Type-Theoretic Replacement and Univalent Completion: Applications and Interpretations

Evan Cavallo
Department of Computer Science and Engineering, University of Gothenburg and Chalmers University of Technology, Gothenburg, Sweden

Thierry Coquand
Department of Computer Science and Engineering, University of Gothenburg and Chalmers University of Technology, Gothenburg, Sweden

Abstract

We study Rijke's type theoretic axiom of replacement, which closes a type-theoretic universe under certain images, and its cousin the axiom of univalent completions, which postulates an extension of each type family to a univalent family. In a long expository section, we survey applications of these two axioms in the literature, from the construction of truncations to the construction of Eilenberg–MacLane spaces to the interpretation of material set theory, making a case for their value as foundational principles. We then give direct, constructive interpretations of the axioms in cubical sets models of type theory and suggest how they can be understood as higher inductive constructions.

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1 Introduction

Rijke's *type theoretic axiom of replacement* for a universe $\mathcal{U}$ states that every map $f : A \to B$ from a $\mathcal{U}$-small type A (i.e., one equivalent to an element of $\mathcal{U}$) to a locally $\mathcal{U}$-small type B (i.e., one whose identity types are $\mathcal{U}$-small) admits a $\mathcal{U}$-small image.

Rijke first states replacement as a *theorem* of intensional type theory with function extensionality and a univalent universe $\mathcal{U}$ closed under pushouts and a natural numbers type [51, Theorem 4.6] and explicitly connects it with set theory's Replacement axiom schema in his dissertation [52, §4.2.3]. His construction is non-trivial, building the image as a sequential colimit of fiberwise joins of f with itself. The special case of replacement where the codomain is an h-set appears earlier, explicitly connected with set theoretic replacement, in Rijke and Spitters's description of the ΠW-pretopos of sets [54]; De Jong and Escardó also discuss this case [27, §3.4] [26, §2.11.4]. The general principle is first taken as an axiom in Rijke's *Introduction to Homotopy Type Theory* [53, Axiom 18.1.8]. Bezem, Buchholtz, Cagne, Dundas, and Grayson's book-in-progress *Symmetry* [10] also isolates the axiom and points out several applications that go beyond the world of h-sets. Although replacement is a relatively weak principle, being derivable from basic higher inductive types, we shall see that it is also surprisingly flexible.

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We work in type theory with intensional Id-types, Σ-types, Π-types with extensionality, and empty and unit types. By a *universe*, we mean one with these type formers. We are mainly interested in *univalent* universes but assume univalence explicitly when we need it.

▶ **Definition 1.** *A map $f : A \to B$ is an* embedding *if for all $a, a' : A$, the action on identities $\mathsf{ap}_f^{a,a'} : (a = a') \to (f\ a = f\ a')$ is an equivalence. Set* $\mathsf{isEmbedding}\ f :\equiv \prod_{a,a'} \mathsf{isEquiv}\ \mathsf{ap}_f^{a,a'}$.

▶ **Definition 2.** *A map $f : A \to B$ is a* surjection *when for every family of propositions P over B, the precomposition map $(- \circ f) : \prod_{b:B} P\ b \to \prod_{a:A} P\ (f\ a)$ is an equivalence.*

▶ Remark 3. Definition 2 quantifies over *all* propositions over B, so it cannot a priori be captured by a type. When we quantify over types, we implicitly quantify over substitutions: $\Gamma \vdash f : A \to B$ is surjective if for each $\sigma : \Gamma' \to \Gamma$ and family of propositions P over $B\sigma$, $(- \circ f\sigma)$ is an equivalence, and our choice of witnesses commutes with subsitution.

Given a (-1)-truncation $\|-\|$, surjectivity of $f : A \to B$ is equivalent to $\prod_{b:B} \|\mathsf{Fiber}_f\ b\|$ where $\mathsf{Fiber}_f\ b :\equiv \sum_{a:A} f\ a =_B b$ [53, Proposition 15.2.3]. This reduction requires that the truncation's eliminator can target type families of arbitrary size, i.e., that the maps $T \to \|T\|$ are surjections in the sense of Definition 2.

▶ **Definition 4.** *An* image *of $f : A \to B$ is a factorization $A \overset{p}{\twoheadrightarrow} X \overset{i}{\hookrightarrow} B$ of f as a surjection followed by an embedding.*

▶ **Definition 5.** *A type is* $\mathcal{U}$-small *when it is equivalent to a type in $\mathcal{U}$ and* locally $\mathcal{U}$-small *when its identity types are $\mathcal{U}$-small. Thus we set* $\mathsf{Small}_{\mathcal{U}}\ A :\equiv \sum_{X:\mathcal{U}} (X \simeq A)$ *and* $\mathsf{LocSmall}_{\mathcal{U}}\ A :\equiv \prod_{a,a':A} \mathsf{Small}_{\mathcal{U}}\ (a =_A a')$.

▶ **Definition 6** (Replacement)**.** *A universe $\mathcal{U}$* has replacement *when for every $\mathcal{U}$-small type A, locally $\mathcal{U}$-small type B, and $f : A \to B$, we have a $\mathcal{U}$-small image of f.*

▶ Remark 7. Rijke [53, Axiom 18.1.8] states replacement assuming (-1)-truncation. In this case, we have the image of $f : A \to B$ as $\sum_{b:B} \|\mathsf{Fiber}_f\ b\|$. Thus, Rijke can formulate replacement as *the image of a map from a $\mathcal{U}$-small type into a locally $\mathcal{U}$-small type is $\mathcal{U}$-small.* We instead bundle the existence and $\mathcal{U}$-smallness of the relevant images in one axiom schema.

Replacement immediately implies the existence of images of maps in $\mathcal{U}$. In fact, assuming $\mathcal{U}$ is univalent, it can be used to define n-images (§ 2.1) and quotients by equivalence relations (§ 2.2) and even implies that $\mathcal{U}$ contains non-sets (§ 2.3).

Replacement is closely related to another principle, the axiom schema of *univalent completions*, which intuitively asks that the "universe of *all* types" behaves like a universe with replacement. Recall that we can formulate *univalence* as a condition on any type family, in such a way that universe $\mathcal{U}$ is univalent precisely when $A : \mathcal{U} \vdash A$ type is univalent.

▶ **Definition 8** (Voevodsky [69, Definition 3.4])**.** *A family $a : A \vdash P(a)$ type is* univalent *if for all $a_0, a_1 : A$ the induced map $a_0 =_A a_1 \to P(a_0) \simeq P(a_1)$ is an equivalence.*

▶ **Definition 9.** *A* cartesian morphism *$(f, e) \colon P \to Q$ between families $a : A \vdash P(a)$ type and $b : B \vdash Q(b)$ type consists of a map $f : A \to B$ and equivalences $e : \prod_{a:A} P(a) \simeq Q(f\ a)$.*

▶ **Definition 10** (Univalent extensions and completions)**.** *A* univalent extension *of a family $a : A \vdash P(a)$ type is a univalent family $\overline{a} : \overline{A} \vdash \overline{P}(\overline{a})$ type together with a cartesian morphism $(q, e) : P \to \overline{P}$. It is a* univalent completion *if $q : A \to \overline{A}$ is a surjection. A type theory* has enough univalent families *if every family admits a univalent extension, and* has univalent completions *if every family admits a univalent completion.*

Univalent completion has been more thoroughly studied from semantic and categorical perspectives, where universes are less casually assumed than in type theory. Van den Berg and Moerdijk [65] show that the simplicial model of homotopy type theory [40] has enough univalent families. They are the first to use the term "univalent completion", to our knowledge; they do not define it, but their theorem takes a fibration $a : A \vdash P(a)$ type to a univalent extension $(f, e) \colon P \to \overline{P}$ with f a cofibration. Stenzel [61, §6] develops the theory of univalent completions in model categories. Gepner and Kock [33] and Rasekh [48] discuss univalent families and completion respectively in $(\infty, 1)$-categories; Anel [5, §A.2–3] continues this study. Uemura [63, Proposition 4.7] shows that ∞-logoi have univalent completions following Rijke's join construction. On the syntactic side, Christensen [21] discusses univalent families in type theory, while Awodey has proposed taking existence of univalent completions as a type theoretic principle at a MURI grant meeting in Pittsburgh in August 2025.

Working in homotopy type theory, we often restrict attention to types in some univalent universe. Such a universe is required for calculations with identity types such as the characterization $\|a =_A a'\|_n \simeq (|a|_{n+1} =_{\|A\|_{n+1}} |a'|_{n+1})$ of truncated identity types. One may assume that every type (or finite collection of types) belongs to some univalent universe (as, e.g., in Rijke [53, Postulate 6.2.1]), mimicking Grothendieck's set theoretic axiom of universes [8, Exposé I, Appendice, §4]. For many applications, however, we do not need these universes to be closed under any type formers. The assumption of enough univalent families can thus decouple uses of univalence from size issues.

▶ **Example 11** (Identities in truncations)**.** Given n- and $(n + 1)$-truncations and enough univalent families, we have $\|a =_A a'\|_n \simeq (|a|_{n+1} =_{\|A\|_{n+1}} |a'|_{n+1})$. By assumption, the family $(a, a') : A \times A \vdash \|a =_A a'\|_n$ type admits a cartesian morphism to a univalent family $u : U \vdash E(u)$ type, so $q : A \times A \to U$ with $\|a =_A a'\|_n \simeq E(q\ (a, a'))$. We then use U as a universe to follow the standard proof [64, Theorem 7.3.12]: we only need that U is univalent and has codes $q\ (a, a')$ for the types $\|a =_A a'\|_n$, not that it is closed under any type formers.

With univalent completions, we can derive many of the same operations as we can get from replacement. To begin with, we have images:

▶ **Example 12.** A type theory with univalent completions has (-1)-truncations.

Proof. Let A type and consider the constant family $a : A \vdash 1$ type. By assumption, we have a univalent $\overline{a} : \overline{A} \vdash E(\overline{a})$ type with $q : A \twoheadrightarrow \overline{A}$ and $1 \simeq E(q\ a)$ for all $a : A$. To show $q : A \twoheadrightarrow \overline{A}$ is the propositional truncation of A, it suffices to check that $\overline{A}$ is a proposition, i.e., $\overline{a} =_{\overline{A}} \overline{a}'$ is contractible for all $\overline{a}, \overline{a}' : \overline{A}$. Since being contractible is a proposition and q is surjective, we can assume $\overline{a} \equiv q\ a$ and $\overline{a}' \equiv q\ a'$ for some $a, a' : A$. In this case $(q\ a =_{\overline{A}} q\ a') \simeq (E(q\ a) \simeq E(q\ a')) \simeq (1 \simeq 1) \simeq 1$ by univalence of E. ◀

▶ **Corollary 13** (cf. Anel [5, Remark A.3.3])**.** *We have univalent completions if and only if we have image factorizations and enough univalent families.*

Proof. If we have univalent completions, then by Example 12 any $f : A \to B$ has as image $\sum_{b:B} \|\mathsf{Fiber}_f\ b\|$. Conversely, if we have enough univalent families, then for any $a : A \vdash P$ type we have a univalent $\overline{a} : \overline{A} \vdash \overline{P}(\overline{a})$ type and $(q, e) : P \to \overline{P}$. Taking an image factorization $q = iq'$, the family $\overline{a}' : \overline{A}' \vdash \overline{P}(i\ \overline{a}')$ type is a univalent completion of P. ◀

Say a *type family in* $\mathcal{U}$ is a pair $(A : \mathcal{U}, P : A \to \mathcal{U})$. When $\mathcal{U}$ is univalent, $P : A \to \mathcal{U}$ is univalent if and only if P is an embedding $A \hookrightarrow \mathcal{U}$. Thus:

▶ **Proposition 14.** *For a type family $P : A \to \mathcal{U}$ in a univalent universe $\mathcal{U}$, a univalent completion of P in $\mathcal{U}$ consists of an image factorization $A \twoheadrightarrow \overline{A} \hookrightarrow \mathcal{U}$ of P with $\overline{A} : \mathcal{U}$.* ◀

Rijke accordingly uses "univalent completion" to mean just such an image [52, Corollary 4.2.18]. Starting from Proposition 14, we now make formal the correspondence between replacement and univalent completion within a universe. This argument was suggested and explained to us by David Wärn.

▶ **Definition 15.** *A universe $\mathcal{U}$* has enough univalent families *if for every family P in $\mathcal{U}$ there is a cartesian morphism $P \to \overline{P}$ to a univalent family in $\mathcal{U}$, and* has univalent completions *when every family in P admits a univalent completion by a family in $\mathcal{U}$.*

▶ Remark 16. Given a universe $\mathcal{U}$, the $\mathcal{U}$-small types form a model of type theory. If this model has univalent completions, it may not imply that $\mathcal{U}$ has univalent completions as a universe: a surjection among $\mathcal{U}$-small types need not be a surjection in the larger theory.

▶ **Proposition 17** (Wärn)**.** *A univalent universe $\mathcal{U}$ has replacement if and only if it has univalent completions.*

Proof. If $\mathcal{U}$ has replacement, then it has univalent completions by Proposition 14. Suppose $\mathcal{U}$ has univalent completions and let $f : A \to B$ where $A : \mathcal{U}$ and B is locally $\mathcal{U}$-small. By Example 12 and local $\mathcal{U}$-smallness of B, f admits an image $X :\equiv \sum_{b:B} \|\mathsf{Fiber}_f\ b\|$. Write $p : A \twoheadrightarrow X$ for the accompanying surjection. We want to show X is $\mathcal{U}$-small.

Since X is locally $\mathcal{U}$-small, we have some $F : X \to \mathcal{U}$ with $F\ x \simeq \mathsf{Fiber}_p\ x$. Taking the univalent completion of $P :\equiv F \circ p : A \to \mathcal{U}$, we have $\overline{A} : \mathcal{U}$ and $\overline{P} : \overline{A} \hookrightarrow \mathcal{U}$ with a surjection $q : A \twoheadrightarrow \overline{A}$ and $F\ (p\ a) \simeq \overline{P}\ (q\ a)$ for $a : A$. By the orthogonality of surjections and embeddings [64, Theorem 7.6.7], there is some $r : X \to \overline{A}$ such that $r \circ p = q$ and $\overline{P} \circ r = F$. We have $X \simeq \sum_{\overline{a}:\overline{A}} \mathsf{Fiber}_r\ \overline{a}$, so it suffices to show that $\mathsf{Fiber}_r\ \overline{a}$ is $\mathcal{U}$-small for $\overline{a} : \overline{A}$. Since being $\mathcal{U}$-small is a proposition and q is surjective, it suffices to check that $\mathsf{Fiber}_r\ (q\ a)$ is $\mathcal{U}$-small for $a : A$. For any $(x, \alpha) : \mathsf{Fiber}_r\ (q\ a)$, we have $(a, \mathbf{refl}) : \mathsf{Fiber}_p\ (p\ a)$, which since $\mathsf{Fiber}_p\ (p\ a) \simeq F\ (p\ a) \simeq \overline{P}\ (q\ a) \simeq \overline{P}\ (r\ x) \simeq F\ x$ induces some $\phi_0\ a\ x\ \alpha : F\ x$. Now

$$\begin{aligned}
\mathsf{Fiber}_r\ (q\ a) &\equiv \textstyle\sum_{x:X} r\ x =_{\overline{A}} q\ a \\
&\simeq \textstyle\sum_{x:X} \sum_{\alpha:r\ x =_{\overline{A}} q\ a} \sum_{\phi:F\ x} (\phi =_{F\ x} \phi_0\ a\ x\ \alpha) && \text{(singleton contractibility)} \\
&\simeq \textstyle\sum_{(x,\phi):\sum_{x:X} F\ x} \sum_{\alpha:r\ x =_{\overline{A}} q\ a} (\phi =_{F\ x} \phi_0\ a\ x\ \alpha) && \text{(rearranging)}
\end{aligned}$$

Since $\sum_{x:X} F\ x$ is equivalent to A and thus $\mathcal{U}$-small, we see that this type is $\mathcal{U}$-small. ◀

We aim to highlight replacement and univalent completion as foundational type theoretic principles. We begin in § 2 with an expository survey of applications of the two in the literature. In this section we do not claim any novelty. In § 3, we give direct semantics for replacement and univalent completion in cubical models of univalent type theory, bypassing Rijke's join construction. Finally, in § 4 we observe how the two can be rendered as instances of higher induction recursion.

2 A survey of applications

2.1 Truncations and images

We have seen in Example 12 that a type theory with univalent completions has propositional truncations. In fact, we can construct n-truncations for every $n \geq 1$. Rijke [51, Theorem 7.1] shows this using replacement; we adapt the argument to use univalent completions.

