Functional Representability in Local Set Theories

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Abstract

In general local set theories (LST), there is an incomplete representability as $(x \mapsto \tau)$ for syntactic functions, i.e. those given by descriptions of the form $\forall x \exists ! y \gamma$, for some formula γ. The well-known equivalence theorem establishes among other things that it is possible to produce a well-termed local set theory that extends any given LST. However, the natural translation between the corresponding languages still does not guarantee strict functional representability, but only up to isomorphisms.

In this report, natural isomorphisms that are compatible with reasoning internally in the original LST are made explicit and explored in more generality. The internal translation along them is exhibited to be compatible with all logical operations; syntactic functions naturally represent themselves in a robust way as described in the main results.

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1 Introduction

As part of the ongoing development of an axiomatic framework for Synthetic Differential Geometry (SDG), McLarty [7] introduced a list of axioms a topos should satisfy to be a good model for SDG. In [10], the authors of this report added an axiom that further describes such a model in terms of the program of Axiomatic Cohesion of Lawvere [6]. In the standard textbooks in the area (see the textbook by Kock [3], Lavendhomme [5], or McLarty [8]), internal reasoning is only indirectly present. One aspect that is hardly mentioned is that of the representability of functions described implicitly within the internal language of a topos, like the ones that commonly occur in the context of SDG. A function f is *represented* if there is a term τ_f in the internal language for which f coincides with

$$(x \mapsto \tau_f), \tag{1}$$

where x is a sequence of variables of the appropriate type.

To set things up, a local language and a local set theory will be considered. A local language is a typed language $\mathcal{L}$ with primitives $\in$, $\{ : \}$ and $=$; ground, product and power type symbols; distinguished type symbols $\mathbf{1}, \mathbf{\Omega}$; and function symbols; together with a natural intuitionistic deduction system. A local set theory (LST) is a collection S of sequents $\Gamma : \alpha$, closed under derivability. Thus, the resulting logic of an LST might still not be classical. A local set theory S produces a category $\mathcal{T}(S)$ with its syntactical sets and functions (called S-sets and S-functions) as in classical Set Theory. Said categories are also toposes.

Conversely, starting with a topos, a local language can be produced – e.g. Mitchell-Bénabou, Zangwill, Lambek-Scott (See [2], [1], or [4]) – whose linguistic topos is equivalent to the starting topos.

There is a lack of representability in general for syntactic functions in local set theories, since a function f might be given simply in terms of its graph $|f|$; without a term τ_f to describe it as in (1). For any such function $f : X \to Y$ – and thus for finitely many – , this can be resolved by changing both the language and the LST by adding a function symbol $\mathbf{f}$ and the corresponding axioms to guarantee that the symbolic function $(x \mapsto \mathbf{f}(x))$ coincides with the given function, when restricted to X, i.e. by adding

$$x \in X : \langle x, \mathbf{f}(x) \rangle \in |f|. \tag{2}$$

This procedure is a basic feature of everyday mathematical practice, yet it formally requires a change of topos every time a new function symbol is added.

The main purpose of this report is to describe a way to deal with this lack of representability for all syntactic functions, by way of suitable reparameterizations of the domain and codomain, in such a way that any expression in the internal language can be effectively translated.

To this effect, recall that the Equivalence Theorem can be proved as follows. For any topos $\mathcal{E}$, denote by $\mathrm{Th}(\mathcal{E})$ the LST associated to it: its type and function symbols are the objects and arrows of $\mathcal{E}$, which are written in bold, i.e. $\boldsymbol{X}$ and $\mathbf{f}$ for the type symbol associated to the object X of $\mathcal{E}$ or the function symbol associated to the arrow f of $\mathcal{E}$. Alternatively, $\mathbf{Symb}(X)$ and $\mathbf{Symb}(f)$ may be used to avoid confusion.

There is a *canonical inclusion* $\iota_{\mathcal{E}} : \mathcal{E} \to \mathcal{T}(\mathrm{Th}(\mathcal{E}))$, given by

$$\iota_{\mathcal{E}} X = U_{\boldsymbol{X}} := \{u_{\boldsymbol{X}} : \top\} \qquad \text{and} \qquad \iota_{\mathcal{E}} f = (x \mapsto \mathbf{f}(x)) : U_{\boldsymbol{X}} \to U_{\boldsymbol{Y}}, \tag{3}$$

where $U_{\boldsymbol{X}}$ is the universal set of type $\boldsymbol{X}$. The Equivalence Theorem (see Proposition II.13.3 in [4]) shows that for any topos $\mathcal{E}$, $\iota_{\mathcal{E}}$ is an equivalence of categories.

Now, a translation from a local set theory S to another one Σ is an assignment (with certain requirements) of a type and a function symbol in Σ for each in S with the evident assignment of the terms in S to terms in Σ of the corresponding types. For any S there is the *natural translation* $\eta_S : S \to \mathrm{Th}(\mathcal{T}(S))$ which sends every type symbol $\mathbf{A}$ of S to $\mathbf{Symb}(U_{\mathbf{A}})$, the type symbol corresponding to the universal set $U_{\mathbf{A}}$, and every function symbol $\mathbf{f}$ to $\mathbf{Symb}((x \mapsto \mathbf{f}(x)))$, the function symbol corresponding to $(x \mapsto \mathbf{f}(x))$.

On the other hand, any translation ζ induces a logical functor $\mathcal{T}(\zeta)$ between the corresponding linguistic toposes. In particular, $\mathcal{T}(\eta_S)$ is given as follows:

$$\mathcal{T}(\eta_S)(X) = \{x_{\eta_S(\mathbf{A})} : \eta_S(\alpha)\} \qquad \text{and} \qquad \mathcal{T}(\eta_S)(f) = (\eta_S(|f|), \eta_S(X), \eta_S(Y)), \tag{4}$$

where $X = \{x_{\mathbf{A}} : \alpha\}$ and $f = (|f|, X, Y)$ with $|f| \subseteq X \times Y$ the graph of f.

By (3) and (4), one has two distinct functors:

$$\iota_{\mathcal{T}(S)}, \mathcal{T}(\eta_S) : \mathcal{T}(S) \to \mathcal{T}(\mathrm{Th}(\mathcal{T}(S))). \tag{5}$$

Notice that the only way for $\mathcal{T}(\eta_S)(X)$ to coincide with $\iota_{\mathcal{T}(S)} X$ is for X to be $U_{\mathbf{A}}$. Likewise, the only way for $\mathcal{T}(\eta_S)(f)$ to coincide with $\iota_{\mathcal{T}(S)} f$ is for f to be of the form $(x \mapsto \mathbf{f}(x))$ for some function symbol of S. Otherwise, these two functors are very much distinct.

Within the proof of the Equivalence Theorem in [1, p.110], a quasi-inverse to ι,

$$\rho_{\mathcal{E}} : \mathcal{T}(\mathrm{Th}(\mathcal{E})) \to \mathcal{E}, \tag{6}$$

is constructed as follows: For objects $X = \{x_{\mathbf{A}} : \alpha\}$, $Y = \{y_{\mathbf{B}} : \beta\}$ in $\mathcal{T}(\mathrm{Th}(\mathcal{E}))$, let i_X and i_Y be the inclusions $(x_{\mathbf{A}} \mapsto x_{\mathbf{A}}) : X \to U_{\mathbf{A}}$ and $(y_{\mathbf{B}} \mapsto y_{\mathbf{B}}) : Y \to U_{\mathbf{B}}$, respectively. Now, let $\rho_{\mathcal{E}} X$ be a corresponding subobject – frequently called *kernel* – of the natural interpretation $[\![\alpha]\!]_x$ of α in $\mathcal{E}$. For $X \overset{\varphi}{\to} Y$ in $\mathcal{T}(\mathrm{Th}(\mathcal{E}))$, $\rho_{\mathcal{E}}\varphi : \rho_{\mathcal{E}} X \to \rho_{\mathcal{E}} Y$ is the unique ψ such that

$$\vdash_{\mathrm{Th}(\mathcal{E})} \langle \boldsymbol{i_X}(u), \boldsymbol{i_Y}(\boldsymbol{\psi}(u))\rangle \in |\varphi|. \tag{7}$$

This derivation shows precisely the extent by which achieving strict representability (as the one obtained from (2)) fails: The representation of φ as an explicit function $(x \mapsto \tau)$ is only fulfilled up to isomorphism. In the case when $\mathcal{E}$ is $\mathcal{T}(S)$ for a local set theory S and φ is $\mathcal{T}(\eta_S)f$, more can be said.

▶ **Theorem A.** *Let $X = \{x_{\mathbf{A}} : \alpha\}$ and $Y = \{y_{\mathbf{B}} : \beta\}$ be objects in $\mathcal{T}(S)$ and let i_X and i_Y be the inclusions $(x_{\mathbf{A}} \mapsto x_{\mathbf{A}}) : X \to U_{\mathbf{A}}$ and $(y_{\mathbf{B}} \mapsto y_{\mathbf{B}}) : Y \to U_{\mathbf{B}}$, respectively. Then,*

$$\vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle \boldsymbol{i_X}(u), \boldsymbol{i_Y}(\mathbf{Symb}(f)(u))\rangle \in |\mathcal{T}(\eta_S)f|. \tag{8}$$

That is, f itself is the only syntactic function for which the following diagram commutes in $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$:

$$\begin{tikzcd} U_{\boldsymbol{X}} \arrow[r, "r_X"] \arrow[d, "(x\mapsto \mathbf{Symb}(f)(x))"'] & \mathcal{T}(\eta_S)X \arrow[d, "\mathcal{T}(\eta_S)f"] \\ U_{\boldsymbol{Y}} \arrow[r, "r_Y"'] & \mathcal{T}(\eta_S)Y, \end{tikzcd} \tag{9}$$

where r_X is the corestriction of $(u \mapsto \boldsymbol{i_X}(u))$ to $\mathcal{T}(\eta_S)X$.

Moreover, the inclusion functor $\iota_{\mathcal{T}(S)}$ and the logical functor induced by the canonical translation η_S are isomorphic:

$$\mathcal{T}(\eta_S) \cong \iota_{\mathcal{T}(S)}. \tag{10}$$

The way local set theories are defined makes their associated toposes have canonical subobjects. In this context, there is an adjunction $\mathcal{T} \dashv \mathrm{Th}$ where η is its unit; the counit is the appropriate restriction, ε, of ρ. By Lemma II.15.3 in [4], one has that $\varepsilon \circ \iota = 1$. From this it follows – as pointed out to us by a kind referee – , the conclusion of (10). However, such proof does not yield (8), which we believe sheds light onto the lack of representability of syntactic functions even in well-termed local set theories.

In order to take advantage of the isomorphisms r_X, the next goal is to describe an internal translation of formulae with variables restricted to X into formulae with unrestricted variables, i.e. whose range is a universal set. The Equivalence Theorem establishes that any object in $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$ is isomorphic to some universal set. The eliminability of descriptions in $\mathrm{Th}\mathcal{T}(S)$ – which is called *well-termedness* (see the last paragraph of Subsection 2.4) – gives that any function f with universal domain $U_{\mathbf{A}}$ in $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$ can indeed be represented by $(x \mapsto \tau_f)$ for some appropriate term τ_f (see Proposition 15). The guaranteed term τ_f is a term in the language of $\mathrm{Th}\mathcal{T}(S)$ which need not be of the form $\eta_S(\sigma)$ for an appropriate term σ in the language of S.

The following result describes the desired translation of formulae along functions with arbitrary codomain but universal domain. This will be applied to r_X, r_Y, etc. correspondingly.

In preparation for this, for any syntactic function $f : X \to Y$ in a local set theory S in a language $\mathcal{L}$ and for any formula θ in $\mathcal{L}$, let $\theta^{\natural}_{f}$ denote[1]

$$\exists y(\langle x, y\rangle \in |f| \wedge \theta).$$

If Γ is a finite set of formulas, then $\Gamma^{\natural}_{f}$ is the finite set $\{\gamma^{\natural}_{f} : \gamma \in \Gamma\}$.

