\nu}$ is as the coefficient of s_λ in the expansion of $s_\mu[s_\nu]$. Here s_μ, s_ν, s_λ denote Schur functions, see the full version of the paper [1] for a precise definition.

▶ **Problem 1** (Plethysm coefficients).
Input: Three partitions λ, μ, ν, encoded as lists of positive integers, in unary.
Task: Compute the plethysm coefficient $a^\lambda_{\mu,\nu}$.

Encoding the partitions in unary is a standard convention in the literature. Thus the input size is $\Theta(|\lambda| + |\mu| + |\nu|)$. We give two concrete examples of plethysm coefficients:

- If both $\mu = (m)$ and $\nu = (d)$ are partitions with a single part, then $\{\mu\}{\downarrow}_H^{\mathrm{GL}(\{\nu\}_H)} = \mathrm{Sym}^m(\mathrm{Sym}^d(V))$ can be interpreted as the space of homogeneous polynomials of degree m in variables that are coefficients on the space of homogeneous polynomials of degree d. This space is central in geometric complexity theory and is the main object in the Foulkes conjecture [24].
- If $\mu = (m)$ and $\nu = (1^d) = (1, \ldots, 1)$, then $\{\mu\}{\downarrow}_H^{\mathrm{GL}(\{\nu\}_H)} = \mathrm{Sym}^m(\mathrm{Alt}^d(V))$ corresponds to the space of polynomials in the coefficients of a wavefunction of d fermions, each having Hilbert space V. This is directly relevant to the N-representability problem in quantum chemistry [2, 31].

The general case is a major problem in algebraic combinatorics as stated in Ref. [48, Problem 9]. Prior work [28] showed that the former example is in #BQP, but already the latter example was out of reach for earlier methods. Here we exhibit an algorithm which proves that the general plethysm problem is in #BQP:

▶ **Theorem 2.** *Problem 1 is in* #BQP*. As a consequence, the problem of deciding if a plethysm coefficient is positive,* $a^{\lambda}_{\mu\nu} > 0$*, is in* QMA*.*

The corresponding quantum algorithm is sketched in Figure 1 and we describe its main ideas in Section 3. The above also implies an analogous result for the *restriction coefficients* r^{μ}_{λ}, defined as the multiplicy of an irreducible S_n-representation (Specht module) in the restriction of an irreducible $\mathrm{GL}(n)$-representation (Weyl module) to the subgroup $S_n \subset \mathrm{GL}(n)$ of permutation matrices. Indeed, the restriction coefficients can be expressed as a positive sum of plethysms: using symmetric functions (see the full version [1] for definitions), we have $r^{\mu}_{\lambda} = \langle s_\mu, s_\lambda[1 + h_1 + h_2 + \cdots]\rangle$. Thus, computing restriction coefficients is in #BQP and their positivity is in QMA, as an immediate corollary of Theorem 2.

We obtain Theorem 2 by specialization of a general theorem that applies to a broad class of multiplicities known as *branching multiplicities (or coefficients)*. They are defined as follows: We consider groups G and H, together with a group homomorphism $\varphi\colon H \to G$. Any representation $\pi\colon G \to \mathrm{GL}(W)$ of G then *restricts* to a representation of H by composition with φ, i.e., $\pi \circ \varphi\colon H \to \mathrm{GL}(W)$. The corresponding branching multiplicities are defined as the multiplicities of irreducible H-representation in the restriction $\pi \circ \varphi$. Here we consider branching multiplicites in the setting where G and H are *products of general linear groups*, and where the group homomorphism φ and the G-representation π are specified succinctly in terms of representation-theoretic data, see the full version of the paper [1] for a formal definition. This encompasses, among others, all coefficients studied in prior work: Kostka numbers, LR coefficients, and Kronecker coefficients, (of course, the first two are even in $\#\mathsf{P} \subset \#\mathsf{BQP}$, while for the latter this is an open problem). It is known that computing branching multiplicities can be done in classical polynomial time when G and H are *fixed*. This holds not just for products of general linear groups, but more generally in the setting of complex reductive (or compact connected) Lie groups [15]. In contrast, here we allow the groups (as well as the representations) to be part of the input. This is substantially more difficult since it involves representations that in general have dimensions that are *exponential* in the input size.

