Improved Parallel Repetition for GHZ-Supported Games via Spreadness

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Abstract

We prove that for any 3-player game $\mathcal{G}$, whose query distribution has the same support as the GHZ game (i.e., all $x, y, z \in \{0, 1\}$ satisfying $x + y + z = 0 \pmod 2$), the value of the n-fold parallel repetition of $\mathcal{G}$ decays exponentially fast:

$$\text{val}(\mathcal{G}^{\otimes n}) \leq \exp(-n^c)$$

for all sufficiently large n, where $c > 0$ is an absolute constant.

We also prove a concentration bound for the parallel repetition of the GHZ game: For any constant $\epsilon > 0$, the probability that the players win at least a $\left(\frac{3}{4} + \epsilon\right)$ fraction of the n coordinates is at most $\exp(-n^c)$, where $c = c(\epsilon) > 0$ is a constant.

In both settings, our work exponentially improves upon the previous best known bounds which were only polynomially small, i.e., of the order $n^{-\Omega(1)}$. Our key technical tool is the notion of *algebraic spreadness* adapted from the breakthrough work of Kelley and Meka (FOCS '23) on sets free of 3-term progressions.

2012 ACM Subject Classification Theory of computation → Computational complexity and cryptography

Keywords and phrases Parallel Repetition, GHZ Game, Algebraic Spreadness

Digital Object Identifier 10.4230/LIPIcs.CCC.2026.6

Related Version *Full Version*: `https://arxiv.org/abs/2602.09290` [46]

Funding *Shachar Lovett*: Supported by Simons Investigator Award #929894 and NSF award CCF-2425349.
Kunal Mittal: Research supported a Simons Investigator Award.

1 Introduction

In a k-player game $\mathcal{G}$, a verifier samples a tuple of questions $(x^{(1)}, \ldots, x^{(k)})$ from a distribution Q. Then, for each $j \in \{1, \ldots, k\}$, the verifier gives the question $x^{(j)}$ to player j, and the player gives back an answer $a^{(j)}$, which depends only on $x^{(j)}$. The verifier now declares whether the players win or lose based on the evaluation of a predicate $V(x, a) \in \{0, 1\}$ depending on the questions $x = (x^{(1)}, \ldots, x^{(k)})$ and answers $a = (a^{(1)}, \ldots, a^{(k)})$. We define the game value, denoted $\text{val}(\mathcal{G})$, as the maximum winning probability (with respect to the distribution Q) over all possible player strategies; see Definitions 14, 15 for formal definitions.

A natural question that arises is: How does the value of the game behave under parallel repetition [29]? The n-fold parallel repetition, denoted $\mathcal{G}^{\otimes n}$, is a game where the players play, and try to win, n independent copies of the game in parallel. More precisely, the verifier samples questions $(x_i^{(1)}, \ldots, x_i^{(k)}) \sim Q$ independently for $i = 1, \ldots, n$, and for each

$j \in \{1, \ldots, k\}$, sends questions $(x_1^{(j)}, \ldots, x_n^{(j)})$ to player j, to which they give back answers $(a_1^{(j)}, \ldots, a_n^{(j)})$. The verifier says the players win if $V((x_i^{(1)}, \ldots, x_i^{(k)}), (a_i^{(1)}, \ldots, a_i^{(k)})) = 1$ for each $i \in \{1, \ldots, n\}$; see Definition 17 for a formal definition.

Note that for any game $\mathcal{G}$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \geq \mathrm{val}(\mathcal{G})^n$, since the players can achieve this value by repeating an optimal strategy in each of the n coordinates. Although one might expect the naïve bound $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq \mathrm{val}(\mathcal{G})^n$ to hold as well, this turns out to be false [28, 22, 26, 55]. Roughly speaking, this failure occurs because the players do not have to treat the n copies of the game independently, and can instead correlate their answers among different copies. Remarkably, it turns out that this failure is intimately connected to the geometry of high-dimensional Euclidean tilings [25, 45, 1, 14].

Parallel repetition of 2-player games is well-understood. Raz [53] showed that for any game with value less than 1, the value of the n-fold parallel repetition decays exponentially in n. Subsequent works have simplified this proof and strengthened the quantitative bounds [38, 6, 52, 56, 21, 12]. These and related works have led to several applications in various domains, including the theory of interactive proofs [8], PCPs and hardness of approximation [24, 2, 3, 4, 7, 23, 36, 43, 44, 35, 18, 20], quantum information [17, 16], and communication complexity [50, 5, 15]. The reader is referred to the survey [54] for more details.

Parallel repetition of multiplayer games is much less understood. The only general bound says that for any game $\mathcal{G}$ with value less than 1, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq 1/\alpha(n)$, where $\alpha(n)$ is a very slowly growing inverse-Ackermann function [58, 30, 51]. Recent work has made substantial progress in understanding special cases of multiplayer games [19, 39, 31, 32, 33, 13, 10, 9, 11]; however, the general question remains wide open, even for 3-player games.

Proving improved parallel repetition for multiplayer games has several potential applications. It is known that a strong parallel repetition theorem for a certain class of multiplayer games implies super-linear lower bounds for non-uniform Turing machines, which is a long-standing open problem in complexity theory [48]. Additionally, parallel repetition in the large answer alphabet regime is equivalent to many problems in high-dimensional extremal combinatorics, such as the density Hales-Jewett problem, and the problem of square-free sets in finite fields [26, 37, 47]. Also, as stated in [19], it is believable that improved understanding of parallel repetition can lead to a better understanding of communication complexity in the number-on-forehead (NOF) model, which is intimately connected to circuit lower bounds.

The focus of this paper is the 3-player GHZ game [34], which proceeds as follows: The verifier samples questions $(x, y, z) \in \{0,1\}^3$ uniformly at random such that $x + y + z = 0 \pmod 2$, and the players' goal is to give answers $a, b, c \in \{0, 1\}$ respectively satisfying $a + b + c \pmod 2 = x \vee y \vee z$. The GHZ game has played a foundational role in quantum information theory, due in part to the fact that quantum strategies can win this game with probability 1, whereas any classical strategy wins with probability at most 3/4. Moreover, the GHZ game satisfies a self-testing property, which says that all quantum strategies achieving value 1 are essentially the same; this has led to applications like entanglement testing and device-independent cryptography [57].

The problem of parallel repetition of the GHZ game has been discussed in several recent works. Dinur, Harsha, Venkat and Yuen [19] extend the 2-player information-theoretic techniques of Raz [53] and prove parallel repetition for a specific class of multiplayer games satisfying a certain connectivity property; they identify the GHZ game as a multiplayer game which is in some sense maximally far from this class of games; regarding the difficulty of the problem, they write

> "We believe that the strong correlations present in the GHZ question distribution represent the "hardest instance" of the multiplayer parallel repetition problem."

Subsequent research established polynomial decay bounds for the n-fold repetition of the GHZ game, showing $\mathrm{val}(\mathrm{GHZ}^{\otimes n}) \le n^{-\Omega(1)}$ [39, 31]. Notably, these results apply to any 3-player game $\mathcal{G}$ whose query distribution Q has the same support as the GHZ game (i.e., $\{(x,y,z) \in \{0,1\}^3 : x+y+z = 0 \pmod 2\}$).

More recently, an exponential decay bound was proven for the GHZ game [13], and then extended to all 3-player XOR games[1] satisfying a certain distributional assumption [9]. However, these works rely heavily on the XOR structure of the game predicate, exploiting it via sophisticated tools from Fourier analysis and additive combinatorics. For instance, in the case of the GHZ game, they observe that the predicate corresponds to a linear equation over the group $\mathbb{Z}/4\mathbb{Z}$. Consequently, these techniques do not extend to general game predicates lacking such XOR structure.

In this work, we improve upon the existing polynomial decay bounds [39, 31] by establishing a stretched exponential bound for all games sharing the query support of the GHZ game. Crucially, unlike the aforementioned works on XOR games [13, 9], our result is agnostic to the answer sets and the game predicate – it solely depends on the support of the query distribution. Our technical approach is distinct as well: we utilize spreadness-based arguments from recent works [42, 41, 40], whose application to the context of parallel repetition is novel.

Our main result is the following:

▶ **Theorem 1** (Parallel Repetition for the GHZ query support)**.** *Let $\mathcal{G}$ be any 3-player game with value $\mathrm{val}(\mathcal{G}) < 1$, whose query distribution has support*

$$\left\{(x,y,z) \in \{0,1\}^3 : x+y+z = 0 \pmod 2\right\}.$$

Then, for all sufficiently large n,[2] it holds that

$$\mathrm{val}(\mathcal{G}^{\otimes n}) \le \exp\left(-n^c\right),$$

where $c > 0$ is an absolute constant.