▶ **Theorem B.** *Let Σ be a well-termed local set theory in a local language $\mathcal{L}$. Let $\mathbf{A}$ and $\mathbf{B}$ be types in $\mathcal{L}$. Let $f : U_{\mathbf{B}} \to X$ be a surjective Σ-function with X determined by the formula α with unique variable of type $\mathbf{A}$, and τ_f a term that represents f. Let γ and Γ be a formula and a finite set of formulae in $\mathcal{L}$. Then,*

$$\Gamma, x \in X \vdash_{\Sigma} \gamma \quad \textit{if and only if} \quad \Gamma(x/\tau_f) \vdash_{\Sigma} \gamma(x/\tau_f).$$

Additionally, if f is also injective, then, for every formula ϑ,

$$\vdash_{\Sigma} \vartheta = \vartheta^{\natural}_{f^{-1}}(x/\tau_f),$$

In this case,

$$\Gamma \vdash_{\Sigma} \gamma \quad \textit{if and only if} \quad \Gamma^{\natural}_{f^{-1}}, x \in X \vdash_{\Sigma} \gamma^{\natural}_{f^{-1}},$$

Furthermore, $(-)^{\natural}_{f^{-1}}$ is coherent with all logical operations.

The proof of Theorem B is the content of Theorems 16 and 18. The coherence result is explained in detail in the latter. When applied to $\Sigma = \mathrm{Th}(\mathfrak{T}(S))$, Theorem B provides an explicit natural parameterization of any $\mathfrak{T}\eta_S(X)$ by the universal $\iota_{\mathfrak{T}(S)}(X)$ preserving all logical operations.

The next result shows that the parameterizations given by the r_X, r_Y, etc. behave nicely with respect to syntactic functions: If one starts with an S-function, its graph can be expressed as $\{\langle x, y\rangle : \gamma\}$ for some appropriate formula γ. By (4), the formula $\mathfrak{T}(\eta_S)\gamma$ determines the function $\mathfrak{T}(\eta_S)f$ in $\mathrm{Th}(\mathfrak{T}(S))$, that is $\{\langle x, y\rangle : \mathfrak{T}(\eta_S)\gamma\}$ is the graph of $\mathfrak{T}(\eta_S)f$.

On the other hand, $\iota_{\mathfrak{T}(S)}(f)$ is the function $(u \mapsto \mathbf{Symb}(f)(u)) : U_{\mathbf{X}} \to U_{\mathbf{Y}}$. By considering the term

$$\tilde{\gamma} \equiv \mathfrak{T}(\eta_S)\gamma(x/\boldsymbol{i}_{\mathbf{X}}(u), y/\boldsymbol{i}_{\mathbf{Y}}(v)), \tag{11}$$

it remains to be verified that $\tilde{\gamma}$ also determines a function from $U_{\mathbf{X}}$ to $U_{\mathbf{Y}}$, i.e.

$$\vdash_{Th(\mathfrak{T}(S))} \forall u \exists! v. \tilde{\gamma}.$$

One would also expect these two functions to agree. The next result verifies both claims. This further allows one to come back to (2) to settle the representability question up to suitable isomorphisms.

▶ **Theorem C.** *Let S be a local set theory in a local language $\mathcal{L}$. Let $f : X \to Y$ be an S-function, and let γ be a formula that determines $|f|$. Then, the graph of the function $(u \mapsto \mathbf{f}(u))$ is determined by $\tilde{\gamma}$ of (11). More precisely,*

$$\vdash_{Th(\mathfrak{T}(S))} (\mathfrak{T}(\eta_S)\gamma(x/\boldsymbol{i}_{\mathbf{X}}(u), y/\boldsymbol{i}_{\mathbf{Y}}(v)))^{\natural}_{r_X^{-1} \times r_Y^{-1}} = \mathfrak{T}(\eta_S)\gamma, \tag{12}$$

and the formula $\mathfrak{T}(\eta_S)\gamma(x/\boldsymbol{i}_{\mathbf{X}}(u), y/\boldsymbol{i}_{\mathbf{Y}}(v))$ indeed determines a function $f^* : U_{\mathbf{X}} \to U_{\mathbf{Y}}$ which furthermore coincides with $(u \mapsto \mathbf{f}(u))$. In other words,

$$\vdash_{Th(\mathfrak{T}(S))} \langle u, \mathbf{Symb}(f)(u)\rangle \in |f^*|. \tag{13}$$

[1] In Definition 10, it is defined differently, yet in Lemma 12 it is proved to be equivalent to this.

The representability thus described allows the user of a local set theory to treat all functions as represented. This gives a far more intuitive way to use the local language in reasoning about objects in toposes, which in turn opens the door to studying axiomatic systems for Synthetic Differential Geometry (or other mathematical theories) while reasoning about them directly in a suitable set theoretical way – albeit with a more general intuitionistic deductive system.

The structure of this report is as follows. In Section 2, we review the necessary notions as well as set the notation. In Section 3, the notion of logical parameterization is investigated in order to prove Theorem B. Lastly, in Section 4, Theorems A and C are proved. The latter is obtained by applying the techniques developed in Section 3.

2 Notations and conventions

This lengthy section serves the purpose of introducing the notation to be used in the sequel. It is essentially a compendium of the results from the textbook of Bell [1]. The expert is compelled to skip it and proceed directly to Section 3 only to come back for clarification. The notation mostly follows that of the Bell's textbook. One important exception is given by $(-)^\natural$ in (19) below. Another exception is that graphs of functions f are denoted by $|f|$ as in the textbook of Lambek and Scott [4].

2.1 Toposes

An elementary topos – henceforth simply *topos* – is a category $\mathcal{C}$ with finite products, a subobject classifier and power objects. A subobject of an object A is an equivalence class $[m]$ of monic maps m with codomain A under an appropriate relation (see page 49 in [1]). A subobject classifier is an object Ω together with a map $\top : 1 \to \Omega$ such that for any monic arrow m there exists a unique characteristic $\chi(m)$ such that the following is a pullback diagram.

$$\begin{tikzcd} \mathrm{dom}(m) \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\ \mathrm{cod}(m) \arrow[r, "\chi(m)"'] & \Omega \end{tikzcd}$$

And every $A \to \Omega$ is of this form.

Given an object A of $\mathcal{C}$, a power object is a pair (PA, e_A) of the object PA and an arrow $e_A : A \times PA \to \Omega$ such that for every $f : A \times B \to \Omega$ there is a unique arrow $\hat{f} : B \to PA$ such that the following diagram commutes.

$$\begin{tikzcd} A \times B \arrow[d, "1_A \times \hat{f}"'] \arrow[dr, "f"] & \\ A \times PA \arrow[r, "e_A"'] & \Omega. \end{tikzcd} \tag{14}$$

In a topos $\mathcal{C}$, given an object A, there is a bijection between the class $\mathrm{Sub}(A)$ of subobjects of A and the class $\mathcal{C}(A, \Omega)$ of arrows $A \to \Omega$ in $\mathcal{C}$. Given a monic arrow m representing a subobject of A, its corresponding characteristic is $\chi(m) : A \to \Omega$; and given an arrow $u : A \to \Omega$ in $\mathcal{C}$, its corresponding subobject is represented by $\bar{u}$. It follows that $u = \chi(\bar{u})$; $\left[\overline{\chi(m)}\right] = [m]$; and that $[m] = [n]$ if and only if $\chi(m) = \chi(n)$.

From this, one can transfer the partial order $\subseteq$ unto $\mathcal{C}(A, \Omega)$, so that given $u, v \in \mathcal{C}(A, \Omega)$, $u \leq v$ if and only if $\bar{u} \subseteq \bar{v}$. In particular, this is equivalent to the existence of an arrow $f : \mathrm{dom}(\bar{u}) \to \mathrm{dom}(\bar{v})$ such that

$$\bar{u} = \bar{v} \circ f. \tag{15}$$

Recall that the maximal characteristic arrow of an object A, denoted by T_A, is $T_A := \chi(1_A) : A \to \Omega$. Wherever A is understood, one writes simply T. Notice that $\overline{T_A} = 1_A$, and that T_A is maximal since $u \leq T_A$ for all $u : A \to \Omega$. Furthermore, recall that $T = v \circ \bar{u}$ if and only if $u \leq v$ and that

$$T_C = v \circ f \text{ if and only if a } g : C \to \mathrm{dom}(\bar{v}) \text{ exists with } f = \bar{v} \circ g. \tag{16}$$

2.2 Local languages

The central definition of this communication is the definition of a local language. A *local language* $\mathcal{L}$ is determined by the specification of the following classes of symbols: a truth-value symbol $\mathbf{\Omega}$ and a unity type symbol $\mathbf{1}$; a possibly empty collection of ground type symbols $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$; a possibly empty collection of function symbols $\mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots$. The type symbols of $\mathcal{L}$ are now defined recursively as follows: $\mathbf{1}, \mathbf{\Omega}$ are type symbols; any ground type symbol is a type symbol; if $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are type symbols, so is the product $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ (with the proviso that, if $n = 1$, then $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ is $\mathbf{A}_1$, and if $n = 0$, then $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ is $\mathbf{1}$); and if $\mathbf{A}$ is a type symbol, so is the power $\mathbf{PA}$.

For each type symbol $\mathbf{A}$, $\mathcal{L}$ contains a denumerable set of symbols $x_{\mathbf{A}}, y_{\mathbf{A}}, z_{\mathbf{A}}, \ldots$ called *variables* of type $\mathbf{A}$. In addition, $\mathcal{L}$ contains the symbol $*$. Each function symbol of $\mathcal{L}$ has a signature assigned of the form $\mathbf{A} \to \mathbf{B}$, where $\mathbf{A}, \mathbf{B}$ are type symbols.

The terms with their corresponding types are recursively defined in the usual way (see Chapter 3 in [1]). In particular, there are primitive symbols $=$, $\in$, $\{:\}$ which yield terms $\sigma = \tau$, $\sigma \in \beta$, $\{x : \alpha\}$, where σ and τ are terms of the same type, β is of the power type of σ and x may or may not be a free variable of α.

Also, $*$ is a term of type $\mathbf{1}$ and there are further auxiliary symbols $()_i$ and $\langle , \rangle$ to deal with finite products. A term of type $\mathbf{\Omega}$ is called a *formula*. Logical operations are also defined recursively (see p. 70 in [1]): Write

$\alpha = \beta$	for	$\alpha \Leftrightarrow \beta$	$\forall \omega . \omega$	for	$\bot$
$* = *$	for	$\top$	$\alpha \Rightarrow \bot$	for	$\neg \alpha$
$\langle \alpha, \beta \rangle = \langle \top, \top \rangle$	for	$\alpha \wedge \beta$	$\forall \omega ((\alpha \Rightarrow \omega \wedge \beta \Rightarrow \omega) \Rightarrow \omega)$	for	$\alpha \vee \beta$
$(\alpha \wedge \beta) \Leftrightarrow \alpha$	for	$\alpha \Rightarrow \beta$	$\forall \omega (\forall x (\alpha \Rightarrow \omega) \Rightarrow \omega)$	for	$\exists x \alpha$
$\{x : \alpha\} = \{x : \top\}$	for	$\forall x \alpha$			

The basic axioms, inference rules and thus all deductions in a local set theory are given in terms of *sequents* $\Gamma : \alpha$, where α is a formula and Γ is a (possibly empty) finite set of formulae. The *basic axioms* are the sequents (see page 71 in [1]): $\alpha : \alpha$ (Tautology); $: x_{\mathbf{1}} = *$ (Unity); $x = y, \alpha(z/x) : \alpha(z/y)$ with x, y free for z in α (Equality); $: (\langle x_1, \ldots, x_n \rangle)_i = x_i$, $: x = \langle (x)_1, \ldots, (x)_n \rangle$ (Products); and $: x \in \{x : \alpha\} \Leftrightarrow \alpha$ (Comprehension).