▶ **Definition 18** (e.g., HoTT Book [64, §7.6])**.** *A map $f : A \to B$ is* n-connected *when for every family of n-types P over B, precomposition $(- \circ f) : \prod_{b:B} P\ b \to \prod_{a:A} P\ (f\ a)$ is an equivalence, and is* n-truncated *if its fibers are n-truncated. An* n-image *is a factorization as an n-connected map followed by an n-truncated map. An* n-truncation *of a type A is an n-image of $A \to 1$.*

▶ **Definition 19.** *$f : A \to B$ is* locally n-connected *if $\mathsf{ap}_f^{a,a'}$ is n-connected for all $a, a' : A$.*

▶ **Lemma 20.** *If f is surjective and locally n-connected, then f is $(n+1)$-connected.*

Proof. Let $f : A \to B$ and let P be a family of $(n+1)$-types over B. We show that precomposition $(- \circ f) : \prod_{b:B} P\ b \to \prod_{a:A} P\ (f\ a)$ is an equivalence by showing that its fibers are contractible. Given $g : \prod_{a:A} P\ (f\ a)$, the fiber of $(- \circ f)$ at g is equivalent to $\prod_{b:B} \sum_{p:P\ b} \prod_{a:A} \prod_{\beta:f\ a=_B b} p =_P^{\beta} g\ a$. We show that $\sum_{p:P\ b} \prod_{a:A} \prod_{\beta:f\ a=_B b} p =_P^{\beta} g\ a$ is contractible for all $b : B$. Since f is surjective and being contractible is a proposition, it is enough to check this when b is $f\ a'$ for some $a' : A$. In this case the type is equivalent to $\sum_{p:P\ (f\ a')} \prod_{a:A} \prod_{\alpha:a=_B a'} p =_P^{\mathsf{ap}_f\ \alpha} g\ a$ by local n-connectedness of f, and the latter type is contractible by two applications of singleton contractibility. ◀

▶ **Example 21** (Truncations, cf. Rijke [51, Theorem 7.1])**.** If we have univalent completions, then we have n-truncations for all $n \geq -2$. Relativizing, if a univalent universe $\mathcal{U}$ has replacement, then $\mathcal{U}$ has n-truncations for all $n \geq -2$.

Proof. For $n = -2$, the n-truncation of a type is simply the unit type. For $n = -1$, we gave an argument in Example 12. Suppose we have n-truncations for some $n \geq -1$; we construct the $(n+1)$-truncation of A. By assumption, $(a, a') : A \times A \vdash \|a =_A a'\|_n$ type has a univalent completion $u : U \vdash E(u)$ type with $q : A \times A \twoheadrightarrow U$. Since $E(q\ (a, a'))$ is n-truncated and q is surjective, $E(u)$ is n-truncated for all $u : U$. Thus U is $(n+1)$-truncated by univalence.

The map $y :\equiv \lambda a.\lambda x.q\ (x, a) : A \to (A \to U)$ has an image $A \overset{p}{\twoheadrightarrow} T \overset{i}{\hookrightarrow} (A \to U)$ by Corollary 13. Since $A \to U$ is $(n+1)$-truncated and i is (-1)-truncated, T is also $(n+1)$-truncated. If $p : A \to T$ is $(n+1)$-connected, then it is our desired $(n+1)$-truncation. Since p is surjective, it suffices by Lemma 20 to show that p is locally n-connected. Because i is an embedding, it is equivalent to show that y is locally n-connected.

The map $\mathsf{ap}_y^{a,a'}$ corresponds, by univalence of E and $E(q\ (a, a')) \simeq \|a =_A a'\|_n$, to the map $a =_A a' \to \prod_{x:A}(\|x =_A a\|_n \simeq \|x =_A a'\|_n)$ sending $\mathbf{refl}$ to the family of identity equivalences. The codomain is an n-type, so $\mathsf{ap}_y^{a,a'}$ factors through $a =_A a' \to \|a =_A a'\|_n$; that map is n-connected, so it suffices to show $\|a =_A a'\|_n \to \prod_{x:A}(\|x =_A a\|_n \simeq \|x =_A a'\|_n)$ is an equivalence. This is an n-truncated version of the Yoneda lemma: by path induction, a family $h : \prod_{x:A}(\|x =_A a\|_n \simeq \|x =_A a'\|_n)$ is uniquely determined by $h\ a\ |\mathbf{refl}|_n : \|a =_A a'\|$. ◀

Assuming a univalent universe $\mathcal{U}$, Rijke describes the n-truncation of $A : \mathcal{U}$ more directly as the replacement of $\lambda a.\lambda a'.\|a' =_A a\|_n : A \to (A \to \mathcal{U})$. As with propositional truncation, n-truncations give us n-images as a corollary [64, Lemma 7.6.4]:

▶ **Example 22** (Higher images)**.** If we have univalent completions, then we have n-images for all $n \geq -2$. If a univalent universe $\mathcal{U}$ has replacement, then $\mathcal{U}$ has n-images for all $n \geq -2$.

2.2 Set quotients

Rijke's stated motive for assuming the type theoretic replacement axiom in his *Introduction to Homotopy Type Theory* [53] is that it enables the construction of quotients of sets (that is, 0-types) by equivalence relations.

▶ **Definition 23** (e.g., Rijke [53, Definition 18.2.1]). *For a type A and type-valued relation $(a, a') : A \times A \vdash R(a, a')$* type*, a* set quotient *of A by R is a set A/R and map $[-] : A \to A/R$ such that (1) $R(a, a')$ implies $[a] =_{A/R} [a']$, and (2) for any set B and $f : A \to B$ such that $R(a, a')$ implies $f\ a =_B f\ a'$ for all $a, a' : A$, there is a unique $f' : A/R \to B$ with $f =_{A \to B} f' \circ [-]$.*

▶ **Example 24** (Set quotients, cf. Rijke [53, §18.1]). If a type theory has univalent completions, then it has set quotients by propositional equivalence relations. Relativizing, if a univalent universe $\mathcal{U}$ has replacement, then $\mathcal{U}$ has set quotients by propositional equivalence relations.

Proof. Let A and a propositional equivalence relation $(a, a') : A \times A \vdash R(a, a')$ type be given. By assumption, R has a univalent completion $u : U \vdash E(u)$ type with $q \colon A \times A \twoheadrightarrow U$. Take the image of $\lambda a.\lambda x.q\ (x, a) : A \to (A \to U)$ to get $A \overset{[-]}{\twoheadrightarrow} A/R \overset{i}{\hookrightarrow} (A \to U)$. We have $([a] =_{A/R} [a']) \simeq (q\ (-, a) =_{A \to U} q\ (-, a')) \simeq (\prod_{x:A} R(x, a) \simeq R(x, a')) \simeq R(a, a')$, where the last step uses that R is an equivalence relation. As $[-]$ is surjective, this makes A/R a set. The equivalence then implies that A/R is the quotient of A by R [53, Theorem 18.2.3]. ◀

▶ **Example 25** (Set pushouts). If we have univalent completions and inductive families, we can construct the reflexive transitive closure of a propositional relation using truncation. Example 24 then gives us set quotients by arbitrary (that is, non-equivalence) propositional relations, equivalently set-truncated pushouts. The work of Van der Weide and Geuvers [66] shows that all finitary set-truncated higher inductive types can be constructed in this fashion.[1]

De Jong and Escardó [27, Theorem 3.29] show that if we restrict our attention to sets, replacement for $\mathcal{U}$ is in fact equivalent to the existence of set quotients by $\mathcal{U}_{\mathsf{prop}}$-valued equivalence relations in $\mathcal{U}$. Still, "set replacement" is a useful abstraction; compare De Jong and Escardó's two proofs that the ordinal of ordinals in a universe $\mathcal{U}$ is closed under A-indexed suprema for $A : \mathcal{U}$ [27, Theorems 5.8 and 5.9]. We can similarly characterize univalent completion for sets using a form of descent (cf. Rijke [52, §2]) for set quotients.

▶ **Definition 26.** *A set quotient A/R has* descent *when for every propositional family $a : A \vdash P(a)$* type *such that $R(a, a')$ implies $P(a) \simeq P(a')$ for all $a, a' : A$, there is a propositional family $c : A/R \vdash \overline{P}(c)$* type *with $\overline{P}([a]) \simeq P(a)$ for $a : A$.*

This principle could also be called *large elimination*, although in that tradition (e.g., Smith [58, §5]) one usually deals in judgmental equations rather than equivalences.

▶ **Proposition 27.** *If every family of propositions admits a cartesian morphism to a univalent family, then every set quotient by a propositional equivalence relation has descent.*

Proof. Let a propositional equivalence relation R be given with set quotient $[-] : A \twoheadrightarrow A/R$ and let $a : A \vdash P(a)$ type be a propositional family such that $P(a) \simeq P(a')$ whenever $R(a, a')$. We have a univalent family $u : U \vdash E(u)$ type with $q : A \to U$ and $E(q\ a) \simeq P(a)$ for $a : A$. For any $a, a' : A$ such that $R(a, a')$, we have $E(q\ a) \simeq P(a) \simeq P(a') \simeq E(q\ a')$ and thus $q\ a =_U q\ a'$ by univalence. Hence there is a unique $q' : A/R \to U$ with $q =_{A \to U} q' \circ [-]$, from which we define the desired family $x : A/R \vdash \overline{P}(x)$ type by setting $\overline{P}(x) :\equiv E(q'\ x)$. ◀

▶ **Proposition 28.** *Every propositional family $a : A \vdash P$* type *admits a univalent completion if and only if every propositional equivalence relation admits a set quotient with descent.*

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[1] Van der Weide and Geuvers assume propositional truncation in addition to set quotients, but the propositional truncation of A can be obtained as its set quotient by the total relation $\lambda\_.\lambda\_.1$.

Proof. The forward direction is Example 24 and Proposition 27. For the converse, let $a : A \vdash P(a)$ type be a family of propositions. The relation $R(a, a') :\equiv P(a) \simeq P(a')$ is valued in propositions, so we have a set quotient $[-] : A \twoheadrightarrow A/R$. By descent, there is a family $c : A/R \vdash \overline{P}(c)$ type such that $\overline{P}([a]) \simeq P(a)$ for $a : A$. To show that $\overline{P}$ is a univalent completion of P, it remains to check that $\overline{P}$ is univalent. By surjectivity of $[-]$, it suffices to check that $[a] =_{A/R} [a'] \to \overline{P}([a]) \simeq \overline{P}([a'])$ is an equivalence for $a, a' : A$, and since this is a map between propositions it is enough to give a map in the opposite direction. Given $\overline{P}([a]) \simeq \overline{P}([a'])$ we have $P(a) \simeq P(a')$, thus $[a] =_{A/R} [a']$ by definition of R. ◀

2.3 Higher types

So far, we have described constructions that preserve h-level or land in sets. However, univalent completion and replacement can also be used to define operations that raise h-level. In fact, starting from univalent completion and some non-propositional set such as a boolean type, we can construct a non-n-type for every $n \geq -2$.

▶ **Example 29** (Delooping automorphism types)**.** For a type X, a univalent completion of $\_ : 1 \vdash X$ type is a type $\mathrm{BAut}(X)$ with surjection $q : 1 \twoheadrightarrow \mathrm{BAut}(X)$ and univalent family $z : \mathrm{BAut}(X) \vdash E(z)$ type such that $E(q \star) \simeq X$. This makes $\mathrm{BAut}(X)$ a *delooping of* $X \simeq X$: a 0-connected type whose point $\mathsf{pt} :\equiv q \star$ has loop space $X \simeq X$. In the formalism of Buchholtz, Van Doorn, and Rijke [15, §3], $\mathrm{BAut}(X)$ exhibits $X \simeq X$ as the carrier of the automorphism ∞-group of X. If X lies in a univalent universe $\mathcal{U}$ with replacement, we get $\mathrm{BAut}(X) : \mathcal{U}$ as the image of $\lambda\_.X : 1 \to \mathcal{U}$ [15, §4.1].

Given a boolean type Bool, the loop space $\Omega(\mathrm{BAut}(\mathsf{Bool}), \mathsf{pt})$ contains two identities from its basepoint to itself. If the points $\mathsf{true}, \mathsf{false} : \mathsf{Bool}$ are distinct – as follows from a universe or large elimination for Bool – then $\mathrm{BAut}(\mathsf{Bool})$ is an example of a non-set.

▶ **Example 30** (Delooping groups)**.** Assume we have univalent completions and let G be a group, in the sense of a set equipped with the usual operations and equations. Writing $|G|$ for its carrier, we have a univalent $z : \mathrm{BAut}(|G|) \vdash E(z)$ type with $E(\mathsf{pt}) \simeq |G|$ per Example 29.

Write $\mathsf{Act}_G(X)$ for the type of left G-actions on a set X. The *principal G-torsor* is the action of G on $|G|$ by multiplication; write $\mathsf{Pr}_G : 1 \to \sum_{z:\mathrm{BAut}(|G|)} \mathsf{Act}_G(E(z))$ for the map that picks out the corresponding action on $E(\mathsf{pt})$ ($\simeq |G|$) and define the *type of G-torsors* $\mathsf{Torsor}(G)$ to be the image of Pr_G. The type of G-torsors is a *delooping of the group G*: a 0-connected 1-groupoid with $\pi_1(\mathsf{Torsor}(G)) \cong G$. Such a type is unique up to equivalence and may also be called the *first Eilenberg–MacLane space*, $K(G, 1)$, associated to G [30]. If G lies in a univalent universe $\mathcal{U}$ with replacement, we can more directly define $\mathsf{Torsor}(G) : \mathcal{U}$ as the image of the principal G-torsor in $\sum_{X:\mathcal{U}} \mathsf{Act}_G(X)$.

As an early example, Bezem, Buchholtz, Grayson, and Shulman [11] defined a circle type in the Rocq UniMath library [71] as the type of $\mathbb{Z}$-torsors. They do not use replacement or univalent completion; they assume a univalent universe $\mathcal{U}$ to define $\mathsf{Torsor}(\mathbb{Z})$ and do not claim that this type is $\mathcal{U}$-small. *Symmetry* [10, §5.5] describes the case of a general G and observes that replacement implies that the delooping is $\mathcal{U}$-small. Licata and Finster [42, §3], who gave the first construction of general Eilenberg–MacLane spaces in a form of HoTT, work with a direct specification of $K(G, 1)$ as a higher inductive type, which they assume lives within a univalent universe.

▶ **Example 31** (Rezk completion)**.** Example 30 is a special case of *Rezk completion.* Let $\mathcal{C}$ be a category: a type of objects $\mathcal{C}_0$, a hom-set $\mathcal{C}_1(x, y)$ for each $x, y : \mathcal{C}$, and identities and compositions satisfying the usual laws. A Rezk completion of $\mathcal{C}$ is a fully faithful and essentially surjective functor $i : \mathcal{C} \to \overline{\mathcal{C}}$ into a *univalent* category $\overline{\mathcal{C}}$: one for which the maps $x = y \to x \cong y$ from equalities to categorical isomorphisms are equivalences for all $x, y : \overline{\mathcal{C}}$ [64, §9.9] [3]. A delooping of a group G is the Rezk completion of G seen as a 1-object category.

The HoTT Book [64, Theorem 9.9.5] shows that for a category $\mathcal{C}$ with $\mathcal{U}$-small hom-sets, a Rezk completion can be defined as the image of the Yoneda map $\mathsf{y} : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{U}_{\mathsf{set}}]$ into the univalent category of $\mathcal{U}_{\mathsf{set}}$-valued presheaves on $\mathcal{C}$. When $\mathcal{U}$ has replacement, the Rezk completion of a $\mathcal{U}$-small category is thus $\mathcal{U}$-small [51, Corollary 5.2]. We can translate this into a construction using univalent completions as in Example 30. Given a category $\mathcal{C}$, take a univalent completion $u : U \vdash E(u)$ type of $(x, y) : \mathcal{C}_0 \times \mathcal{C}_0 \vdash \mathcal{C}_1(x, y)$ type with surjection $q : \mathcal{C}_0 \times \mathcal{C}_0 \twoheadrightarrow U$. We have a univalent category $\mathcal{U}_{\mathcal{C}}$ with $(\mathcal{U}_{\mathcal{C}})_0 :\equiv U$ and $(\mathcal{U}_{\mathcal{C}})_1(u, u') :\equiv (E(u) \to E(u'))$. The category $[\mathcal{C}^{op}, \mathcal{U}_{\mathcal{C}}]$ of functors from $\mathcal{C}^{op}$ to $\mathcal{U}_{\mathcal{C}}$ is also univalent, and the image of the map $\lambda x.q\,(-, x) : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{U}_{\mathcal{C}}]$ is a Rezk completion of $\mathcal{C}$.