We complement this result with a concentration bound for games with the GHZ query distribution. To the best of our knowledge, the exponential decay bounds established in prior works [13, 10] do not imply such concentration. We prove the following:

▶ **Theorem 2** (A Concentration Bound; restated and proved as Theorem 42)**.** *Let $\mathcal{G}$ be any 3-player game with value $\mathrm{val}(\mathcal{G}) < 1$, whose query distribution is uniform over the set*

$$\left\{(x,y,z) \in \{0,1\}^3 : x+y+z = 0 \pmod 2\right\}.$$

Then, for every constant $\epsilon > 0$, there exists $c = c(\epsilon) > 0$, such that for all sufficiently large n,[3] the probability that the players can win at least $\mathrm{val}(\mathcal{G}) + \epsilon$ fraction of the n coordinates in the game $\mathcal{G}^{\otimes n}$ is at most $\exp\left(-n^c\right)$.

As a direct consequence, we obtain a concentration bound for the standard GHZ game.

▶ **Corollary 3.** *For every constant $\epsilon > 0$, there exists $c = c(\epsilon) > 0$, such that the probability of the players winning at least $\left(\frac{3}{4} + \epsilon\right) n$ coordinates in the game $\mathrm{GHZ}^{\otimes n}$ is at most $\exp\left(-n^c\right)$.*

[1] In a 3-player XOR game, the answers a, b, c of the three players lie in some finite Abelian group H, and the predicate is of the form $a + b + c = \varphi(x,y,z)$, for some function φ mapping questions (x,y,z) of the players to elements of H.

[2] i.e., $n \ge N$, where N is a constant depending on the game $\mathcal{G}$.

[3] i.e., $n \ge N$, where N is a constant depending on the game $\mathcal{G}$ and the parameter ϵ.

It remains an interesting open problem to establish analogous concentration bounds (as in Theorem 2) for games over the GHZ support where the underlying distribution is not uniform.[4]

Organization

In Section 2, we give an overview of our proofs. In Section 3, we establish some preliminaries. In Section 4, we define a notion of pseudorandomness, called *algebraic spreadness*, that will be useful throughout this paper; we also show how to decompose arbitrary sets into components satisfying this spreadness condition. In Section 5, we show that any *diagonal-product* set composed of algebraically spread sets is *uniformly covered* by *squares*. Finally, in Section 6, we use the results of the previous sections and prove parallel repetition for games with the GHZ query support (Theorem 1). In Section 7, we prove our concentration bound (Theorem 2).

2 Overview

In this section, we outline the proof of our main result (Theorem 1).

2.1 General Inductive Framework and High-Level Approach

We begin by introducing a general inductive framework for parallel repetition. Our proof follows the basic setup established by Raz for parallel repetition of 2-player games [53].

Let $\mathcal{G}$ be a 3-player game, whose query distribution Q has support[5]

$$\mathrm{supp}(Q) = \left\{(x, y, z) \in \mathbb{F}_2^3 : x + y + z = 0\right\}.$$

Here, we identified $\{0, 1\}$ with the finite field $\mathbb{F}_2$. For the purpose of parallel repetition, we may assume without loss of generality that Q is the uniform distribution over its support (see Lemma 18).[6] Now, consider the game $\mathcal{G}^{\otimes n}$, and consider any strategy for the 3 players in this game. For each $i \in [n]$, let Win_i be the event that the players win the i^{th} coordinate. By the chain rule, for any permutation $i_1, i_2, \ldots, i_n$ of $[n]$, we can write the winning probability as

$$\Pr[\mathrm{Win}_1 \wedge \mathrm{Win}_2 \wedge \cdots \wedge \mathrm{Win}_n] = \prod_{k=1}^{n} \Pr[\mathrm{Win}_{i_k} \mid \mathrm{Win}_{i_1} \wedge \cdots \wedge \mathrm{Win}_{i_{k-1}}].$$

To bound this, we proceed inductively. Assuming the players have won a set of coordinates $i_1, \ldots, i_k$, we aim to identify a *hard coordinate* $i \in [n]$ whose conditional winning probability is at most $1 - \Omega(1)$. Formally, we wish to prove the following condition:[7]

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[4] We remark that Theorem 2, combined with a reduction similar to Lemma 18 (also see Footnote 6), already implies the following: the probability that the players win at least $1 - \beta + \epsilon$ fraction of the n coordinates is at most $\exp(-n^{\Omega_\epsilon(1)})$, where $\beta = \beta(\mathcal{G}) > 0$ is a constant.

[5] The framework applies to all k-player games, but we restrict to 3 player games with the GHZ queries.

[6] Roughly speaking, this is because any distribution Q *contains* a small copy of the uniform distribution over $\mathrm{supp}(Q)$, and hence winning n copies under the distribution Q is at least as hard as winning $\Omega(n)$ copies under the the uniform distribution.

[7] Conditioning on the event $\mathrm{Win}_{i_1} \wedge \cdots \wedge \mathrm{Win}_{i_k}$ can introduce complex correlations among the players' inputs. However, this event depends deterministically on the questions and answers of the players in coordinates $i_1, \ldots, i_k$, and hence instead of conditioning on the event $\mathrm{Win}_{i_1} \wedge \cdots \wedge \mathrm{Win}_{i_k}$, we can condition on typical questions and answers to the players in coordinates $i_1, \ldots, i_k$. This induces a product event on the players' input.

Inductive Step: For every product event $\mathcal{E} = E \times F \times G \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n$, with measure $\Pr[\mathcal{E}] \geq \alpha$, there exists a coordinate $i \in [n]$ such that $\Pr[\mathrm{Win}_i \mid \mathcal{E}] \leq 1 - \Omega(1)$.

Establishing this condition for $\alpha = \exp(-n^{\Omega(1)})$ implies the desired bound on $\mathrm{val}(\mathcal{G}^{\otimes n})$ via the inductive strategy above (see Lemma 19).

For the remainder of this overview, consider any product event $\mathcal{E} = E \times F \times G \subseteq (\mathbb{F}_2^n)^3$, with measure $\Pr[\mathcal{E}] \geq \alpha = \exp(-n^{\Omega(1)})$. Our goal is to identify a coordinate $i \in [n]$ that remains hard to win when the inputs are conditioned on $\mathcal{E}$. The proof proceeds in two main steps:

1. **Identify Hard Sets:** We define a class of sets called *squares*, such that if the input distribution is restricted to a square, many coordinates are hard to win.
2. **Distributional Approximation:** We show that the conditional distribution $Q^{\otimes n}|\mathcal{E}$ can be approximated (in ℓ_1 distance) by a convex combination of such square distributions.

Combining these steps establishes that $\Pr[\mathrm{Win}_i \mid \mathcal{E}] \leq 1 - \Omega(1)$ for randomly chosen $i \in [n]$.

2.2 Step 1: Squares are Hard

Since the third player's input in $\mathcal{G}^{\otimes n}$ is fully determined by the inputs to the first two players (inputs $(x, y, z) \in \mathrm{supp}(Q)^n \subseteq (\mathbb{F}_2^n)^3$ satisfy $z = x + y$), we can analyze the game primarily by only looking at inputs of the first two players. The restriction of a product event $\mathcal{E} = E \times F \times G$ to the first two players is captured by a *diagonal-product set*, defined as follows:

▶ **Definition 4** (Diagonal-Product Set). *Given sets $X, Y, Z \subseteq \mathbb{F}_2^n$, we define the corresponding diagonal-product set, denoted $S(X, Y, Z)$, as*

$$S(X, Y, Z) = \{(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n : x \in X, y \in Y, x + y \in Z\}.$$

Sampling inputs $(x, y, z) \sim Q^{\otimes n}|\mathcal{E}$ is the same as sampling $(x, y) \sim S(E, F, G)$ uniformly and setting $z = x + y$.

We now define our hard sets, that are *squares* in $\mathbb{F}_2^n \times \mathbb{F}_2^n$:

▶ **Definition 5** (Square). *A* square $s_{x,y,w} \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n$*, for $x, y, w \in \mathbb{F}_2^n$, is the set*

$$s_{x,y,w} = \{(x, y), (x + w, y), (x, y + w), (x + w, y + w)\}.$$

▶ Remark 6. Given a square $s \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n$, suppose we wish to represent it as $s = s_{x,y,w}$. Note that the width w is uniquely determined by the square s. If $w = 0$, the square contains a single point, in which case (x, y) is uniquely determined. If $w \neq 0$, the square $s_{x,y,w}$ has 4 different representations, given by $s_{x,y,w} = s_{x+w,y,w} = s_{x,y+w,w} = s_{x+w,y+w,w}$.

The crucial property of a square is its local hardness: Consider a square $s = s_{x_0,y_0,w}$, and a coordinate $i \in [n]$ such that $w_i \neq 0$ (a *non-trivial* coordinate).[8] Suppose the inputs (x, y, z) to the three players are sampled from the square s as follows: let $(x, y) \sim s$ be chosen uniformly at random, and let $z = x + y$. Under this distribution, no strategy of the players can win coordinate i with probability more than $\mathrm{val}(\mathcal{G})$. Roughly speaking, this is true since this distribution "looks exactly the same" as the base distribution Q, and hence the players can embed a single copy of the game $\mathcal{G}$ into coordinate i of this distribution; see Lemma 38 for formal details of this step.[9]

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[8] A coordinate is non-trivial if and only if the points in s do not have a constant value in this coordinate.