The inference rules are the following:

Thinning $$\frac{\Gamma : \alpha}{\beta, \Gamma : \alpha}$$

Cut $$\frac{\Gamma : \alpha \qquad \alpha, \Gamma : \beta}{\Gamma : \beta} \text{ (free variables in } \alpha \text{ free in } \Gamma \text{ or } \beta)$$

Substitution $$\frac{\Gamma : \alpha}{\Gamma(x/\tau) : \alpha(x/\tau)} \ (\tau \text{ free for } x \text{ in } \Gamma \text{ and } \alpha)$$

Extensionality $$\frac{\Gamma : x \in \sigma \Leftrightarrow x \in \tau}{\Gamma : \sigma = \tau} \ (x \text{ not free in } \Gamma, \sigma, \tau)$$

Equivalence $$\frac{\alpha, \Gamma : \beta \qquad \beta, \Gamma : \alpha}{\Gamma : \alpha \Leftrightarrow \beta}$$

For any collection S of sequents, a *proof from* S is a finite tree with its vertex at the bottom whose nodes are correlated downwards by direct consequence by one of the rules of inference and topmost nodes are correlated with either a basic axiom or a member of S. The sequent at the vertex is the *conclusion* of the proof. A sequent $\Gamma : \alpha$ is *derivable from* S, and write $\Gamma \vdash_S \alpha$ provided there is a proof from S of which the sequent $\Gamma : \alpha$ is the conclusion.

2.3 Local set theories

A *local set theory* (LST) in $\mathcal{L}$ is a collection S of sequents which is closed under derivability: $\Gamma \vdash_S \alpha$ if and only if $(\Gamma : \alpha)$ is in S. Clearly, any collection of sequents generates an LST. A local set theory is said to be consistent if it is not the case that $\vdash_S \bot$.

Closed terms of power type are called $\mathcal{L}$-sets. Capital letters $A, B, \dots, Z$ will be used to denote $\mathcal{L}$-sets and the usual abbrevations will be used: $\forall x \in X.\alpha$ for $\forall x(x \in X \Rightarrow \alpha)$; $\exists x \in X.\alpha$ for $\exists x(x \in X \wedge \alpha)$; $\exists! x \in X.\alpha$ for $\exists! x(x \in X \wedge \alpha)$; and $\{x \in X : \alpha\}$ for $\{x : x \in X \wedge \alpha\}$.

The expected set operations $\subseteq, \cap, \cup \dots$ can be defined in the usual way for sets of the same type. For each type $\mathbf{A}$ there is a universal set $\{x_{\mathbf{A}} : \top\}$ (resp. empty set $\{x_{\mathbf{A}} : \bot\}$) denoted by $U_{\mathbf{A}}$ or A (resp. $\varnothing_{\mathbf{A}}$ or $\varnothing$). For arbitrary sets, products and disjoint unions can be defined in the usual way. Also, function spaces $\{u : u \subseteq Y \times X \wedge \forall y \in Y \exists! x \in X.\langle y, x\rangle \in u\}$ are denoted by X^Y. Lastly, the familiar $\{\tau\}$ and $\{\tau : \alpha\}$ symbolize $\{x : x = \tau\}$ and $\{z : \exists x_1 \cdots \exists x_n(z = \tau \wedge \alpha)\}$, respectively (see 3.10 in [1]).

An S-set is an equivalence class of $\mathcal{L}$-sets under the relation $\sim_S$, given by $X \sim_S Y$ if and only if $\vdash_S X = Y$. Representative $\mathcal{L}$-sets will sometimes be regarded as S-sets.

An S-function $X \xrightarrow{f} Y$ is a triplet of S-sets $(|f|, X, Y)$ such that $\vdash_S |f| \in Y^X$, where $|f|$ is called the graph of f as per usual.

▶ **Proposition 1.** *Let $X, Y, |f|$ be S-sets with $\vdash_S |f| \subseteq X \times Y$. Then, $x \in X \vdash_S \exists! y \in Y(\langle x, y\rangle \in |f|)$ if and only if $x \in X \vdash_S \exists y \in Y(\langle x, y\rangle \in |f|)$ and $\langle x, y\rangle \in |f|, \langle x, y'\rangle \in |f| \vdash_S y = y'$.*

Proof. Straightforward. ◀

For S-functions $X \xrightarrow{f} Y \xrightarrow{g} Z$ the composition is defined through its graph. Namely,

$$|g \circ f| = \{\langle x, z\rangle : \exists y(\langle x, y\rangle \in |f| \wedge \langle y, z\rangle \in |g|)\}. \tag{17}$$

A consequence of extensionality is that for S-functions $X \overset{f}{\to} Y, X \overset{g}{\to} Y$,

$$f \text{ is equal to } g \text{ if and only if } x \in X \vdash_S \langle x, y\rangle \in |f| \Leftrightarrow \langle x, y\rangle \in |g|. \tag{18}$$

▶ Remark 2. It is verified staring in page 85 in [1] that the collection of S-sets and S-functions forms a category $\mathcal{T}(S)$ that is also a topos, the *linguistic topos* of S.

2.4 Representability

Let X and Y be S-sets and τ be a term for which

$$\langle x_1, \ldots, x_n\rangle \in X \vdash_S \tau \in Y,$$

with all the free variables of τ contained within the $x_1, \ldots, x_n$. It is immediate to see that $\{\langle\langle x_1, \ldots, x_n\rangle, \tau\rangle : \langle x_1, \ldots, x_n\rangle \in X\}$ is the graph of a function which will be denoted by $(\langle x_1, \ldots, x_n\rangle \mapsto \tau)$.

Not every S-function is representable in this way. However, for an S-function $f : X \to \Omega$, let[2]

$$f^\natural(x) := \langle x, \top\rangle \in |f|. \tag{19}$$

Then, $x \in X \vdash \langle x, f^\natural(x)\rangle \in |f|$, and therefore

▶ **Proposition 3** ([1], 3.16.4)**.**

(i) $X \xrightarrow{x \mapsto f^\natural(x)} \Omega$ *is* f*,*

(ii) $\vdash_S (\langle x_1, \ldots, x_n\rangle \mapsto \alpha)^\natural(\langle x_1, \ldots, x_n\rangle) \Leftrightarrow \alpha.$

An LST is said to be *well-termed* if whenever $\vdash_S \exists! x\alpha$, there is a term τ such that its free variables are that of α except x and $\vdash_S \alpha(x/\tau)$; an LST for which every syntactic set is isomorphic to a universal is said to be *well-typed.* For a given LST S, let $\mathbf{C}^*(S)$ be the subcategory of S-sets of the form $U_{\mathbf{A}}$ and S-functions of the form $(x \mapsto \tau)$ with τ a term. It follows that S is well-termed if and only if $\mathbf{C}^*(S) \hookrightarrow \mathcal{T}(S)$ is full; and well-typed if and only it is essentially surjective (see remark in page 113 in [1]).

2.5 Interpretations

An *interpretation* of a local language $\mathcal{L}$ is a pair $(\mathcal{E}, I)$, with $\mathcal{E}$ a topos and I an assignment to each type symbol $\mathbf{\Xi}$ of an object $\mathbf{\Xi}_I = \Xi$ in $\mathcal{E}$ in a way compatible with products and powers that further assigns to $\mathbf{\Omega}$ a subobject classifier $\mathbf{\Omega}_I = \Omega$; to $\mathbf{1}$ a terminal object $\mathbf{1}_I = 1$; and to a function symbol $\mathbf{f}$ of signature $\mathbf{A} \to \mathbf{B}$ an arrow in $\mathcal{E}$ of the form $\mathbf{f}_I : \mathbf{A}_I \to \mathbf{B}_I$.

Given any interpretation, terms are further interpreted as arrows in the usual way. Let τ be a term of type $\mathbf{B}$ and let $\boldsymbol{x} = x_1, \ldots, x_n$ be different variables of types $\mathbf{A}_1, \ldots, \mathbf{A}_n$ including all free variables of τ. Define

$$[\![\tau]\!]_{I, x_1, \ldots, x_n} = [\![\tau]\!]_{I, \boldsymbol{x}} = [\![\tau]\!]_{\boldsymbol{x}}$$

as an arrow $A_1 \times \cdots \times A_n \to B$ recursively (see page 92 in [1]).

In particular, if τ is a term all of whose variables are among $z_1, \ldots, z_m$ and $\sigma_1, \ldots, \sigma_m$ are terms such that σ_i is free for z_i in τ for each $i = 1, \ldots, m$; then, for each interpretation of $\mathcal{L}$ in a topos $\mathcal{E}$,

$$[\![\tau(\boldsymbol{z}/\boldsymbol{\sigma})]\!]_{\boldsymbol{x}} = [\![\tau]\!]_{\boldsymbol{z}} \circ [\![\langle \sigma_1, \ldots, \sigma_m\rangle]\!]_{\boldsymbol{x}} \tag{20}$$

where $\boldsymbol{x}$ includes all the free variables of $\sigma_1, \ldots, \sigma_m$.

[2] Bell uses the symbol f^* but here it is reserved for the parameterizations of Section 3.

The interpretation of a finite collection Γ of formulae, $\llbracket \Gamma \rrbracket_{I,\boldsymbol{x}}$, is the interpretation of its conjunction.

Finally, if β is another formula and $\boldsymbol{x} = x_1, \ldots, x_n$ is the list of the free variables of $\Gamma \cup \{\beta\}$, then write

$$\Gamma \vDash_I \beta \quad \text{or} \quad \Gamma \vDash_{\mathcal{E}} \beta \quad \text{instead of} \quad \llbracket \Gamma \rrbracket_{I,\boldsymbol{x}} \leq \llbracket \beta \rrbracket_{I,\boldsymbol{x}}. \tag{21}$$

A sequent $\Gamma : \beta$ is *valid* under I (or in $\mathcal{E}$) if $\Gamma \vDash_I \beta$. It is clear that $\vDash_I \beta$ if and only if $\llbracket \beta \rrbracket_{\boldsymbol{x}} = T$. From which if $\mathcal{E}$ is degenerate then $\vDash_I \alpha$ for every formula α in $\mathcal{L}$.

Given a local set theory S, recall that by Remark 2, the linguistic topos has objects the S-sets and arrows the syntactic functions. Now, the canonical interpretation $(\mathcal{T}(S), K_S)$ of its local language is the one that assigns $\mathbf{A}_{K_S} = U_{\mathbf{A}}$ for each type symbol $\mathbf{A}$ and $\mathbf{f}_{K_S} = U_{\mathbf{A}} \xrightarrow{x \mapsto \mathbf{f}(x)} U_{\mathbf{B}}$ for each function symbol $\mathbf{f}$ with signature $\mathbf{A} \to \mathbf{B}$. It follows that (see 3.28 in [1])

$$\Gamma \vDash_{K_S} \alpha \quad \text{if and only if} \quad \Gamma \vdash_S \alpha, \tag{22}$$

and that (see 3.27 in [1])

$$\llbracket \tau \rrbracket_{K_S,\boldsymbol{x}} = (\boldsymbol{x} \mapsto \tau). \tag{23}$$

In particular, the following expected result holds.

▶ **Lemma 4.** *Let S be a local set theory in a local language $\mathcal{L}$. Let α be a formula with a unique free variable x of type $\mathbf{A}$. The following is a pullback diagram in $\mathcal{T}(S)$.*

$$\begin{tikzcd} \{x : \alpha\} \arrow[r] \arrow[d, "(x \mapsto x)"'] & 1 \arrow[d, "\top"] \\ A \arrow[r, "\llbracket \alpha \rrbracket_x", "(x \mapsto \alpha)"'] & \Omega \end{tikzcd} \tag{24}$$

Proof. Straightforward. ◀

▶ Remark 5. The map $(x \mapsto x) : X \longrightarrow A$ will frequently be denoted by i_X.

