Choice Principles and Hypercompletion in HoTT

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Abstract

Building on work of Anel and Barton [1], we will introduce the construction of a hypercompletion modality [8] in HoTT under the assumption that countable choice holds. Central results have been formalized in Cubical Agda [5] and will be referenced where appropriate.

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1 Introduction

We work in homotopy type theory [9]. We make ready use of the elimination rule (J-rule) for intensional identity types, the univalence axiom, and higher inductive types. Modalities were studied intensively in the setting of HoTT by Rijke, Shulman and Spitters [8]. We will briefly revisit some of their terminology and key theorems in the preliminaries section, alongside a few other established HoTT results and definitions that we will need going forward. After that we will spend some time on the concepts of countable products and countable inverse limits in HoTT. Some of that material has been formalized by the author. The two sections subsequent to that contain the main results of this work and are fully formalized. There is a description of the choice principles identified by Anel and Barton [1], their relationship to Postnikov towers, and a proof that a certain operation is a modality which presents the hypercomplete types. We conclude with an application to the setting of synthetic Stone duality, unformalized as of the time of writing.

The question whether a hypercompletion operation can be defined in HoTT was posed in [8]. The work presented here (esp. the formalized portion) is a partial resolution to that question in the case where we are allowed to assume further classical axioms (variants of countable or dependent choice).

2 Preliminaries

We follow the notation of (e.g.) [10]. So if P is a type-family over A, then the product type is $(a : A) \to P(a)$, and the sum type is $(a : A) \times P(a)$. We will write $\mathcal{U}$ for the universe of all (small) types, $\mathcal{U}^{\leq -1}$ for the type of propositions and $\mathcal{U}^{\leq n}$ for the type of n-types. We will write $|\bullet|_n$ for the universal map $X \to \|X\|_n$ for any type X.

▶ **Theorem 1** (Fundamental theorem of identity types). *If A is pointed by a and $P : A \to \mathcal{U}$ is a family of types such that $P(a)$ is pointed, and $(x : A) \times P(x)$ is contractible with the point of $P(a)$ as the center of contraction, then $P(x) \simeq (x = a)$ naturally in x.*

Proof. Theorem 11.2.2 in [7]. ◀

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▶ **Definition 2** (Pullback). *If $f : A \to B \leftarrow C : g$ is a cospan of maps, then the pullback $A \times_B C$ is the type $(a : A) \times (c : C) \times (f(a) = g(c))$.*

The following definitions, propositions and examples come from [8].

▶ **Definition 3** (Modality). *Suppose $\bigcirc : \mathcal{U} \to \mathcal{U}$ is an operation on the universe with a unit $\eta : (X : \mathcal{U}) \to X \to \bigcirc X$, we say that the pair $(\bigcirc, \eta)$ is a (uniquely eliminating) modality if the map*

$$\lambda f. f \circ \eta_X : ((x : \bigcirc X) \to \bigcirc P(x)) \to ((x : X) \to \bigcirc P(\eta_X(x)))$$

is an equivalence for all $X : \mathcal{U}$ and $P : \bigcirc X \to \mathcal{U}$.

▶ **Definition 4** (Nullification). *If $(\bigcirc, \eta)$ is a modality, and X is a type, we say that X is nullified by the modality if $\bigcirc X$ is contractible.*

▶ **Definition 5.** *If $(\bigcirc, \eta)$ is a modality and $f : A \to B$ is a map, we say that f is nullified by the modality if all its fibers are nullified in the sense of the previous definition.*

▶ **Definition 6** (Σ-closed reflective subuniverse). *A Σ-closed reflective subuniverse is a triple $(M, \bigcirc, \eta)$ where $\bigcirc$ and η are an operation and a unit as before, and M is a function taking types to propositions, such that*

1. *$M(\bigcirc X)$ holds for all types X*
2. *For any type X and any type B such that $M(B)$, the map*

   $$\lambda f. f \circ \eta_X : (\bigcirc X \to B) \to (X \to B)$$

   is an equivalence
3. *If X is any type, and $P : X \to \mathcal{U}$, and $M(X)$, and $M(P(x))$ for all $x : X$, then*

   $$M((x : X) \times P(x))$$

▶ **Proposition 7** (Functoriality of $\bigcirc$). *If $(M, \bigcirc, \eta)$ is a reflective subuniverse, then there is an operation $(A, B : \mathcal{U}) \to (A \to B) \to (\bigcirc A \to \bigcirc B)$ which for convenience we shall label $\bigcirc$ also. This is functorial in the sense that*

- *$\bigcirc(\mathrm{id}_A) = \mathrm{id}_{\bigcirc A}$, and*
- *$\bigcirc(g \circ f) = \bigcirc g \circ \bigcirc f$*

Proof. Lemma 1.21 in [8]. ◀

▶ **Proposition 8.** *If $(M, \bigcirc, \eta)$ and $(M', \bigcirc', \eta')$ are two Σ-closed reflective subuniverses, then they are equal in the type of reflective closed subuniverses if and only if (the proposition) $(M(X) \Leftrightarrow M'(X))$ is inhabited for all types X.*