▶ **Theorem 3.** *Computing branching multiplicities for products of general linear groups is in* #BQP*. As a consequence, deciding their positivity is in* QMA*.*

We prove Theorem 3 in the full version of the paper [1] by giving a quantum algorithm that is perfectly sound and complete (if the gate set is such that the circuit can be implemented perfectly), similarly to the algorithms in Refs. [5, 28]. In particular, this shows that the positivity problem is in QMA_1 if there exists a universal gate set such that the high-dimensional version of the quantum Schur transform on $(\mathbb{C}^d)^{\otimes N}$ can be implemented exactly for any choice of N and d. We discuss the main technical ideas behind our quantum algorithms in Section 3. Moreover, in the full version of the paper [1], we describe a general approach for showing that representation-theoretic multiplicities are in #BQP that subsumes our quantum algorithms as well as those of the prior works [5, 28] and explains their relation.

As discussed, the plethysm coefficients and in fact most representation-theoretic coefficients are #P-hard, and it is a central open question to determine when they are in #P, see Refs. [41, 45]. By Theorem 2, a negative answer to this question would imply a breakthrough separation between the complexity classes #P and #BQP. To gain further insight into their computational complexity, we complement our quantum results by showing that the branching multiplicities are also in GapP extending earlier results in Refs. [10, 42, 44].