2.6 Internal language and local set theory of a topos

Let $\mathcal{E}$ be a topos with specified terminal $1_{\mathcal{E}}$, subobject classifier $\Omega_{\mathcal{E}}$, products and power objects. The internal language of $\mathcal{E}$ is the local language $\mathcal{L}(\mathcal{E})$ whose ground type symbols are precisely the objects X of $\mathcal{E}$, denoted by $\mathbf{X}$, and whose function symbols the arrows f of $\mathcal{E}$, denoted by $\mathbf{f}$, with signature $\mathbf{dom(f)} \to \mathbf{cod(f)}$. The obvious natural interpretation $(\mathcal{E}, H_{\mathcal{E}})$ of $\mathcal{L}(\mathcal{E})$ is given by $\mathbf{X}_{H_{\mathcal{E}}} = X$ and $\mathbf{f}_{H_{\mathcal{E}}} = f$. Equality, membership and comprehension in $\mathcal{L}(\mathcal{E})$ are written in bold; i.e. $\boldsymbol{=}, \boldsymbol{\in}, \boldsymbol{\{\, : \,\}}$.

The local set theory of a topos, $\mathrm{Th}(\mathcal{E})$, is determined by the requirement that $\Gamma : \alpha$ is in $\mathrm{Th}(\mathcal{E})$ if $\Gamma \vDash_{H_{\mathcal{E}}} \alpha$. It follows that (see 3.30 in [1])

$$\Gamma \vdash_{\mathrm{Th}(\mathcal{E})} \alpha \quad \text{if and only if} \quad \Gamma \vDash_{H_{\mathcal{E}}} \alpha. \tag{25}$$

This interpretation is compatible with one's intuition, e.g. for any $\mathcal{E}$-arrow f (see 3.33 in [1]),

$$f \text{ is monic if and only if } \mathbf{f}(x) \boldsymbol{=} \mathbf{f}(y) \vdash_{\mathrm{Th}(\mathcal{E})} x \boldsymbol{=} y. \tag{26}$$

As a consequence of (20), [1, 3.23.2], and (16) explicit term representation can be described categorically as follows.

▶ **Theorem 6** ([1] 3.34). *The following are equivalent:*

(i) $\vdash_{\mathrm{Th}(\mathcal{E})} \alpha(y_1/\tau_1, \ldots, y_n/\tau_n)$,

(ii) $\overline{[\![\alpha]\!]_{H_{\mathcal{E}},\boldsymbol{y}}} \circ f = [\![\langle \tau_1, \ldots, \tau_n \rangle]\!]_{H_{\mathcal{E}},\boldsymbol{x}}$ *for some arrow f, where $y_1, \ldots, y_n$ are the free variables of α and τ_i is free for y_i in α for each i.*

▶ **Theorem 7** ([1] 3.35). *If m is monic in $\mathcal{E}$, then*

$$\vdash_{\mathrm{Th}(\mathcal{E})} \boldsymbol{\chi(m)}(x) \Leftrightarrow \exists y.x = \boldsymbol{m}(y).$$

$\mathrm{Th}(\mathcal{E})$ is both well-termed and well-typed as the following two results show.

▶ **Theorem 8** ([1] 3.36). *If $\vdash_{\mathrm{Th}(\mathcal{E})} \exists! y\alpha$, then there exists a unique $\mathcal{E}$-arrow f such that*

$$\vdash_{\mathrm{Th}(\mathcal{E})} \alpha(y/\mathbf{f}(\langle x_1, \ldots, x_n \rangle)),$$

where $x_1, \ldots, x_n, y$ are the free variables of α.

Finally, passing from $\mathcal{E}$ to the linguistic topos of $\mathrm{Th}(\mathcal{E})$ gives no additional information. Indeed, the following equivalence theorem holds.

▶ **Theorem 9** ([1] 3.37). *Every topos $\mathcal{E}$ is equivalent to the linguistic topos $\mathcal{T}(\mathrm{Th}(\mathcal{E}))$ of its local set theory.*

2.7 Translations

A *translation* $\kappa : \mathcal{L} \to \Lambda$ between local languages $\mathcal{L}$ and Λ is an assignment of a type (resp. function) symbol in Λ to each type (resp. function) symbol in $\mathcal{L}$ in a way compatible with products, powers, truth-value and unity types. Terms are translated in the expected way as to preserve the basic axioms. Between LSTs, S and Σ in $\mathcal{L}$ and Λ respectively, κ is said to be a translation if for every sequent $\Gamma : \alpha$ of $\mathcal{L}$, $\Gamma \vdash_S \alpha$ implies that $\kappa\Gamma \vdash_\Sigma \kappa\alpha$. If under the same conditions $\Gamma \vdash_S \alpha$ if and only if $\kappa\Gamma \vdash_\Sigma \kappa\alpha$, then κ is called *conservative*.

Any translation $\kappa : \mathcal{L} \to \Lambda$ induces a logical functor $\mathcal{T}(\kappa)$ (one that preserves the specified terminal object, products, subobject classifier, power objects and evaluation arrows) between the associated linguistic toposes.

For any local set theory S with underlying local language $\mathcal{L}$, let the *canonical translation* $\eta_S : S \to \mathrm{Th}(\mathcal{T}(S))$ be the assignment to each type symbol $\boldsymbol{\Xi}$ of the type symbol associated to $U_{\boldsymbol{\Xi}}$ and to $\mathbf{f}$ of the function symbol associated to $(x \mapsto \mathbf{f}(x))$. It follows that it is a conservative translation and that

$$[\![\tau]\!]_{K_S,\boldsymbol{x}} = [\![\eta_S(\tau)]\!]_{H_{\mathcal{T}(S)},\eta_S\boldsymbol{x}}. \tag{27}$$

Whenever there is no risk of confusion one can represent the type symbol associated to a type symbol $\mathbf{A}$ with the same letter; in essence writing $\mathbf{A}$ for $\eta_S(\mathbf{A})$. In particular, given an S-set $X := \{x_{\mathbf{A}} : \alpha\}$, $\eta_S(X)$ is described by $\{\eta_S(x_{\mathbf{A}}) : \eta_S(\alpha)\}$, and since η_S is conservative, there is no loss in representing $\eta_S(X) = X$. The same will be assumed for a syntactical function $f = (|f|, X, Y)$. However, $(x \mapsto \mathbf{f}(x))$ need not be an S-function, since the function symbol $\mathbf{f}$ need not be in $\mathcal{L}$, only in $\mathcal{L}(\mathcal{T}(S))$. That is, $(x \mapsto \mathbf{f}(x)) : \boldsymbol{X} \to \boldsymbol{Y}$ is an arrow in $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$, not in $\mathcal{T}(S)$.

3 Parameterizations

The main purpose of this section is to prove Theorem B. To this effect, the central idea will be the translation of arbitrary formulas along arbitrary syntactic functions. This is introduced in the following definition. Next, some of its properties will be studied; in particular, an explicit description will be given in Lemma 12.

▶ **Definition 10.** *Let S be a local set theory on a local language $\mathcal{L}$. Let $f : X \to Y$ be an S-function with X of type* **PA** *and Y of type* **PB**. *Let ϑ be an arbitrary formula, x a variable of type* **A** *not free in ϑ, and y a variable of type* **B**. *Let $\boldsymbol{y}$ be a finite sequence of variables containing the free variables of ϑ and y. Denote by $\boldsymbol{x}$ the sequence of variables obtained from $\boldsymbol{y}$ by replacing y with the variable x. For any formula ϑ in $\mathcal{L}$, its* preimage translation by f, *or simply* f-translation, *is the formula*

$$\vartheta^{\natural}_{\boldsymbol{y},f,x} := ([\![\vartheta]\!]_{K,\boldsymbol{y}} \circ 1 \times i_Y \circ 1 \times f)^{\natural}(\boldsymbol{x}), \tag{28}$$

where for any function g, the term $g^{\natural}(x) := \langle x, \top\rangle \in |g|$ is given by 19.

▶ Remark 11. Under the same hyphotheses of Definition 10, one has that $\vartheta^{\natural}_{\boldsymbol{y},f,x} \vdash_S x \in X$.

▶ **Lemma 12.** *Let S be a local set theory in a local language $\mathcal{L}$. Let $f : X \to Y$ be an S-function with X of type* **PA** *and Y of type* **PB**. *Then $\vdash_S \vartheta^{\natural}_{\boldsymbol{y},f,x} \Leftrightarrow \exists y(\langle x, y\rangle \in |f| \wedge \vartheta)$.*

Proof. First, to verify $\vartheta^{\natural}_{\boldsymbol{y},f,x} \vdash_S \exists y.\langle x, y\rangle \in |f| \wedge \vartheta$, by (28) and (19), is the same as to verify

$$\langle\langle x_1, \dots, x_n, x\rangle, \top\rangle \in |(\boldsymbol{y} \mapsto \vartheta) \circ 1 \times i_Y \circ 1 \times f| \vdash_S \exists y'(\langle x, y'\rangle \in |f| \wedge \vartheta(y/y')),$$

where $x_1, \dots, x_n, y$ contain the variables of ϑ. Notice that the replacement of y by y' requires y' to be free for y in ϑ. Equivalently,

$$\begin{aligned}\exists z \exists z'.\langle\langle x_1, \dots, x_n, x\rangle, z\rangle \in |1 \times f| \wedge \langle z, z'\rangle \in |1 \times i_Y| \wedge \langle z', \top\rangle \in |(\boldsymbol{y} \mapsto \vartheta)|&\\ \vdash_S \exists y'(\langle x, y'\rangle \in |f| \wedge \vartheta(y/y')).&\end{aligned}$$

By [1, 3.7.2], it suffices to verify

$$\begin{aligned}\langle\langle x_1, \dots, x_n, x\rangle, z\rangle \in |1 \times f| \wedge \langle z, z'\rangle \in |1 \times i_Y| \wedge \langle z', \top\rangle \in |(\boldsymbol{y} \mapsto \vartheta)|&\\ \vdash_S \exists y'(\langle x, y'\rangle \in |f| \wedge \vartheta(y/y')).&\end{aligned} \tag{29}$$

To this effect, let $x'_1, \dots, x'_n$ and $x''_1, \dots, x''_n$ and $x'''_1, \dots, x'''_n$ be variables of the same types as $x_1, \dots, x_n$, respectively. Let x' be of the same type as x, and y'', y''' be different from but of the same type as y. Let z, z' be variables of the same type as $\langle x_1, \dots, x_n, y\rangle$. Then,

$$\begin{aligned}\langle\langle x_1, \dots, x_n, x\rangle, z\rangle = \langle\langle x'_1, \dots, x'_n, x'\rangle, \langle x'_1, \dots, x'_n, y'\rangle\rangle \wedge\\ x'_1 \in A_1 \wedge \dots \wedge x'_n \in A_n \wedge \langle x', y'\rangle \in |f| \wedge\\ \langle z, z'\rangle = \langle\langle x''_1, \dots, x''_n, y''\rangle, \langle x''_1, \dots, x''_n, y''\rangle\rangle \wedge\\ y'' \in Y \wedge x''_1 \in A_1 \wedge \dots \wedge x''_n \in A_n \wedge\\ \langle z', \top\rangle = \langle\langle x'''_1, \dots, x'''_n, y'''\rangle, \vartheta\rangle \wedge\\ x'''_1 \in A_1 \wedge \dots \wedge x'''_n \in A_n \wedge y''' \in B \vdash_S\\ \langle x, y'\rangle \in |f| \wedge \vartheta = \top\end{aligned}$$