Proof. Theorem 1.18 in [8]. ◀

▶ **Definition 9.** *A (pullback-)stable orthogonal factorization system is a pair of predicates*

$$L, R : (A, B : \mathcal{U}) \to (A \to B) \to \mathcal{U}^{\leq -1}$$

such that

- *For any $f : A \to B$, and $g : B \to C$, if $L(f)$ and $L(g)$, then $L(g \circ f)$, and likewise for R.*
- *If $f : A \to B$ is an isomorphism, then $L(f)$ and $R(f)$.*

- *For all $f : A \to B$, the type*

$$(C : \mathcal{U}) \times (g : A \to C) \times (h : C \to B) \times (f = hg) \times L(g) \times R(h)$$

of factorizations of the map into an L-map and an R-map is contractible, and such factorization diagrams are preserved by pullbacks.

▶ **Proposition 10.** *If (L, R) is a stable orthogonal factorization system, then R is precisely the type of maps which satisfy right-lifting with respect to all maps in L and vice versa.*

Proof. Lemma 1.46 in [8]. ◀

▶ **Theorem 11** (Modalities and reflective subuniverses)**.** *The type of modalities is equivalent to the types of*

1. *Σ-closed reflective subuniverses*
2. *Stable orthogonal factorization systems*

where, in particular

1. *M is the proposition that $\eta(X)$ is an equivalence, if this holds we say that X is a modal type.*
2. *The maps L are those which are nullified by the modality. The maps in R are those with modal fibers; the units η always lie in L.*

Hence, in particular, a modality is uniquely determined by the subuniverse of types such that η is an equivalence.

Proof. Using Theorems 1.13, 1.16, 1.34 and 1.54 in [8]. ◀

▶ **Example 12** (n-Truncation)**.** Fix a natural number $n : \mathbb{N}$, the pair $(\lambda X.\|X\|_n, \lambda X.| \bullet |_n)$ is a modality.

The corresponding subuniverse is the type of n-truncated types, and the left maps in the corresponding factorization system are the n-connected maps.

▶ **Definition 13** (Left-exact modality)**.** *If $(\bigcirc, \eta)$ is a modality, and $B \to A \leftarrow C$ is a cospan of types, then there is a canonical map*

$$(\bigcirc B \times_{\bigcirc A} \bigcirc C) \to \bigcirc (B \times_A C)$$

We say that $(\bigcirc, \eta)$ is a left-exact modality if this map is an equivalence for all spans.

One consequence is that the maps which are nullified by the modality coincide with the maps whose image under $\bigcirc$ is an equivalence.

▶ **Proposition 14.** *The 0-truncation modality is not left-exact. This proposition generalizes to all n.*

Proof. Consider the pullback square:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d] \\ \mathbf{1} \arrow[r] & S^1 \end{tikzcd}$$

$\|\mathbb{Z}\|_0 = \mathbb{Z}$, but $\|S^1\|_0 = \mathbf{1}$, so we would be asking for an equivalence between $\mathbf{1}$ and $\mathbb{Z}$.

More generally we can take a pullback square:

$$\begin{tikzcd} K(\mathbb{Z}, n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d] \\ \mathbf{1} \arrow[r] & K(\mathbb{Z}, n+1) \end{tikzcd}$$

And similar reasoning applies. ◀

Before returning to modalities, we'll list some more useful facts about connected maps.

▶ **Proposition 15.** *If*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r, "v"] & Y \end{tikzcd}$$

commutes, and the right-most map is n-truncated, and the left-most map is n-connected, then the type of fillers for the square is contractible. In fact, this characterizes n-connected maps.

Proof. Refer to Proposition 10, Theorem 11 and Example 12. ◀

▶ **Proposition 16.** *If A is n-connected and $P(a)$ is n-connected for each $a : A$, then $(a : A) \times P(a)$ is also n-connected.*

Proof. Follows immediately from Theorem 7.3.9 in [9]. ◀

▶ **Proposition 17.** *If $f : A \to B$ is n-connected, then $\|f\|_n$ is an equivalence. The converse is not necessarily true. However, $f : \|A\|_n \to \|B\|_n$ is an equivalence if and only if f is n-connected.*

Proof. For an example of a map such that $\|f\|_0$ is an equivalence, but the map is not 0-connected, see the proof of Proposition 14. The map $\mathbf{1} \to S^1$ is as specified. ◀

▶ **Proposition 18.** *Suppose $f = g \circ h$ and f and h are n-connected, then so is g.*

Proof. Immediate using the orthogonal factorization system. ◀

▶ **Proposition 19.** *If A is $(n+1)$-connected, then $x =_A y$ is n-connected for all $x, y : A$. If f is $(n+1)$-connected, then all $\mathrm{ap}(f)$ are n-connected.*

As a partial converse: if A is merely inhabited and $x =_A y$ are all n-connected, then A is $(n+1)$-connected. If f is surjective (i.e. (-1)-connected) and all $\mathrm{ap}(f)$ are n-connected, then f is $(n+1)$-connected.