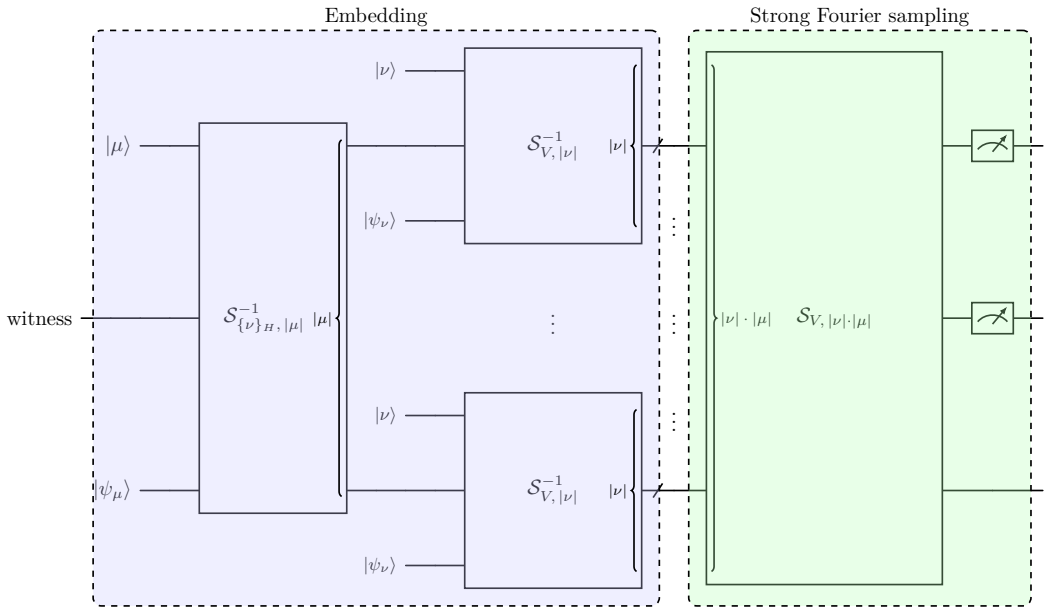

Figure 1 Quantum circuit for the plethysm coefficient $a^\lambda_{\mu,\nu}$ (Theorem 2). The witness is a state in the space $\{\mu\}_{\mathrm{GL}(\{\nu\}_H)}$. The first inverse Schur transform acts on the space $\{\nu\}_H^{\otimes|\mu|}$, taking an input in the Schur basis and transforming it in the computational basis, consisting of $|\mu|$ registers, each isomorphic to $\{\nu\}_H$. Each inverse Schur transform from the second layer acts on $V^{\otimes|\nu|}$, again transforming from the Schur basis to the computational basis, consisting of $|\nu|$ registers of dimension n. After the final Schur transform, taking an input in the computational basis of $V^{\otimes|\mu||\nu|}$ and transforming it into the Schur basis, we measure the registers corresponding to the partition and the Weyl module. We accept if and only if the former returns outcome λ and the latter p_0, where $|p_0\rangle \in \{\lambda\}_H$ is an arbitrary fixed basis state of the Weyl module.

▶ **Theorem 4.** *Computing branching multiplicities for products of general linear groups is in* GapP.

We prove Theorem 4 in the full version of the paper [1], and we also present a more specialized algorithm for computing the plethysm coefficients. In particular we obtain a polynomial time algorithm when the number of parts of λ and the size of μ are fixed, extending the results of Ref. [47].

3 Methods

We discuss the main ideas behind our results, focusing on the special case of plethysm coefficients for concreteness (Theorem 2). For further details see the full version of the paper [1].

Quantum algorithm for plethysm coefficients

Our algorithm is depicted in circuit form in Figure 1. Let λ, μ, ν be partitions and $V = \mathbb{C}^n$. We may assume that the number of parts of the partitions and n are such that the corresponding Weyl modules are nonzero, otherwise the plethysm coefficient $a^\lambda_{\mu,\nu} = 0$. Our approach uses Schur-Weyl duality to embed the H-module $\{\mu\}\downarrow^G_H$ inside $V^{\otimes|\nu|\cdot|\mu|}$, where $G = \mathrm{GL}(\{\nu\}_H)$ and $H = \mathrm{GL}(V)$ as before. This is achieved as follows: We first embed the G-module $\{\mu\}_G$

in a tensor power of $\{\nu\}_H$, which is the defining module of $G = \mathrm{GL}(\{\nu\}_H)$. We then consider each $\{\nu\}_H$ as a H-module and embed it into a tensor power of V, which is the defining module of $H = \mathrm{GL}(V)$. Since $\{\mu\}\downarrow_H^G$ is nothing but the restriction of the G-module $\{\mu\}_G$ to H, we obtain the following chain of H-equivariant inclusions

$$\{\mu\}\downarrow_H^G \hookrightarrow (\{\nu\}_H)^{\otimes|\mu|} \hookrightarrow V^{\otimes|\nu|\cdot|\mu|}. \tag{3.1}$$

In order to implement these embeddings, we employ *inverse* quantum Schur transforms, as these allow us to map from the *Schur basis* of the tensor power spaces to the standard basis (blue box in Figure 1). After the embeddings, we implement a final (forward) quantum Schur transform on all $|\nu| \cdot |\mu|$ registers and measure in the Schur basis – accepting if and only if we obtain the partition λ, meaning that the input state was in the λ-isotypic component of $\{\mu\}\downarrow_H^G$, and if we obtain an arbitrary but fixed basis state p_0 on the Weyl module register (green box in Figure 1). As inclusions are H-equivariant, they preserve the dimension of the isotypic component, and they act as the identity on the Weyl module. As such, the entire circuit implements a projective measurement, and the dimension of the space of witnesses that are accepted with probability one (assuming the Schur transform is implemented exactly) is precisely the multiplicity of $\{\lambda\}_H$ in $\{\mu\}\downarrow_H^G$. By Equation (2.1), this multiplicity is the plethysm coefficient $a^\lambda_{\mu,\nu}$. See the full version of the paper [1] for more details and for the analysis of the algorithm soundness, completeness and complexity.