▶ **Example 32** (Groupoid quotients). Another special case of Rezk completion (and generalization of Example 30) is the *groupoid quotient* of a type A by a groupoid $R : A \to A \to \mathcal{U}_{\mathsf{set}}$ (in the sense of a family defining a category (A, R) with all morphisms invertible). Sojakova [60] specifies the groupoid quotient as a higher inductive type; Veltri and Van der Weide [68] point out that its universal property is satisfied by the Rezk completion of the category (A, R). Veltri and Van der Weide moreover show that a general class of finitary groupoid-truncated higher inductive types can be derived from groupoid quotients assuming inductive families.

We expect that the trend indicated by Examples 24 and 32 continues, i.e., that univalent completion can be used to construct n-groupoid quotients for all n. We will not pursue this here. We will, however, observe individual examples of proper n-types for all n:

▶ **Example 33** (Eilenberg–MacLane spaces). For a group G, the nth *Eilenberg–MacLane space* [30] $K(G, n)$ is the unique $(n-1)$-connected type satisfying $\Omega^n(K(G, n), \mathsf{pt}) \simeq G$. These were first constructed by in homotopy type theory Licata and Finster [42], who used a bespoke higher inductive type for $K(G, 1)$ and truncated suspensions for higher $K(G, n)$. Assuming only a univalent universe $\mathcal{U}$ and that G lives in $\mathcal{U}$, Buchholtz, Christensen, Flaten, and Rijke [14] and Wärn [73] give alternative constructions of the Eilenberg–MacLane spaces that are $\mathcal{U}$-small when $\mathcal{U}$ has replacement.

The first space $K(G, 1)$ is the delooping of G from Example 30. For $n \geq 1$, Buchholtz et al. construct $K(G, n+1) :\equiv \sum_{X:\mathcal{U}} \|K(G, n) \simeq X\|_0$ [14, Proposition 5.9]. Replacement implies these are $\mathcal{U}$-small: any connected and locally $\mathcal{U}$-small type A is $\mathcal{U}$-small, as it merely admits a point $1 \to A$ and is then equivalent to the image of that point. If we assume univalent completion instead of a universe, we can instead define $K(G, n+1) :\equiv \sum_{z:\mathrm{BAut}(K(G,n))} \|K(G, n) \simeq E(z)\|_0$ using the family E from Example 29. Similar arguments apply to Wärn's construction [73, Footnote to Definition 1].

In particular, univalent completion and a natural numbers type together imply the existence of types that are not n-truncated for any n, such as $\prod_{n:\mathbb{N}} K(\mathbb{Z}, n)$. We refer to Buchholtz et al. [14] and Wärn [73] for further applications of replacement to constructing deloopings.

2.4 Higher replacement

The concept of image being an instance of the more general concept of n-image, replacement admits a natural generalization to axiom schemata of "n-replacement" for $n \geq -2$. Following Christensen [22], say that a type A is 0-*locally* $\mathcal{U}$-*small* if it is $\mathcal{U}$-small and $(n+1)$-*locally small* when its identity types are n-small.

▶ **Definition 34.** *For $n \geq 2$, we say that $\mathcal{U}$* has n-replacement *when every $f : A \to B$ where A is $\mathcal{U}$-small and B is $(n+2)$-locally $\mathcal{U}$-small admits a $\mathcal{U}$-small n-image.*

Replacement is then (-1)-replacement. Christensen [22, Proposition 2.2] observes that in type theory with pushouts, Rijke's join construction proves not only (-1)-replacement but n-replacement for all n. Gylterud and Stenholm [35, Proposition 17] abstract from the join construction to derive n-replacement from (-1)-replacement. Their argument assumes an enveloping univalent universe ("Type"), but we can also make do with univalent completions:

▶ **Example 35** (Higher replacement, cf. Gylterud and Stenholm [35, Proposition 17])**.** Given univalent completions and a univalent universe $\mathcal{U}$ with replacement (i.e., closed under univalent completions), $\mathcal{U}$ has n-replacement for all $n \geq -2$.

Proof. Let $f : A \to B$ where A is $\mathcal{U}$-small and B and $(n+2)$-locally $\mathcal{U}$-small. By Example 22, f has an n-image $A \overset{p}{\twoheadrightarrow} X \overset{i}{\hookrightarrow} B$. Since i is n-truncated, X is also $(n+2)$-locally $\mathcal{U}$-small. Thus it is equivalent to show that any $(n+2)$-locally $\mathcal{U}$-small type X admitting an n-connected map $p : A \to X$ from a $\mathcal{U}$-small type A is $\mathcal{U}$-small. We show this by induction on n.

The case $n = -2$ is trivial and the case $n = -1$ holds by replacement. For $n \geq 0$, let $p : A \to X$ be as above. By replacement, it is enough to show that X is locally $\mathcal{U}$-small. Since p is in particular surjective, we can show that all identity types in X are $\mathcal{U}$-small by showing that $p\ a =_X p\ a'$ is $\mathcal{U}$-small for all $a, a' : A$. This follows by induction hypothesis, since for $a, a' : A$ the map $\mathsf{ap}_p^{a,a'} : (a =_A a') \to (p\ a =_X p\ a')$ is an $(n-1)$-connected map from a $\mathcal{U}$-small type to a $(n-1)$-locally $\mathcal{U}$-small type. We use univalent completions for the fact that an n-connected map induces $(n-1)$-connected maps on identity types: since truncations exist, a map g is k-connected if and only if the k-truncations of its fibers are contractible [64, Lemma 7.5.7], and with this characterization the fact follows from Example 11. ◀

As a consequence (cf. Christensen [22, Theorem 2.6, Proof 2]), given univalent completions and a univalent universe $\mathcal{U}$ with replacement, a type A is $\mathcal{U}$-small as soon as there is some $n \geq -1$ for which $\|A\|_n$ is $\mathcal{U}$-small and A is $(n+1)$-locally $\mathcal{U}$-small.

2.5 Set theory

Type theoretic replacement is named by analogy to the Replacement Axiom Schema of set theory. In Zermelo–Fraenkel set theory, Replacement states that for any functional first-order formula $\phi(a, b)$ and set A, there is a set B such that $b \in B$ exactly when $\phi(a, b)$ for some $a \in A$. Replacement was proposed, by at least Fraenkel [32] and Skolem [57], after it was observed that Zermelo's set theory [74] cannot define sets of cardinality $\geq \aleph_\omega$. With Replacement, one can define sets such as $\bigcup\{\beth_0, \beth_1, \ldots\}$ that have at least this cardinality: we obtain $\{\beth_0, \beth_1, \ldots\}$ by Replacement as the image of $\{0, 1, \ldots\}$ under the mapping $\alpha \mapsto \beth_\alpha$. Hallett [36, §8.2] credits Von Neumann [72] with recognizing the importance of Replacement to the theory of transfinite arithmetic. We refer to Van Heijenoort [67], Hallett [36, §8.2], Ferreirós [31, §§XI.1–2], and Kanamori [38] for more of the set theoretic history.

This motivating example in set theory, going from $\{0, 1, \ldots\}$ to $\{\beth_0, \beth_1, \ldots\}$, does not have an immediate analogue in a structural foundation such as type theory. However, the applications of type theoretic replacement do parallel applications of set theoretic replacement in predicative constructive set theories such as Aczel's CZF [1]. For example, Aczel and Rathjen [2, Lemma 4.3.5] use Replacement to define quotients, as we have done in Definition 23 and Example 25. Kanamori [38, §1] also highlights the construction of quotient sets as an instance of the "Replacement motif" in classical set theory. Gylterud and Stenholm [34, 35] connect type theoretic and set theoretic replacement in their work on (constructive, predicative) material set theory in type theory:

▶ **Example 36** (Replacement for iterated sets)**.** Aczel's universe $V_\infty :\equiv \mathsf{W}_{A:\mathcal{U}} A$ [1, §3], defined using a W-type [44], is a model of his constructive set theory CZF. Gylterud [34] defines a universe of *iterative sets* in univalent type theory as the subtype V of V_∞ carved out by a predicate $\mathsf{isIterativeSet} : V_\infty \to \mathcal{U}$, which is defined by induction so that $\mathsf{isIterativeSet}\ (\mathsf{sup}\ a\ f) :\equiv \mathsf{isEmbedding}\ f \times \prod_{a:A} \mathsf{isIterativeSet}\ (f\ a)$. The universe V interprets Myhill's Constructive Set Theory (CST) [46]. Unlike Aczel's model of CZF in V_∞, Gylterud's model interprets equality by V's identity type. To show that V models Pairing, Union, and Replacement, Gylterud assumes $\mathcal{U}$-small images of maps from $\mathcal{U}$-small types into the locally $\mathcal{U}$-small V. As V is a set, set quotients in $\mathcal{U}$ would suffice [27, Theorem 3.29].

Gylterud and Stenholm [35] define universes of *iterative n-types* by replacing $\mathsf{isEmbedding}\ f$ in $\mathsf{isIterativeSet}$ with a requirement that f is $(n-1)$-connected. To prove a form of Replacement for the universe of iterative n-types, they assume type theoretic replacement [35, §4.1] and use in particular that it implies $(n-1)$-replacement (Example 35).

▶ Remark 37. In some constructive set theories, Replacement is derived from a stronger Collection or Strong Collection axiom schema; see Rathjen [49] for some discussion of the added power of Collection. Rijke and Spitters [54, Definition 4.7] state a Collection axiom for sets in homotopy type theory. Gylterud [34, §7.1] considers two type theoretic renderings of Strong Collection for his V. One follows from type theoretic replacement for $\mathcal{U}$; for the other, he formulates a collection principle for $\mathcal{U}$ that implies it [34, Definition 7.2].

Still, in impredicative foundations one customarily constructs quotients using Power Set. Analogously, Rijke and Spitters [54] observe that Voevodsky's *propositional resizing* axioms [70] [64, §3.5] imply type theoretic replacement for sets:

▶ **Proposition 38** (Rijke and Spitters [54, Lemma 3.28])**.** *If $\mathcal{U}$ contains a univalent universe of* all *propositions, i.e., a univalent family $p : \Omega \vdash \mathsf{El}_\Omega(p) : \mathcal{U}_{\mathsf{prop}}$ such that every proposition is equivalent to one in Ω, then every $f : A \to B$ where A is $\mathcal{U}$-small and B is a set has a $\mathcal{U}$-small image.*

Proof. Combine the equivalence of set replacement and set quotients [27, Theorem 3.29] with Voevodsky's construction of set quotients from propositional resizing [4, §2.5] [54, §3.4]. ◀

Contrary to what its set theoretic name suggests, then, type theoretic replacement has no power as an axiom among the sets of type theory with resizing – it is simply a theorem.

▶ Remark 39. Martin-Löf [43] writes of his type theory that "there are axioms for universes (in the sense of category theory) which link the generation of objects and types and play somewhat the same role for the present theory as does the replacement axiom for Zermelo–Fraenkel set theory." Streicher [62] also emphasizes the connection between Replacement and universes. Montague [45] and Lévy [41, Theorem 6] connect Replacement with *reflection principles*, which can be likened to the existence of enough universes. Returning to the example of $\{\beth_0, \beth_1, \ldots\}$, a universe $\mathcal{U}$ enables the definition of a family $\beth : \mathbb{N} \to \mathcal{U}$ by recursion with $\beth_0 :\equiv \mathbb{N}$ and $\beth_{n+1} :\equiv (\beth_n \to 2)$. In the absence of universes, large eliminators à la Smith [58] can play a similar role. Although Rijke's axiom of replacement is irrelevant to the definability of $\beth$, we have seen that univalent completion (and by analogy replacement) can also be seen as providing "enough universes" with some property, in this case univalence.

3 Replacement and univalent completion in cubical models

Cubical interpretations provide constructive semantics of homotopy type theory [23, 7, 47, 6]. Their univalent universes are closed under pushouts [25, 17], so have replacement by Rijke's join construction [51, Theorem 4.6]. Here we give more direct constructive justifications for replacement and univalent completion. Besides providing intuition for the meaning of the principles in these models, these will point to connections to higher inductive and universe constructions in prior work. We first recall background on cubical models in § 3.1, followed by a free algebra construction in § 3.2. In § 3.3 we give an interpretation of (-1)-truncation previously sketched by Cherubini, Coquand, and Hutzler [20, §8.3]. This case study motivates our justifications of replacement in § 3.4 and univalent completion in § 3.5.

3.1 Preliminaries

First, we fix some categorical notation. Given a small category $\mathcal{C}$, we write $\widehat{\mathcal{C}}$ for presheaves on $\mathcal{C}$. For $X \in \widehat{\mathcal{C}}$ and $I \in \mathcal{C}$, we write $X(I) \in \mathbf{Set}$ for the value of X at I. For $f\colon X \to Y$ in $\widehat{\mathcal{C}}$, we write $f_I\colon X(I) \to Y(I)$ for the value of f at I. We write $\mathbf{y} : \mathcal{C} \to \widehat{\mathcal{C}}$ for the Yoneda embedding $\mathbf{y}I \coloneqq \mathcal{C}(-, I)$. A monomorphism $m\colon X \rightarrowtail Y$ in $\widehat{\mathcal{C}}$ is *levelwise decidable* when m_I is a decidable monomorphism for $I \in \mathcal{C}$.

For the remainder of §3, we assume the following basic data, with which we can build a homotopical model of type theory following Angiuli, Brunerie, Coquand, Harper, Favonia, and Licata [6].

▶ **Setting 40.** We fix a finite product category $\mathcal{C}$ whose hom-sets have decidable equality. We require:

- A representable *interval*: an object $\mathbb{I} \in \mathcal{C}$ with disjoint points $0, 1\colon 1 \to \mathbb{I}$.
- A *cofibration classifier*: a levelwise decidable mono $\widetilde{\Phi} \rightarrowtail \Phi$ in $\widehat{\mathcal{C}}$, together with maps $(\equiv)\colon \mathbf{y}\mathbb{I} \times \mathbf{y}\mathbb{I} \to \Phi$, $\bot \in 1 \to \Phi$, and $(\vee)\colon \Phi \times \Phi \to \Phi$ fitting in pullback squares

$$\begin{tikzcd} \mathbf{y}\mathbb{I} \arrow[r] \arrow[d, tail, "(\mathrm{id}_{\mathbf{y}\mathbb{I}}, \mathrm{id}_{\mathbf{y}\mathbb{I}})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\Phi} \arrow[d, tail] \\ \mathbf{y}\mathbb{I} \times \mathbf{y}\mathbb{I} \arrow[r, "(\equiv)"'] & \Phi \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\Phi} \arrow[d, tail] \\ 1 \arrow[r, "\bot"'] & \Phi \end{tikzcd} \qquad \begin{tikzcd} (\widetilde{\Phi} \times \Phi) \cup (\Phi \times \widetilde{\Phi}) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\Phi} \arrow[d, tail] \\ \Phi \times \Phi \arrow[r, "(\vee)"'] & \Phi. \end{tikzcd}$$

  We write $r \equiv r'$ for $(\equiv)_I(r, r')$, likewise $\bot$ for $\bot_I(\star)$ and $\phi \vee \phi'$ for $(\vee)_I(\phi, \phi')$.
- For each $I \in \mathcal{C}$ and $\phi \in \Phi(I)$, a *choice of finite presentation* C_ϕ *of* ϕ: a finite set $C_\phi \subseteq \mathcal{C}/I$ of morphisms into I such that for every $\alpha\colon J \to I$, $\phi\alpha$ belongs to $\widetilde{\Phi}(J)$ if and only if $\alpha = \alpha'\beta$ for some $\alpha' \in C_\phi$ and morphism β of $\mathcal{C}$.

A cofibration $\phi \in \Phi(I)$ corresponds by the Yoneda lemma to a morphism $\phi\colon \mathbf{y}I \to \Phi$, and pulling back $\widetilde{\Phi} \rightarrowtail \Phi$ along this morphism yields a levelwise decidable mono $\phi^*\widetilde{\Phi} \rightarrowtail \mathbf{y}I$. That is, each $\Phi(I)$ specifies a class of subobjects of $\mathbf{y}I$, which we call *cofibrations*. The operations $(\equiv)$, $\bot$, and $(\vee)$ encode closure properties of this class. The finite presentations tell us that each cofibration $\phi \in \Phi(I)$ has a representation as a finite union of "atomic" cofibrations. We use these to avoid depending on quotients in our metatheory and to ensure that our inductive constructions converge (see Construction 55).

Setting 40 suffices for our purposes, but we would need more to interpret all of univalent type theory; we refer to Angiuli et al. [6] for details. We use Angiuli et al.'s style of model, which is characterized by the use of "diagonal cofibrations" $r \equiv r'$ and a decomposition of filling into *homogeneous filling* and *coercion*. Our constructions adapt straightforwardly to Cohen, Coquand, Huber, and Mörtberg's style of model [23, 25] and could be described uniformly for both styles using "weak" filling [18], but for simplicity we focus on one style.