Proof. The first statement follows from Theorem 7.3.12 in [9].

According to (amongst others) Leclerc [3, p. 8], the converse is due to Coquand. If A satisfies the hypothesis of the converse, then $\|A\|_{n+1}$ is a proposition, using the theorem cited above. Then $\|A\|_{n+1} \simeq \|A\|_{-1}$ and the latter type is contractible by assumption. The corresponding statement for maps follows readily. ◀

▶ **Proposition 20.** *Suppose $f = g \circ h$ and f and g are $(n+1)$-connected, then h is n-connected.*

Proof. Follows from Proposition 19. ◀

▶ **Corollary 21.** *If $\|f\|_{n+1}$ is an equivalence, then f is n-connected.*

Back to modalities, drawing still on [8, esp. Section 3].

▶ **Example 22** (Open modality). If P is a proposition, then there is a modality whose operation is

$$\lambda X.\,(P \to X)$$

This modality is left-exact.

▶ **Definition 23** (Topological modality). *A modality is called topological if there is some (type-indexed) family of propositions P such that the modal types X (hence the whole modality) are characterized by the lifting properties for the squares*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbf{1} \arrow[r, equal] & \mathbf{1} \end{tikzcd}$$

Clearly each P is nullified by the modal operator.

Clearly the open modality is topological.

▶ **Proposition 24.** *Topological modalities are lex.*

Proof. Corollary 3.12 in [8]. ◀

▶ **Definition 25.** *A map f is called ∞-connected if it is n-connected for every n.*

▶ **Proposition 26.** *If A, B are types, $P : A \to \mathcal{U}$, and $Q : B \to \mathcal{U}$, and there are maps $f : A \to B$, and $g_a : P(a) \to Q(f(a))$ for all $a : A$, each of which is ∞-connected, then $\lambda(a,p).(f(a), g_a(p)) : (a : A) \times P(a) \to (b : B) \times Q(b)$ is ∞-connected.*

Proof. The fiber is $(a : A) \times (p : P(a)) \times (q : f(a) = b) \times (g_a(p) =^q p')$. This is equivalent to $(a : A) \times (q : f(a) = b) \times (p : P(a)) \times (g_a(p) =^q p')$ and that this is connected follows from the hypothesis. ◀

▶ **Definition 27.** *A type X is called hypercomplete if it has right lifting against all ∞-connected maps.*

▶ **Proposition 28.** *Having left lifting against all maps with hypercomplete fibers characterizes ∞-connected maps.*

Proof. We are trying to prove the bi-implication:

$$f \text{ is } \infty\text{-connected} \;\Leftrightarrow\; f \text{ has left lifting against all maps with hypercomplete fibers}$$

The left-to-right direction is immediate.

For the right-to-left direction, observe that the fibers of the map

$$\|f\|_{n+1} : \|X\|_{n+1} \to \|Y\|_{n+1}$$

will be $(n+1)$-truncated, hence hypercomplete. So the type of fillers for the square:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \|X\|_{n+1} \arrow[d, "\|f\|_{n+1}"] \\ Y \arrow[r] & \|Y\|_{n+1} \end{tikzcd}$$

is contractible, so there is some $u : Y \to \|X\|_{n+1}$, this map can be factorized like so:

$$\begin{tikzcd} Y \arrow[rr, "u"] \arrow[dr] & & \|X\|_{n+1} \\ & \|Y\|_{n+1} \arrow[ur, "u'"] & \end{tikzcd}$$

and it is straightforward to show that the map u' is an inverse for $\|f\|_{n+1}$. So the theorem follows using Corollary 21. ◀

▶ **Definition 29** (Hypercompletion Modality). *A hypercompletion modality if it exists is a modality whose modal types are precisely the hypercomplete types.*

In the corresponding orthogonal factorization system, the left maps will be precisely the ∞-connected maps. The right maps will be those with hypercomplete fibers.

By Proposition 8 the hypercompletion is unique if it exists. Obviously there is some analogy between hypercompletion and truncation, but there are important differences too.

▶ **Proposition 30** (Hypercompletion is lex). *If the hypercompletion modality exists, then the units must be ∞-connected. This implies that a hypercompletion modality must be left exact.*

Proof. Suppose f is nullified by the modal operator, then it is ∞-connected, in particular η is nullified by the modal operator. By Proposition 28.