Generalization to branching multiplicities

Our result for the branching problem for products of general linear groups (Theorem 3) is a natural generalization of this idea. The quantum algorithm is formally described and analyzed in detail in the full version of the paper [1]. One conceptual contribution, detailed in the full version, is to identify an appropriate definition of the computational problem. This is subtle, as it involves objects that will in general be exponentially large and hence need to be succinctly specified in order to obtain a nontrivial problem that captures the plethysm coefficients and other classical coefficients as special cases. On the algorithmic side, we employ a similar sequence of embeddings as before, this time on products of tensor powers of the defining modules, and implemented by a sequence of parallel quantum circuits. In our general setting, the defining modules for G need no longer be irreducible as H-modules, requiring us to implement intermediate measurements in order to realize the subsequent embeddings. See the full version of the paper [1] for further details.

Comparison with prior work

Previous results on the quantum computation of Kronecker and special plethysm coefficient employed the formulation of these coefficients in terms of the symmetric group S_n and rely on Beals' quantum Fourier transform for S_n [3]. In principle, any plethysm coefficient can be computed as branching multiplicities of representations of wreath products of symmetric groups. However, as pointed out in Ref. [33], a direct use of Beals' quantum Fourier transform leads to an exponential blowup in general. As such it is not immediately clear how to generalize the symmetric group approach to the general plethysm problem (Problem 1). We overcome this challenge by working with the representation theory of the general linear group. In this regard we may view this approach as the "Schur-Weyl dual" of prior approaches to treat representation-theoretic multiplicities. We emphasize that the local dimensions in our setting (such as the dimension of $\{\nu\}_H$) can be *exponentially large*. Accordingly, we require a version of the quantum Schur transform that only depends polylogarithmically

on the local dimensions. This has recently been achieved in Ref. [7], see also Ref. [25]. In contrast, for the Kronecker coefficients the relevant local dimension are polynomial and hence the original quantum Schur transform suffices. We discuss quantum algorithms for the Schur transform in more detail in the full version of the paper [1].

Unified approach towards multiplicities

Despite the difference between these approaches there is a unified picture that captures both and which applies even more broadly, to the general problem of the multiplicity of an irreducible H-module V_λ in an H-module V, for some group H. Consider the following two-step procedure:

1. Embed the H-module V we wish to decompose into an H-module V_{model}.
2. Apply strong Fourier sampling for V_{model} and accept if the outcome consists of is the label of the irreducible representation V_λ, together with a fixed basis state $|p\rangle \in V_\lambda$.

It is clear that if these two steps can be efficiently implemented, then the multiplicity problem is in #BQP, and the corresponding decision problem is in QMA. Remarkably, this captures both the approach of this work and the prior works:

- For the plethysm problem, $V = \{\mu\}{\downarrow}_H^G$ and $V_{\text{model}} = (\mathbb{C}^n)^{\otimes|\nu|\cdot|\mu|}$. The first (embedding) step is achieved by the two layers of inverse Schur transforms (blue box in Figure 1), while the second (strong Fourier sampling) step is achieved by the (forward) Schur transform and subsequent measurements (green box in Figure 1). Our more general algorithm for the branching problem of products of general linear groups likewise fits this picture.
- The key ingredient of Refs. [5, 28] is the *generalized phase estimation* circuit [25], which likewise admits a natural interpretation in this framework. Here, $V_{\text{model}} = \mathbb{C}[H] \otimes V$, where $\mathbb{C}[H]$ is the group algebra and H acts on $\mathbb{C}[H]$ but not on V. The first (embedding) step is realized using the controlled group action of V (that is, the unitary $|h\rangle\,|\psi\rangle \mapsto |h\rangle\,|h \cdot \psi\rangle$ on $\mathbb{C}[H] \otimes V$), and the second (strong Fourier sampling) step is implemented using the Fourier transform of the group. [2] This procedure applies whenever the controlled group action and the Fourier transform of the group can be efficiently implemented, for further details see the full version of the paper [1].