[9] We remark that the set of inputs $\{(x, y, x + y) : (x, y) \in s\}$ was introduced in the work [31], where they call this set a *bow-tie*. We also note that squares correspond to the maximal *forbidden-subgraphs* inside the GHZ query distribution (in the sense that no player strategy can win on all four points of a square), and the largest size of square-free sets in $\mathbb{F}_2^n \times \mathbb{F}_2^n$ exactly captures the value of the n-fold parallel repetition of the GHZ query distribution in the large answer alphabet regime [47].

2.3 Step 2: Approximating the Distribution

The second, and more technically demanding, step of the proof is to show that the distribution $Q^{\otimes n}|\mathcal{E}$ is well-approximated by a mixture of square distributions as above. Equivalently, we wish to approximate the uniform distribution over the diagonal-product set $S = S(E, F, G)$, denoted U_S, by a convex combination of uniform distributions over squares. Specifically, we analyze the distribution μ generated as follows: sample a square $s \subseteq S$ uniformly at random, and output an element uniformly at random from s.

Note that a priori it is not even clear that the set S contains squares, and that the distribution μ is well-defined. However, we utilize the concept of *algebraic spreadness*, a pseudorandomness notion introduced in the breakthrough work of Kelley and Meka on bounds for sets without 3-term arithmetic progressions [42]. They show that algebraically spread subsets behave like random subsets in a certain sense (see Definition 20 and Theorem 21); in our setting, we show that if all E, F, G are algebraically spread, the set S contains the expected density of squares. Our approximation argument proceeds in two stages:

Uniformization

Given arbitrary sets E, F, G, we decompose the diagonal-product set $S(E, F, G)$ into disjoint components $S(E_1, F_1, G_1) \cup S(E_2, F_2, G_2) \cdots \cup S(E_T, F_T, G_T)$ (plus a negligible remainder), such that within each component, the sets E_i, F_i, G_i are all algebraically spread. This generalizes the recent decomposition technique of [40], which achieved algebraic spreadness for only two out of the three sets and used it to prove bounds on corner-free sets in finite fields. We extend this to all three sets via a careful recursive argument; see Section 4.2 for more details.

Counting and Approximation

Assuming that all of E, F, G are algebraically spread, we demonstrate that they behave like random sets of the same density, and that the square-sampling distribution μ approximates the uniform distribution U_S in ℓ_1 distance. We prove this by establishing tight bounds on the ℓ_2 norm (collision probability) of the distribution μ. A key technical tool here is a recent graph-counting result of [27], based on a pseudorandomness notion called combinatorial spreadness, which we show holds in our setting. See Lemma 33 and Proposition 34 for more details on this step.

Why Algebraic Spreadness?

A crucial difference between our work and previous polynomial decay bounds [31] lies in the choice of the pseudorandomness property for the sets E, F, G. Prior works use the concept of Fourier-uniformity (pseudorandomness against linear tests), and yield bounds when the density $\alpha = \Pr[\mathcal{E}]$ is at least polynomial, i.e., of the order $n^{-O(1)}$. The use of algebraic spreadness enables us to surpass this barrier and get bounds even when the density $\alpha = \exp(-n^{\Omega(1)})$ is exponentially smaller. However, this improvement incurs a significant technical cost. Both the decomposition and approximation steps are standard under Fourier uniformity, whereas establishing them under algebraic spreadness constitutes the main technical contribution of this work.

3 Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the set of natural numbers. For $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

3.1 Probability Distributions

Let P be a distribution (over an underlying finite set Ω, which is usually clear from context). We use $\mathrm{supp}(P) = \{\omega \in \Omega : P[\omega] > 0\}$ to denote the support of the distribution P. For an event $E \subseteq \Omega$ with $P[E] > 0$, we use $P|E$ to denote the conditional probability distribution P conditioned on E.

For distributions P and Q over a set Ω, the ℓ_1-distance between them is defined as

$$\|P - Q\|_1 = \sum_{\omega \in \Omega} |P[\omega] - Q[\omega]| .$$

We state a useful tail bound on the sum of independent random variables:

▶ **Fact 7** (Chernoff Bounds, see [49] for reference). *Let $X_1, \ldots, X_n \in \{0, 1\}$ be independent random variables each with mean μ, and let $X = \sum_{i=1}^{n} X_i$. Then, for all $\delta \in (0, 1)$,*

$$\Pr[X \leq (1 - \delta)\mu n] \leq e^{-\frac{\delta^2 \mu n}{2}},$$

$$\Pr[X \geq (1 + \delta)\mu n] \leq e^{-\frac{\delta^2 \mu n}{3}}.$$

We also state another useful lemma:

▶ **Lemma 8.** *Let $t, n \in \mathbb{N}$, $t < n$. Let μ be the uniform distribution on $[n]$, and let ν be the uniform distribution on $[n - t]$. Then, $\|\mu - \nu\|_1 = 2t/n$.*

Proof. We have $\|\mu - \nu\|_1 = \sum_{i=1}^{n-t} \left(\frac{1}{n-t} - \frac{1}{n}\right) + \sum_{i=n-t+1}^{n} \frac{1}{n} = \frac{2t}{n}$. ◀

3.2 Vector Spaces over GF(2)

Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{F}_2$, with the uniform measure; often we shall have $\mathcal{V} = \mathbb{F}_2^n$ for some $n \in \mathbb{N}$.

▶ **Definition 9** (Inner Product). *For functions $f, g : \mathcal{V} \to \mathbb{R}$, define their inner product as*

$$\langle f, g \rangle = \mathop{\mathbb{E}}_{x \sim \mathcal{V}}[f(x)g(x)],$$

where $x \sim \mathcal{V}$ denotes that x is uniformly chosen from $\mathcal{V}$.

▶ **Definition 10** (L^p norm). *For $f : \mathcal{V} \to \mathbb{R}$, and $p \geq 1$, we define*

$$\|f\|_p = \mathbb{E}[f^p]^{1/p}.$$

▶ **Definition 11** (Convolution). *For $f, g : \mathcal{V} \to \mathbb{R}$, we define their convolution as*

$$(f * g)(x) = \mathop{\mathbb{E}}_{y \sim \mathcal{V}}[f(y)g(x + y)].$$

We observe the following simple fact:

▶ **Fact 12.** *For functions $f, g, h : \mathcal{V} \to \mathbb{R}$,*

$$\langle f * g, h\rangle = \langle f * h, g\rangle = \langle g * h, f\rangle = \mathop{\mathbb{E}}_{x,y\sim\mathcal{V}}[f(x)g(y)h(x+y)].$$

▶ **Definition 13** (Density Function)**.** *For a nonempty set $A \subseteq \mathcal{V}$, we define its density function as*

$$\varphi_A = \frac{\mathbb{1}_A}{\mathbb{E}[\mathbb{1}_A]} = \frac{\mathbb{1}_A}{|A| / |\mathcal{V}|}.$$

Note that $\mathbb{E}[\varphi_A] = 1$.

3.3 3-Player Games and Parallel Repetition

We overview some basic definitions regarding multiplayer games. We shall restrict our focus to 3-player games.

▶ **Definition 14** (3-Player Game)**.** *A 3-player game $\mathcal{G}$ is a tuple $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ Q, V_{pred})$, where the question sets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and the answer sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are finite sets, Q is a probability distribution over $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, and $V_{pred} : (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}) \times (\mathcal{A} \times \mathcal{B} \times \mathcal{C}) \to \{0, 1\}$ is a predicate.*

The game $\mathcal{G}$ proceeds as follows: A verifier samples questions $(X, Y, Z) \sim Q$; then, the verifier sends X to player 1, Y to player 2, and Z to player 3, to which the players respond back with answers $A \in \mathcal{A},\ B \in \mathcal{B},\ C \in \mathcal{C}$ respectively. Finally, the verifier declares that the players win if and only if $V_{pred}((X, Y, Z), (A, B, C)) = 1$.

▶ **Definition 15** (Game Value)**.** *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ Q, V_{pred})$ be a 3-player game. The value of the game $\mathcal{G}$, denoted $\mathrm{val}(\mathcal{G})$, is defined as*

$$\mathrm{val}(\mathcal{G}) = \max_{f,g,h} \Pr_{(X,Y,Z)\sim Q} \left[V_{pred}\left((X, Y, Z), (f(X), g(Y), h(Z))\right) = 1\right],$$

where the maximum is over player strategies $f : \mathcal{X} \to \mathcal{A},\ g : \mathcal{Y} \to \mathcal{B},\ h : \mathcal{Z} \to \mathcal{C}$.

▶ **Fact 16.** *The game value is unchanged even if the players are allowed to use public and private randomness, since there always exists some optimal fixed values of the random strings.*

Next, we define the parallel repetition of a 3-player game, which corresponds to playing n independent copies of the game in parallel.