The canonical map $A \times_B C \to \bigcirc A \times_{\bigcirc B} \bigcirc C$ is given by $\lambda(x, y, p).(\eta(x), \eta(y), \mathrm{ap}_\eta(p))$, and ap_η is ∞-connected because η is by Proposition 19, as are the other two components. A triple of ∞-connected maps is ∞-connected by Proposition 26. There is nothing else to show. ◀

▶ **Definition 31.** *A modality is called cotopological if all propositions it nullifies were already contractible. This implies, the only modality that is both topological and cotopological is trivial.*

▶ **Proposition 32** (Hypercompletion is cotopological). *Any proposition nullified by hypercompletion is trivial.*

3 Countable limits

▶ **Definition 33.** *A (countable) family is a function $\mathbb{N} \to \mathcal{U}$.*

▶ **Definition 34.** *A (countable) tower is a family A together with a dependent family of functions, $a : (n : \mathbb{N}) \to A_{n+1} \to A_n$.*

▶ **Definition 35.** *The product of a family is the type of dependent functions*

$$\prod A := (n : \mathbb{N}) \to A_n$$

▶ **Definition 36.** *The limit of a tower is the type*

$$\lim(A, a) := \left(x : \prod A\right) \times ((n : \mathbb{N}) \to a_n(x(n+1)) = x(n))$$

Abusing notation a bit, if $x : \lim(A, a)$, we will often write x for the first component and q_x for the second component. We might suppress the map in the tower and just write $\lim A$ when it's clear from context.

▶ **Proposition 37.** *If (A_{n+k}, a_{n+k}) is the tower obtained from A by shifting k places up, then* $\lim(A_{n+k}, a_{n+k}) = \lim(A, a)$.

Proof. There is an obvious map $\lim(A, a) \to \lim(A_{n+k}, a_{n+k})$, which has contractible fibers as extensions of a term in the latter limit are essentially unique by definition. ◀

▶ **Definition 38.** *If X is a type, there is a constant tower given by X and the identity. Its limit is X.*

▶ **Proposition 39.** *If (A, a) is a tower, X is a type, $f : A_0 \to X$, then there is another tower A' s.t. $A'_{n+1} = A_n$, and $A'_0 = X$. The limit of the new tower is* $\lim(A, a)$ *still.*

▶ **Definition 40.** *A map of families is a family of maps.*

▶ **Definition 41.** *Tower maps or ladders are families of maps $f_n : A_n \to B_n$ such that all the squares commute.*

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & B_{n+1} \arrow[d] \\
A_n \arrow[r, "f_n"] \arrow[d] & B_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Write $\hom((A, a), (B, b))$ *for the type of such ladders.*

Abusing notation a bit, if $f : \hom((A, a), (B, b))$ we will often write f for the first component and p_f for the second component.

▶ **Proposition 42.** *If $(A, a), (B, b)$ are two towers then*

$$((A, a) = (B, b)) \simeq ((f : \hom((A, a), (B, b))) \times (n : \mathbb{N}) \to \mathrm{isEquiv}(f_n))$$

Proof. We will apply the fundamental theorem of identity types (Theorem 1).

The family $P := \lambda(B, b).((f : \hom((A, a), (B, b))) \times (n : \mathbb{N}) \to \mathrm{isEquiv}(f_n))$ is such that:

1. $P(A, a)$ is pointed, by the ladder of identity maps on the A_n.
2. We have the following chain of natural equivalences:

$$\begin{aligned}
&(B : \mathbb{N} \to \mathcal{U}) \times (b : (n : \mathbb{N}) \to B_{n+1} \to B_n) \\
&\quad \times ((f, e) : \mathbb{N} \to (A_n \simeq B_n)) \times (p_f : (n : \mathbb{N}) \to f_n \circ a_n = b_n \circ f_{n+1}) \\
&\simeq ((B : \mathbb{N} \to \mathcal{U}) \times (f : (n : \mathbb{N}) \to B_n \simeq A_n)) \\
&\quad \times \big((b : (n : \mathbb{N}) \to B_{n+1} \to B_n) \times ((n : \mathbb{N}) \to b_n = f_n a_n f_{n+1}^{-1})\big)
\end{aligned}$$

which is contractible.

So we may apply the fundamental theorem of identity types and the proposition follows. ◀

▶ **Proposition 43.** *Tower maps induce maps between limits which are equivalences if their respective components are.*

Proof. If $x = (x, q_x)$ is a term of type $\lim A$, then $\lambda n.f_n(x(n))$ is a term of type $\prod B$, and for each $n : \mathbb{N}$, $p_f(n) \cdot \mathrm{ap}_{f_n}(q_x(n)) : b_n(f_{n+1}(x(n+1))) =_{B_n} f_n(x_n)$, so $\lambda n.f_n(x(n))$ forms a term of type $\lim B$.

By Proposition 42, there is an equivalence between the identity type of the type of towers, and the type of ladders of equivalences between the towers. So, we can appeal to the J-rule and the observation that the above construction on the identity ladder yields the identity map on the limit. ◀

▶ **Proposition 44.** *If X is a type and for each $x : X$, we have a tower $P(x)$, then there is a tower whose types are*

$$(x : X) \to P_n(x)$$

and whose arrows come from composition $(x : X) \to P_{n+1}(x) \to P_n(x)$

There is an equivalence of types

$$((x : X) \to \lim P(x)) \simeq \lim((x : X) \to P_n(x))$$

given by postcomposition with the projection maps from $\lim P(x)$.