We may also combine these two ideas. For example, the restriction coefficients r_λ^μ can be captured by an embedding step that first uses an inverse Schur transform (as in the first bullet point) to realize $\{\lambda\}_{\mathrm{GL}(n)}$ in $V = (\mathbb{C}^n)^{\otimes|\lambda|}$, followed by GPE (as in the second bullet point), noting that the controlled group action of S_n on $\mathbb{C}^n$ can be efficiently implemented. We further expand on this general perspective in the full version of the paper [1].

References

1 Matthias Christandl, Aram W. Harrow, Greta Panova, Pietro M. Posta, and Michael Walter. Plethysm is in #BQP, full version, 2026. arXiv:2602.08441.

2 M. Altunbulak and A. Klyachko. The Pauli principle revisited. *Communications in Mathematical Physics*, 282(1):287–322, 2008. doi:10.1007/s00220-008-0552-z.

3 Robert Beals. Quantum computation of Fourier transforms over symmetric groups. In *Proceedings of the 29th Annual ACM Symposium on Theory of Computing (STOC '97)*, pages 48–53. ACM, 1997. doi:10.1145/258533.258548.

[2] In Ref. [5] the authors use *weak* Fourier sampling and therefore prove membership in #BQP for a *scaled* version of the Kronecker coefficients (namely, the Kronecker coefficients multiplied by the dimension of corresponding irreducible representations).

4 R. E. Borland and K. Dennis. The conditions on the one-matrix for three-body fermion wavefunctions with one-rank equal to six. *Journal of Physics B: Atomic and Molecular Physics*, 5(1):7–15, 1972. doi:10.1088/0022-3700/5/1/009.

5 Sergey Bravyi, Anirban Chowdhury, David Gosset, Vojtěch Havlíček, and Guanyu Zhu. Quantum complexity of the Kronecker coefficients. *PRX Quantum*, 5(1):010329, 2024. doi:10.1103/PRXQuantum.5.010329.

6 Michel Brion. Stable properties of plethysm: on two conjectures of Foulkes. *Manuscripta Mathematica*, 80:347–371, 1993. doi:10.1007/BF03026558.

7 Adam Burchardt, Jiani Fei, Dmitry Grinko, Martin Larocca, Maris Ozols, Sydney Timmerman, and Vladyslav Visnevskyi. High-dimensional quantum Schur transforms, 2025. doi:10.48550/arXiv.2509.22640.

8 Peter Bürgisser, Matthias Christandl, and Christian Ikenmeyer. Even partitions in plethysms. *Journal of Algebra*, 328(1):322–329, 2011. doi:10.1016/j.jalgebra.2010.10.031.

9 Peter Bürgisser, Matthias Christandl, Ketan Mulmuley, and Michael Walter. Membership in moment polytopes is in NP and coNP. *SIAM Journal on Computing*, 46(3):972–991, 2017. doi:10.1137/15M1048859.

10 Peter Bürgisser and Christian Ikenmeyer. The complexity of computing Kronecker coefficients. In *FPSAC'08 - 20th International Conference on Formal Power Series and Algebraic Combinatorics*, 2008. doi:10.46298/dmtcs.3622.

11 Peter Bürgisser and Christian Ikenmeyer. Geometric complexity theory and tensor rank. In *Proceedings of the Forty-Third Annual ACM Symposium on Theory of Computing (STOC '11)*, pages 509–518. ACM, 2011. doi:10.1145/1993636.1993704.

12 Peter Bürgisser and Christian Ikenmeyer. Deciding positivity of Littlewood–Richardson coefficients. *SIAM Journal on Discrete Mathematics*, 27(4):1639–1681, 2013. doi:10.1137/120892532.

13 Peter Bürgisser, Christian Ikenmeyer, and Jesko Hüttenhain. Permanent versus determinant: not via saturations. *Proceedings of the American Mathematical Society*, 145(3):1247–1258, 2017. doi:10.1090/proc/13310.