▶ **Definition 17** (Parallel Repetition)**.** *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ Q, V_{pred})$ be a 3-player game. We define its n-fold repetition as $\mathcal{G}^{\otimes n} = (\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n,\ \mathcal{A}^n \times \mathcal{B}^n \times \mathcal{C}^n,\ Q^{\otimes n}, V_{pred}^{\otimes n})$. The distribution $Q^{\otimes n}$ is the n-fold product of the distribution Q with itself, i.e., $Q^{\otimes n}[(x, y, z)] = \prod_{i=1}^{n} Q[(x_i, y_i, z_i)]$ for each $x \in \mathcal{X}^n, y \in \mathcal{Y}^n, z \in \mathcal{Z}^n$. The predicate $V_{pred}^{\otimes n}$ is defined as $V_{pred}^{\otimes n}((x, y, z), (a, b, c)) = \bigwedge_{i=1}^{n} V_{pred}((x_i, y_i, z_i), (a_i, b_i, c_i))$.*

Some Basic Results on Parallel Repetition

We state a lemma from [26], which shows that it suffices to prove parallel repetition in the case when the game's distribution is uniform over its support:

▶ **Lemma 18** ([32, Lemma 3.14])**.** *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ Q, V_{pred})$ be a 3-player game such that $\mathrm{val}(\mathcal{G}) < 1$. Let $\tilde{\mathcal{G}} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ U, V_{pred})$, where U is the uniform distribution over $\mathrm{supp}(Q)$. Then, $\mathrm{val}(\tilde{\mathcal{G}}) < 1$, and there exists a constant $c = c(\mathcal{G}) > 0$, such that for every sufficiently large $n \in \mathbb{N}$,*

$$\mathrm{val}(\mathcal{G}^{\otimes n}) \le 2 \cdot \mathrm{val}(\tilde{\mathcal{G}}^{\otimes \lfloor cn \rfloor}).$$

We state an inductive parallel repetition criterion from [53]:

▶ **Lemma 19** ([11, Lemma B.1]). *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ Q, V_{pred})$ be a 3-player game, and consider its n-fold repetition $\mathcal{G}^{\otimes n} = (\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n,\ \mathcal{A}^n \times \mathcal{B}^n \times \mathcal{C}^n,\ Q^{\otimes n}, V_{pred}^{\otimes n})$ for some sufficiently large $n \in \mathbb{N}$. Fix optimal strategies for the 3 players in this game, and for each $i \in [n]$, let Win_i be the event that this strategy wins the i^{th} coordinate of the game.*

Let $\epsilon > 0$ be a constant, and $\alpha \in (0, 1]$, $\alpha \geq 2^{-n}$ be such that the following condition holds: For every product event $E \times F \times G \subseteq \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$ with $\Pr_{Q^{\otimes n}}[E \times F \times G] \geq \alpha$, there exists a coordinate $i \in [n]$ such that $\Pr[\mathrm{Win}_i \mid E \times F \times G] \leq 1 - \epsilon$.

Then, for some constant $c = c(\mathcal{G})$, it holds that $\mathrm{val}(\mathcal{G}^{\otimes n}) \leq \alpha^c$.

4 Algebraic Spreadness and Uniformization

In this section, we overview a notion of pseudorandomness called algebraic spreadness. We also show how to decompose arbitrary sets into components that satisfy this spreadness definition.

4.1 Algebraic Spreadness

▶ **Definition 20** (Algebraic Spreadness). *Let $\mathcal{V} \subseteq \mathbb{F}_2^n$ be an affine subspace. We say that a subset $A \subseteq \mathcal{V}$ is (r, ϵ)-algebraically spread within $\mathcal{V}$ if for all affine subspaces $\mathcal{V}' \subseteq \mathcal{V}$ satisfying $\dim(\mathcal{V}') \geq \dim(\mathcal{V}) - r$, it holds that*

$$\frac{|A \cap \mathcal{V}'|}{|\mathcal{V}'|} \leq (1 + \epsilon) \cdot \frac{|A|}{|\mathcal{V}|}.$$

We state a useful result about spread subsets.

▶ **Theorem 21.** *Let $d \geq 1, \epsilon \in (0, 1/4)$. Then, there exists a sufficiently large integer $r = d^8 \epsilon^{-O(1)}$, and a sufficiently small $\delta = \Omega(\epsilon)$, such that the following holds: Suppose $A, B, C \subseteq \mathbb{F}_2^n$ are sets each of size at least $2^{-d} \cdot |\mathbb{F}_2^n|$. Then,*

(i) *([42, Proposition 2.16]) If at least two of A, B, C are (r, δ)-algebraically spread within $\mathbb{F}_2^n$, then*[10]

$$|\langle \varphi_A * \varphi_B,\ \varphi_C \rangle - 1| \leq \epsilon.$$

(ii) *([42, Proposition 4.10]) If at least one of A, B, C is (r, δ)-algebraically spread within $\mathbb{F}_2^n$, then*

$$\langle \varphi_A * \varphi_B,\ \varphi_C \rangle \leq 1 + \epsilon.$$

4.2 Uniformization

In this section, we describe a procedure that takes an arbitrary set $X \subseteq \mathbb{F}_2^n$ and approximately decomposes it into components that are algebraically spread. More generally, we find a simultaneous decomposition for three sets X, Y, Z, compatible with the diagonal-product:

[10] For example, if both A, B are algebraically spread within $\mathbb{F}_2^n$, we have $|\langle \varphi_A * \varphi_B,\ \varphi_C \rangle - 1| \leq 2\|\varphi_A * \varphi_B - 1\|_d \leq \epsilon$, where the first inequality follows from Hölder's inequality, and the second inequality follows from [42, Proposition 2.16].

▶ **Proposition 22.** *Let $r \in \mathbb{N}$, $\epsilon, \eta \in (0, 1/10)$, and let $X, Y, Z \subseteq \mathcal{V}$ for a linear subspace $\mathcal{V} \subseteq \mathbb{F}_2^n$. Define $\alpha = |S(X, Y, Z)| / |\mathcal{V}|^2$. Then, there exists $C_\eta = (1/\eta)^{O(1)} \geq 1$ (that depends only on η), an integer $T \in \mathbb{N}$, and for each $i \in [T]$, a linear subspace $\mathcal{V}_i \subseteq \mathcal{V}$, points $x_i, y_i \in \mathcal{V}/\mathcal{V}_i$, and subsets $X_i \subseteq x_i + \mathcal{V}_i$, $Y_i \subseteq y_i + \mathcal{V}_i$, $Z_i \subseteq x_i + y_i + \mathcal{V}_i$, such that:*

1. $\dim(\mathcal{V}_i) \geq \dim(\mathcal{V}) - r\epsilon^{-3} \log_2(4/\alpha)^{C_\eta}$ *for all $i \in [T]$.*
2. *$S(X_1, Y_1, Z_1), \ldots, S(X_T, Y_T, Z_T)$ are disjoint subsets of $S(X, Y, Z)$ such that*
$$\left|S(X, Y, Z) \setminus \cup_{i=1}^{T} S(X_i, Y_i, Z_i)\right| \leq \eta \left|S(X, Y, Z)\right|.$$
The diagonal-product sets are defined as in Definition 4.
3. *For all $i \in [T]$, we have $|X_i|, |Y_i|, |Z_i| \geq 2^{-(\log_2(4/\alpha))^{C_\eta}} \cdot |\mathcal{V}_i|$, and X_i, Y_i, Z_i are (r, ϵ)-algebraically spread within $x_i + \mathcal{V}_i, y_i + \mathcal{V}_i, x_i + y_i + \mathcal{V}_i$ respectively.*

The proof of the above proposition is based on [40, Section 5.3], with the exception that we work to make all the 3 sets spread (instead of just 2). As a result, we obtain a weaker dependence on the parameter η compared to their two set version; however, this is inconsequential in our setting, as we use this only for constant η. In the following subsections, we first show how to decompose one set, then two sets, and finally three sets.

4.2.1 Uniformization for 1 Set

We first show how to decompose a single set X into spread components. Before that, we show that any large set contains a spread set in it.

▶ **Lemma 23** (Existence of a Spread Subset). *Let $r \in \mathbb{N}$, $\epsilon \in (0, 1)$, and let $X \subseteq \mathcal{V}$ for a linear subspace $\mathcal{V} \subseteq \mathbb{F}_2^n$. Define $\alpha_X = |X| / |\mathcal{V}|$. Then, there exists an affine subspace $\mathcal{V}' \subseteq \mathcal{V}$ such that:*

1. $\dim(\mathcal{V}') \geq \dim(\mathcal{V}) - r\epsilon^{-1} \log_2(1/\alpha_X)$.
2. *The set $X' = X \cap \mathcal{V}'$ satisfies $\frac{|X'|}{|\mathcal{V}'|} \geq \alpha_X$, and X' is (r, ϵ)-algebraically spread within $\mathcal{V}'$.*

Proof. The reader is referred to the full version of this paper [46]. ◀

Using the above lemma, we show how to decompose a single set into spread components.