Proof. This is a consequence of the so-called "type theoretic axiom of choice" not to be confused with the axioms of choice we discuss below

$$\begin{aligned}
(x : X) \to \lim P(x) &\simeq (x : X) \to \Big((x' : \prod P(x)) \times a_n(x'(n+1)) = x'(n)\Big) \\
&\simeq (x' : (x : X) \to \prod P(x)) \\
&\quad \times ((x : X) \to (n : \mathbb{N}) \to a_n(x'(x)(n+1)) = x'(x)(n)) \\
&\simeq (x' : \prod((x : X) \to P(x))) \\
&\quad \times ((n : \mathbb{N}) \to a_n \circ x'(n+1) = x'(n)) \\
&\simeq \lim((x : X) \to P(x))
\end{aligned}$$

If f is a function in $(x : X) \to \lim P(x)$, then the equivalence on the second line sends it to $(\lambda x.f(x), \lambda x.q_{f(x)})$, which is composed with the projections through the equivalence on the third line as claimed. ◀

▶ **Proposition 45.** *A lifting problem*

$$\begin{array}{ccc} Y & \longrightarrow & \lim A \\ \big\downarrow & & \\ Z & & \end{array}$$

has a solution if and only if each of the lifting problems

$$\begin{array}{ccc} Y & \longrightarrow & A_n \\ \big\downarrow & & \\ Z & & \end{array}$$

has a solution.

Proof. Immediate from Proposition 44. This has been formalized in [5] (link). ◀

▶ **Proposition 46.** *If* $x, y : \prod A$*, then* $(x = y) \simeq \prod(x_n = y_n)$*.*

Proof. This is just function extensionality [9]. ◀

▶ **Proposition 47.** *If* $x, y : \lim A$*, then there is a tower whose types are* $x_n =_{A_n} y_n$*, and whose maps are* $\lambda(p : x_{n+1} = y_{n+1}).q_x(n)^{-1} \cdot \mathrm{ap}_{a_n}(p) \cdot q_y(n)$*, and we have:*

$$(x = y) \simeq \lim(x_n = y_n)$$

Proof. By well-known facts about the identity types of sum types, we have the following:

$$(x = y) \simeq (p : x = y) \times q_x =^p q_y$$

which by function extensionality becomes

$$(p : x = y) \times ((n : \mathbb{N}) \to q_x(n) =^p q_y(n))$$

Now using the J-rule, we can show that for any ($x, y : \lim A$, and) $p : x = y$

$$(q_x(n) =^p q_y(n)) \simeq (q_x(n)^{-1} \cdot \mathrm{ap}_{a_n}(p_{n+1}) \cdot q_y(n) = p_n)$$

which implies that our original type is in fact equivalent to:

$$(p : \prod x_n = y_n)((n : \mathbb{N}) \to q_x(n)^{-1} \cdot \mathrm{ap}_{a_n}(p_{n+1}) \cdot q_y(n) = p_n)$$

Which is precisely the limit of the tower defined before, which completes the proof. ◀

▶ **Proposition 48.** *If* $A_n \to B_n \leftarrow C_n$ *is a family of cospans, then* $\prod A \to \prod B \leftarrow \prod C$ *is a cospan of products, and* $\prod A \times_{\prod B} \prod C \simeq \prod(A_n \times_{B_n} C_n)$*.*

▶ **Proposition 49.** *If* $(A, a), (B, b), (C, c)$ *are three families of maps, and there are ladders* $f : A \to B$ *and* $g : C \to B$*, then there is a tower* $A_n \times_{B_n} C_n$*, and the limit of this tower is equivalent to the pullback* $\lim A \times_{\lim B} \lim C$

Proof. If $(A, a), (B, b), (C, c)$ are three towers, and $f_n : A_n \to B_n$ and $g_n : C_n \to B_n$ are two tower-maps, then there are maps $A_{n+1} \times_{B_{n+1}} C_{n+1} \to A_n \times_{B_n} C_n$ given by $\lambda(x, y, p).(a_n(x), c_n(y), p_f(n)^{-1} \cdot \mathrm{ap}_{b_n}(p) \cdot p_g(n))$. We will write H for the function in the third argument, for convenience.