14 Peter Bürgisser, Christian Ikenmeyer, and Greta Panova. No occurrence obstructions in geometric complexity theory. *Journal of the American Mathematical Society*, 32(1):163–193, 2019. doi:10.1090/jams/908.

15 Matthias Christandl, Brent Doran, and Michael Walter. Computing multiplicities of Lie group representations. In *2012 IEEE 53rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 639–648. IEEE, 2012. doi:10.1109/FOCS.2012.43.

16 Matthias Christandl, Aram W. Harrow, and Graeme Mitchison. On nonzero Kronecker coefficients and what they tell us about spectra. *Communications in Mathematical Physics*, 270(3):575–585, 2007. doi:10.1007/s00220-006-0157-3.

17 Matthias Christandl, Aram W. Harrow, and Michael Walter. Presentation at the algorithms and complexity in algebraic geometry reunion workshop. Presented by M. Walter at the Simons Institute for the Theory of Computing, Berkeley, 2015.

18 Matthias Christandl and Graeme Mitchison. The spectra of quantum states and the Kronecker coefficients of the symmetric group. *Communications in Mathematical Physics*, 261(3):789–797, 2006. doi:10.1007/s00220-005-1435-1.

19 A. J. Coleman and V. I. Yukalov. *Reduced Density Matrices: Coulson's Challenge*. Lecture Notes in Chemistry, Vol. 72. Springer Berlin Heidelberg, 2000. doi:10.1063/1.1359715.

20 Laura Colmenarejo, Rosa Orellana, Franco Saliola, Anne Schilling, and Mike Zabrocki. The mystery of plethysm coefficients. *Open Problems in Algebraic Combinatorics (Proc. Sympos. Pure Math.)*, 110:275–292, 2024. doi:10.48550/arXiv.2208.07258.

21 Melanie de Boeck, Rowena Paget, and Mark Wildon. Plethysms of symmetric functions and highest weight representations. *Transactions of the American Mathematical Society*, 374(11):8013–8043, 2021. doi:10.1090/tran/8481.

22 Julian Dörfler, Christian Ikenmeyer, and Greta Panova. On geometric complexity theory: Multiplicity obstructions are stronger than occurrence obstructions. *SIAM Journal on Applied Algebra and Geometry*, 4(2):354–376, 2020. doi:10.1137/19M1287638.

23 Nick Fischer and Christian Ikenmeyer. The computational complexity of plethysm coefficients. *Computational Complexity*, 29:1–43, 2020. doi:10.1007/s00037-020-00198-4.

24 H. O. Foulkes. Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form. *Journal of the London Mathematical Society*, 1(3):205–209, 1950. doi:10.1112/jlms/s1-25.3.205.

25 Aram W. Harrow. *Applications of coherent classical communication and the Schur transform to quantum information theory*. PhD thesis, Massachusetts Institute of Technology, 2005. arXiv:quant-ph/0512255.

26 Christian Ikenmeyer, Ketan Mulmuley, and Michael Walter. On vanishing of Kronecker coefficients. *Computational Complexity*, 26:949–992, 2017. doi:10.1007/s00037-017-0158-y.

27 Christian Ikenmeyer and Greta Panova. Rectangular Kronecker coefficients and plethysms in geometric complexity theory. *Advances in Mathematics*, 319:40–66, 2017. doi:10.1016/j.aim.2017.08.024.

28 Christian Ikenmeyer and Sathyawageeswar Subramanian. A remark on the quantum complexity of the Kronecker coefficients, 2023. arXiv:2307.02389.

29 Thomas Kahle and Mateusz Michałek. Plethysm and lattice point counting. *Foundations of Computational Mathematics*, 16(5):1241–1261, 2016. doi:10.1007/s10208-015-9275-7.