▶ **Lemma 24** (Uniformization for 1 Set). *Let $r \in \mathbb{N}$, $\epsilon, \eta \in (0, 1)$. Let $X \subseteq \mathcal{V}$ for a linear subspace $\mathcal{V} \subseteq \mathbb{F}_2^n$. Then, there exists $T \in \mathbb{N}$, and for each $i \in [T]$, an affine subspace $\mathcal{V}_i \subseteq \mathcal{V}$ and a subset $X_i \subseteq \mathcal{V}_i$, such that:*

1. $\dim(\mathcal{V}_i) \geq \dim(\mathcal{V}) - r\epsilon^{-1} \log_2(1/\eta)$ *for all $i \in [T]$.*
2. *$X_1, \ldots, X_T$ are disjoint subsets of X such that $\left|X \setminus \cup_{i \in [T]} X_i\right| \leq \eta |\mathcal{V}|$.*
3. *For all $i \in [T]$, $|X_i| \geq \eta |\mathcal{V}_i|$ and X_i is (r, ϵ)-algebraically spread within $\mathcal{V}_i$.*

Proof. The reader is referred to the full version of this paper [46]. ◀

4.2.2 Uniformization for 2 Sets

Next, we state a lemma about decomposing the product $X \times Y$ of two sets X, Y into spread components.

▶ **Lemma 25** (Uniformization for 2 Sets, [40, Lemma 5.9]). *Let $r \in \mathbb{N}$, $\epsilon, \eta \in (0, 1/10)$, and let $X, Y \subseteq \mathcal{V}$ for a linear subspace $\mathcal{V} \subseteq \mathbb{F}_2^n$. Define $\alpha_{XY} = |X| |Y| / |\mathcal{V}|^2$. Then, there exists $T \in \mathbb{N}$, and for each $i \in [T]$, a linear subspace $\mathcal{V}_i \subseteq \mathcal{V}$, points $x_i, y_i \in \mathcal{V}/\mathcal{V}_i$, and subsets $X_i \subseteq x_i + \mathcal{V}_i$, $Y_i \subseteq y_i + V_i$, such that:*

1. $\dim(\mathcal{V}_i) \geq \dim(\mathcal{V}) - O\left(r\epsilon^{-2}\log_2(1/\alpha_{XY})^2\log_2(1/\eta) + r\epsilon^{-2}\log_2(1/\eta)^5\right)$ *for all* $i \in [T]$.
2. $X_1 \times Y_1, \ldots, X_T \times Y_T$ *are disjoint subsets of* $X \times Y$ *such that*

$$\left|(X \times Y) \setminus \cup_{i\in[T]}(X_i \times Y_i)\right| \leq \eta\,|X \times Y|$$

3. *For all* $i \in [T]$*, we have* $|X_i \times Y_i| \geq \kappa \cdot |\mathcal{V}_i|^2$ *for* $\kappa \geq 2^{-O(\log_2(1/\eta)^2)} \cdot \alpha_{XY}$*, and* X_i, Y_i *are* (r,ϵ)*-algebraically spread within* $x_i + \mathcal{V}_i, y_i + \mathcal{V}_i$ *respectively.*

4.2.3 Uniformization for 3 Sets

We shall prove Proposition 22 and show how to decompose three sets into spread components, compatible with the diagonal-product. We start by stating a simple fact about the size of a diagonal-product $S(X,Y,Z)$, which along with Theorem 21 allows us to control this size when at least one or two of X, Y, Z are spread.

▶ **Fact 26.** *Let* $X, Y, Z \subseteq \mathcal{V}$ *for a linear subspace* $\mathcal{V} \subseteq \mathbb{F}_2^n$*, and let* $\alpha_X = |X|/|\mathcal{V}|$*,* $\alpha_Y = |Y|/|\mathcal{V}|$*,* $\alpha_Z = |Z|/|\mathcal{V}|$*. Then,* $|S(X,Y,Z)| = \alpha_X\alpha_Y\alpha_Z \cdot \langle \varphi_X * \varphi_Y,\ \varphi_Z\rangle$*.*

First, we show a one round partitioning result for $S(X,Y,Z)$:

▶ **Lemma 27.** *Let* $\eta \in (0, 1/50)$*, and let* $X, Y, Z \subseteq \mathcal{V}$ *for a linear subspace* $\mathcal{V} \subseteq \mathbb{F}_2^n$*. Define* $\alpha = |S(X,Y,Z)|/|\mathcal{V}|^2$*. Then, there exists an integer* $r' = O(\log_2(1/(\eta\alpha))^{16})$*, and constant* $\epsilon' = \Omega(1)$*, such that the following hold:*

Let $r \in \mathbb{N}$*,* $\epsilon \in (0, 1/10)$ *satisfy* $r \geq r'$*,* $\epsilon \leq \epsilon'$*. Then, there exists* $T \in \mathbb{N}$ *, a set* $\mathcal{G} \subseteq [T]$*, and for each* $i \in [T]$*, a linear subspace* $\mathcal{V}_i \subseteq \mathcal{V}$*, points* $x_i, y_i \in \mathcal{V}/\mathcal{V}_i$*, and subsets* $X_i \subseteq x_i + \mathcal{V}_i, Y_i \subseteq y_i + \mathcal{V}_i, Z_i \subseteq x_i + y_i + \mathcal{V}_i$*, such that*
1. $\dim(\mathcal{V}_i) \geq \dim(\mathcal{V}) - O\left(r\epsilon^{-3}\log_2(1/(\eta\alpha))^6\right)$ *for all* $i \in [T]$.
2. $S(X_1,Y_1,Z_1), \ldots, S(X_T,Y_T,Z_T)$ *are disjoint subsets of* $S(X,Y,Z)$ *such that*

$$\left|S(X,Y,Z) \setminus \cup_{i=1}^{T} S(X_i,Y_i,Z_i)\right| \leq 4\eta\,|S(X,Y,Z)|\,.$$

3. $|S(X_i,Y_i,Z_i)| \geq 2^{-O(\log_2(1/(\eta\alpha))^2)} \cdot |\mathcal{V}_i|^2$ *for all* $i \in [T]$.
4. *For all* $i \in \mathcal{G}$*, the sets* X_i, Y_i, Z_i *are* (r,ϵ)*-algebraically spread within* $x_i + \mathcal{V}_i, y_i + \mathcal{V}_i, x_i + y_i + \mathcal{V}_i$ *respectively.*
5. $\sum_{i\in\mathcal{G}} |S(X_i,Y_i,Z_i)| \geq \frac{1}{10} \cdot |S(X,Y,Z)|$.

Proof. The reader is referred to the full version of this paper [46]. ◀

We write down a simpler version of the above lemma, with the assumption on r, ϵ removed:

▶ **Corollary 28.** *Let* $r \in \mathbb{N}$*,* $\epsilon, \eta \in (1/10)$*, and let* $X, Y, Z \subseteq \mathcal{V}$ *for a linear subspace* $\mathcal{V} \subseteq \mathbb{F}_2^n$*. Define* $\alpha = |S(X,Y,Z)|/|\mathcal{V}|^2$*. Then, there exists* $T \in \mathbb{N}$ *, a set* $\mathcal{G} \subseteq [T]$*, and for each* $i \in [T]$*, a linear subspace* $\mathcal{V}_i \subseteq \mathcal{V}$*, points* $x_i, y_i \in \mathcal{V}/\mathcal{V}_i$*, and subsets* $X_i \subseteq x_i + \mathcal{V}_i, Y_i \subseteq y_i + \mathcal{V}_i, Z_i \subseteq x_i + y_i + \mathcal{V}_i$*, such that*
1. $\dim(\mathcal{V}_i) \geq \dim(\mathcal{V}) - O\left(r\epsilon^{-3}\log_2(1/(\eta\alpha))^6 + \epsilon^{-3}\log_2(1/(\eta\alpha))^{22}\right)$ *for all* $i \in [T]$.
2. $S(X_1,Y_1,Z_1), \ldots, S(X_T,Y_T,Z_T)$ *are disjoint subsets of* $S(X,Y,Z)$ *such that*

$$\left|S(X,Y,Z) \setminus \cup_{i=1}^{T} S(X_i,Y_i,Z_i)\right| \leq \eta\,|S(X,Y,Z)|\,.$$

3. $|S(X_i,Y_i,Z_i)| \geq 2^{-O(\log_2(1/(\eta\alpha))^2)} \cdot |\mathcal{V}_i|^2$ *for all* $i \in [T]$.
4. *For all* $i \in \mathcal{G}$*, the sets* X_i, Y_i, Z_i *are* (r,ϵ)*-algebraically spread within* $x_i + \mathcal{V}_i, y_i + \mathcal{V}_i, x_i + y_i + \mathcal{V}_i$ *respectively.*
5. $\sum_{i\in\mathcal{G}} |S(X_i,Y_i,Z_i)| \geq \frac{1}{10} \cdot |S(X,Y,Z)|$.