One instance of Setting 40 takes $\mathcal{C}$ to be the cartesian cube category, $\mathbb{I}$ to be the 1-cube, and Φ to be the levelwise decidable subobject classifier Ω_{ldec} [6, §3.2]. Here $\Omega_{\mathrm{ldec}}(I)$ is the set of levelwise decidable subobjects of $\mathbf{y}I$; see Orton and Pitts [47, Definition 8.3]. The cube category admits an Eilenberg–Zilber structure (see, e.g., Campion [16, Theorem 8.12]) which implies that each $\phi \in \Omega_{\mathrm{ldec}}(I)$ has a finite presentation by the set of monomorphisms $J \rightarrowtail I$ that factor through ϕ. Another instance, sketched by Coquand and Höfer [24], takes $\mathcal{C}$ to be the category of finite posets with an inductively generated cofibration classifier.

We now describe the interpretation of type theory in $\widehat{\mathcal{C}}$ induced by the data of Setting 40.

▶ **Definition 41.** *A* context *is an object of* $\widehat{\mathcal{C}}$*. A* (non-fibrant) type *over* $\Gamma \in \widehat{\mathcal{C}}$ *is a presheaf over the category of elements* $\int_{\mathcal{C}} \Gamma$*; write* $\mathrm{Ty}(\Gamma) \coloneqq \widehat{\int_{\mathcal{C}} \Gamma}$*. A* term *over* $A \in \mathrm{Ty}(\Gamma)$ *is a global section* $a \colon 1 \to A$*, i.e., a family of elements* $a(I,\gamma) \in A(I,\gamma)$ *for all* $I \in \mathcal{C}$ *and* $\gamma \in \Gamma(I)$ *satisfying* $a(I,\gamma)\alpha = a(J,\gamma\alpha)$ *for* $\alpha \colon J \to I$*. Write* $\mathrm{Ter}(\Gamma; A)$ *for the collection of terms of* A*.*

Despite the notation, $\mathrm{Ty}(\Gamma)$ is not the interpretation of the typing judgment. Rather, syntactic types are interpreted as *fibrant types*, which are elements of $\mathrm{Ty}(\Gamma)$ equipped with *filling operations*. Before defining these, we first fix some notation.

▶ Notation 42. For $A \in \mathrm{Ty}(\Gamma)$, we write $\mathsf{p}_A : \Gamma.A \to \Gamma$ for the extended context and projection and $\mathsf{q}_A \in \mathrm{Ter}(\Gamma.A; A\mathsf{p}_A)$ for the variable. For $\sigma \colon \Delta \to \Gamma$, $A \in \mathrm{Ty}(\Gamma)$, and $t \in \mathrm{Ter}(\Delta; A\sigma)$, we write $(\sigma, t) \colon \Delta \to \Gamma.A$ for the extended substitution. For $\sigma \colon \Delta \to \Gamma$ and $f \colon A \to B\sigma$ in $\mathrm{Ty}(\Delta)$, we write $\sigma.f \colon \Delta.A \to \Gamma.B$ for the induced substitution.

We use similar notation for "interval variables". For $I \in \mathcal{C}$, we write $\mathsf{p}_{\mathbb{I}} \colon I \times \mathbb{I} \to I$, $\mathsf{q}_{\mathbb{I}} \colon I \times \mathbb{I} \to \mathbb{I}$ for the projections; for $\Gamma \in \widehat{\mathcal{C}}$, we write $\mathsf{p}_{\mathbb{I}} \colon \Gamma \times \mathbf{y}\mathbb{I} \to \Gamma$, $\mathsf{q}_{\mathbb{I}} \colon \Gamma \times \mathbf{y}\mathbb{I} \to \mathbf{y}\mathbb{I}$. For $r \colon I \to \mathbb{I}$, we write $\underline{r} \coloneqq (\mathrm{id}_I, r) \colon I \to I \times \mathbb{I}$; for $r \colon \Gamma \to \mathbf{y}\mathbb{I}$ we write $\underline{r} \coloneqq (\mathrm{id}_\Gamma, r) \colon \Gamma \to \Gamma \times \mathbf{y}\mathbb{I}$.

The interpretation of types involves the concept of *partial elements*, which are elements of a presheaf or type defined on some subobject of a cube given by a cofibration.

▶ Notation 43. We use the following notations for partial elements:

Sets of partial elements: Given $I \in \mathcal{C}$ and $\phi \in \Phi(I)$, we write $\Gamma(I \upharpoonright \phi)$ for the set of families $(\gamma_\alpha \in \Gamma(J) \mid \alpha \colon J \to I \text{ s.t. } \phi\alpha \in \widetilde{\Phi}(J))$ satisfying $\gamma_\alpha\beta = \gamma_{\alpha\beta}$ for all $K \xrightarrow{\beta} J \xrightarrow{\alpha} I$. Given $\gamma \in \Gamma(I \upharpoonright \phi)$ and $\alpha \colon J \to I$, we define $\gamma\alpha \in \Gamma(J \upharpoonright \phi\alpha)$ by $(\gamma\alpha)_\beta \coloneqq \gamma_{\alpha\beta}$. Given $\gamma \in \Gamma(I)$, we define $\gamma \upharpoonright \phi \in \Gamma(I \upharpoonright \phi)$ by $(\gamma \upharpoonright \phi)_\alpha \coloneqq \gamma\alpha$. When ϕ has a finite presentation by some single $\alpha_0 \colon J_0 \to I$ and we have $\gamma \in \Gamma(J_0)$, we may write $\gamma \in \Gamma(I \upharpoonright \phi)$. Given $A \in \mathrm{Ty}(\Gamma)$, $I \in \mathcal{C}$, $\phi \in \Phi(I)$, and $\gamma \in \Gamma(I \upharpoonright \phi)$, we similarly write $A(I \upharpoonright \phi, \gamma)$ for the set of families $(a_\alpha \in A(J, \gamma_\alpha) \mid \alpha \colon J \to I \text{ s.t. } \phi\alpha \in \widetilde{\Phi}(J))$ such that $a_\alpha\beta = a_{\alpha\beta}$ for all α, β.

Definition by cases: For ϕ_0, ϕ_1 and $a_k \in A(I \upharpoonright \phi_k, \gamma)$ for each k, we write $[\phi_0 \mapsto a_0 \mid \phi_1 \mapsto a_1]$ for the unique $a \in A(I \upharpoonright (\phi_0 \vee \phi_1), \gamma)$ with $a \upharpoonright \phi_k = a_k$ for each k, should such an element exist (i.e., if $(a_0)_\alpha = (a_1)_\alpha$ whenever both sides are defined).

Maps: Given $f \colon A \to B$ in $\mathrm{Ty}(\Gamma)$, we write $f_{I \upharpoonright \Phi, \gamma} \colon A(I \upharpoonright \phi, \gamma) \to B(I \upharpoonright \phi, \gamma)$ for the action on partial elements. In general, when we apply some operation (e.g., a pair projection) to a partial element, we mean to apply the operation levelwise to the family.

Context extension: Given $\Gamma \in \widehat{\mathcal{C}}$ and $\phi \colon \Gamma \to \Phi$, we write $\Gamma.\phi \in \widehat{\mathcal{C}}$ for the presheaf $(\Gamma.\phi)(I) \coloneqq \{\gamma \in \Gamma(I) \mid \phi_I(\gamma) \in \widetilde{\Phi}(I)\}$ and $\mathsf{p}_\phi \colon \Gamma.\phi \to \Gamma$ for the inclusion.

▶ **Definition 44.** *A* filling operation $\mathfrak{a}$ *for* $A \in \mathrm{Ty}(\Gamma)$ *is an operation implementing the rule*

$$\frac{\gamma \in \Gamma(I \times \mathbb{I}) \quad r \in \mathbf{y}\mathbb{I}(I) \quad a_0 \in A(I, \gamma\underline{r}) \quad \phi \in \Phi(I) \quad a \in A(I \times \mathbb{I} \upharpoonright \phi\mathsf{p}_{\mathbb{I}}, \gamma) \quad a\underline{r} = a_0 \upharpoonright \phi}{\mathfrak{a}_{I,\gamma}\ r\ a_0\ \phi\ a \in A(I \times \mathbb{I}, \gamma)}$$

and equations $(\mathfrak{a}_{I,\gamma}\ r\ a_0\ \phi\ a) \upharpoonright \phi = a$ *and* $(\mathfrak{a}_{I,\gamma}\ r\ a_0\ \phi\ a)\underline{r} = a_0$ *stably under substitution, i.e., with* $(\mathfrak{a}_{I,\gamma}\ r\ a_0\ \phi\ a)(\alpha \times \mathbb{I}) = \mathfrak{a}_{J,\gamma(\alpha\times\mathbb{I})}\ (r\alpha)\ (a_0\alpha)\ (\phi\alpha)\ (a(\alpha \times \mathbb{I}))$ *for all* $\alpha \colon J \to I$*.*

▶ **Definition 45.** *A* fibrant type *over* $\Gamma \in \widehat{\mathcal{C}}$ *is a pair* $(A, \mathfrak{a})$ *of an* $A \in \mathrm{Ty}(\Gamma)$ *and a filling operation* $\mathfrak{a}$ *for* A*. We write* $\mathrm{Ty}^{\mathrm{f}}(\Gamma)$ *for the collection of fibrant types over* Γ*. We may leave the projection to* $\mathrm{Ty}(\Gamma)$ *implicit and write* A *for both a fibrant type and its underlying type.*

The filling operation can be decomposed into two simpler operations. This is key to the interpretation of higher inductive types [25, 17] and will be similarly important for us.

▶ **Definition 46.** *A* homogeneous filling operation $\mathfrak{a}$ *for* $A \in \mathrm{Ty}(\Gamma)$ *is an operation implementing the rule*

$$\frac{\gamma \in \Gamma(I) \qquad r \in \mathbf{y}\mathbb{I}(I) \qquad a_0 \in A(I,\gamma) \qquad \phi \in \Phi(I) \qquad a \in A(I \times \mathbb{I} \restriction \phi \mathrm{p}_{\mathbb{I}}, \gamma \mathrm{p}_{\mathbb{I}}) \qquad a\underline{r} = a_0 \restriction \phi}{\mathfrak{a}_{I,\gamma}\ r\ a_0\ \phi\ a \in A(I \times \mathbb{I}, \gamma \mathrm{p}_{\mathbb{I}})}$$

and equations $(\mathfrak{a}_{I,\gamma}\ r\ a_0\ \phi\ a) \restriction \phi = a$ *and* $(\mathfrak{a}_{I,\gamma}\ r\ a_0\ \phi\ a)\underline{r} = a_0$ *stably under substitution. A* coercion operation $\mathfrak{a}$ *for* $A \in \mathrm{Ty}(\Gamma)$ *is an operation implementing the rule*

$$\frac{\gamma \in \Gamma(I \times \mathbb{I}) \qquad r \in \mathbf{y}\mathbb{I}(I) \qquad a_0 \in A(I, \gamma\underline{r})}{\mathfrak{a}_{I,\gamma}\ r\ a_0 \in A(I \times \mathbb{I}, \gamma)}$$

and equation $(\mathfrak{a}_{I,\gamma}\ r\ a_0)\underline{r} = a_0$ *stably under substitution.*

A filling operation $\mathfrak{a}$ on A immediately restricts to a homogeneous filling operation $\mathfrak{a}^{\mathrm{h}}$ and coercion operation $\mathfrak{a}^{\mathrm{c}}$. Conversely, given homogeneous filling and coercion operations for $A \in \mathrm{Ty}(\Gamma)$, we can derive a filling operation for A [6, §2.7].

▶ Notation 47. We reuse type-theoretic notation such as Σ and Π for the corresponding operations on semantic types; see [23, §8.2] or [6, §3] for their definitions. For $A \in \mathrm{Ty}(\Gamma)$ and $a_0, a_1 \in \mathrm{Ter}(\Gamma; A)$, $\mathtt{Path}_A\ a_0\ a_1 \in \mathrm{Ty}(\Gamma)$ is the type of paths from a_0 to a_1: $(\mathtt{Path}_A\ a_0\ a_1)(I,\gamma) = \{p \in A(I \times \mathbb{I}, \gamma \mathrm{p}_{\mathbb{I}}) \mid p\underline{0} = a_0 \text{ and } p\underline{1} = a_1\}$. We use $\mathtt{Path}_A\ a_0\ a_1$ as the semantic analogue of the identity type $a_0 =_A a_1$. The path type at a fibrant A does not quite interpret the computation rule of Martin-Löf's J eliminator, but under appropriate assumptions is equivalent to a type that does [23, §9.1] [17, §3.3]. These basic type constructors give us semantic equivalents of compound types, such as types of fibers and equivalences, which we write in `monospace` (e.g., $\mathtt{Fiber}_f$, $A \simeq B$, $\mathtt{isProp}\ A$, etc.).

▶ Remark 48. *Cubical type theories* [23, 7] extend Martin-Löf type theory with judgments inspired by the interval and cofibrations of cubical models: an *interval term* judgment $\Gamma \vdash r : \mathbb{I}$, and a *cofibration* judgment $\Gamma \vdash \phi\ \mathsf{cof}$ supporting operations $(\equiv)$, $\bot$, and $(\vee)$. Partial elements are specified by restricting contexts by cofibrations: $\Gamma, \phi \vdash a : A$ means A and a are defined on the part of Γ where ϕ holds. Given a total type A and partial $\Gamma, \phi \vdash u : A$, the notation $\Gamma \vdash a : A\ [\phi \mapsto u]$ means that $\Gamma \vdash a : A$ and $\Gamma, \phi \vdash a \equiv u : A$, i.e., a is a total element extending u. Space does not permit us to develop the cubical type theory corresponding to our semantic arguments, but we will present a few rules that would appear in such a theory in order to aid the reader.

3.2 Constructing free algebras

We interpret (-1)-truncation, replacement, and completion as least cubical sets closed under some constructors. Each can be expressed as a free algebra for a pointed endofunctor in a different category: (-1)-truncation in $\mathrm{Ty}(\Gamma)$, replacement in a slice category $\mathrm{Ty}(\Gamma)/B$, and completion in a category of type families. We describe here a free algebra construction that we can apply in each case, specializing a result in Cavallo and Sattler [19, Theorem 2.3.27, Remark 2.3.8]. This construction uses only colimits in **Set** that can be constructed from coproducts, avoiding a dependency on quotients, and is an alternative to Coquand, Huber, and Mörtberg's approach to constructing higher inductive types without quotients [25].

▶ **Definition 49.** *A wide subcategory* $\mathcal{M} \hookrightarrow \mathcal{E}$ has cobase changes *if (a) every span of the form* $B \leftarrow A \overset{\in\mathcal{M}}{\to} C$ *in* $\mathcal{E}$ *admits a pushout* $B \to D \leftarrow C$*, and (b) for every such pushout, the map* $B \to D$ *is also in* $\mathcal{M}$*.*

▶ **Proposition 50.** *The wide subcategory of decidable monos in* **Set** *has cobase changes, and every* ω*-chain of decidable monos in* **Set** *has a colimit in* **Set***.*

Proof. Given a span $B \xleftarrow{f} A \overset{m}{\rightarrowtail} C$ where m is a decidable mono, we have a pushout $B \to B \sqcup (C \setminus A) \leftarrow C$ where $B \to B \sqcup (C \setminus A)$ is the coproduct inclusion and $C \to B \sqcup (C \setminus A)$ is the composite of $C \cong A \sqcup (C \setminus A)$ with $f \sqcup (C \setminus A) \colon A \sqcup (C \setminus A) \to B \sqcup (C \setminus A)$. If $A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots$ is a chain of decidable monos, then the coproduct $\coprod_{n\in\mathbb{N}}(A_n \setminus A_{n-1})$ (taking A_{-1} to be the empty set) is its colimit. ◀

▶ Remark 51. In set theory, constructing a coproduct $\coprod_{n\in\mathbb{N}} S_n$ uses Replacement, namely to form the set $\{(n, S_n) \mid n \in \mathbb{N}\}$ whose union is that coproduct. Countable Replacement, i.e., Replacement for class functions with domain $\mathbb{N}$, is not provable in Zermelo set theory but is weaker than full Replacement: within ZFC, the Von Neumann universe V_α is a model of Zermelo set theory for any limit ordinal $\alpha > \omega$ (see, e.g., Smullyan and Fitting [59, Chapter 7, Theorem 4.1]) and models Countable Replacement when α has cofinality $> \omega$. In particular, V_{ω_1} (where ω_1 is the first uncountable ordinal) models Countable Replacement.

In a constructive set theory such as CZF, the existence of $\coprod_{n\in\mathbb{N}} S_n$ is likewise guaranteed by Replacement. In a type theoretic metatheory, we must either assume that $\mathbb{N}$ has large elimination or else assume a universe and take **Set** to be the category of sets in that universe.