By the substitution rule, replacing y with y', and [1, 3.7.1]

$$\begin{aligned}\langle\langle x_1,\ldots,x_n,x\rangle,z\rangle = \langle\langle x_1',\ldots,x_n',x'\rangle,\langle x_1',\ldots,x_n',y'\rangle\rangle\wedge\\ x_1'\in A_1\wedge\cdots\wedge x_n'\in A_n\wedge\langle x',y'\rangle\in|f|\wedge\\ \langle z,z'\rangle=\langle\langle x_1'',\ldots,x_n'',y''\rangle,\langle x_1'',\ldots,x_n'',y''\rangle\rangle\wedge\\ y''\in Y\wedge x_1''\in A_1\wedge\cdots\wedge x_n''\in A_n\wedge\\ \langle z',\top\rangle=\langle\langle x_1''',\ldots,x_n''',y'''\rangle,\vartheta\rangle\wedge\\ x_1'''\in A_1\wedge\cdots\wedge x_n'''\in A_n\wedge y'''\in B\vdash_S\\ \langle x,y'\rangle\in|f|\wedge\vartheta(y/y')=\top\vdash_S\\ \langle x,y'\rangle\in|f|\wedge\vartheta(y/y')\vdash_S\\ \exists y'(\langle x,y'\rangle\in|f|\wedge\vartheta(y/y'));\end{aligned}$$

wherefrom, by [1, 3.7.2],

$$\begin{aligned}\langle\langle x_1,\ldots,x_n,x\rangle,z\rangle = \langle\langle x_1',\ldots,x_n',x'\rangle,\langle x_1',\ldots,x_n',y'\rangle\rangle\wedge\\ x_1'\in A_1\wedge\cdots\wedge x_n'\in A_n\wedge\langle x',y'\rangle\in|f|\wedge\\ \langle z,z'\rangle=\langle\langle x_1'',\ldots,x_n'',y''\rangle,\langle x_1'',\ldots,x_n'',y''\rangle\rangle\wedge\\ y''\in Y\wedge x_1''\in A_1\wedge\cdots\wedge x_n''\in A_n\wedge\\ \exists x_1'''\cdots\exists x_n'''\exists y'''(\langle z',\top\rangle=\langle\langle x_1''',\ldots,x_n''',y'''\rangle,\vartheta\rangle\wedge\\ x_1'''\in A_1\wedge\cdots\wedge x_n'''\in A_n\wedge y'''\in B)\vdash_S\\ \exists y'(\langle x,y'\rangle\in|f|\wedge\vartheta(y/y'));\end{aligned}$$

thus,

$$\begin{aligned}\langle\langle x_1,\ldots,x_n,x\rangle,z\rangle = \langle\langle x_1',\ldots,x_n',x'\rangle,\langle x_1',\ldots,x_n',y'\rangle\rangle\wedge\\ x_1'\in A_1\wedge\cdots\wedge x_n'\in A_n\wedge\langle x',y'\rangle\in|f|\wedge\\ \langle z,z'\rangle=\langle\langle x_1'',\ldots,x_n'',y''\rangle,\langle x_1'',\ldots,x_n'',y''\rangle\rangle\wedge\\ y''\in Y\wedge x_1''\in A_1\wedge\cdots\wedge x_n''\in A_n\wedge\\ \langle z',\top\rangle\in|(\boldsymbol{y}\mapsto\vartheta)|\vdash_S\\ \exists y'(\langle x,y'\rangle\in|f|\wedge\vartheta(y/y')).\end{aligned}$$

Following the same process, one obtains (29).

Conversely,

$$\langle x,y\rangle\in|f|\wedge\vartheta\vdash_S\vartheta\vdash_S\vartheta=\top\vdash_S\langle\langle x_1,\ldots,x_n,y\rangle,\top\rangle=\langle\langle x_1,\ldots,x_n,y\rangle,\vartheta\rangle;$$

wherefrom, by [1, 3.7.4],

$$\langle x,y\rangle\in|f|\wedge\vartheta\vdash_S\exists x_1'\cdots\exists x_n'\exists y'(\langle\langle x_1,\ldots,x_n,y\rangle,\top\rangle=\langle\langle x_1',\ldots,x_n',y'\rangle,\vartheta(\boldsymbol{y}/\boldsymbol{y'})\rangle),$$

so that $\langle x,y\rangle\in|f|\wedge\vartheta\vdash_S\langle\langle x_1,\ldots,x_n,y\rangle,\top\rangle\in|(\boldsymbol{y}\mapsto\vartheta)|$. Similarly,

$$\langle x,y\rangle\in|f|\wedge\vartheta\vdash_S\langle\langle x_1,\ldots,x_n,y\rangle,\langle x_1,\ldots,x_n,y\rangle\rangle\in|1\times i_Y|\quad\text{and}$$

$$\langle x,y\rangle\in|f|\wedge\vartheta\vdash_S\langle\langle x_1,\ldots,x_n,x\rangle,\langle x_1,\ldots,x_n,y\rangle\rangle\in|1\times f|.$$

Therefore,

$$\begin{aligned}\langle x,y\rangle\in|f|\wedge\vartheta\vdash_S&\langle\langle x_1,\ldots,x_n,y\rangle,\top\rangle\in|(\boldsymbol{y}\mapsto\vartheta)|\wedge\\ &\langle\langle x_1,\ldots,x_n,y\rangle,\langle x_1,\ldots,x_n,y\rangle\rangle\in|1\times i_Y|\\ &\wedge\langle\langle x_1,\ldots,x_n,x\rangle,\langle x_1,\ldots,x_n,y\rangle\rangle\in|1\times f|.\end{aligned}$$

Thus, by [1, 3.7.4], $\langle x,y\rangle\in|f|\wedge\vartheta\vdash_S\langle\langle x_1,\ldots,x_n,x\rangle,\top\rangle\in|(\boldsymbol{y}\mapsto\vartheta)\circ 1\times i_Y\circ 1\times f|$. This, together with [1, 3.7.2], yields that $\exists y(\langle x,y\rangle\in|f|\wedge\vartheta)\vdash_S\vartheta^{\natural}_{\boldsymbol{y},f,x}$. ◀

This lemma has a couple of immediate consequences.

▶ **Corollary 13.** *Let S be an LST in a local language $\mathcal{L}$. Let $f : X \to Y$ be an S-function with X of type* **PA** *and Y of type* **PB**, *and ϑ a formula in $\mathcal{L}$. Let $\boldsymbol{y}$ be a finite sequence of variables containing those of ϑ, as well as y. Let $\boldsymbol{y'}$ be a subsequence of $\boldsymbol{y}$ still containing the free variables of ϑ, as well as y. Then, $\vdash_S \vartheta^{\natural}_{\boldsymbol{y},f,x} = \vartheta^{\natural}_{\boldsymbol{y'},f,x}$.*

Under the same conditions, given that one can add superfluous variables, denote $\vartheta^{\natural}_{\boldsymbol{y},f,x}$ as $\vartheta^{\natural}_{y,f,x}$; if from context one can infer the fixed variables, denote it as $\vartheta^{\natural}_{f}$; and if the S-function f is clear from context, denote it simply as $\vartheta^{\natural}$.

▶ **Corollary 14.** *Let S be an LST in a local language $\mathcal{L}$. Let $f : X \to Y$ be an S-function with X of type* **PA** *determined by α and Y of type* **PB**, *and ϑ a formula in $\mathcal{L}$ not having y as a free variable. Then, $\vdash_S \vartheta^{\natural}_{f} = \vartheta \wedge x \in X$.*

Proof. From Lemma 12 and [1, 3.7.6], one obtains $\vartheta^{\natural}_{f} \vdash_S \vartheta$ and, by Remark 11, $\vartheta^{\natural}_{f} \vdash_S \vartheta \wedge x \in X$. Conversely, it follows that $\vartheta \wedge x \in X \vdash_S \vartheta$ and $x \in X \vdash_S \exists! y(\langle x, y\rangle \in |f|) \vdash_S \exists y(\langle x, y\rangle \in |f|)$. Wherefrom, by [1, 3.7.6],

$$\vartheta \wedge x \in X \vdash_S \exists y(\langle x, y\rangle \in |f|) \wedge \vartheta \vdash_S \exists y(\langle x, y\rangle \in |f| \wedge \vartheta) \vdash_S \vartheta^{\natural}_{f}.$$ ◀

For a well-termed local set theory S in a local language $\mathcal{L}$, there is functional representability from universals even if the codomain is not universal.

▶ **Proposition 15.** *Let S be a well-termed LST in a local language $\mathcal{L}$. Let* **A** *and* **B** *be types. Let $f : U_{\mathbf{B}} \to X$ be an S-function with $X := \{x : \alpha\}$ with α a formula with a unique free variable x of type* **A**. *Then f is the corestriction to X of $(u \mapsto \tau_f)$ for some term τ_f in $\mathcal{L}$ and $\vdash_S \tau_f \in X$. In this case, τ_f* represents *f.*

Proof. Consider the S-function $i_X \circ f$. Since S is well-termed, $i_X \circ f = (u \mapsto \tau_f)$ for some term τ_f. Thus, by (23), $\overline{[\![\alpha]\!]_{K,x}} \circ f = [\![\tau_f]\!]_{K,u}$; wherefrom, $[\![\alpha]\!]_{K,x} \circ [\![\tau_f]\!]_{K,u} = T_B$. This, together with (20) and [1, 3.23.2], $\vDash_K \alpha(x/\tau_f)$. So that by (22), $\vdash_S \alpha(\tau_f)$; thus, $\vdash_S \tau_f \in X$.

Lastly, let $g := (|(u \mapsto \tau_f)|, B, X)$. Since i_X is monic and $i_X \circ g = (u \mapsto \tau_f)$, the conclusion follows. ◀

One can now proceed to prove Theorem B. The statement is split into Theorem 16 and Theorem 18. The former provides the logical framework to the set-theoretical functional representability of Section 4.

▶ **Theorem 16.** *Let S be a well-termed local set theory in a local language $\mathcal{L}$. Let* **A** *and* **B** *be types in $\mathcal{L}$. Let $f : U_{\mathbf{B}} \to X$ be a surjective S-function with X determined by the formula α with unique variable of type* **A**, *and τ_f a term that represents f. Let γ and Γ be a formula and a finite set of formulae in $\mathcal{L}$. Then,*

$$\Gamma, x \in X \vdash_S \gamma \quad \textit{if and only if} \quad \Gamma(x/\tau_f) \vdash_S \gamma(x/\tau_f).$$

Furthermore, if f is injective, then, for every formula ϑ,

$$\vdash_S \vartheta = \vartheta^{\natural}_{f^{-1}}(x/\tau_f).$$

And in this case,

$$\Gamma \vdash_S \gamma \quad \textit{if and only if} \quad \Gamma^{\natural}_{f^{-1}}, x \in X \vdash_S \gamma^{\natural}_{f^{-1}}.$$

▶ **Lemma 17.** *Let S be a local set theory in a local language $\mathcal{L}$. Let ξ, γ be formulae in $\mathcal{L}$. Let $x_1, \dots, x_n$ be variables of types $\mathbf{A}_1, \dots, \mathbf{A}_n$, containing the free variables of ξ and γ. Let*

$$Z := \{\langle x_1, \dots, x_n\rangle : \xi\}.$$

Then, $\xi \vdash_S \gamma$ if and only if the following diagram commutes in $\mathcal{T}(S)$:

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i_Z"'] & 1 \arrow[d, "\top"] \\ A_1 \times \cdots \times A_n \arrow[r, "(\boldsymbol{x} \mapsto \gamma)"'] & \Omega. \end{tikzcd}$$

Proof. Let $W := \{\langle x_1, \dots, x_n\rangle : \gamma\}$. Then,

$$\begin{aligned} \xi \vdash_S \gamma \text{iff} \quad & \xi \vDash_K \gamma && \text{(by (22))} \\ \text{iff} \quad & [\![\xi]\!]_{K,\boldsymbol{x}} \le [\![\gamma]\!]_{K,\boldsymbol{x}} && \text{(by (21))} \\ \text{iff} \quad & \text{there is an arrow } f : Z \to W \text{ such that } i_Z = i_W \circ f \end{aligned}$$

which completes the proof. ◀

Proof of Theorem 16. Without loss of generality, assume $\Gamma = \{\xi\}$ and that u, of type $\mathbf{B}$, is neither in ξ nor in γ. Replace an empty Γ with $\{\top\}$.