Proof. We plug in the parameter $\eta/5$ in Lemma 27 to get $r' = O(\log_2(1/(\eta\alpha))^{16})$, and $\epsilon' = \Omega(1)$. Then, we use the lemma with parameters $r_0 = r + r'$ and $\epsilon_0 = \epsilon \cdot \epsilon'$, noting that any set which is (r_0, ϵ_0)-algebraically spread is also (r, ϵ)-algebraically spread. ◀

Finally, we complete the main result of this section:

Proof of Proposition 22. The reader is referred to the full version of this paper [46]. ◀

5 Uniform Square Covers

In this section, we show that any *diagonal-product* set (see Definition 4) composed of algebraically spread sets (see Definition 20) is *uniformly covered* by *squares* (see Definition 5). In Section 5.1, we introduce a combinatorial notion of spreadness that will be useful for us; in Section 5.2, we show that algebraically spread sets satisfy this definition (in a specific way); finally, in Section 5.3, we prove Proposition 34, our square covering result.

5.1 Combinatorial Spreadness and Graph Counts

We state some useful definitions from [41]:

▶ **Definition 29** (Combinatorial Spreadness). *Let X, Y be finite sets, and let $f : X \times Y \to \{0,1\}$. We say that f is (r,ϵ)-combinatorially spread if for every $S \subseteq X$, $T \subseteq Y$ satisfying $|S \times T| \geq 2^{-r} |X \times Y|$, it holds that*

$$\mathop{\mathbb{E}}_{(x,y)\sim S\times T}[f(x,y)] \leq (1+\epsilon)\,\mathbb{E}[f],$$

where $\mathbb{E}[f] = \mathbb{E}_{(x,y)\sim X\times Y}[f(x,y)]$.

▶ **Definition 30** (Lower Bounded Left-Marginals). *Let X, Y be finite sets, and let $f : X \times Y \to \{0,1\}$. We say that f has (r,ϵ)-lower bounded left-marginals if*

$$\Pr_{x\sim X}\left[\mathop{\mathbb{E}}_{y\sim Y}[f(x,y)] \leq (1-\epsilon)\cdot \mathbb{E}[f]\right] \leq 2^{-r}.$$

We state a result about graph counts in spread sets:

▶ **Theorem 31** ([27, Theorem 2.1]). *For any $k \in \mathbb{N}$, $\epsilon \in (0,1)$, there exists sufficiently small $\gamma = \gamma(\epsilon,k) > 0$, and sufficiently large $C = C(\epsilon,k) \in \mathbb{N}$, such that the following holds:*

Let $H = ([k], E)$ be an oriented graph, where $(i,j) \in E$ implies $i < j$. Let $X_1, \dots, X_k$ be finite sets; let $d \geq 1$, and for each $(i,j) \in E$, let $f_{ij} : X_i \times X_j \to \{0,1\}$ be a function satisfying:

1. $\mathbb{E}[f_{ij}] \geq 2^{-d}$,
2. *f_{ij} is (Cd^2, γ)-combinatorially spread, and*
3. *f_{ij} has (Cd, γ)-lower bounded left-marginals.*

Then, it holds that

$$\left|\mathop{\mathbb{E}}_{x_1\sim X_1,\dots,x_k\sim X_k}\left[\prod_{(i,j)\in E} f_{ij}(x_i,x_j)\right] - \prod_{(i,j)\in E}\mathbb{E}[f_{ij}]\right| \leq \epsilon \cdot \prod_{(i,j)\in E}\mathbb{E}[f_{ij}].$$

5.2 Algebraic Spreadness implies Graph Counts

We show that a certain function corresponding to algebraically spread sets is both combinatorially spread and has lower bounded left-marginals.

▶ **Lemma 32.** *Let $d, r \geq 1, \epsilon \in (0, 1/4)$. Then, there exists a sufficiently large integer $s = (r^8 + d^8) \cdot \epsilon^{-O(1)}$, and a sufficiently small $\delta = \Omega(\epsilon)$, such that the following holds:*

Let $X, Y, Z \subseteq \mathbb{F}_2^n$ be subsets each of density at least 2^{-d}, and such that Y, Z are (s, δ)-algebraically spread. Then, the function $f : X \times Y \to \{0, 1\}$, given by $f(x, y) = \mathbb{1}[x + y \in Z]$, satisfies:

1. *$|\mathbb{E}[f] - \alpha_Z| \leq \epsilon \cdot \alpha_Z$, where $\alpha_Z = |Z| / 2^n$,*
2. *f is (r, ϵ)-combinatorially spread, and*
3. *f has (r, ϵ)-lower bounded left-marginals.*

Proof. The reader is referred to the full version of this paper [46]. ◀

5.3 Squares inside Spread Diagonal-Product Sets

In this subsection, we prove our square covering result. We start by proving the following counting lemma:

▶ **Lemma 33.** *Let $\epsilon \in (0, 1/4)$ be a constant, and let $d \geq 1$. Then, there exists a sufficiently large integer $r = O_\epsilon(d^{16})$ and sufficiently small $\delta = \delta(\epsilon) > 0$ such that the following holds:*

Let $X, Y, Z \subseteq \mathbb{F}_2^n$ be each of density at least 2^{-d}, and such that all of X, Y, Z are (r, δ)-algebraically spread. Let $S = S(X, Y, Z)$ be as in Definition 4, and let $\Gamma : \mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{R}$ be the function mapping (x, y) to the normalized number of squares (see Definition 5) in S containing (x, y), i.e.,

$$\Gamma(x, y) = \mathop{\mathbb{E}}_{w \sim \mathbb{F}_2^n} [\mathbb{1}[s_{x,y,w} \subseteq S]] .$$

Define $\alpha_X = \mathbb{E}[\mathbb{1}_X], \alpha_Y = \mathbb{E}[\mathbb{1}_Y], \alpha_Z = \mathbb{E}[\mathbb{1}_Z]$. Then, it holds that

1. *$\left||S| \cdot 2^{-2n} - \alpha_X \alpha_Y \alpha_Z\right| \leq \epsilon \cdot \alpha_X \alpha_Y \alpha_Z$.*
2. *$\left|\|\Gamma\|_1 - \alpha_X^2 \alpha_Y^2 \alpha_Z^2\right| \leq \epsilon \cdot \alpha_X^2 \alpha_Y^2 \alpha_Z^2$.*
3. *$\left|\|\Gamma\|_2^2 - \alpha_X^3 \alpha_Y^3 \alpha_Z^3\right| \leq \epsilon \cdot \alpha_X^3 \alpha_Y^3 \alpha_Z^3$.*

Proof. The reader is referred to the full version of this paper [46]. ◀

With the above, we are ready to prove the main result of this section.

▶ **Proposition 34.** *Let $\epsilon \in (0, 1/4)$ be a constant, and let $d \geq 1$. Then, there exists a sufficiently large integer $r = O_\epsilon(d^{16})$ and sufficiently small $\delta = \delta(\epsilon) > 0$ such that the following holds:*

Let $X, Y, Z \subseteq \mathbb{F}_2^n$, be each of density at least 2^{-d}, and such that all of X, Y, Z are (r, δ)-algebraically spread. Let $S = S(X, Y, Z)$ be as in Definition 4, and let $\mathcal{T}$ denote the set of all squares in S (see Definition 5); formally,

$$\mathcal{T} = \left\{(x, y, w) \in (\mathbb{F}_2^n)^3 : s_{x,y,w} \subseteq S\right\} .$$

Define $\alpha_X = \mathbb{E}[\mathbb{1}_X], \alpha_Y = \mathbb{E}[\mathbb{1}_Y], \alpha_Z = \mathbb{E}[\mathbb{1}_Z]$. Then, it holds that

1. *$\left||S| \cdot 2^{-2n} - \alpha_X \alpha_Y \alpha_Z\right| \leq \epsilon \cdot \alpha_X \alpha_Y \alpha_Z$.*
2. *$\left||\mathcal{T}| \cdot 2^{-3n} - \alpha_X^2 \alpha_Y^2 \alpha_Z^2\right| \leq \epsilon \cdot \alpha_X^2 \alpha_Y^2 \alpha_Z^2$.*

3. *Let μ be the distribution on S obtained as follows: Pick a random square in S, by picking uniformly at random $(x, y, w) \sim \mathcal{T}$; then, output a uniformly random element from the set $s_{x,y,w}$. It holds that*

$$\|\mu - U_S\|_1 \leq \epsilon,$$

where U_S denotes the uniform distribution over S.

Proof. The reader is referred to the full version of this paper [46]. ◀

6 Parallel Repetition for Games with the GHZ Query Support

This section is devoted to the proof of Theorem 1. By Lemma 18, it suffices to consider the case where Q is the uniform distribution over $\mathrm{supp}(Q)$. We prove the following:

▶ **Theorem 35.** *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ Q, V_{pred})$ be any 3-player game with $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathbb{F}_2$, and with Q the uniform distribution over*

$$\mathrm{supp}(Q) = \left\{(x, y, z) \in \mathbb{F}_2^3 : x + y + z = 0\right\},$$

and such that $\mathrm{val}(\mathcal{G}) < 1$. Then, for all sufficiently large n,[2] *it holds that*

$$\mathrm{val}(\mathcal{G}^{\otimes n}) \leq \exp\left(-n^c\right),$$

where $c > 0$ is an absolute constant.