▶ **Corollary 52.** *For any small category* $\mathcal{C}$*, the class of levelwise decidable monos in* $\widehat{\mathcal{C}}$ *has cobase changes and every* ω*-chain of levelwise decidable monos in* $\widehat{\mathcal{C}}$ *has a colimit in* $\widehat{\mathcal{C}}$*.*

Proof. Colimits in presheaves are computed levelwise, so this follows from Proposition 50. ◀

▶ **Construction 53** (Free algebras)**.** Let $\mathcal{M} \hookrightarrow \mathcal{E}$ be a wide subcategory such that $\mathcal{M}$ is closed under cobase change and unions and every ω-chain in $\mathcal{M}$ has a colimit in $\mathcal{E}$. Let $\mathsf{T} = (T, \tau)$ be a pointed endofunctor on $\mathcal{E}$. If T restricts to a pointed endofunctor on $\mathcal{M}$, τ is cartesian, and T preserves colimits of ω-chains in $\mathcal{M}$, then for $A \in \mathcal{E}$ we have a free T-algebra on A.

Proof. Given $A \in \mathcal{E}$, we first define sequences $(A_n \in \mathcal{E})_{n\in\mathbb{N}}$, $(f_n \colon A_n \overset{\in\mathcal{M}}{\to} A_{n+1})_{n\in\mathbb{N}}$, and $(a_n \colon TA_n \to A_{n+1})_{n\in\mathbb{N}}$ such that $a_n \tau_{A_n} = f_n$ and $f_{n+1} a_n = a_{n+1}(Tf_n)$. Set $A_0 \coloneqq A$, $A_1 \coloneqq TA$, $f_0 \coloneqq \tau_A$, and $a_0 \coloneqq \mathrm{id}_{TA}$. Given A_n, A_{n+1}, f_n, and a_n, take a pushout to define

$$\begin{tikzcd}
A_{n+1} \sqcup_{A_n} TA_n \arrow[r, "{[\tau_{A_{n+1}}, Tf_n]}"] \arrow[d, "{[\mathrm{id}_{A_{n+1}}, a_n]}"'] & TA_{n+1} \arrow[d, dashed, "a_{n+1}"] \\
A_{n+1} \arrow[r, dashed, "f_{n+1}"'] & A_{n+2} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the upper left object is the pushout of $A_{n+1} \xleftarrow{f_n} A_n \xrightarrow{\tau_A} TA_n$. The bottom right pushout exists because $[\tau_{A_{n+1}}, Tf_n]$ is the union of $\tau_{A_{n+1}}$ and Tf_n and thus in $\mathcal{M}$. By closure under cobase change, f_{n+1} also belongs to $\mathcal{M}$ as required.

Set $A_\omega \coloneqq \mathrm{colim}_{n<\omega} A_n$. By the assumption that T preserves colimits of ω-chains in $\mathcal{M}$, there is a unique $a_\omega \colon TA_\omega \to A_\omega$ such that $a_\omega(T\iota_n) = \iota_{n+1} a_n$ for all $n \in \mathbb{N}$. It follows from the universal property that $a_\omega \tau_{A_\omega} = \mathrm{id}_{A_\omega}$, making (A_ω, a_ω) a T-algebra.

We claim that $\iota_0 \colon A \to A_\omega$ exhibits (A_ω, a_ω) as the free T-algebra on A. Suppose that $(B, b \colon TB \to B)$ is a T-algebra and we have some $g \colon A \to B$. We first show by induction that there is a unique sequence $(g_n \colon A_n \to B)_{n\in\mathbb{N}}$ such that $g_0 = g$ and $g_{n+1} a_n = b(Tg_n)$

(which also implies $g_{n+1}f_n = g_n$) for all $n \in \mathbb{N}$. Set $g_0 \coloneqq g$ and $g_1 \coloneqq b(Tg_0)$. Given unique g_i for $i \leq n+1$, the universal property of A_{n+2} induces a unique $g_{n+2} \colon A_{n+2} \to B$ such that $g_{n+2}f_{n+1} = g_{n+1}$ and $g_{n+2}a_{n+1} = b(Tg_{n+1})$. Now there is a unique $g_\omega \colon A_\omega \to B$ such that $g_\omega \iota_n = g_n$ for all $n \in \mathbb{N}$. This g_ω defines a morphism of T-algebras from $(A_\omega, a_\omega) \to (B, b)$, and its uniqueness follows from uniqueness of the g_n. ◀

By definition, the free algebras of Construction 53 are preserved by any functor that preserves the relevant colimits. We will use the following in particular to check that our interpretations of replacement and completion are stable under the substitution.

▶ **Proposition 54.** *For $k \in \{0,1\}$, let $\mathcal{M}_k \hookrightarrow \mathcal{E}_k$ and $\mathsf{T}_k = (T_k, \tau_k)$ on $\mathcal{E}_k$ be given satisfying the hypotheses of Construction 53. If $F \colon \mathcal{E}_0 \to \mathcal{E}_1$ preserves the assumed colimits in $\mathcal{E}_0$ and we have $\theta \colon FT_0 \cong T_1F$ with $\theta \circ F\tau_0 = \tau_1 F$, then F sends free T_0-algebras to free T_1-algebras. If F* strictly *preserves the assumed choices of colimits and θ is an identity, then F strictly sends the constructed free T_0-algebra on $A \in \mathcal{E}_0$ to the constructed free T_1-algebra on FA.* ◀

3.3 Propositional truncation

We motivate our semantics of univalent completion and replacement with a special case: (-1)-truncation. We recall an interpretation of (-1)-truncation that first appeared in Cherubini, Coquand, and Hutzler [20, §8.3] and has also been used by Bocquet [12, §2.5]. Said interpretation can be translated to a type former of cubical type theory, in which case its constructors are given as follows:

$$\frac{a : A}{|a| : \|A\|} \qquad\qquad \frac{t_0 : \|A\| \qquad \phi\ \mathsf{cof} \qquad \phi \vdash t : \|A\|}{\mathsf{ext}\ t_0\ [\phi \mapsto t] : \|A\|\ [\phi \mapsto t]}$$

That is, every element of A gives an element of $\|A\|$, and if $\|A\|$ is inhabited by some $t_0 : \|A\|$ then every partial element $\phi \vdash t : \|A\|$ extends to a total element $\mathsf{ext}\ t_0\ [\phi \mapsto t] : \|A\|$. Cohen, Coquand, Huber, and Mörtberg [23, §5.1] show that a fibrant type B is contractible if and only if every partial element in B extends to a total element. Thus, by the equivalence $\texttt{isProp}\ B \simeq (B \to \texttt{isContr}\ B)$, a type B is a proposition if and only if for every point of B, every partial element in B can be extended to a total element [20, §8.3].

We now translate this type-theoretic description into a semantic definition using Construction 53. After constructing $\|A\|$ as a cubical set, we first prove it is fibrant and then that it supports the universal property (i.e., eliminator) for the (-1)-truncation.

▶ **Construction 55** (Truncation)**.** For every $A \in \mathrm{Ty}(\Gamma)$, we have $\|A\| \in \mathrm{Ty}(\Gamma)$ generated by

$$\frac{a \in A(I,\gamma)}{\texttt{in}\ a \in \|A\|(I,\gamma)} \qquad\qquad \frac{t_0 \in \|A\|(I,\gamma) \qquad \phi \in \Phi(I) \qquad t \in \|A\|(I \restriction \phi, \gamma)}{\texttt{ext}\ t_0\ \phi\ t \in \|A\|(I,\gamma)}$$

$$\texttt{ext}\ t_0\ \phi\ t \restriction \phi = t \qquad\quad (\texttt{in}\ a)\alpha = \texttt{in}\ (a\alpha) \qquad\quad (\texttt{ext}\ t_0\ \phi\ t)\alpha = \texttt{ext}\ (t_0\alpha)\ (\phi\alpha)\ (t\alpha)$$

in the sense that every $B \in \mathrm{Ty}(\Gamma)$ interpreting this signature admits a unique interpretation-preserving $\|A\| \to B$. The construction is stable under substitution into Γ.

Proof. For every $A \in \mathrm{Ty}(\Gamma)$ and $(I,\gamma) \in \int_{\mathcal{C}} \Gamma$, define

$$TA(I,\gamma) \coloneqq A(I,\gamma) \sqcup \{(a_0, \phi, a) \mid a_0 \in A(I,\gamma), \phi \in \Phi(I) \setminus \widetilde{\Phi}(I), a \in A(I \restriction \phi, \gamma)\}.$$

Write $\iota_0(a)$ for elements in the left summand and $\iota_1(a_0,\phi,a)$ for elements in the right summand. The $TA(I,\gamma)$ define a $TA \in \mathrm{Ty}(\Gamma)$ with action by $\alpha\colon J \to I$ given by $\iota_0(a)\alpha := \iota_0(a\alpha)$, $\iota_1(a_0,\phi,a)\alpha := \iota_0(a_\alpha)$ if $\phi\alpha \in \widetilde{\Phi}(J)$, and $\iota_1(a_0,\phi,a)\alpha := \iota_1(a_0\alpha,\phi\alpha,a\alpha)$ otherwise.

The mapping $A \mapsto TA$ extends to a pointed endofunctor $\mathsf{T} = (T,\tau)$ with unit $\tau_A\colon A \rightarrowtail TA$ given by the inclusions ι_0. The unit is levelwise decidable and cartesian. Thanks to our choices of finite presentations, any levelwise decidable $A \rightarrowtail B$ induces decidable monos $TA(I,\gamma) \rightarrowtail TB(I,\gamma)$: we can decide if an element $\iota_1(b_0,\phi,b)$ in TB belongs to TA by testing whether b_0 belongs to A and whether the components of b on the finite presentation for ϕ belong to A. Finally, if $A\colon \omega \to \mathrm{Ty}(\Gamma)$ is an ω-chain of levelwise decidable monos, then the maps $\mathrm{colim}_{n<\omega}\, TA_n(I,\gamma) \to T(\mathrm{colim}_{n<\omega}\, A_n)(I,\gamma)$ are isomorphisms: given $\iota_0(a)$ we have some $n < \omega$ for which a lies in a_n, and given $\iota_1(a_0,\phi,a) \in T(\mathrm{colim}_{n<\omega}\, A_n)(I,\gamma)$ we can use our finite presentation for ϕ to identify some $n < \omega$ for which both a_0 and a lie in A_n.

Apply Construction 53 with T and the class of levelwise decidable monos (using Corollary 52) and take $\|A\| \in \mathrm{Ty}(\Gamma)$ to be the free T-algebra on $A \in \mathrm{Ty}(\Gamma)$. The map $A \to \|A\|$ provides the constructor `in`. We define `ext` $t_0\ \phi\ t$ to be t_{id} when $\phi \in \widetilde{\Phi}(I)$ and use the T-algebra structure when $\phi \in \Phi(I) \setminus \widetilde{\Phi}(I)$. Stability under substitution is Proposition 54. ◀

▶ Remark 56. This is an alternative to Cohen, Coquand, Huber, and Mörtberg's semantics for (-1)-truncation [23, 25], in which the role of `ext` is played by two constructors: `squash`, which introduces a path between each pair of elements, and `hcomp`, which implements homogeneous filling directly. The merging of these roles in `ext` parallels Cohen et al.'s use of Glue *types* to interpret the univalence axiom and fibrancy of the universe, as opposed to Angiuli, Favonia, and Harper's [7] use of separate V *types* [7, §5.6] (for univalence) and hcom *types* [7, §6] (for fibrancy). This parallel will show itself more clearly in our semantics of univalent completion.

▶ **Lemma 57.** *For $A \in \mathrm{Ty}(\Gamma)$ we have a homogeneous filling operation $\mathfrak{h}$ on $\|A\|$, stably under substitution.*

Proof. Define $\mathfrak{h}_{I,\gamma}\ r\ t_0\ \phi\ t :=$ `ext` $(t_0\mathrm{p}_{\mathbb{I}})\ (\phi\mathrm{p}_{\mathbb{I}})\ t$. ◀

▶ Notation 58. For $A \in \mathrm{Ty}(\Gamma \times \mathbf{y}\mathbb{I})$, write $\prod_{\mathbb{I}} A \in \mathrm{Ty}(\Gamma)$ for dependent product over the interval, $(\prod_{\mathbb{I}} A)\,(I,\gamma) := A(I \times \mathbb{I}, \gamma\mathrm{p}_{\mathbb{I}})$. Write $\mathtt{ev}_r\colon (\prod_{\mathbb{I}} A) \to A\underline{r}$ for evaluation at $r\colon \Gamma \to \mathbf{y}\mathbb{I}$.

▶ **Lemma 59.** *For each $A \in \mathrm{Ty}(\Gamma \times \mathbf{y}\mathbb{I})$, coercion operation $\mathfrak{a}$ for A, and $r\colon \Gamma \to \mathbf{y}\mathbb{I}$, we have a choice of section $\mathtt{coe}^{A,r}\colon \|A\|\underline{r} \to \prod_{\mathbb{I}}\|A\|$ of $\mathtt{ev}_r$ in $\mathrm{Ty}(\Gamma)$, stably under substitution. As a corollary, we have a coercion operation $\mathfrak{c}$ on $\|A\|$ defined by $\mathfrak{c}_{I,\gamma}\ r\ t_0 := \mathtt{coe}^{A\gamma,r}_{I,\mathrm{id}_I}(t_0)$.*

Proof. We define $\mathtt{coe}^{A,r}$ by universal property of $\|A\|\underline{r}$, i.e., recursion: set $\mathtt{coe}^{A,r}_{I,\gamma}(\mathtt{in}\ a) :=$ `in` $(\mathfrak{a}_{I,\gamma}\ (r_I(\gamma))\ a)$ and $\mathtt{coe}^{A,r}_{I,\gamma}(\mathtt{ext}\ t_0\ \phi\ t) := \mathtt{ext}\ (\mathtt{coe}^{A,r}_{I,\gamma}(t_0))\ (\phi\mathrm{p}_{\mathbb{I}})\ (\mathtt{coe}^{A,r}_{I\restriction\phi,\gamma}(t))$. ◀

▶ **Theorem 60** (Fibrancy of truncation)**.** *Construction 55 extends to a substitution-stable $\|-\|\colon \mathrm{Ty}^{\mathrm{f}}(\Gamma) \to \mathrm{Ty}^{\mathrm{f}}(\Gamma)$.* ◀

For $\|A\|$ to be the (-1)-truncation of A, it suffices that $\|A\|$ is a proposition and that for any fibrant proposition P with $A \to P$, we have $\|A\| \to P$ [53, Remark 14.1.3].

▶ **Lemma 61** (Cherubini, Coquand, and Hutzler [20, §8.3])**.** *For every $A \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$, each $p \in \mathrm{Ter}(\Gamma; \mathtt{isProp}\ A)$ induces an operation $\mathtt{ext}_p$ sending $a_0 \in A(I,\gamma)$, $\phi \in \Phi(I)$, and $a \in A(I \restriction \phi, \gamma)$ to an element $\mathtt{ext}_p\ a_0\ \phi\ a \in A(I,\gamma)$ satisfying $(\mathtt{ext}_p\ a_0\ \phi\ a) \restriction \phi = a$. Conversely, any such operation induces an inhabitant of $\mathrm{Ter}(\Gamma; \mathtt{isProp}\ A)$.* ◀

The `ext` constructor thus makes $\|A\|$ a proposition. For mapping out, we have:

▶ **Theorem 62** (Universal property of $\|A\|$)**.** *For every $A \in \mathrm{Ty}(\Gamma)$ and proposition $P \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ with a map $A \to P$, we have a map $\|A\| \to P$, stably under substitution.*

Proof. Let $P \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$, $p \in \mathtt{isProp}\ A$, and $f\colon A \to P$. We define $f'\colon \|A\| \to P$ by recursion, setting $f'_{I,\gamma}(\mathtt{in}\ a) \coloneqq f_{I,\gamma}(a)$ and $f'_{I,\gamma}(\mathtt{ext}\ t_0\ \phi\ t) \coloneqq \mathtt{ext}_p\ (f'_{I,\gamma}(t))\ \phi\ (f'_{I\restriction\phi,\gamma}(t))$. ◀

3.4 Replacement

Following the pattern for (-1)-truncation, we will interpret replacement of a map $f : A \to B$ as the closure of A under an "extension" operation. In a cubical type theory, given B type, $\beta : \mathsf{LocSmall}_{\mathcal{U}} B$, $A : \mathcal{U}$, and $f : A \to B$, the type $\mathsf{Rep}_{B,\beta}(A, f) : \mathcal{U}$ would be specified beginning as follows:

$$\frac{t : \mathsf{Rep}_{B,\beta}(A,f)}{\mathsf{out}\ t : B} \qquad \frac{a : A}{\mathsf{in}\ a : \mathsf{Rep}_{B,\beta}(A,f)} \qquad \mathsf{out}\ (\mathsf{in}\ a) \equiv f\ a : B$$

$$\frac{t_0 : \mathsf{Rep}_{B,\beta}(A,f) \qquad \phi\ \mathsf{cof} \qquad \phi \vdash w : \sum_{t:\mathsf{Rep}_{B,\beta}(A,f)} (\beta\ (\mathsf{out}\ t)\ (\mathsf{out}\ t_0)).1}{\mathsf{ext}\ t_0\ \ [\phi \mapsto w] : \mathsf{Rep}_{B,\beta}(A,f)\ [\phi \mapsto w.1]}$$

$$\mathsf{out}\ (\mathsf{ext}\ t_0\ \ [\phi \mapsto w]) \equiv \mathsf{hcom}^{0\to 1}_B\ \ [\phi \mapsto \cdots]\ \ (\cdots) : B$$

Here the type $(\beta\ (\mathsf{out}\ t)\ (\mathsf{out}\ t_0)).1$ is a $\mathcal{U}$-small equivalent to $\mathsf{out}\ t =_B \mathsf{out}\ t_0$. The reduction of $\mathsf{out}\ (\mathsf{ext}\ t_0\ \ [\phi \mapsto w])$ and hcom uses homogeneous filling in B [6, §2.7], as we see below. Now ext can be roughly understood as forcing the fiber types $\sum_{t:\mathsf{Rep}_{B,\beta}(A,f)} \mathsf{out}\ t =_B \mathsf{out}\ t_0$ to be propositions, which has the effect of making out an embedding.