Assume that $\xi, x \in X \vdash_S \gamma$. By substitution, $\xi(x/\tau_f), \tau_f \in X \vdash_S \gamma(\tau_f)$. By Proposition 15, it follows from the Cut Rule that $\xi(x/\tau_f) \vdash_S \gamma(\tau_f)$.

Conversely, assume $\xi(x/\tau_f) \vdash_S \gamma(\tau_f)$. Let $x_1, \dots, x_n$ be variables of types $\mathbf{A}_1, \dots, \mathbf{A}_n$, resp., containing the free variables of ξ and α, with $x_n = x$. Let

$$Z := \{\langle x_1, \dots, x_n\rangle : \xi \wedge \alpha\} \quad \text{and} \quad Z' := \{\langle x_1, \dots, x_{n-1}, u\rangle : \xi(x/\tau_f)\}.$$

Thus, the following are pullback diagrams:

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "i_{Z'}"'] & 1 \arrow[d, "\top"] \\ A_1 \times \cdots \times A_{n-1} \times B \arrow[r, "(\boldsymbol{x'} \mapsto \xi(x/\tau_f))"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d, "i_Z"'] & 1 \arrow[d, "\top"] \\ A_1 \times \cdots \times A_n \arrow[r, "(\boldsymbol{x} \mapsto \xi \wedge \alpha)"'] & \Omega, \end{tikzcd}$$

with $i_{Z'} = (\boldsymbol{x'} \mapsto \boldsymbol{x'})$ and $i_Z = (\boldsymbol{x} \mapsto \boldsymbol{x})$.

Now, since $\langle x_1, \dots, x_{n-1}, x\rangle \in Z \vdash_S \xi \wedge \alpha \vdash_S \alpha \vdash_S x \in X$, let $j_Z := (|i_Z|, Z, A_1 \times \cdots \times X)$. Then $i_Z = 1 \times i_X \circ j_Z$, and the diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[d, "j_Z"'] & & 1 \arrow[d, "\top"] \\ A_1 \times \cdots \times A_{n-1} \times X \arrow[r, "1 \times i_X"'] & A_1 \times \cdots \times A_n \arrow[r, "(\boldsymbol{x} \mapsto \xi \wedge \alpha)"'] & \Omega \end{tikzcd}$$

is a pullback diagram too. So is the following:

$$\begin{tikzcd} Z' \arrow[rrr] \arrow[d, "i_{Z'}"'] & & & 1 \arrow[d, "\top"] \\ A_1 \times \cdots \times B \arrow[r, "1 \times f"] \arrow[rr, bend right, "1 \times (u \mapsto \tau_f)"'] \arrow[rrr, bend right=40, "(\boldsymbol{x'} \mapsto \xi(x/\tau_f))"'] & A_1 \times \cdots \times X \arrow[r, "1 \times i_X"] & A_1 \times \cdots \times A_n \arrow[r, "(\boldsymbol{x} \mapsto \xi \wedge \alpha)"] & \Omega. \end{tikzcd}$$

Consider now the following diagram:

$$\begin{tikzcd}
Z' \arrow[r, dotted, "f'"] \arrow[d, "i_{Z'}"'] \arrow[rrr, bend left] & Z \arrow[rr] \arrow[d, "j_Z"] & & 1 \arrow[d, "\top"] \\
A_1 \times \cdots \times B \arrow[r, "1\times f"'] & A_1 \times \cdots \times X \arrow[r, "1\times i_X"'] & A_1 \times \cdots \times A_n \arrow[r, "(\boldsymbol{x}\mapsto \xi\wedge\alpha)"'] & \Omega.
\end{tikzcd}$$

Since the right square is a pullback and the exterior rectangle commutes, there is a unique $f' : Z' \to Z$ making the diagram commute. Since the exterior diagram is a pullback, the left square is too. Now, since f is epic, f is surjective; thus, so is $1 \times f$ and thus also epic. Therefore, f' is epic since in a topos the pullback of an epic is epic.

Now consider the following diagram:

$$\begin{tikzcd}
Z' \arrow[r, "f'"] \arrow[d, "i_{Z'}"'] & Z \arrow[rr] \arrow[d, "j_Z"] & & 1 \arrow[d, "\top"] \\
A_1 \times \cdots \times B \arrow[r, "1\times f"] \arrow[rrr, bend right, "(\boldsymbol{x'}\mapsto\gamma(x/\tau_f))"'] & A_1 \times \cdots \times X \arrow[r, "1\times i_X"] & A_1 \times \cdots \times A_n \arrow[r, "(\boldsymbol{x}\mapsto\gamma)"] & \Omega.
\end{tikzcd}$$

Since $\xi(x/\tau_f) \vdash_S \gamma(x/\tau_f)$, the external diagram commutes by Lemma 17. Since the left square commutes, $(\boldsymbol{x} \mapsto \gamma) \circ i_Z \circ f' = \top \circ !_Z \circ f'$.

Since f' is epic, the right square commutes; therefore by Lemma 17, $\xi \wedge \alpha \vdash_S \gamma$ and thus $\xi, x \in X \vdash_S \gamma$.

Lastly, suppose that f is furthermore monic. Let ϑ be a formula the free variables of which are within $x_1, \ldots, x_{n-1}, u$. Define

$$\mathfrak{X} := \{\langle x_1, \ldots, x_{n-1}, u\rangle : \vartheta\} \qquad\qquad N_{\mathfrak{X}} := \{\langle x_1, \ldots, x_{n-1}, x\rangle : \vartheta^\natural\}.$$

Recall that $[\![\vartheta]\!]_{K,x_1,\ldots,x_{n-1},u} = (\langle x_1, \ldots, x_{n-1}, u\rangle \mapsto \vartheta)$ by (23), and that by Proposition 3, $(\boldsymbol{x'} \mapsto \vartheta) \circ 1 \times f^{-1} = (\boldsymbol{x} \mapsto \vartheta^\natural_{f^{-1}})$. Consider the following diagram:

$$\begin{tikzcd}
N_{\mathfrak{X}} \arrow[r, dashed, "g"] \arrow[d, "j_{N_{\mathfrak{X}}}"'] \arrow[rr, bend left] & \mathfrak{X} \arrow[r] \arrow[d, "i_{\mathfrak{X}}"] & 1 \arrow[d, "\top"] \\
A_1 \times \cdots \times X \arrow[r, "1\times f^{-1}"'] & A_1 \times \cdots \times B \arrow[r, "(\boldsymbol{x'}\mapsto\vartheta)"'] & \Omega.
\end{tikzcd}$$

Again, since the right square is a pullback and the exterior diagram commutes, there is a unique g making the whole diagram commute. Since the rectangle is a pullback (the external diagram was a pullback already), the left square is too, so that $N_{\mathfrak{X}} \cong \mathfrak{X}$.

Now, considering the following commutative diagram,

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, dashed] \arrow[d, "i_{\mathfrak{X}}"'] & N_{\mathfrak{X}} \arrow[r, "g"] \arrow[d, "j_{N_{\mathfrak{X}}}"'] & \mathfrak{X} \arrow[r] \arrow[d, "i_{\mathfrak{X}}"] & 1 \arrow[d, "\top"] \\
A_1 \times \cdots \times B \arrow[r, "1\times f"] & A_1 \times \cdots \times X \arrow[r, "1\times f^{-1}"] \arrow[rr, bend right, "(\boldsymbol{x}\mapsto\vartheta^\natural)"'] & A_1 \times \cdots \times B \arrow[r, "(\boldsymbol{x'}\mapsto\vartheta)"] & \Omega,
\end{tikzcd}$$

one clearly obtains $\vdash_S \vartheta^\natural(\tau_f) = \vartheta$, since f is the corestriction to X of $(u \mapsto \tau_f)$. From here, $\xi \vdash_S \gamma$ if and only if $\xi^\natural, x \in X \vdash_S \gamma^\natural$, since $\xi^\natural(\tau_f) \vdash_S \gamma^\natural(\tau_f)$ if and only if $\xi^\natural, x \in X \vdash_S \gamma^\natural$, by the first part of the theorem. ◀

▶ **Theorem 18.** *Let S be a well-termed local set theory in a local language $\mathcal{L}$. Let $f : U_{\mathbf{B}} \to X$ be an S-bijection, with X determined by a formula α with unique variable of type $\mathbf{A}$. Let τ_f be a term representing f and let ϑ and γ be formulae in $\mathcal{L}$. Then,*

$$\vdash_S (\vartheta \Box \gamma)^\natural_{f^{-1}} = \vartheta^\natural_{f^{-1}} \Box \gamma^\natural_{f^{-1}},$$

where $\Box$ is any binary logical connective. Furthermore,

$$\vdash_S (\neg\vartheta)^\natural_{f^{-1}} = \neg\vartheta^\natural_{f^{-1}},$$

and if u is a variable of type $\mathbf{B}$ free in ϑ and x is not free in ϑ, then

$$\vdash_S \forall u\vartheta = \forall x \in X.\vartheta^\natural_{f^{-1}} \text{ and } \vdash_S \exists u\vartheta = \exists x \in X.\vartheta^\natural_{f^{-1}}.$$

Proof. Throughout this proof, denote $(-)^\natural_{\langle x_1,\ldots,x_{n-1},u\rangle,f^{-1},x}$ with $(-)^\natural_{f^{-1}}$, where $x_1,\ldots,x_{n-1}$ are the appropriate variables. By the previous theorem,

$$\vdash_S (\vartheta\Box\gamma)^\natural_{f^{-1}}(\tau_f) = \vartheta\Box\gamma, \quad \vdash_S \vartheta^\natural_{f^{-1}}(\tau_f) = \vartheta, \quad \vdash_S \gamma^\natural_{f^{-1}}(\tau_f) = \gamma.$$

Wherefrom, $\vdash_S (\vartheta\Box\gamma)^\natural_{f^{-1}}(\tau_f) = \vartheta^\natural_{f^{-1}}(\tau_f)\Box\gamma^\natural_{f^{-1}}(\tau_f)$. Now, by Proposition 15, $f = (|(u \mapsto \tau_f)|, B, X)$. So that

$$(\boldsymbol{x} \mapsto (\vartheta\Box\gamma)^\natural_{f^{-1}}) \circ 1 \times f = (\boldsymbol{x} \mapsto \vartheta^\natural_{f^{-1}}\Box\gamma^\natural_{f^{-1}}) \circ 1 \times f; \tag{30}$$

thus since f is iso, $1 \times f$ is too. Therefore,

$$(\boldsymbol{x} \mapsto (\vartheta\Box\gamma)^\natural_{f^{-1}}) \text{ is equal to } (\boldsymbol{x} \mapsto \vartheta^\natural_{f^{-1}}\Box\gamma^\natural_{f^{-1}}).$$

Notice that both functions go from $A_1 \times \cdots \times X$ to Ω; however,

$$N_{(\vartheta\Box\gamma)^\natural_{f^{-1}}} := \{\langle x_1,\ldots,x\rangle : (\vartheta\Box\gamma)^\natural_{f^{-1}}\}, \qquad N_{\vartheta^\natural_{f^{-1}}\Box\gamma^\natural_{f^{-1}}} := \{\langle x_1,\ldots,x\rangle : \vartheta^\natural_{f^{-1}}\Box\gamma^\natural_{f^{-1}}\}$$

are each the pullback object of the S-functions $(\boldsymbol{x} \mapsto (\vartheta\Box\gamma)^\natural_{f^{-1}})$ and $(\boldsymbol{x} \mapsto \vartheta^\natural_{f^{-1}}\Box\gamma^\natural_{f^{-1}})$ from $A_1 \times \cdots \times A$ to Ω, so that $\vdash_S (\vartheta\Box\gamma)^\natural_{f^{-1}} = \vartheta^\natural_{f^{-1}}\Box\gamma^\natural_{f^{-1}}$. Likewise, $\vdash_S (\neg\vartheta)^\natural_{f^{-1}} = \neg\vartheta^\natural_{f^{-1}}$.