The remainder of this section is devoted to the proof of the above theorem. Towards this, we fix a 3-player game $\mathcal{G}$ as in the statement of the theorem; in particular, we must have that $\mathrm{val}(\mathcal{G}) \leq 3/4$.

The main step of the proof is to prove that the game $\mathcal{G}^{\otimes n}$ has a hard coordinate when the inputs to the players are conditioned on a product event with sufficiently large measure. We carry this out in two parts: first, in Section 6.1, we prove this statement assuming that each of the three sets in the product event are well spread (as in Definition 20) inside some large subspace of $\mathbb{F}_2^n$; then, in Section 6.2, we prove the statement for general product events, using our uniformization strategy.

6.1 Hard Coordinate under Spread Product Events

We consider the game $\mathcal{G}^{\otimes n} = (\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n,\ \mathcal{A}^n \times \mathcal{B}^n \times \mathcal{C}^n,\ Q^{\otimes n}, V_{pred}^{\otimes n})$ for some sufficiently large $n \in \mathbb{N}$. Fix any strategies for the 3 players in this game, and for each $i \in [n]$, let $\mathrm{Win}_i \subseteq \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$ be the event that this strategy wins the i^{th} coordinate of the game.

▶ **Lemma 36.** *For any $1 \leq d \leq o(n)$, and constant $\epsilon \in (0, 1/4)$, there exists a (sufficiently large) integer $r = O_\epsilon(d^{16})$, and a (sufficiently small) constant $0 < \delta = \delta(\epsilon) < 1$, such that the following holds:*

Let $\mathcal{V} \subseteq \mathbb{F}_2^n$ be a linear subspace of dimension $\dim(\mathcal{V}) \geq n - o(n)$, and let $E, F, G \subseteq \mathcal{V}$ be such that each of them is (r, δ)-algebraically spread within $\mathcal{V}$ (see Definition 20), and such that each of the densities $\frac{|E|}{|\mathcal{V}|}, \frac{|F|}{|\mathcal{V}|}, \frac{|G|}{|\mathcal{V}|} \geq 2^{-d}$. Then,

$$\mathop{\mathbb{E}}_{i \sim [n]} \left[\Pr_{Q^{\otimes n}}\left[\mathrm{Win}_i \mid E \times F \times G\right]\right] \leq \frac{1}{2} + \frac{\mathrm{val}(\mathcal{G})}{2} + \epsilon.$$

The remainder of this subsection is devoted to the proof of the Lemma 36. We fix some $1 \leq d \leq o(n)$, and constant $\epsilon \in (0, 1/4)$, and let $r = O_\epsilon(d^{16})$, $\delta = \delta(\epsilon) > 0$ be so that Proposition 34 holds with the parameters $d, \frac{\epsilon}{2}$. Let $\mathcal{V} \subseteq \mathbb{F}_2^n$ be a linear subspace of dimension $\dim(\mathcal{V}) \geq n - o(n)$, and let $E, F, G \subseteq \mathcal{V}$ be each (r, δ)-algebraically spread within $\mathcal{V}$, and such that each of the densities $\alpha_E = \frac{|E|}{|\mathcal{V}|}$, $\alpha_F = \frac{|F|}{|\mathcal{V}|}$, $\alpha_G = \frac{|G|}{|\mathcal{V}|}$ is at least 2^{-d}.

We start by proving a lemma that demonstrates the usefulness of squares. It says that the no strategy for the game $\mathcal{G}^{\otimes n}$ can win on non-trivial coordinates of squares.

▶ **Definition 37** (Non-trivial coordinates)**.** *For a square $s = s_{x,y,w} \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n$, as in Definition 5, we define its set of* non-trivial coordinates *as*

$$\mathcal{I}_s = \{i \in [n] : w_i \neq 0\}.$$

This is well-defined since w is uniquely determined by the square s (see Remark 6).

▶ **Lemma 38.** *Let s be a square, and let $i \in \mathcal{I}_s$ be a non-trivial coordinate in s. Then,*

$$\Pr_{(x,y)\sim s}[(x, y, x + y) \in \mathrm{Win}_i] \leq \mathrm{val}(\mathcal{G}) < 1.$$

Proof. The reader is referred to the full version of this paper [46]. ◀

Next, we show that the spreadness of $E, F, G \subseteq \mathcal{V}$ guarantees that $S = S(E, F, G)$ (see Definition 4) is covered uniformly by squares. For this, let $\mathcal{T}$ denote the set of all squares contained in S; formally,

$$\mathcal{T} = \left\{(x, y, w) \in \mathcal{V}^3 : s_{x,y,w} \subseteq S\right\}.$$

Let U_S denote the uniform distribution over S. Also, let μ be the distribution on S obtained as follows: Pick a random square in S, by picking uniformly at random $(x, y, w) \sim \mathcal{T}$; then, output a uniformly random element from the set $s_{x,y,w}$.

▶ **Lemma 39.** *We have that*

1. $\left|\frac{|S|}{|\mathcal{V}|^2} - \alpha_E\alpha_F\alpha_G\right| \leq \frac{\epsilon}{2} \cdot \alpha_E\alpha_F\alpha_G$.
2. $\left|\frac{|\mathcal{T}|}{|\mathcal{V}|^3} - \alpha_E^2\alpha_F^2\alpha_G^2\right| \leq \frac{\epsilon}{2} \cdot \alpha_E^2\alpha_F^2\alpha_G^2$.
3. $\|\mu - U_S\|_1 \leq \frac{\epsilon}{2}$.

Proof. This follows directly by Proposition 34, applied with respect to $\mathcal{V} \cong \mathbb{F}_2^{\dim(\mathcal{V})}$. Note that the definition of squares is *coordinate-free*, and hence the proposition is applicable. ◀

Before completing the proof of Lemma 36, we show that a random square inside $S = S(E, F, G)$ contains many non-trivial coordinates.

▶ **Lemma 40.** *Let $(x, y, w) \sim \mathcal{T}$, and let $s = s_{x,y,w}$. Then,*

$$\mathbb{E}\,|\mathcal{I}_s| \geq \frac{n}{2} \cdot (1 - o(1)).$$

Proof. The reader is referred to the full version of this paper [46]. ◀

Finally, we complete the proof.

Proof of Lemma 36. For any $i \in [n]$, and $S = S(E, F, G)$, we can write

$$\begin{aligned}\Pr_{Q^{\otimes n}}[\mathrm{Win}_i \mid E \times F \times G] &= \Pr_{x,y\sim\mathbb{F}_2^n}[(x, y, x+y) \in \mathrm{Win}_i \mid (x, y, x+y) \in E \times F \times G] \\ &= \Pr_{x,y\sim\mathbb{F}_2^n}[(x, y, x+y) \in \mathrm{Win}_i \mid (x, y) \in S] \\ &= \Pr_{(x,y)\sim S}[(x, y, x+y) \in \mathrm{Win}_i].\end{aligned}$$

By Lemma 39, this can be bounded as

$$\Pr_{Q^{\otimes n}}[\mathrm{Win}_i \mid E \times F \times G] \le \mathop{\mathbb{E}}_{(x_0,y_0,w)\sim\mathcal{T}} \mathop{\mathbb{E}}_{(x,y)\sim s_{x_0,y_0,w}} [\mathbb{1}[(x, y, x+y) \in \mathrm{Win}_i]] + \frac{\epsilon}{2}. \tag{1}$$

Observe that for a fixed square $(x_0, y_0, w) \in \mathcal{T},\ s = s_{x_0,y_0,w}$, by Lemma 38, we can bound

$$\mathop{\mathbb{E}}_{(x,y)\sim s_{x_0,y_0,w}} [\mathbb{1}[(x, y, x+y) \in \mathrm{Win}_i]] \le 1 - (1 - \mathrm{val}(\mathcal{G})) \cdot \mathbb{1}[i \in \mathcal{I}_s].$$

Plugging this into the above, and taking expectation over $i \sim [n]$, we obtain

$$\begin{aligned}\mathop{\mathbb{E}}_{i\sim[n]}\left[\Pr_{Q^{\otimes n}}[\mathrm{Win}_i \mid E \times F \times G]\right] &\le \mathop{\mathbb{E}}_{i\sim[n]} \mathop{\mathbb{E}}_{(x_0,y_0,w)\sim\mathcal{T}} \left[1 - (1 - \mathrm{val}(\mathcal{G})) \cdot \mathbb{1}\left[i \in \mathcal{I}_{s_{x_0,y_0,w}}\right]\right] + \frac{\epsilon}{2} \\ &= 1 - \frac{(1 - \mathrm{val}(\mathcal{G}))}{n} \mathop{\mathbb{E}}_{(x_0,y_0,w)\sim\mathcal{T}} |\mathcal{I}_{s_{x_0,y_0,w}}| + \frac{\epsilon}{2}. \\ &\le 1 - (1 - \mathrm{val}(\mathcal{G})) \cdot \frac{1}{2} \cdot (1 - o(1)) + \frac{\epsilon}{2} \qquad \text{(by Lemma 40)} \\ &\le \frac{1}{2} + \frac{\mathrm{val}(\mathcal{G})}{2} + \frac{\epsilon}{2} + o(1) \le \frac{1}{2} + \frac{\mathrm{val}(\mathcal{G})}{2} + \epsilon.\end{aligned}$$