Now,

$$\begin{aligned} \lim A \times_{\lim B} \lim C &\simeq (x : \lim A) \times (y : \lim C) \times f(x) = g(y) \\ &\simeq (x : \lim A) \times (y : \lim C) \times \lim(f_n(x_n) = g_n(y_n)) \end{aligned}$$

Using Proposition 47, then the proposition follows by the type-theoretic axiom of choice. Observe that the term $\lambda n.f_n(x(n))$ of type $\lim B$ has for its second component the term

$p_f(n) \cdot \mathrm{ap}_{f_n}(q_x(n)) : b_n(f_{n+1}(x(n+1))) = f_n(x(n))$, similarly for $\lambda n.g_n(y(n))$

$$\begin{aligned}
&(x : \lim A) \times (y : \lim C) \times \lim(f_n(x_n) = g_n(y_n)) \\
&\simeq (x : \prod A) \times (q_x : (n : \mathbb{N}) \to a(x(n+1)) = x(n)) \times \\
&\quad (y : \prod C) \times (q_y : (n : \mathbb{N}) \to c(y(n+1)) = y(n)) \times \\
&\quad (p : \prod f_n(x(n)) = g_n(y(n))) \times \\
&\quad ((n : \mathbb{N}) \to \mathrm{ap}_{f_n}(q_x(n))^{-1} \cdot p_f(n)^{-1} \cdot p(n+1) \cdot p_g(n) \cdot \mathrm{ap}_{g_n}(q_y(n)) = p(n)) \\
&\simeq (x : \prod A) \times (y : \prod C) \times (p : \prod f_n(x(n)) = g_n(y(n))) \times \\
&\quad ((n : \mathbb{N}) \to (q_x : a(x(n+1)) = x(n)) \times (q_y : c(y(n+1)) = y(n)) \times \\
&\quad (H(p(n+1)) =^{\mathrm{ap}_f(q_x), \mathrm{ap}_g(q_y)} p(n))) \\
&\simeq (x : \prod A) \times (y : \prod C) \times (p : \prod f_n = g_n) \times \\
&\quad ((n : \mathbb{N}) \to (a(x(n+1)), c(y(n+1)), H(p(n+1))) = (x(n), y(n), p(n))) \\
&\simeq \lim(\lambda n.A_n \times_{B_n} C_n, \lambda n.\lambda(x, y, p).(a(x), c(y), H(p)))
\end{aligned}$$

Using the fact about pullbacks of products above, since the map of the triples is exactly as given here. ◀

4 Choice principles

The following definitions and propositions appear in [1]. They have been formalized [5].

▶ **Definition 50.** *Anel-Barton choice principles*

We say that countable choice of dimension $\leq d$ holds if whenever A, B are families of types, and $f : A_n \to B_n$ is a family of $(k+d)$-connected maps between them, then the map $\prod f : \prod A \to \prod B$ is k-connected.

We say that dependent choice of dimension $\leq d$ holds if whenever $(A, a), (B, b)$ are towers, and $f : A_n \to B_n$ is a ladder of $(k+d)$-connected maps between them, then the map $\lim f : \lim A \to \lim B$ is k-connected.

▶ **Proposition 51.** *Countable choice of dimension $\leq d$ holds if and only if, whenever A is a family of $(k+d)$-connected types, then $\prod A$ is k-connected.*

Proof. The fiber of f is the product of the fibers of the f_n by Proposition 48. In the other direction consider the constant family on $\mathbf{1}$. ◀

▶ **Proposition 52.** *Dependent choice of dimension $\leq d$ holds if and only if, whenever (A, a) is a family of $(k+d)$-connected types, then $\lim(A, a)$ is k-connected.*

Proof. The fiber of f is the limit of the fibers of f_n using Proposition 49. In other direction consider constant family on $\mathbf{1}$. ◀

▶ **Proposition 53.** *Dependent choice of dimension $\leq d$ implies countable choice of the same dimension*

Proof. $\prod A_n$ is the limit of $A_1 \times \cdots \times A_k$, the latter are $d+n$ connected if each of the A_i are. ◀

▶ **Proposition 54.** *Countable choice of dimension $\leq d$ implies dependent choice of dimension $\leq d+1$.*

Proof. Suppose the hypothesis holds, and (A, a) is a tower whose types are all $n + d + 1$ connected. Then $\prod A$ is $n+1$ connected. For $x : \prod A$, each identity type $a_n(x(n+1)) = x(n)$ is $n+d$ connected, because of Proposition 19, so $(n : \mathbb{N}) \to a_n(x(n+1)) = x(n)$ is n-connected by countable choice. It follows that the limit, being a sum of n-connected types over an $n+1$ connected type is itself n-connected. There is nothing else to show. ◀

5 Postnikov effectiveness and hypercompletion

The Postnikov decomposition of a space was introduced by Postnikov [6]. It was subsequently generalized to define a certain class of diagrams in an arbitrary higher topos [4]. The definition can be internalized into HoTT like so:

▶ **Definition 55** (Postnikov tower). *A tower (A, a) is called a Postnikov tower if each A_n is n-truncated and each a_n is n-connected.*

▶ **Definition 56** (The Postnikov tower of a type). *If X is a type, the Postnikov tower of X is the tower whose types are $\|X\|_n$ and whose morphisms come from the universal property of truncations.*

▶ **Definition 57.** *We write $\eta_X : X \to \lim \|X\|_n$ for the unique morphism from X to the limit of the Postnikov tower of X such that $\lambda x.\eta_X(x)_n = |\bullet|_n$ for all n.*

▶ **Definition 58** (Postnikov effectiveness). *If (A, a) is a Postnikov tower, then by the universal property, there exist maps $\|\lim(A, a)\|_n \to A_n$, these are maps ϵ_n such that $\epsilon_n \circ |\bullet|_n = \lambda x.x_n$ the projection map. In fact these maps form a ladder, and we say Postnikov effectiveness holds if they are all equivalences.*

▶ **Proposition 59** (Anel-Barton [1]). *Anel-Barton dependent choice implies Postnikov effectiveness.*

Proof. Throughout we write $p_n : \lim(A, a) \to A_n$ for the projection map in an arbitrary tower.