Now let ϑ be a formula with u among its free variables and let $x_1,\ldots,x_{n-1}$ contain the rest of the free variables of ϑ. Then, by [1, 3.4.4], $\forall u\vartheta \vdash_S \vartheta$. Thus, by Theorem 16 and Remark 11, $(\forall u\vartheta)^\natural_{f^{-1}} \vdash_S \vartheta^\natural_{f^{-1}}$; so that by Corollary 14 $\vdash_S (\forall u\vartheta)^\natural_{f^{-1}} = (\forall u\vartheta \wedge x \in X)$. Therefore, $\forall u\vartheta \wedge x \in X \vdash_S \vartheta^\natural_{f^{-1}}$. From here, $\forall u\vartheta \vdash_S x \in X \Rightarrow \vartheta^\natural_{f^{-1}}$, and thus, by [1, 3.4.2], $\forall u\vartheta \vdash_S \forall x \in X.\vartheta^\natural_{f^{-1}}$.

Conversely, by [1, 3.4.4] $\forall x \in X.\vartheta^\natural_{f^{-1}} \vdash_S x \in X \Rightarrow \vartheta^\natural_{f^{-1}}$. From here, $x \in X, \forall x \in X.\vartheta^\natural_{f^{-1}} \vdash_S \vartheta^\natural_{f^{-1}}$. Thus by Theorem 16, $(\forall x \in X.\vartheta^\natural_{f^{-1}})(\tau_f) \vdash_S \vartheta$.

As u is no longer free in $\forall x \in X.\vartheta^\natural_{f^{-1}}$, since – by Lemma 12 – $\vartheta^\natural_{\langle x_1,\ldots,x_{n-1},u\rangle,f^{-1},x}$ binds u, it follows (by [1, 3.4.2]) that $\forall x \in X.\vartheta^\natural_{f^{-1}} \vdash_S \forall u\vartheta$, and thus $\vdash_S \forall u\vartheta = \forall x \in X.\vartheta^\natural_{f^{-1}}$.

Finally, by [1, 3.7.1], $\vartheta \vdash_S \exists u\vartheta$. Thus

$$\begin{aligned} x \in X, \vartheta^\natural_{f^{-1}} &\vdash_S (\exists u\vartheta)^\natural_{f^{-1}} && \text{(by Theorem 16)} \\ &\vdash_S \exists u\vartheta \wedge x \in X && \text{(by Corollary 14)} \\ &\vdash_S \exists u\vartheta. \end{aligned}$$

Therefore, by [1, 3.7.2], $\exists x \in X.\vartheta^\natural_{f^{-1}} \vdash_S \exists u\vartheta$.

Conversely, by [1, 3.7.1], $x \in X \wedge \vartheta^{\natural}_{f^{-1}} \vdash_S \exists x \in X.\vartheta^{\natural}_{f^{-1}}$. Now, by Theorem 16, $\vartheta \vdash_S (\exists x \in X.\vartheta^{\natural}_{f^{-1}})(\tau_f)$, thus $\vartheta \vdash_S \exists x \in X.\vartheta^{\natural}_{f^{-1}}$.

Since u is not free in $\exists x \in X.\vartheta^{\natural}_{f^{-1}}$, by [1, 3.7.2], it follows that $\exists u\vartheta \vdash_S \exists x \in X.\vartheta^{\natural}_{f^{-1}}$ and $\vdash_S \exists u\vartheta = \exists x \in X.\vartheta^{\natural}_{f^{-1}}$ ◀

4 Functional representability

The main purpose of this section is to provide proofs for Theorems A and C. This section is a more *set theoretical* counterpart to the previous section which analyzed the logical aspects of the reparameterizations studied therein.

▶ **Theorem A.** *Let* $X = \{x_{\mathbf{A}} : \alpha\}$ *and* $Y = \{y_{\mathbf{B}} : \beta\}$ *be objects in* $\mathcal{T}(S)$ *and let* i_X *and* i_Y *be the inclusions* $(x_{\mathbf{A}} \mapsto x_{\mathbf{A}}) : X \to U_{\mathbf{A}}$ *and* $(y_{\mathbf{B}} \mapsto y_{\mathbf{B}}) : Y \to U_{\mathbf{B}}$*, respectively. Then,*

$$\vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle \boldsymbol{i_X}(u), \boldsymbol{i_Y}(\mathbf{Symb}(f)(u))\rangle \in |\mathcal{T}(\eta_S)f|. \tag{8}$$

That is, f *itself is the only syntactic function for which the following diagram commutes in* $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$*:*

$$\begin{tikzcd} U_{\boldsymbol{X}} \arrow[r, "r_X"] \arrow[d, "(x\mapsto\mathbf{Symb}(f)(x))"'] & \mathcal{T}(\eta_S)X \arrow[d, "\mathcal{T}(\eta_S)f"] \\ U_{\boldsymbol{Y}} \arrow[r, "r_Y"'] & \mathcal{T}(\eta_S)Y, \end{tikzcd} \tag{9}$$

where r_X *is the corestriction of* $(u \mapsto \boldsymbol{i_X}(u))$ *to* $\mathcal{T}(\eta_S)X$*.*

Moreover, the inclusion functor $\iota_{\mathcal{T}(S)}$ *and the logical functor induced by the canonical translation* η_S *are isomorphic:*

$$\mathcal{T}(\eta_S) \cong \iota_{\mathcal{T}(S)}. \tag{10}$$

▶ Remark 19. Since $\vdash_{\mathrm{Th}(\mathcal{T}(S))} \exists! v.\langle \boldsymbol{i_X}(u), \boldsymbol{i_Y}(v)\rangle \in |f|$, by [1, 3.36], f is the unique S-function that satisfies (8).

The following results will be needed in the proof of Theorem A. Their conclusions are the expected.

▶ **Lemma 20.** *Let* $\mathcal{E}$ *be a topos and* α *be a formula in* $\mathrm{Th}(\mathcal{E})$ *with a single free variable. Then,* $\vdash_{\mathrm{Th}(\mathcal{E})} x \in \{x : \alpha\} \Leftrightarrow \exists u.x = \boldsymbol{i}(u)$*, where* i *is* $\overline{[\![\alpha]\!]_x}$*.*

Proof. Since $i = \overline{[\![\alpha]\!]_x}$, it follows that $[\![\alpha]\!]_x = \chi(i)$. From which,

$$[\![\boldsymbol{\chi}(\boldsymbol{i})(x)]\!]_x = \chi(i) \circ [\![x]\!]_x = \chi(i) = [\![\alpha]\!]_x = [\![x \in \{x : \alpha\}]\!]_x \qquad \text{(by (25))};$$

wherefrom, by [1, 3.23.3], $\vDash_{H_{\mathcal{E}}} \boldsymbol{\chi}(\boldsymbol{i})(x) \Leftrightarrow x \in \{x : \alpha\}$. Then, $\vdash_{\mathrm{Th}(\mathcal{E})} \boldsymbol{\chi}(\boldsymbol{i})(x) \Leftrightarrow x \in \{x : \alpha\}$, and thus, $\vdash_{\mathrm{Th}(\mathcal{E})} \exists u.x = \boldsymbol{i}(u) \Leftrightarrow x \in \{x : \alpha\}$. ◀

▶ **Lemma 21.** *Let* S *be a local set theory in a local language* $\mathcal{L}$*. Let* X *be an* S*-set determined by the formula* α*. Then,*

$$\vdash_{\mathrm{Th}(\mathcal{T}(S))} \alpha(x/\boldsymbol{i_X}(u)), \tag{32}$$

with u *a variable of type* $\boldsymbol{X}$*; and thus*

$$\vdash_{\mathrm{Th}(\mathcal{T}(S))} \boldsymbol{i_X}(u) \in X. \tag{33}$$

Furthermore, one has that the arrow $r_X : U_{\boldsymbol{X}} \to X$, the corestriction of $(u \mapsto \boldsymbol{i_X}(u))$ to X, is an isomorphism. (Recall that $\boldsymbol{X}$ is the type symbol associated with X in $\mathcal{L}(\mathcal{T}(S))$ and that $\boldsymbol{i_X}$ is a function symbol not present in $\mathcal{L}$).

Proof. By Lemma 4, the diagram (24) is a pullback diagram in $\mathcal{T}(S)$, and hence in $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$ (where x is a variable of type $\mathbf{A}$, since the following translation is being used: $\eta_S x_{\mathbf{A}} = x_{\eta_S \mathbf{A}} = x_{\mathbf{A}}$, and $[\![\alpha]\!]_{K_S,x} = [\![\alpha]\!]_{H_{\mathcal{T}(S)},x}$, by (27)).

Now, since $\overline{[\![\alpha]\!]_{H_{\mathcal{T}(S)},x}} \circ r = i_X = [\![\boldsymbol{i_X}(u)]\!]_{H_{\mathcal{T}(S)},u}$ for some isomorphism r in $\mathcal{T}(S)$, (32) and (33) follow from [1, 3.34]. Lastly, to see that r_X is an isomorphism, define $|h|$ as the $\mathrm{Th}(\mathcal{T}(S))$-set $\{\langle x, u\rangle : \boldsymbol{i_X}(u) = x\}$. As the notation suggests, h is a function and, thus it is clearly the inverse of r_X. Indeed, by Lemma 20 and the fact that i_X is monic, $\vdash_{\mathrm{Th}(\mathcal{T}(S))} x \in \{x : \alpha\} \Leftrightarrow \exists! u. x = \boldsymbol{i_X}(u)$; wherefrom, $\vdash_{\mathrm{Th}(\mathcal{T}(S))} x \in X \Rightarrow \exists! u. x = \boldsymbol{i_X}(u)$, and thus h is a $\mathrm{Th}(\mathcal{T}(S))$-function $X \to U_{\boldsymbol{X}}$. ◀

Proof of Theorem A. Under the canonical translation η_S, one can regard S as a subtheory of $\mathrm{Th}(\mathcal{T}(S))$ and the associated logical functor $\mathcal{T}(\eta_S) : \mathcal{T}(S) \hookrightarrow \mathcal{T}(\mathrm{Th}\mathcal{T}(S))$ as the inclusion. Let $f : X \to Y$ be an arbitrary S-function and denote by $\mathbf{f}$ its corresponding function symbol $\mathbf{Symb}(f)$ in $\mathrm{Th}(\mathcal{T}(S))$.