To state replacement, we first need a universe; to interpret type theory with a universe, we assume a universe in our metatheory.

▶ **Assumption 63** (for § 3.4)**.** We assume a fixed $\mathcal{V} \in \mathbf{Set}$ such that the class of $\mathcal{V}$-small sets (that is, sets isomorphic to an element of $\mathcal{V}$) is closed under finite limits and coproducts indexed by $\mathcal{V}$-small sets and contains all finite sets, $\mathbb{N}$, and $\Phi(I) \setminus \widetilde{\Phi}(I)$ for all $I \in \mathcal{C}$.

▶ **Definition 64.** *For $\Gamma \in \widehat{\mathcal{C}}$, write $\mathrm{Ty}_{\mathcal{V}}(\Gamma)$ for the category of presheaves on $\int_{\mathcal{C}} \Gamma$ valued in elements of $\mathcal{V}$, and $\mathrm{Ty}^{\mathrm{f}}_{\mathcal{V}}(\Gamma)$ for the category of such presheaves equipped with a filling operation. Write $\mathtt{U}_{\mathcal{V}} \in \mathrm{Ty}(\Gamma)$ for the* universe of fibrant types $\mathtt{U}_{\mathcal{V}}(I, \_) \coloneqq \mathrm{Ty}^{\mathrm{f}}_{\mathcal{V}}(\mathbf{y}I)$. *Given $A \in \mathrm{Ter}(\Gamma; \mathtt{U}_{\mathcal{V}})$, write $\mathtt{El}_{\mathcal{V}}\ A \in \mathrm{Ty}(\Gamma)$ for its decoding. Abbreviate $\mathtt{LocSmall}_{\mathcal{V}} \coloneqq \mathtt{LocSmall}_{\mathtt{U}_{\mathcal{V}}}$.*

Assumption 63 is enough to interpret replacement in $\mathtt{U}_{\mathcal{V}}$, but closing $\mathtt{U}_{\mathcal{V}}$ under other type formers may require more. Additional assumptions on Φ are also needed to make $\mathtt{U}_{\mathcal{V}}$ fibrant and univalent; see Cohen et al. [23, §8.2] or Angiuli et al. [6, §§2.12 & 3.5]. We do not need univalence of $\mathtt{U}_{\mathcal{V}}$ here, but, as we have seen, most *uses* of replacement depend on it.

▶ **Definition 65.** *Given $(A, f) \in \mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$ and $\beta \in \mathrm{Ter}(\Gamma; \mathtt{LocSmall}_{\mathcal{V}}\ B)$, define the small family of fibers $(\mathtt{Fib}_{\mathcal{V}}\ \beta\ f)(b) \coloneqq \sum_{a:A} \mathtt{El}_{\mathcal{V}}\ ((\beta\ (f\ a)\ b).1) \in \mathrm{Ty}_{\mathcal{V}}(\Gamma.B)$. Given $a \in A(I, \gamma)$ and $p \in B(I \times \mathbb{I}, \gamma\mathrm{p}_{\mathbb{I}})$ with $p\underline{0} = f_{I,\gamma}(a)$, write $(\!|a, p|\!) \in (\mathtt{Fib}_{\mathcal{V}}\ \beta\ f)(I, (\gamma, p\underline{1}))$ for the corresponding element of the small fiber. Given $w \in (\mathtt{Fib}_{\mathcal{V}}\ \beta\ f)(I, (\gamma, b))$, write $\mathtt{ptOf}\ w \in A(I, \gamma)$ and $\mathtt{pathOf}\ w \in B(I \times \mathbb{I}, \gamma\mathrm{p}_{\mathbb{I}})$ for the corresponding point and path.*

▶ **Construction 66** (Replacement). Let $(B, \mathfrak{b}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ and $\beta \in \mathrm{Ter}(\Gamma; \mathtt{LocSmall}_{\mathcal{V}}\ B)$. For every pair $(A, f) \in \mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$, we have $(\mathtt{Rep}_{B,\beta}(A, f), \mathtt{out}) \in \mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$ generated by

$$\frac{a \in A(I,\gamma)}{\mathtt{in}\ a \in (\mathtt{Rep}_{B,\beta}(A,f))(I,\gamma)} \qquad \frac{\begin{array}{c} t_0 \in (\mathtt{Rep}_{B,\beta}(A,f))(I,\gamma) \\ \phi \in \Phi(I) \qquad w \in (\mathtt{Fib}_{\mathcal{V}}\ \beta\ \mathtt{out})(I \upharpoonright \phi, (\gamma, \mathtt{out}_{I,\gamma}(t_0))) \end{array}}{\mathtt{ext}\ t_0\ \phi\ w \in (\mathtt{Rep}_{B,\beta}(A,f))(I,\gamma)}$$

$$(\mathtt{in}\ a)\alpha = \mathtt{in}\ (a\alpha) \qquad (\mathtt{ext}\ t_0\ \phi\ w)\alpha = \mathtt{ext}\ (t_0\alpha)\ (\phi\alpha)\ (w\alpha) \qquad (\mathtt{ext}\ t_0\ \phi\ w) \upharpoonright \phi = w.1$$

$$\mathtt{out}_{I,\gamma}(\mathtt{in}\ a) = f_{I,\gamma}(a) \qquad \mathtt{out}_{I,\gamma}(\mathtt{ext}\ t_0\ \phi\ w) = (\mathfrak{b}^{\mathrm{h}}_{I,\gamma}\ 1\ (\mathtt{out}_{I,\gamma}(t_0))\ \phi\ (\mathtt{pathOf}\ w))\underline{0}$$

in the sense that each $F \in \mathrm{Ty}(\Gamma)/B$ interpreting this signature admits a unique interpretation-preserving map $(\mathtt{Rep}_{B,\beta}(A, f), \mathtt{out}) \to F$. The construction is stable under substitution.

Proof. For $(A, f) \in \mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$, take $\partial(A, f)(I, \gamma) \in \mathcal{V}$ for $(I, \gamma) \in \int_{\mathcal{C}} \Gamma$ to be the set of triples (a_0, ϕ, w) for $a_0 \in A(I, \gamma)$, $\phi \in \Phi(I) \setminus \widetilde{\Phi}(I)$, and $w \in (\mathtt{Fib}_{\mathcal{V}}\ \beta\ f)(I \upharpoonright \phi, (\gamma, f_{I,\gamma}(a_0)))$. To get $\partial(A, f)(I, \gamma)$ in $\mathcal{V}$, we use our choices of finite presentations and the closure of $\mathcal{V}$-small sets under $(\Phi(I) \setminus \widetilde{\Phi}(I))$-indexed coproducts and finite limits. We define $T_0(A, f) \in \mathrm{Ty}_{\mathcal{V}}(\Gamma)$ by setting $T_0(A, f)(I, \gamma) \coloneqq A(I, \gamma) \sqcup \partial(A, f)(I, \gamma) \in \mathcal{V}$ and defining the action by $\alpha\colon J \to I$ by $\iota_0(a)\alpha \coloneqq \iota_0(a\alpha)$, $\iota_1(a_0, \phi, w)\alpha \coloneqq \iota_0(w_\alpha.1)$ if $\phi\alpha \in \widetilde{\Phi}(J)$, and $\iota_1(a_0, \phi, w)\alpha \coloneqq \iota_1(a_0\alpha, \phi\alpha, w\alpha)$ if $\phi\alpha \notin \widetilde{\Phi}(J)$. We then define $T_1(A, f)\colon T_0(A, f) \to B$ by cases: $T_1(A, f)_{I,\gamma}(\iota_0(a)) \coloneqq f_{I,\gamma}(a)$ and $T_1(A, f)_{I,\gamma}(\iota_1(t_0, \phi, w)) \coloneqq (\mathfrak{b}^{\mathrm{h}}_{I,\gamma}\ 1\ (\mathtt{out}_{I,\gamma}(t_0))\ \phi\ (\mathtt{pathOf}\ w))\underline{0}$.

We get a pointed endofunctor $\mathsf{T} = (T, \tau)$ on $\mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$ with $T(A, f) \coloneqq (T_0(A, f), T_1(A, f))$ and point τ given by the inclusions ι_0. Write $\mathcal{M} \hookrightarrow \mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$ for the wide subcategory of $m\colon (A, f) \to (A', f')$ such that $m\colon A \to A'$ is a levelwise decidable mono in $\mathrm{Ty}_{\mathcal{V}}(\Gamma)$ with complements $A'(I, \gamma) \setminus A(I, \gamma)$ in $\mathcal{V}$. As colimits in $\mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$ are computed in $\mathrm{Ty}_{\mathcal{V}}(\Gamma)$, it follows from the proof of Corollary 52 and our assumptions on $\mathcal{V}$ that $\mathcal{M}$ is closed under cobase change and every ω-chain in $\mathcal{M}$ has a colimit in $\mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$. The unit τ is cartesian and valued in $\mathcal{M}$. As in Construction 55, T preserves $\mathcal{M}$ and colimits of ω-chains in $\mathcal{M}$.

We can thus apply Construction 53 with $\mathcal{M}$ and T. Take $(\mathtt{Rep}_{B,\beta}(A, f), \mathtt{out}) \in \mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$ to be the free T-algebra on (A, f), interpreting the signature as in Construction 55. As the inclusion of $\mathrm{Ty}_{\mathcal{V}}(\Gamma)/B$ into $\mathrm{Ty}(\Gamma)/B$ preserves the relevant colimits, it follows from Proposition 54 that $(\mathtt{Rep}_{B,\beta}(A, f), \mathtt{out})$ is also the free object supporting the required signature in $\mathrm{Ty}(\Gamma)/B$. Stability under substitution also follows from Proposition 54. ◀

Construction 66 brings to mind Dybjer and Setzer's description of inductive recursive definitions as initial algebras in slice categories [29]; we return to this connection in § 4.

▶ Remark 67. Like the $\mathtt{ext}$ constructor of the (-1)-truncation, the $\mathtt{ext}$ constructor for the replacement has two purposes: to provide a homogeneous filling operation and to make $\mathtt{out}$ into an embedding. Like the (-1)-truncation (see Remark 56), there is an alternative construction that divides $\mathtt{ext}$ into two constructors: one which reflects paths in B between elements in the image of $\mathtt{out}$ to paths in $\mathtt{Rep}_{B,\beta}(A, f)$, and one which closes $\mathtt{Rep}_{B,\beta}(A, f)$ under homogeneous filling. We leave the formulation of this alternative as an exercise.

To define homogeneous filling with $\mathtt{ext}$, we use the following path-scaling operation.

▶ **Lemma 68.** *Let $(A, \mathfrak{a}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$. Given $\gamma \in \Gamma(I)$, $r \in \mathbf{y}\mathbb{I}(I)$, and $p \in A(I \times \mathbb{I}, \gamma\mathrm{p}_{\mathbb{I}})$, we have $\mathtt{scale}\ \mathfrak{a}\ r\ p \in A(I \times \mathbb{I} \times \mathbb{I}, \gamma\mathrm{p}_{\mathbb{I}}\mathrm{p}_{\mathbb{I}})$ satisfying $(\mathtt{scale}\ \mathfrak{a}\ r\ p)\underline{0} = p$, $(\mathtt{scale}\ \mathfrak{a}\ r\ p)\underline{1} = pr\underline{r}\mathrm{p}_{\mathbb{I}}$, and $(\mathtt{scale}\ \mathfrak{a}\ r\ p)(\underline{r} \times \mathbb{I}) = pr\underline{r}\mathrm{p}_{\mathbb{I}}$, stably under substitution.*

Proof. Define $\texttt{scale}\ \mathfrak{a}\ r\ p$ to be the reindexing by $I \times \langle(.2),(.1)\rangle \colon I \times \mathbb{I} \times \mathbb{I} \to I \times \mathbb{I} \times \mathbb{I}$ of $\mathfrak{a}^{\mathrm{h}}_{I\times\mathbb{I},\gamma\mathsf{p}_\mathbb{I}}\ (r\mathsf{p}_\mathbb{I})\ (p\underline{r}\mathsf{p}_\mathbb{I})\ (\mathsf{q}_\mathbb{I} \equiv 0 \vee \mathsf{q}_\mathbb{I} \equiv 1)\ [(\mathsf{q}_\mathbb{I} \equiv 0)\mathsf{p}_\mathbb{I} \mapsto p \mid (\mathsf{q}_\mathbb{I} \equiv 1)\mathsf{p}_\mathbb{I} \mapsto p\underline{r}\mathsf{p}_\mathbb{I}]$. ◀

▶ **Lemma 69.** *Let $(B,\mathfrak{b}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ and $\beta \in \mathrm{Ter}(\Gamma; \texttt{LocSmall}_\mathcal{V}\ B)$. For $(A,f) \in \mathrm{Ty}_\mathcal{V}(\Gamma)/B$ we have a homogeneous filling operation $\mathfrak{h}$ on $\texttt{Rep}_{B,\beta}(A,f)$, stably under substitution.*

Proof. In order to define $\mathfrak{h}_{I,\gamma}\ r\ t_0\ \phi\ t$, we first combine t_0 and t into a single element $t_+ := [\mathsf{q}_\mathbb{I} \equiv r\mathsf{p}_\mathbb{I} \mapsto t_0\mathsf{p}_\mathbb{I} \mid \phi\mathsf{p}_\mathbb{I} \mapsto t] \in \texttt{Rep}_{B,\beta}(A,f)(I \times \mathbb{I} \upharpoonright (\mathsf{q}_\mathbb{I} \equiv r\mathsf{p}_\mathbb{I} \vee \phi\mathsf{p}_\mathbb{I}), \gamma\mathsf{p}_\mathbb{I})$. We then take $\mathfrak{h}_{I,\gamma}\ r\ t_0\ \phi\ t := \texttt{ext}\ (t_0\mathsf{p}_\mathbb{I})\ (\mathsf{q}_\mathbb{I} \equiv r\mathsf{p}_\mathbb{I} \vee \phi\mathsf{p}_\mathbb{I})\ (\!|t_+, \texttt{scale}\ \mathfrak{b}\ r\ (\texttt{out}\ t_+)|\!)$. ◀

▶ Notation 70. We use the following notation for coercion inside of $\mathtt{U}_\mathcal{V}$: for $(A,\mathfrak{a}) \in \mathtt{U}_\mathcal{V}(I\times\mathbb{I},\gamma)$, $r \in \mathbf{y}\mathbb{I}(I)$, and $a_0 \in A(I, \gamma\underline{r})$, write $\texttt{coe}^r_{\in\mathcal{V}}\ (A,\mathfrak{a})\ a_0 := \mathfrak{a}^{\mathrm{c}}_{I,\mathrm{id}_{I\times\mathbb{I}}}\ r\ a_0 \in A(I \times \mathbb{I}, \gamma)$.