◀

6.2 Hard Coordinate under General Product Events

Now, we work with general product events. Consider the game $\mathcal{G}^{\otimes n} = (\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n,\ \mathcal{A}^n \times \mathcal{B}^n \times \mathcal{C}^n,\ Q^{\otimes n}, V_{pred}^{\otimes n})$ for some sufficiently large $n \in \mathbb{N}$. Fix any strategies for the 3 players in this game, and for each $i \in [n]$, let $\mathrm{Win}_i \subseteq \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$ be the event that this strategy wins the i^{th} coordinate of the game. We prove that:

▶ **Lemma 41.** *For any constant $\epsilon \in (0, 1/4)$, there exists a constant $c = c(\epsilon) > 0$, such that the following holds: For any sets $E, F, G \subseteq \mathbb{F}_2^n$ with $\Pr_{Q^{\otimes n}}[E \times F \times G] \ge 2^{-n^c}$, it holds that*

$$\mathop{\mathbb{E}}_{i\sim[n]}\left[\Pr_{Q^{\otimes n}}[\mathrm{Win}_i \mid E \times F \times G]\right] \le \frac{1}{2} + \frac{\mathrm{val}(\mathcal{G})}{2} + \epsilon.$$

Proof. The reader is referred to the full version of this paper [46]. ◀

With the above, the proof of our main theorem is direct:

Proof of Theorem 35. The theorem follows by combining Lemma 19 with Lemma 41, applied with the parameter choice $\epsilon = 0.1$, recalling that $\mathrm{val}(\mathcal{G}) \le 3/4$. ◀

Proof of Theorem 1. The theorem follows by combining Theorem 35 and Lemma 18. ◀

7 A Concentration Bound

We also prove a concentration bound for the GHZ game:

▶ **Theorem 42.** *Let $\mathcal{G} = (\mathcal{X} \times \mathcal{Y} \times \mathcal{Z},\ \mathcal{A} \times \mathcal{B} \times \mathcal{C},\ Q, V_{pred})$ be any 3-player game with $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathbb{F}_2$, and with Q the uniform distribution over*

$$\mathrm{supp}(Q) = \left\{(x, y, z) \in \mathbb{F}_2^3 : x + y + z = 0\right\},$$

and such that $\mathrm{val}(\mathcal{G}) < 1$. Then, for every constant $\epsilon \in (0, 1/4)$, there exists a constant $c = c(\epsilon) > 0$, such that the following holds for all sufficiently large n:[3]

Consider the game $\mathcal{G}^{\otimes n}$, and consider any strategies for the 3 players in this game. Then, if Z denotes the number of coordinates that the players win, it holds that

$$\Pr\left[Z \geq (\mathrm{val}(\mathcal{G}) + \epsilon) \cdot n\right] \leq \exp\left(-n^c\right).$$

The remainder of this section is devoted to the proof of the above theorem. Towards this, we fix a game $\mathcal{G}$ as in the statement of the theorem. Consider the game $\mathcal{G}^{\otimes n} = (\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n,\ \mathcal{A}^n \times \mathcal{B}^n \times \mathcal{C}^n,\ Q^{\otimes n}, V_{pred}^{\otimes n})$ for some sufficiently large $n \in \mathbb{N}$. Also, fix any strategies for the 3 players in this game, and for each $i \in [n]$, let $\mathrm{Win}_i \subseteq \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$ be the event that this strategy wins the i^{th} coordinate of the game. For each $S \subseteq [n]$, we use Win_S to denote the event $\wedge_{i \in S} \mathrm{Win}_i$.

7.1 An Improved Bound for Product Events

The first step in our proof is to prove an improved version of Lemma 41, where we don't lose a "factor of 2." We prove the following:

▶ **Lemma 43.** *For any constant $\epsilon \in (0, 1/4)$, there exists $c = c(\epsilon) > 0$, such that the following holds: For any sets $E, F, G \subseteq \mathbb{F}_2^n$ with $\Pr_{Q^{\otimes n}}[E \times F \times G] \geq 2^{-n^c}$, it holds that*

$$\mathop{\mathbb{E}}_{i \sim [n]}\left[\Pr_{Q^{\otimes n}}\left[\mathrm{Win}_i \mid E \times F \times G\right]\right] \leq \mathrm{val}(\mathcal{G}) + \epsilon.$$

The proof of the above lemma exactly follows the proof in Section 6, so we only mention how to improve it. We note that the only place where we lost a factor of 2 in the bound was in the proof of Lemma 36; the proof of Lemma 41, using uniformization, goes through as is if an improved version of Lemma 36 is proved. Now, in Lemma 36, we proceed in exactly the same manner upto Equation (1), which says (after taking expectation over $i \sim [n]$)

$$\mathop{\mathbb{E}}_{i \sim [n]}\left[\Pr_{Q^{\otimes n}}\left[\mathrm{Win}_i \mid E \times F \times G\right]\right] \leq \mathop{\mathbb{E}}_{i \sim [n]} \mathop{\mathbb{E}}_{(x_0, y_0, w) \sim \mathcal{T}} \mathop{\mathbb{E}}_{(x,y) \sim s_{x_0, y_0, w}}\left[\mathbb{1}\left[(x, y, x + y) \in \mathrm{Win}_i\right]\right] + \frac{\epsilon}{2}.$$

The factor 2 loss now occurred because when $(x_0, y_0, w) \sim \mathcal{T}$, the event $w_i = 1$ occurs with probability roughly $1/2$ (see Lemma 40), and we can only apply Lemma 38 when this holds. To improve upon this bound, we will replace the distribution with one where we are already conditioning on the event $w_i = 1$, and only then apply Lemma 38; this requires proving a stronger version of Lemma 40, which says that the distribution of (x, y) in the above statement remains unaffected even when the square is sampled conditioned on $w_i = 1$. Formally, we prove:

▶ **Lemma 44.** *Recall the distribution μ on S obtained as follows: choose $(x_0, y_0, w) \sim \mathcal{T}$, and let $(x, y) \sim s_{x_0, y_0, w}$. For any $i \in [n]$, define the distribution ν_i as follows: choose $(x_0, y_0, w) \sim \mathcal{T}$ conditioned on $w_i = 1$, and let $(x, y) \sim s_{x_0, y_0, w}$. Then, we have*

$$\mathop{\mathbb{E}}_{i \sim [n]} \left\|\mu - \nu_i\right\|_1 \leq o(1).$$

Proof. The reader is referred to the full version of this paper [46]. ◀

Finally, we complete the proof of the improved bound:

Proof of Lemma 43. As observed before, it suffices to improve Lemma 36, and we have

$$\mathop{\mathbb{E}}_{i\sim[n]}\left[\Pr_{Q^{\otimes n}}[\mathrm{Win}_i \mid E\times F\times G]\right] \leq \mathop{\mathbb{E}}_{i\sim[n]}\ \mathop{\mathbb{E}}_{(x_0,y_0,w)\sim\mathcal{T}}\ \mathop{\mathbb{E}}_{(x,y)\sim s_{x_0,y_0,w}}[\mathbb{1}\left[(x,y,x+y)\in\mathrm{Win}_i\right]]+\frac{\epsilon}{2}.$$

By Lemma 44 and Lemma 38, the above is at most

$$\begin{aligned}\mathop{\mathbb{E}}_{i\sim[n]}\ \mathop{\mathbb{E}}_{(x_0,y_0,w)\sim\mathcal{T}|w_i=1}\ \mathop{\mathbb{E}}_{(x,y)\sim s_{x_0,y_0,w}}[\mathbb{1}\left[(x,y,x+y)\in\mathrm{Win}_i\right]]+\frac{\epsilon}{2}+o(1) &\leq \mathrm{val}(\mathcal{G})+\frac{\epsilon}{2}+o(1)\\ &\leq \mathrm{val}(\mathcal{G})+\epsilon.\end{aligned}$$ ◀

7.2 Random Sets are Hard under Parallel Repetition

We prove a strengthening of Theorem 35, by using an argument similar to the proof of Lemma 19.

▶ **Lemma 45.** *For every constant $\epsilon\in(0,1/4)$, there exists $c=c(\epsilon)>0$, such that for every integer $1\leq t\leq n^c$, it holds that*

$$\mathop{\mathbb{E}}_{|S|=t}\left[\Pr_{Q^{\otimes n}}[\mathrm{Win}_S]\right]\leq 2\cdot(\mathrm{val}(\mathcal{G})+\epsilon)^t,$$

where $\mathbb{E}_{|S|=t}$ denotes that the expectation is over a uniformly chosen subset $S\subseteq[n]$ of size t.

Proof. The reader is referred to the full version of this paper [46]. ◀

7.3 Proof of the Concentration Bound

Finally, we complete the proof of our concentration bound, using standard arguments [52]:

Proof of Theorem 42. The reader is referred to the full version of this paper [46]. ◀

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