30 Alexander A. Klyachko. Quantum marginal problem and representations of the symmetric group, 2004. arXiv:quant-ph/0409113.

31 Alexander A. Klyachko. Quantum marginal problem and N-representability, 2005. arXiv:quant-ph/0511102.

32 Allen Knutson and Terence Tao. The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture. *Journal of the American Mathematical Society*, 12(4):1055–1090, 1999. doi:10.1090/s0894-0347-99-00299-4.

33 Martin Larocca and Vojtech Havlicek. Quantum algorithms for representation-theoretic multiplicities, 2024. arXiv:2407.17649.

34 D. E. Littlewood. Polynomial concomitants and invariant matrices. *Journal of the London Mathematical Society*, s1-11(1):49–55, 1936. doi:10.1112/jlms/s1-11.1.49.

35 D. E. Littlewood. Products and plethysms of characters with orthogonal, symplectic and symmetric groups. *Canadian Journal of Mathematics*, 10:17–32, 1958. doi:10.4153/CJM-1958-002-7.

36 Dudley Ernest Littlewood and Archibald Read Richardson. Group characters and algebra. *Philosophical Transactions of the Royal Society of London. Series A*, 233:99–141, 1934. doi:10.1098/rsta.1934.0015.

37 Ketan Mulmuley, Hariharan Narayanan, and Milind Sohoni. Geometric complexity theory III: on deciding nonvanishing of a Littlewood–Richardson coefficient. *Journal of Algebraic Combinatorics*, 36(1):103–110, 2012. doi:10.1007/s10801-011-0325-1.

38 Francis D. Murnaghan. On the analysis of the Kronecker product of irreducible representations of the symmetric group. *American Journal of Mathematics*, 60(3):761–784, 1938. doi:10.1073/pnas.41.7.515.

39 Rosa Orellana, Franco Saliola, Anne Schilling, and Mike Zabrocki. Plethysm and the algebra of uniform block permutations. *Algebraic Combinatorics*, 5(5):1165–1203, 2022. doi:10.5802/alco.243.

40 Rosa Orellana, Franco Saliola, Anne Schilling, and Mike Zabrocki. From quasi-symmetric to Schur expansions with applications to symmetric chain decompositions and plethysm. *The Electronic Journal of Combinatorics*, 31(4):P4.23, 2024. doi:10.37236/12950.

41 Igor Pak. What is a combinatorial interpretation? In *Open Problems in Algebraic Combinatorics*, volume 110 of *Proc. Sympos. Pure Math.*, pages 191–260. Amer. Math. Soc., 2024. doi:10.1090/pspum/110/02007.

42 Igor Pak and Greta Panova. On the complexity of computing Kronecker coefficients. *Computational Complexity*, 26:1–36, 2017. doi:10.1007/s00037-015-0109-4.

43 Igor Pak, Greta Panova, and Joshua P. Swanson. A combinatorial interpretation for certain plethysm and Kronecker coefficients, 2025. Unpublished manuscript.

44 Igor Pak and Colleen Robichaux. Signed combinatorial interpretations in algebraic combinatorics, 2024. doi:10.48550/arXiv.2406.13902.

45 Greta Panova. Computational complexity in algebraic combinatorics. In *Current Developments in Mathematics*. International Press Boston, 2023. doi:10.4310/CDM.250429173831.

46 Greta Panova. Complexity and asymptotics of structure constants. In *Open Problems in Algebraic Combinatorics*, volume 110 of *Proc. Sympos. Pure Math.*, pages 61–86. Amer. Math. Soc., 2024. doi:10.1090/pspum/110/02016.

47 Greta Panova. Polynomial time classical versus quantum algorithms for representation theoretic multiplicities, 2025. doi:10.48550/arXiv.2502.20253.

48 Richard P. Stanley. Positivity problems and conjectures in algebraic combinatorics. *Mathematics: Frontiers and Perspectives*, pages 295–319, 2000. doi:10.5281/zenodo.18768172.