▶ **Lemma 71.** *Let $(B,\mathfrak{b}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma \times \mathbf{y}\mathbb{I})$ and $\beta \in \mathrm{Ter}(\Gamma \times \mathbf{y}\mathbb{I}; \texttt{LocSmall}_\mathcal{V}\ B)$. For every $(A,f) \in \mathrm{Ty}_\mathcal{V}(\Gamma \times \mathbf{y}\mathbb{I})/B$, coercion operation $\mathfrak{a}$ for A, and $r \colon \Gamma \to \mathbf{y}\mathbb{I}$, we have a choice of section $\texttt{coe}^r \colon (\texttt{Rep}_{B,\beta}(A,f))\underline{r} \to \prod_\mathbb{I} \texttt{Rep}_{B,\beta}(A,f)$ of $\texttt{ev}_r$ in $\mathrm{Ty}(\Gamma)$, stably under substitution. As a corollary, we have a coercion operation $\mathfrak{c}$ on $\texttt{Rep}_{B,\beta}(A,f)$.*

Proof. We define $\texttt{coe}^r$ as a map $((\texttt{Rep}_{B,\beta}(A,f))\underline{r}, \texttt{out}) \to (\prod_\mathbb{I} \texttt{Rep}_{B,\beta}(A,f), \texttt{out} \circ \texttt{ev}_r)$ in $\mathrm{Ty}(\Gamma)/B\underline{r}$ by universal property, i.e., recursion: set $\texttt{coe}^r_{I,\gamma}(\texttt{in}\ a) := \texttt{in}\ (\mathfrak{a}^{\mathrm{c}}_{I,\gamma}\ (r_I(\gamma))\ a)$ and

$$\texttt{coe}^r_{I,\gamma}(\texttt{ext}\ t_0\ \phi\ (t,u)) \quad := \quad \texttt{ext}\ (\texttt{coe}^r_{I,\gamma}(t_0))\ (\phi\mathsf{p}_\mathbb{I})\ (\texttt{coe}^r_{I\upharpoonright\phi,\gamma}(t), \texttt{coe}^{r_I(\gamma)}_{\in\mathcal{V}}\ (\beta'.1)\ u)$$

where $\beta' := \beta(I \times \mathbb{I} \upharpoonright \phi\mathsf{p}_\mathbb{I}, (\gamma\mathsf{p}_\mathbb{I}, \mathsf{q}_\mathbb{I}))\ (\texttt{out}\ \texttt{coe}^r_{I\upharpoonright\phi,\gamma}(t))\ (\texttt{out}\ \texttt{coe}^r_{I,\gamma}(t_0)))$. By uniqueness, this definition satisfies $\texttt{ev}_r \circ \texttt{coe}^r = \mathrm{id}$ and is stable under substitution. ◀

▶ **Theorem 72** (Fibrancy of replacement). *For $B \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ and $\beta \in \mathrm{Ter}(\Gamma; \texttt{LocSmall}_\mathcal{V}\ B)$, replacement extends to a substitution-stable $\texttt{Rep}_{B,\beta} \colon \mathrm{Ty}^{\mathrm{f}}_\mathcal{V}(\Gamma)/B \to \mathrm{Ty}^{\mathrm{f}}_\mathcal{V}(\Gamma)/B$.* ◀

Finally, we check that $\texttt{Rep}_{B,\beta}(A,f)$ defines an image factorization of f.

▶ **Theorem 73** ($\texttt{out}$ is an embedding). *For $(B,\mathfrak{b}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$, $\beta \in \mathrm{Ter}(\Gamma; \texttt{LocSmall}_\mathcal{V}\ B)$, and $(A,f) \in \mathrm{Ty}^{\mathrm{f}}_\mathcal{V}(\Gamma)/B$, $\texttt{out} \colon \texttt{Rep}_{B,\beta}(A,f) \to B$ is an embedding.*

Proof. A map $g : C \to D$ is an embedding if and only if it is (-1)-truncated, i.e., its fibers are propositions [64, Lemma 7.6.2]. It is equivalent to show that given $c : C$, the fiber of g at $g\ c$ is contractible. We show in the semantics that in every such fiber of $\texttt{out}$ (in or rather its equivalent small fibers), every partial element extends to a total element [23, §5.1].

Given $t_0 \in (\texttt{Rep}_{B,\beta}(A,f))(I,\gamma)$ and $w \in (\texttt{Fib}_\mathcal{V}\ \beta\ \texttt{out})(I \upharpoonright \phi, (\gamma, \texttt{out}_{I,\gamma}(t_0)))$, the element $(\!|\texttt{ext}\ t_0\ \phi\ w, \mathfrak{b}^{\mathrm{h}}_{I,\gamma}\ 1\ (\texttt{out}_{I,\gamma}(t_0))\ \phi\ (\texttt{pathOf}\ w)|\!)$ extends $(\!|\texttt{ptOf}\ w, \texttt{pathOf}\ w|\!)$ to a total element. As the type of small fibers is fibrant by Theorem 72, we can correct this extension along the path from $(\!|\texttt{ptOf}\ w, \texttt{pathOf}\ w|\!)$ to w to obtain an extension of w. ◀

▶ **Theorem 74** ($\texttt{in}$ is surjective). *For $(B,\mathfrak{b}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$, $\beta \in \mathrm{Ter}(\Gamma; \texttt{LocSmall}_\mathcal{V}\ B)$, and $(A,f) \in \mathrm{Ty}^{\mathrm{f}}_\mathcal{V}(\Gamma)/B$, $\texttt{in} \colon \texttt{Rep}_{B,\beta}(A,f) \to B$ is a surjection.*

Proof. By Theorem 73, it suffices to show that each fiber of f over an element in the image of $\texttt{out}$ is merely inhabited. We construct $s \colon (\texttt{Rep}_{B,\beta}(A,f), \texttt{out}) \to (\Sigma\ B\ \|\texttt{Fiber}_f\|, (.1))$ in $\mathrm{Ty}^{\mathrm{f}}(\Gamma)/B$ by recursion. Define $s_{I,\gamma}(\texttt{in}\ a) := (f_{I,\gamma}(a), \texttt{in}\ (a, \texttt{refl}_{f_{I,\gamma}(a)}))$. On $\texttt{ext}\ t_0\ \phi\ w$, we take the first component to be $s_{I,\gamma}(\texttt{ext}\ t_0\ \phi\ w).1 := (\mathfrak{b}^{\mathrm{h}}_{I,\gamma}\ 1\ (\texttt{out}\ t_0)\ \phi\ (\texttt{pathOf}\ w))\underline{0}$, as required by the definition of $\texttt{out}$ on $\texttt{ext}$. For the second, note that we have a path from $s_{I,\gamma}(\texttt{ext}\ t_0\ \phi\ w).1$ to $\texttt{out}\ t_0$, namely $\mathfrak{b}^{\mathrm{h}}_{I,\gamma}\ 1\ (\texttt{out}\ t_0)\ \phi\ (\texttt{pathOf}\ w)$. Coercing $s_{I,\gamma}(t_0).2$ backward along this path gives us a mere inhabitant of the fiber at $s_{I,\gamma}(\texttt{ext}\ t_0\ \phi\ w).1$. By Lemma 61, we can correct it to get a new element with the required boundary on ϕ. ◀

Combining Construction 66, and Theorems 72–74, we have the fibrant image factorization specified by replacement. The desired $\mathcal{V}$-smallness of the image type is achieved in Construction 66, using the (local) $\mathcal{V}$-smallness of the inputs.

3.5 Univalent completion

Finally, the interpretation of univalent completion closely follows the pattern of replacement. In a cubical type theory, given A type and $a : A \vdash P(a)$ type, their univalent completion $\mathsf{U}_{A,P}$ type and $t : \mathsf{U}_{A,P} \vdash \mathsf{El}_{A,P}(t)$ type would be populated as follows:

$$\frac{a : A}{\mathsf{in}\ a : \mathsf{U}_{A,P}} \qquad \frac{t_0 : \mathsf{U}_{A,P} \qquad \phi\ \mathsf{cof} \qquad \phi \vdash w : \sum_{t:\mathsf{U}_{A,P}} \mathsf{El}_{A,P}(t) \simeq \mathsf{El}_{A,P}(t_0)}{\mathsf{ext}\ t_0\ [\phi \mapsto w] : \mathsf{U}_{A,P}\ [\phi \mapsto w.1]}$$

$$\mathsf{El}_{A,P}(\mathsf{in}\ a) \equiv P(a)\ \mathsf{type} \qquad \mathsf{El}_{A,P}(\mathsf{ext}\ t_0\ [\phi \mapsto w]) \equiv \mathsf{Glue}\ [\phi \mapsto \cdots]\ (\cdots)\ \mathsf{type}$$

The reduction for $\mathsf{El}_{A,P}(\mathsf{ext}\ t_0\ [\phi \mapsto w])$ uses so-called Glue types [23, §6] [6, §2.11]. To interpret univalent completion, we need the collection of all types to itself be "fibrant and univalent". While we cannot speak of identities between large types in type theory, we can speak of *paths* between cubical types: a path between types is an element of $\mathrm{Ty}^{\mathrm{f}}(\Gamma \times \mathbf{y}\mathbb{I})$. The fibrancy and univalence of $\mathrm{Ty}^{\mathrm{f}}(\Gamma)$ with respect to this notion of path is captured by its closure under the `Glue` type former introduced by Cohen, Coquand, Huber, and Mörtberg [23]. There are various interpretations of `Glue` types in the literature using different conditions on $\mathbb{I}$ and Φ, so for generality's sake we will simply assume they are given to us. We refer to Angiuli et al. [6, Theorem 2] for sufficient conditions to fulfill this hypothesis.

▶ **Assumption 75** (for § 3.5). Given $A_0 \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$, $A \in \mathrm{Ty}^{\mathrm{f}}(\Gamma.\phi)$, and $e \in \mathrm{Ter}(\Gamma.\phi; A \simeq A_0\mathrm{p}_\phi)$, we have $\mathtt{Glue}\ A_0\ \phi\ (A,e) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ and $\mathtt{unglue} \in \mathrm{Ter}(\Gamma; \mathtt{Glue}\ A_0\ \phi\ (A,e) \simeq A_0)$ such that $(\mathtt{Glue}\ A_0\ \phi\ (A,e))\mathrm{p}_\phi = A$ and $(\mathtt{unglue})\mathrm{p}_\phi = e$, stably under substitution.

▶ **Definition 76.** *For* $\Gamma \in \widehat{\mathcal{C}}$*, write* $\mathrm{Fam}^{\mathrm{f}}(\Gamma)$ *for the category whose objects are pairs* (A,P) *with* $A \in \mathrm{Ty}(\Gamma)$ *and* $P \in \mathrm{Ty}^{\mathrm{f}}(\Gamma.A)$ *and whose morphisms* $f\colon (A,P) \to (A',P')$ *are morphisms* $f\colon A \to A'$ *in* $\mathrm{Ty}(\Gamma)$ *such that* $P = P'(\mathrm{id}_\Gamma.f)$*.*

▶ **Definition 77.** *For* $(A,P) \in \mathrm{Fam}^{\mathrm{f}}(\Gamma)$*,* $\mathtt{Equiv}_P \in \mathrm{Ty}(\Gamma.A.A\mathrm{p}_A)$ *is the family informally written* $\mathtt{Equiv}_P(a,a') :\equiv (P(a') \simeq P(a))$*. Set* $\mathtt{Sing}_P :\equiv \Sigma\ (A\mathrm{p}_A)\ \mathtt{Equiv}_P \in \mathrm{Ty}(\Gamma.A)$*.*

▶ Notation 78. Given $A \in \mathrm{Ty}(\Gamma)$, write $A^+ \in \mathrm{Ty}(\Gamma)$ for the type of partial elements $A^+(I,\gamma) :\equiv \{(\phi,a) \mid \phi \in \Phi(I), a \in A(I \restriction \phi, \gamma)\}$.

▶ Notation 79. Given $(A,P) \in \mathrm{Fam}^{\mathrm{f}}(\Gamma)$, we write $\mathtt{Glue}_P \in \mathrm{Ty}(\Gamma.A.\mathtt{Sing}_P^+)$ for the family given informally by $(\mathtt{Glue}_P)(a,(\phi,w)) :\equiv \mathtt{Glue}\ P(a)\ \phi\ (P(w.1), w.2)$. We then write $\mathtt{unglue}_P \in \mathrm{Ter}(\Gamma.A.\mathtt{Sing}_P^+; \mathtt{Glue}_P \simeq P\mathrm{p}_{\mathtt{Sing}_P^+})$ for the associated instance of `unglue`.

▶ **Construction 80** (Univalent completion). Let $\Gamma \in \widehat{\mathcal{C}}$. For $(A,P) \in \mathrm{Fam}^{\mathrm{f}}(\Gamma)$, we have $(\mathtt{U}_P, \mathtt{El}_P) \in \mathrm{Fam}^{\mathrm{f}}(\Gamma)$ generated by the rules and equations

$$\frac{\gamma \in \Gamma(I) \qquad a \in A(I,\gamma)}{\mathtt{in}\ a \in \mathtt{U}_P(I,\gamma)} \qquad \frac{\gamma \in \Gamma(I) \qquad t_0 \in \mathtt{U}_P(I,\gamma) \qquad z \in \mathtt{Sing}^+_{\mathtt{El}_P}(I,(\gamma,t_0))}{\mathtt{ext}\ t_0\ z \in \mathtt{U}_P(I,\gamma)}$$

$$(\mathtt{ext}\ t_0\ (\phi,w)) \restriction \phi = w.1 \qquad (\mathtt{in}\ a)\alpha = \mathtt{in}\ (a\alpha) \qquad (\mathtt{ext}\ t_0\ z)\alpha = \mathtt{ext}\ (t_0\alpha)\ (z\alpha)$$

$$\begin{aligned} \mathtt{El}_P(I,(\gamma,\mathtt{in}\ a)) &= P(I,(\gamma,a)) \\ \mathtt{El}_P(I,(\gamma,\mathtt{ext}\ t_0\ z)) &= (\mathtt{Glue}_{\mathtt{El}_P})(I,(\gamma,t_0,z)) \end{aligned}$$

in the sense that any $F \in \mathrm{Fam}^{\mathrm{f}}(\Gamma)$ interpreting the signature admits a unique interpretation-preserving $(\mathtt{U}_P, \mathtt{El}_P) \to F$. The construction is stable under substitution.

Proof. Given $(A,P) \in \mathrm{Fam}^{\mathrm{f}}(\Gamma)$, take $\partial(A,P)(I,\gamma)$ for $(I,\gamma) \in \int_{\mathcal{C}} \Gamma$ to be the set of pairs (a_0, z) where $a_0 \in A(I,\gamma)$ and $z = (\phi, a) \in \mathtt{Sing}^{+}_{P}(I,(\gamma,a_0))$ with $\phi \notin \widetilde{\Phi}(I)$. Set $T_0(A,P)(I,\gamma) \coloneqq A(I,\gamma) \sqcup \partial(A,P)(I,\gamma)$. The sets $T_0(A,P)(I,\gamma)$ define a presheaf $T_0(A,P) \in \mathrm{Ty}(\Gamma)$ with action by $\alpha\colon J \to I$ given by $\iota_0(a)\alpha \coloneqq \iota_0(a\alpha)$, $\iota_1(a,(\phi,w))\alpha \coloneqq \iota_0(w_\alpha.1)$ if $\phi\alpha \in \widetilde{\Phi}(J)$, and $\iota_1(a,(\phi,w))\alpha \coloneqq \iota_1(a_0\alpha,(\phi\alpha,w\alpha))$ if $\phi\alpha \notin \widetilde{\Phi}(J)$.

Define $T_1(A,P) \in \mathrm{Ty}(\Gamma.T_0(A,P))$ by cases, setting $T_1(A,P)(I,(\gamma,\iota_0(a))) \coloneqq P(I,(\gamma,a))$ and $T_1(A,P)(I,(\gamma,\iota_1(a_0,z))) \coloneqq \mathtt{Glue}_P(I,(\gamma,a_0,z))$. The type $T_1(A,P)$ extends to a fibrant type: given filling inputs, we compute a filler either in P or in $\mathtt{Glue}_P$ depending on which case is applicable, and the boundary equation for $\mathtt{Glue}$ types (in Assumption 75) ensures that the result is coherent with respect to reindexing.

Set $T(A,P) \coloneqq (T_0(A,P), T_1(A,P)) \in \mathrm{Fam}^{\mathrm{f}}(\Gamma)$; then T with the inclusions ι_0 defines a pointed endofunctor $\mathsf{T} = (T,\tau)$ on $\mathrm{Fam}^{\mathrm{f}}(\Gamma)$. Write $\mathcal{M} \hookrightarrow \mathrm{Fam}^{\mathrm{f}}(\Gamma)$ for the wide subcategory of $m\colon (A,P) \to (A',P')$ for which $m\colon A \to A'$ is a levelwise decidable mono in $\mathrm{Ty}(\Gamma)$. To compute the colimit of an ω-chain $(A_n,P_n)\colon \omega \to \mathrm{Fam}^{\mathrm{f}}(\Gamma)$ in $\mathcal{M}$, we take the first component A_ω to be the colimit of the A_n's, computed using Corollary 52, then define the values and filling operation of $P_\omega \in \mathrm{Ty}^{\mathrm{f}}(\mathrm{id}_\Gamma.A_\omega)$ by case analysis so that substituting along $\iota_n\colon A_n \to A_\omega$ yields $P_\omega(\mathrm{id}_\Gamma.\iota_n) = P_n$. We compute the cobase change of a map in $\mathcal{M}$ similarly.