We fix a natural number $d : \mathbb{N}$, and assume that Anel-Barton dependent choice of dimension $\leq d$ holds, we also fix a Postnikov tower (A, a). We observe that we have a ladder of maps $f_n : \|\lim A_k\|_n \to A_n$, which is such that $f_n \circ |\bullet|_n^{\lim A_k} = p_n : \lim A_k \to A_n$.

Observe, it suffices to show that each p_n is n-connected, since then f_n factors an n-connected map through an n-connected map, implying that f_n itself is n-connected, which, since its source and target are both n-truncated, implies that f_n is an equivalence.

So, fix n, and find m such that $m \geq n+d$. Consider the restricted diagram A_{k+m}, and the constant diagram A_m. There is a ladder of maps between these diagrams $q_{n'} : A_{n'+m} \to A_m$, where $q_{n'} = a_m \circ \cdots \circ a_{n'+m-1}$. Each $q_{n'}$ is at least $m \geq n + d$ connected, so the map $\lim q_k : \lim A_{k+m} \to \lim_k A_m$ is at least n-connected. But $\lim A_{k+m}$ is none other than $\lim A_k$, $\lim_k A_m$ is none other than A_m, and $\lim q_k$ is none other than p_m. Hence, p_m is at least n-connected.

Finally, we observe that $p_n = a_n \circ \cdots \circ a_m \circ p_m$, and each of these maps are at least n-connected, so p_n is at least n-connected. Which completes the proof. ◀

▶ **Proposition 60.** *If Postnikov effectiveness holds then, by definition, there is a family of equivalences $\epsilon_n : \|\lim \|X\|_k\|_n \to \|X\|_n$ for any space X. As an equivalence, ϵ_n has an inverse ϵ_n^{-1}, and this inverse is none other than $\|\eta_X\|_n$.*

Proof. We write p_n for the projection map, as in the previous proof.

We have the following equations:

$$|\bullet|_n^X = p_n \circ \eta_X \qquad p_n = \epsilon_n \circ |\bullet|_n^{\lim \|X\|_k}$$

by definition.

The second equation implies that

$$|\bullet|_n^{\lim \|X\|_k} = \epsilon_n^{-1} \circ p_n$$

Whence

$$\begin{aligned} \epsilon_n^{-1} \circ |\bullet|_n^X &= \epsilon_n^{-1} \circ p_n \circ \eta_X \\ &= |\bullet|_n^{\lim \|X\|_k} \circ \eta_X \end{aligned}$$

Which implies

$$\epsilon_n^{-1} = \|\eta_X\|_n$$

by the universal property of the n-truncation. ◀

▶ **Proposition 61.** *Postnikov effectiveness implies that the operation sending a type to the limit of its Postnikov tower is a modality. Which we call the Postnikov limit modality.*

Proof. We want to check that

$$\lambda f.f \circ \eta_X : ((x : \lim \|X\|_k) \to \lim \|P(x)\|_n) \to ((x : X) \to \lim \|P(\eta_X(x))\|_n)$$

is an equivalence for all $P : \lim \|X\|_k \to \mathcal{U}$, for all X.

Since limits commute with products (Proposition 44), we see it is enough to check that

$$\lambda f.f \circ \eta_X : ((x : \lim \|X\|_k) \to \|P(x)\|_n) \to ((x : X) \to \|P(\eta_X(x))\|_n)$$

for all n.

Since the type of n-types is an $(n+1)$-type, the map $\lambda x.\|P(x)\|_n : \lim \|X\|_k \to \mathcal{U}^{\leq n}$ factors through $|\bullet|_{n+1} : \lim \|X\|_k \to \|\lim \|X\|_k\|_{n+1}$, say by $\lambda x.Q(x) : \|\lim \|X\|_k\|_{n+1} \to \mathcal{U}^{\leq n}$, and since n-types are, in particular, $(n+1)$-truncated, we see that the universal property implies it suffices to check that

$$\lambda f.f \circ \|\eta_X\|_{n+1} : ((x : \|\lim \|X\|_k\|_{n+1}) \to Q(x)) \to ((x : \|X\|_{n+1}) \to Q(\|\eta_X\|_{n+1}(x)))$$

is an equivalence for all $Q : \|\lim \|X\|_k\|_{n+1} \to \mathcal{U}^{\leq n}$

But this is trivial because $\|\eta_X\|_{n+1}$ is an equivalence by Proposition 60. ◀