The first step of the proof is to verify that the following diagram in $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$ is commutative:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd, "(x \mapsto x)"'] & Y \arrow[d, "(y \mapsto y)"] \\
& B \arrow[d, "(z \mapsto \{z\})"] \\
A \arrow[r, "(x \mapsto \{y : \langle x, y\rangle \in |f|\})"'] & PB.
\end{tikzcd}
\tag{34}
$$

In other words, this amounts to $\vdash_{\mathrm{Th}(\mathcal{T}(S))} |(x \mapsto \{y : \langle x, y\rangle \in |f|\})| = |(y \mapsto \{y\}) \circ f|$. To see this, first observe that

$$
\begin{aligned}
\langle x, z\rangle \in |f|, u \in \{z : \langle x, z\rangle \in |f|\} &\vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, z\rangle \in |f| \wedge \langle x, u\rangle \in |f| \\
&\vdash_{\mathrm{Th}(\mathcal{T}(S))} u = z \qquad\qquad \text{(by Proposition 1)} \\
&\vdash_{\mathrm{Th}(\mathcal{T}(S))} u \in \{z\},
\end{aligned}
$$

and that

$$
\begin{aligned}
\langle x, z\rangle \in |f|, u \in \{z\} &\vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, z\rangle \in |f| \wedge u = z \\
&\vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, u\rangle \in |f| \\
&\vdash_{\mathrm{Th}(\mathcal{T}(S))} u \in \{z : \langle x, z\rangle \in |f|\};
\end{aligned}
$$

so that, by the equivalence and extensionality rules,

$$
\langle x, z\rangle \in |f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \{z : \langle x, z\rangle \in |f|\} = \{z\}. \tag{35}
$$

Since

$$
\langle z, b\rangle \in |(y \mapsto \{y\})| \vdash_{\mathrm{Th}(\mathcal{T}(S))} b = \{z\}, \tag{36}
$$

it follows from (36) and (35) that

$$\langle x, z\rangle \in |f| \wedge \langle z, b\rangle \in |(y \mapsto \{y\})| \vdash_{\mathrm{Th}(\mathcal{T}(S))} b = \{z : \langle x, z\rangle \in |f|\};$$

thus, by [1, 3.7.2], $\exists z(\langle x, z\rangle \in |f| \wedge \langle z, b\rangle \in |(y \mapsto \{y\})|) \vdash_{\mathrm{Th}(\mathcal{T}(S))} b = \{z : \langle x, z\rangle \in |f|\}$. By (17) and the cut rule,

$$\langle x, b\rangle \in |(y \mapsto \{y\}) \circ f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, b\rangle = \langle x, \{z : \langle x, z\rangle \in |f|\}\rangle.$$

So that, by

$$\begin{aligned}\langle x, b\rangle \in |(y \mapsto \{y\}) \circ f| &\vdash_{\mathrm{Th}(\mathcal{T}(S))} x \in X \\ &\vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, \{z : \langle x, z\rangle \in |f|\}\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})|,\end{aligned}$$

it follows that

$$\langle x, b\rangle \in |(y \mapsto \{y\}) \circ f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})|. \tag{37}$$

To see the converse of (37), notice that

$$\langle x, z\rangle \in |f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} z \in Y \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle z, \{z\}\rangle \in |(y \mapsto \{y\})|$$

yields

$$\langle x, z\rangle \in |f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, z\rangle \in |f| \wedge \langle z, \{z\}\rangle \in |(y \mapsto \{y\})|. \tag{38}$$

Now, since $\langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})| \vdash_{\mathrm{Th}(\mathcal{T}(S))} b = \{z : \langle x, z\rangle \in |f|\}$, it follows from (35) and (38) that

$$\langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})|, \langle x, z\rangle \in |f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, z\rangle \in |f| \wedge \langle z, b\rangle \in |(y \mapsto \{y\})|;$$

wherefrom, by [1, 3.7.1],

$$\begin{aligned}\langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})|, \langle x, z\rangle \in |f| &\vdash_{\mathrm{Th}(\mathcal{T}(S))} \\ &\exists z(\langle x, z\rangle \in |f| \wedge \langle z, b\rangle \in |(y \mapsto \{y\})|),\end{aligned}$$

and thus $\langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})|, \langle x, z\rangle \in |f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, b\rangle \in |(y \mapsto \{y\}) \circ f|$, so that, by [1, 3.7.2],

$$\langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})|, \exists z(\langle x, z\rangle \in |f|) \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, b\rangle \in |(y \mapsto \{y\}) \circ f|.$$

However,

$$\begin{aligned}\langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})| &\vdash_{\mathrm{Th}(\mathcal{T}(S))} x \in X \\ &\vdash_{\mathrm{Th}(\mathcal{T}(S))} \exists! z(\langle x, z\rangle \in |f|) \\ &\vdash_{\mathrm{Th}(\mathcal{T}(S))} \exists z(\langle x, z\rangle \in |f|),\end{aligned}$$

and thus

$$\langle x, b\rangle \in |(x \mapsto \{z : \langle x, z\rangle \in |f|\})| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle x, b\rangle \in |(y \mapsto \{y\}) \circ f|. \tag{39}$$

Now, by the equivalence rule applied to (37) and (39), it follows that (34) commutes.

Under the natural interpretation of $\mathrm{Th}(\mathcal{T}(S))$ in $\mathcal{T}(S)$ – denoted by $[\![\tau]\!]_{\boldsymbol{x}} = [\![\tau]\!]_{H_{\mathcal{T}(S)},\boldsymbol{x}}$, where τ is a term of $\mathrm{Th}(\mathcal{T}(S))$ – it follows from (23) and (27) that the following diagram is the same as (34) and that it also commutes:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "1"'] & & Y \arrow[d, "{[\![\{\boldsymbol{i_Y}(v)\}]\!]_v}"] \\
X \arrow[rr, "{[\![\{y:\langle \boldsymbol{i_X}(u),y\rangle\in|f|\}]\!]_u}"] \arrow[dr, "{[\![\boldsymbol{i_X}(u)]\!]_u}"'] & & PB; \\
& A \arrow[ur, "{[\![\{y:\langle x,y\rangle\in|f|\}]\!]_x.}"'] &
\end{tikzcd}$$

Since $[\![\mathbf{f}(u)]\!]_u = f$, it follows that $[\![\{y : \langle \boldsymbol{i_X}(u), y\rangle \in |f|\}]\!]_u = [\![\{\boldsymbol{i_Y}(\mathbf{f}(u))\}]\!]_u$; and, by [1, 3.23.3],

$$\vDash_{H_{\mathcal{T}(S)}} \{y : \langle \boldsymbol{i_X}(u), y\rangle \in |f|\} = \{\boldsymbol{i_Y}(\mathbf{f}(u))\}.$$

Therefore, by (25), $\vdash_{\mathrm{Th}(\mathcal{T}(S))} \{y : \langle \boldsymbol{i_X}(u), y\rangle \in |f|\} = \{\boldsymbol{i_Y}(\mathbf{f}(u))\}$. Thus, since $\{\boldsymbol{i_Y}(\mathbf{f}(u))\}$ is by definition $\{y : y = \boldsymbol{i_Y}(\mathbf{f}(u))\}$, by [1, 3.4.3],

$$\vdash_{\mathrm{Th}(\mathcal{T}(S))} y = \boldsymbol{i_Y}(\mathbf{f}(u)) \Leftrightarrow \langle \boldsymbol{i_X}(u), y\rangle \in |f|. \tag{40}$$

From this,

$$\begin{aligned}\langle \boldsymbol{i_X}(u), y\rangle \in |f| &\vdash_{\mathrm{Th}(\mathcal{T}(S))} y = \boldsymbol{i_Y}(\mathbf{f}(u)) \wedge \langle \boldsymbol{i_X}(u), y\rangle \in |f| \\ &\vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle \boldsymbol{i_X}(u), \boldsymbol{i_Y}(\mathbf{f}(u))\rangle \in |f|.\end{aligned}$$

So that, by [1, 3.7.2], $\exists y(\langle \boldsymbol{i_X}(u), y\rangle \in |f|) \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle \boldsymbol{i_X}(u), \boldsymbol{i_Y}(\mathbf{f}(u))\rangle \in |f|$.

On the other hand, by (33),

$$\vdash_{\mathrm{Th}(\mathcal{T}(S))} \boldsymbol{i_X}(u) \in X \vdash_{\mathrm{Th}(\mathcal{T}(S))} \exists! y(\langle \boldsymbol{i_X}(u), y\rangle \in |f|) \vdash_{\mathrm{Th}(\mathcal{T}(S))} \exists y(\langle \boldsymbol{i_X}(u), y\rangle \in |f|).$$

wherefrom, by the cut rule, (8) follows.

Equation (8) is equivalent to the commutativity of (9) in $\mathcal{T}(\mathrm{Th}(\mathcal{T}(S)))$. The horizontal isomorphisms are clearly natural. ◀

Lastly, the proof of Theorem C is divided in the next two theorems.

▶ **Theorem 22.** *Let S be a local set theory in a local language $\mathcal{L}$. Let $f : X \to Y$ be an S-function, and let γ be a formula that determines $|f|$. Then*

$$\vdash_{Th(\mathcal{T}(S))} (\gamma(x/\boldsymbol{i_X}(u), y/\boldsymbol{i_Y}(v)))^{\natural}_{r_X^{-1}\times r_Y^{-1}} = \gamma. \tag{41}$$

Furthermore,

$$(\{\langle u, v\rangle : \gamma(x/\boldsymbol{i_X}(u), y/\boldsymbol{i_Y}(v))\}, U_{\mathbf{X}}, U_{\mathbf{Y}})$$

is a $\mathrm{Th}(\mathcal{T}(S))$*-function henceforth denoted by* f^**.*

Proof. Recall from Lemma 21 that r_X and r_Y are the corestrictions of $(u \mapsto \boldsymbol{i_X}(u))$ and $(v \mapsto \boldsymbol{i_Y}(v))$ to X and Y, respectively. Denote by $|f^*|$ the set $\{\langle u, v\rangle : \gamma(x/\boldsymbol{i_X}(u), y/\boldsymbol{i_Y}(v))\}$.

The equation (41) follows from the fact that the following is a pullback of pullbacks:

$$\begin{tikzcd}
{|f^*|} \arrow[r, "\cong"] \arrow[d, "i_{|f^*|}"'] & {|f|} \arrow[rr] \arrow[d, "j_{|f|}"'] & & 1 \arrow[d, "\top"] \\
U_{\mathbf{X}} \times U_{\mathbf{Y}} \arrow[r, "r_X \times r_Y", "\cong"'] \arrow[rr, bend right, "(\langle u,v\rangle \mapsto \langle \boldsymbol{i}_X(u), \boldsymbol{i}_Y(v)\rangle)"'] \arrow[rrr, bend right=40, "(\langle u,v\rangle \mapsto \gamma(\boldsymbol{i}_X(u), \boldsymbol{i}_Y(v)))"'] & X \times Y \arrow[r, "i_X \times i_Y"] & A \times B \arrow[r, "(\langle x,y\rangle \mapsto \gamma)"] & \Omega,
\end{tikzcd}$$

the proof of which is *mutatis mutandis* already within the proof of Theorem 16. A direct application of Theorem 18 proves that f^* is a function. ◀

▶ **Theorem 23.** *Let S be a local set theory in a local language $\mathcal{L}$. Let X and Y be S-sets, with α and β the formulae determining them, resp., and let $f : X \to Y$ be an S-function with γ the formula that determines $|f|$. Let f^* be the function obtained in Theorem 22. Then,*

$$\vdash_{Th(\mathcal{T}(S))} \langle u, \mathbf{f}(u)\rangle \in |f^*|. \tag{42}$$

In other words, that $f^* = (u \mapsto \mathbf{f}(u))$.

Proof. It is immediate that $\langle \boldsymbol{i}_{\mathbf{X}}(u), \boldsymbol{i}_{\mathbf{Y}}(\mathbf{f}(u))\rangle \in |f| \vdash_{\mathrm{Th}(\mathcal{T}(S))} \langle u, \mathbf{f}(u)\rangle \in |f^*|$, thus (42) follows by Theorem A. Finally, since by Lemma 22 both are functions, the conclusion now follows. ◀

References

1 J. L. Bell. *Toposes and local set theories*, volume 14 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1988. An introduction, Oxford Science Publications.

2 P. T. Johnstone. *Topos theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1977. London Mathematical Society Monographs, Vol. 10.

3 Anders Kock. *Synthetic differential geometry*, volume 333 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, second edition, 2006. doi:10.1017/CBO9780511550812.

4 J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.

5 René Lavendhomme. *Basic concepts of synthetic differential geometry*, volume 13 of *Kluwer Texts in the Mathematical Sciences*. Kluwer Academic Publishers Group, Dordrecht, 1996. Translated from the 1987 French original, Revised by the author. doi:10.1007/978-1-4757-4588-7.

6 F. William Lawvere. Axiomatic cohesion. *Theory Appl. Categ.*, 19:No. 3, 41–49, 2007.

7 Colin McLarty. Elementary axioms for canonical points of toposes. *J. Symbolic Logic*, 52(1):202–204, 1987. doi:10.2307/2273873.

8 Colin McLarty. *Elementary categories, elementary toposes*, volume 21 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1992. Oxford Science Publications. doi:10.1093/oso/9780198533924.001.0001.

9 Enrique Ruiz Hernández and Pedro Solórzano. On logical parameterizations and functional representability in local set theories, 2021. arXiv:2104.07405.

10 Enrique Ruiz-Hernández and Pedro Solórzano. Connectedness through decidable quotients. *Cah. Topol. Géom. Différ. Catég.*, 66(3):65–78, 2025.