As in Constructions 55 and 66, T and $\mathcal{M}$ are suitable inputs to our free algebra construction, Construction 53. We take $(\mathtt{U}_P, \mathtt{El}_P)$ to be the free T-algebra on (A,P) and get stability under substitution from Proposition 54. ◀

The proof that $\mathtt{U}_P$ has homogeneous filling is similar to those we have seen in previous sections, but is also essentially an application of the known fact that a universe closed under $\mathtt{Glue}$ types has homogeneous filling [23, §7.1] [6, §2.12]: by definition of $\mathtt{El}_P$, the $\mathtt{ext}$ constructor closes $\mathtt{U}_P$ under $\mathtt{Glue}$ types. The proof that $\mathtt{El}_P$ is a univalent family is likewise an instance of the derivation of univalence from $\mathtt{Glue}$ types [23, §7.2] [6, §2.12].

▶ **Lemma 81.** *Let $(P,\mathfrak{p}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma.A)$. For $\gamma \in \Gamma(I)$, $r \in \mathbf{y}\mathbb{I}(I)$, and $a \in A(I \times \mathbb{I}, \gamma\mathrm{p}_{\mathbb{I}})$, we have $\simeq\!\mathtt{scale}\ \mathfrak{p}\ r\ a \in \mathtt{Equiv}_P(I \times \mathbb{I}, (\gamma\mathrm{p}_{\mathbb{I}}, a, a\underline{r}\mathrm{p}_{\mathbb{I}}))$, stably under substitution.*

Proof. Because P is fibrant, $\mathtt{Equiv}_P$ is also fibrant; write $\mathfrak{e}$ for its filling operator. Set $\simeq\!\mathtt{scale}\ \mathfrak{p}\ r\ a \coloneqq \mathfrak{e}^{\mathrm{c}}_{I,(\gamma\mathrm{p}_{\mathbb{I}},a,a\underline{r}\mathrm{p}_{\mathbb{I}})}\ r\ (\mathtt{ide}_P(I,(\gamma,a\underline{r})))$ where $\mathtt{ide}_P$ is the identity equivalence. ◀

▶ **Theorem 82** (Fibrancy of univalent completion). *The mapping $(A,P) \mapsto (\mathtt{U}_P, \mathtt{El}_P)$ extends to a substitution-stable mapping $\mathrm{Ty}^{\mathrm{f}}(\Gamma) \times_{\mathrm{Ty}(\Gamma)} \mathrm{Fam}^{\mathrm{f}}(\Gamma) \to \mathrm{Ty}^{\mathrm{f}}(\Gamma) \times_{\mathrm{Ty}(\Gamma)} \mathrm{Fam}^{\mathrm{f}}(\Gamma)$.*

Proof. Let $(A,\mathfrak{a}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ and $(P,\mathfrak{p}) \in \mathrm{Ty}^{\mathrm{f}}(\Gamma.A)$; we define homogeneous filling and coercion for $\mathtt{U}_P$. Write $\overline{\mathfrak{p}}$ for the filling operator for $\mathtt{El}_P$.

For the homogeneous filling $\mathfrak{h}_{I,\gamma}\ r\ t_0\ \phi\ t$, we define t_+ from t_0 and t as in the proof of Lemma 69, then take $\mathtt{ext}\ (t_0\mathrm{p}_{\mathbb{I}})\ (\phi\mathrm{p}_{\mathbb{I}} \vee \mathrm{q}_{\mathbb{I}} \equiv r\mathrm{p}_{\mathbb{I}}, \simeq\!\mathtt{scale}\ \overline{\mathfrak{p}}\ r\ t_+)$ for the filler.

For $r\colon \Gamma \to \mathbf{y}\mathbb{I}$, we have $\mathtt{ev}_r\colon (\prod_{\mathbb{I}} \mathtt{U}_P, \mathtt{El}_P(\underline{r}.\mathtt{ev}_r)) \to (\mathtt{U}_P\underline{r}, \mathtt{El}_P(\underline{r}.\mathrm{id}_{A\underline{r}}))$ in $\mathrm{Fam}^{\mathrm{f}}(\Gamma)$. We build a section $\mathtt{coe}^r$ using the universal property from Construction 80, which gives a coercion operator for $\mathtt{U}_P$ (cf. Lemma 71). Set $\mathtt{coe}^r_{I,\gamma}(\mathtt{in}\ a) \coloneqq \mathtt{in}\ (\mathfrak{a}^{\mathrm{c}}_{I,\gamma}\ (r_I(\gamma))\ a)$ and

$$\mathtt{coe}^r_{I,\gamma}(\mathtt{ext}\ t_0\ (\phi,(t,e))) \quad\coloneqq\quad \mathtt{ext}\ (\mathtt{coe}^r_{I,\gamma}(t_0))\ (\phi\mathrm{p}_{\mathbb{I}}, (\mathtt{coe}^r_{I\restriction\phi,\gamma}(t), \overline{\mathfrak{e}}^{\mathrm{c}}_{I\restriction\phi,\gamma'}\ (r_I(\gamma))\ e))$$

where $\overline{\mathfrak{e}}$ is the filling operator for $\mathtt{Equiv}_{\mathtt{El}_P}$ and $\gamma' \coloneqq (\gamma, \mathtt{coe}^r_{I\restriction\phi,\gamma}(t), \mathtt{coe}^r_{I,\gamma}(t_0))$. ◀

▶ **Theorem 83** (Univalence). *For $A \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ and $P \in \mathrm{Ty}^{\mathrm{f}}(\Gamma.A)$, the family $\mathtt{El}_P$ is univalent.*

Proof. It suffices to show that $\mathtt{Sing}_{\mathtt{El}_P} \in \mathrm{Ty}(\Gamma.\mathtt{U}_P)$ is contractible (cf. [23, Corollary 11]). We show partial elements extend: given a partial element $(\phi,(t,e)) \in \mathtt{Sing}^{+}_{\mathtt{El}_P}(I,(\gamma,t_0))$, we have the total element $(\mathtt{ext}\ t_0\ (\phi,(t,e)), \mathtt{unglue}_{\mathtt{El}_P}(I,(\gamma,t_0,(\phi,(t,e))))) \in \mathtt{Sing}_{\mathtt{El}_P}(I,(\gamma,t_0))$. ◀

▶ **Theorem 84** (Surjectivity). *For $A \in \mathrm{Ty}^{\mathrm{f}}(\Gamma)$ and $P \in \mathrm{Ty}^{\mathrm{f}}(\Gamma.A)$, $\mathtt{in}\colon A \to \mathtt{U}_P$ is a surjection.*

Proof. By Theorem 83, it suffices to construct a section in $\mathrm{Fam}^{\mathrm{f}}(\Gamma)$ of the projection $(\Sigma\ (\mathtt{U}_P)\ \|\Sigma\ (A\mathtt{p}_{\mathtt{U}_P})\ (\mathtt{Equiv}_{\mathtt{El}_P}(\mathrm{id}_{\Gamma.\mathtt{U}_P}.\mathtt{in}))\|, \mathtt{El}_P \circ (.1)) \to (\mathtt{U}_P, \mathtt{El}_P)$. We can do so using the universal property of $(\mathtt{U}_P, \mathtt{El}_P)$: as in Theorem 74, we send $\mathtt{in}\ a$ to $(\mathtt{in}\ a, \mathtt{in}\ (a, \mathtt{ide}))$, then handle the case of $\mathtt{ext}\ t_0\ z$ using the propositionality of the truncation. ◀

4 Replacement as an instance of higher induction recursion

Propositional truncation can be presented as a *higher inductive type* [64, §6], i.e., as inductively generated by point and path constructors: $\|A\|$ is generated by $\mathsf{in} : A \to \|A\|$ and $\mathsf{squash} : \prod_{t,t':\|A\|} t =_{\|A\|} t'$ [64, §6.9]. We saw in § 2 that univalent completion and replacement can be used to construct many higher inductive types, and in § 3 we have justified the two by inductive constructions. Now we situate them among established forms of induction by observing that the two are instances of "higher induction recursion".

An *inductive recursive definition*, as formulated by Dybjer [28], introduces an inductive type I simultaneously with a recursive function from I into some existing type. The power of induction recursion [29] lies in its allowing the inductive constructors to refer to the recursive function. We know of no framework for *higher* inductive recursion in the literature. Still, we can naively extrapolate what a signature and elimination principle for a higher inductive recursive type would be and see that the results indeed capture replacement and completion.

▶ **Definition 85** (Inductive recursive replacement). *For B type, $\beta : \mathsf{LocSmall}_{\mathcal{U}}\ B$, $A : \mathcal{U}$, and $f : A \to B$, we specify replacement by the following higher inductive recursive signature:*

$$\begin{array}{ll}
\mathsf{Rep}_{B,\beta}(A,f) : \mathcal{U} & \mathsf{out} : \mathsf{Rep}_{B,\beta}(A,f) \to B \\
\mathsf{in} : A \to \mathsf{Rep}_{B,\beta}(A,f) & \mathsf{out}\ (\mathsf{in}\ a) :\equiv f\ a \\
\mathsf{quo} : \prod_{t,t'} (\beta\ (\mathsf{out}\ t)\ (\mathsf{out}\ t')).1 \to t = t' & \mathsf{ap}^{c,c'}_{\mathsf{out}}\ (\mathsf{quo}\ c\ c'\ q) :\equiv (\beta\ (\mathsf{out}\ c)\ (\mathsf{out}\ c')).2\ q
\end{array}$$

The semantic $\mathtt{Rep}$ and $\mathtt{out}$ of § 3.4 interpret this signature, using the fact that $\mathtt{out}$ is an embedding (Theorem 73) for the interpretation of quo and its out computation rule. The equation for $\mathsf{out}\ (\mathsf{in}\ a)$ holds strictly in said interpretation, while the equation for $\mathsf{ap}^{c,c'}_{\mathsf{out}}\ (\mathsf{quo}\ c\ c'\ q)$ holds up to a path. If we understand Definition 85 as a signature in *cubical* type theory, with ap defined as application under an interval variable, then the alternative interpretation mentioned in Remark 67 would interpret both equations as strict equalities.

Naively combining frameworks for higher inductive [9, 25, 17, 39] and inductive recursive definitions, the eliminator for this signature would have the following rule and equations:

$$\frac{t : \mathsf{Rep}_{B,\beta}(A,f) \vdash C(t)\ \mathsf{type} \qquad x : A \vdash n : C(\mathsf{in}\ x) \qquad z, z' : \mathsf{Rep}_{B,\beta}(A,f), c : C(z), c' : C(z'), r : (\beta\ (\mathsf{out}\ z)\ (\mathsf{out}\ z')).1 \vdash q : c =^{C}_{\mathsf{quo}\ z\ z'\ r} c'}{t : \mathsf{Rep}_{B,\beta}(A,f) \vdash \mathsf{elim}_C\ [\mathsf{in}\ x \mapsto n \mid \mathsf{quo}\ (z,c)\ (z',c')\ r \mapsto q]\ t : C(t)}$$

$$\mathsf{elim}_C\ [\mathsf{in}\ x \mapsto n \mid \mathsf{quo}\ (z,c)\ (z',c')\ r \mapsto q]\ (\mathsf{in}\ a) = n[a/x]$$

$$\mathsf{apd}^{t,t'}_{\mathsf{elim}_C\ [\mathsf{in}\ x \mapsto n \mid \mathsf{quo}\ (z,c)\ (z',c')\ r \mapsto q]}(\mathsf{quo}\ t\ t'\ u) = q[t/z, t'/z', \mathsf{elim}\ \cdots\ t/c, \mathsf{elim}\ \cdots\ t'/c', u/r]$$

The quo case asks for a family of dependent paths [64, §6.2] and its reduction rule uses the action of dependent functions on paths. Because the relation β (i.e., B's identity type) is reflexive, this premise can be instantiated if and only if C is a family of propositions. The eliminator thus expresses precisely that $\mathsf{in} : A \to \mathsf{Rep}_{B,\beta}(A,f)$ is a surjection in the sense of Definition 2, and hence that $\mathsf{out} \circ \mathsf{in}$ is an image factorization of f.

We can similarly cast univalent completion as a higher inductive definition of a type $\mathsf{U}_{A,P}$ together with a recursive definition of a family $u : \mathsf{U}_{A,P} \vdash \mathsf{El}_{A,P}(u)$ type. Dybjer's formulation of inductive recursive types does allow for such recursive definitions of large type families. It is less obvious how to integrate path constructors, because we cannot speak of paths between types in HoTT (although we can in cubical type theories). We opt to replace paths with equivalences and ap with $\mathsf{subst}_P^{a,a'} : (a =_A a') \to P(a) \simeq P(a')$.

▶ **Definition 86** (Inductive recursive univalent completion)**.** *For A type and $a : A \vdash P(a)$ type, we specify univalent completion by the following higher inductive recursive signature:*

$$\begin{array}{ll} \mathsf{U}_{A,P}\ \mathsf{type} & u : \mathsf{U}_{A,P} \vdash \mathsf{El}_{A,P}(u)\ \mathsf{type} \\ \mathsf{in} : A \to \mathsf{U}_{A,P} & \mathsf{El}_{A,P}(\mathsf{in}\ a) :\simeq P(a) \\ \mathsf{ua} : \prod_{u,u'} \mathsf{El}_{A,P}(u) \simeq \mathsf{El}_{A,P}(u') \to u = u' & \mathsf{subst}^{u,u'}_{\mathsf{El}_{A,P}(-)}\ (\mathsf{ua}\ u\ u'\ e) := e \end{array}$$

The existence of type formers $\mathsf{U}_{A,P}$ and $\mathsf{El}_{A,P}(-)$ with the constructors and equations-up-to-equivalence/path above is equivalent to existence of enough univalent families. Elimination for $\mathsf{U}_{A,P}$, formulated as for replacement above, is then equivalent to surjectivity of in.

Inductive recursive definitions have always been associated with the construction of universes [28]. Definition 86 is only a slight variation on Shulman's observation [56] that higher induction recursion can define univalent universes: we simply replace closure under type formers such as Σ or Π with a single "type former" $\mathsf{in} : A \to \mathsf{U}_{A,P}$. Sattler and Vezzosi [55, §2.1] mention a similar higher inductive recursive definition.

▶ Remark 87. Replacement holds in univalent type theory with pushouts [51], so the proof-theoretic strength of the types $\mathsf{Rep}_{B,\beta}(A, f)$ and $\mathsf{U}_{A,P}$ is much less than that of general induction recursion [29, 50, 13]. We may note that the constructors of $\mathsf{Rep}_{B,\beta}(A, f)$ and $\mathsf{U}_{A,P}$ are finitely branching – whereas a constructor such as $\Sigma : \prod_{A:\mathsf{U}}(\mathsf{El}\ A \to \mathsf{U}) \to \mathsf{U}$ is not – and that the convergence of our free algebra constructions at stage ω depends on this.

We remark, however, that Definitions 85 and 86 are *not* instances of what Hancock, McBride, Ghani, Malatesta, and Altenkirch call *small induction recursion* [37], where the size of the target of the recursion is no larger than that of the inductive type. At least in extensional type theory, small induction recursion reduces to indexed induction. Here, however, it is precisely the point that $\mathsf{Rep}_{B,\beta}(A, f)$ can be smaller than B.

5 Conclusion

We have seen that replacement and univalent completion are versatile and natural principles. We have taken a tour through their uses within type theory, and we have made a case for their foundational character by introducing direct interpretations and a comparison with induction recursion. We conclude with a couple of remarks on future work.

We saw that univalent completion yields both set and groupoid quotients (Examples 24 and 32). One suspects that completion is in fact equivalent to the existence of general quotients by higher equivalence relations. First, we need a meaningful definition of "higher equivalence relation". This is the subject of work in preparation by David Wärn.

On the other hand, we know thanks to Rijke [51] that the power of replacement is bounded above by the power of pushouts and $\mathbb{N}$. It is natural to wonder whether replacement is *strictly* less powerful than pushouts and, if so, whether there are *useful* models that only support replacement. If this is the case, it is a state of affairs exclusive to the higher setting: set quotients suffice to encode set pushouts. It may be that higher categories with "quotients but not colimits" appear naturally, but have been missed because they have no analogue in 1-category theory.

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