▶ **Proposition 62** (Postnikov limit modality is hypercompletion). *Assuming Postnikov effectiveness, the modal types for the Postnikov limit modality are precisely the hypercomplete types.*

Proof. Assuming Postnikov effectiveness, the map $X \to \lim \|X\|_k$ is ∞-connected. This we can see from the equation:

$$|\bullet|_{n+1}^{\lim \|X\|_k} \circ \eta_X = |\bullet|_{n+1}^X \circ p_{n+1}$$

Which shows that η factors an $(n+1)$-connected map through an $(n+1)$-connected map, so must itself be n-connected by Proposition 20.

If A_n is a Postnikov tower, then $\lim A_k$ is hypercomplete, this follows from the behavior of limits and lifting problems, see Proposition 45.

So, assuming Postnikov effectiveness, it follows that if X is hypercomplete, then η must be an equivalence – as it is an ∞-connected map between hypercomplete types. And, if η is an equivalence, then X is hypercomplete by univalence (as it is equivalent to a hypercomplete type). ◀

Thus, we have a hypercompletion modality under the assumption that (e.g.) countable choice holds.

6 SSD dependent choice

In the paper [2], a framework is developed for reasoning about the category of "light condensed anima". The setting goes by the name of synthetic Stone duality and has applications to condensed mathematics. They introduce a choice principle which we state below. We can prove Postnikov effectiveness from this choice principle, hence derive a characterization of the hypercompletion operation in their setting. The results in this section were arrived at in conversation with Hugo Moeneclaey and have not (yet) been formalized.

▶ **Definition 63.** *SSD dependent choice*

If (A, a) is a family of types and a_n are all (-1)-connected, then $p_0 : \lim A \to A_0$ is (-1)-connected.

▶ **Proposition 64.** *SSD choice implies that if (A, a) is a family of types, and the a_n are all n-connected, then $p_0 : \lim A \to A_0$ is n-connected.*

Proof. Using Proposition 19 it suffices to show that p_0 is (-1)-connected, and, for each $x, y : \lim A$, $\mathrm{ap}_{p_0} : (x = y) \to (x_0 = y_0)$ is $(n-1)$-connected. The first is true by definition of SSD dependent choice.

Using Proposition 47, we know that $(x = y)$ is a limit of the diagram

$$((x_n = y_n), \lambda p.q_x(n)^{-1} \cdot \mathrm{ap}_{a_n}(p) \cdot q_y(n))$$

and ap_{p_0} is the projection map. Using induction, it suffices to assume that the result of the proposition holds for $(n-1)$, and then it follows immediately that ap_{p_0} is $(n-1)$-connected.

There is nothing else to show. ◀

▶ **Proposition 65.** *SSD dependent choice implies dependent choice of dimension ≤ 1.*

Proof. If (A, a) is a tower of $n+1$-connected types, then

$$\ldots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow \mathbf{1}$$

is a tower whose limit is just the same as $\lim(A, a)$, and whose morphisms are all now n-connected as they factor the $n+1$-connected maps into the unit like so: $A_{n+1} \to A_n \to \mathbf{1}$. So the map $\lim(A, a) \to \mathbf{1}$ is n-connected as well by SSD dependent choice. ◀

▶ **Proposition 66.** *Dependent choice of dimension ≤ 0 implies SSD dependent choice.*

Proof. Consider the ladder between the diagram (A_{n+1}, a_{n+1}), and the constant diagram A_0. By hypothesis all these maps are (-1)-connected, so the map of limits, which is precisely the projection is (-1)-connected. ◀

▶ **Proposition 67.** *SSD dependent choice implies countable choice of dimension ≤ 0.*

Proof. As before, $\prod A$ is limit of $\mathbf{1} \times A_1 \times \cdots \times A_k$, the projection maps to $\mathbf{1}$ are all of course n-connected if all the A_k are. So the map from the limit is also n-connected by foregoing. ◀

▶ **Proposition 68.** *SSD choice implies Postnikov effectiveness.*

Proof. Suppose (A, a) is a Postnikov tower.

By results in Section 3, $\lim A$ is also the limit of the towers, (A_{n+k}, a_{n+k}). These maps, a_{n+k}, are all at least k-connected, whence it follows that $p_k : \lim A \to A_k$ is k-connected, and the remainder of the proof proceeds like in the other case. ◀

▶ **Corollary 69.** *In the setting of synthetic Stone duality, the hypercompletion operation exists and is given by* $\lambda X. \lim \|X\|$.

7 Conclusion

We have introduced some choice principles, derived from the work of Anel-Barton [1], and the synthetic Stone duality framework [2] and described their relationship to the form of the hypercompletion operator. Along the way we proved a number of useful propositions about countable inverse limits.

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