Mendler Dialgebras and Recursion Schemes of Mixed Variance

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Abstract

We introduce the notion of *Mendler dialgebra* and provide a categorical semantics of recursion schemes of mixed variance in its terms. The Mendler-style approach – which employs second-order inference rules – to recursion schemes of mixed variance was championed by Uustalu and Vene in [48] using dinatural transformations, and we generalize their methods to include a variety of new recursion schemes including those presented by Ahn-Sheard in [3], and Stump et al. in [45]. We give sufficient criteria for reducibility of Mendler dialgebras to Lambek algebras [27] which correspond to first-order inference rules. A similar reduction in elementary terms for the special case of the systems studied in [48] had been given by the authors, but our approach differs in that we use the language of two-sided fibrations [43] to express the reduction for our generalization. This reveals that the "diagonal" of every two-sided fibration with a fibered initial object is isomorphic to a category of Lambek algebras; a dual version concerning reduction to Lambek coalgebras, as well as to bialgebras (inserters), [28] is given. We also discuss various properties and examples of Mendler dialgebras, including higher-order abstract syntax which is paradigmatic for definitions of mixed variance. Generally, we regard the paper as a contribution to a more systematic understanding of the relation between higher-order and first-order inference rules.

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1 Introduction

Mendler algebras were introduced in context of second-order lambda calculus in [35] as part of a recursion scheme for iteration on inductively defined types. While there are many specialized forms of recursion (e.g. iteration, primitive recursion, recursion allowing for certain nestings in the recursive calls), and assorted notions of *algebra*, one can distinguish two major styles of recursive definitions; speaking categorically[1] these are: recursion where the signature f is explicitly required to be functorial, the *Lambek style* [27], and recursion where functoriality is no explicit requirement, the *Mendler style* [35].

To explain the difference, let $f : \mathcal{C} \to \mathcal{C}$ be an endofunctor on some category $\mathcal{C}$, and recall that a *Lambek algebra* is a pair (X, α) consisting of an object X (the carrier) of $\mathcal{C}$, and a morphism $\alpha : f(X) \to X$. By contrast, a *Mendler algebra* is a pair (X, ϕ) where again $X : \mathcal{C}$, but $\phi : [-, X] \to [f-, X]$ is a natural transformation between the hom functors $[-, X]$ and $[f-, X]$. A morphism between such Lambek, respectively Mendler, algebras is a morphism

[1] The same distinction can be discussed in set theory as we will briefly do in Section 2, and in syntactic deduction calculi.

between the carriers commuting with the algebra structures (respectively given by α and ϕ) in the obvious way. Speaking 2-categorically, the difference is thus between algebra-structures as 1-morphisms vs. algebra-structures as 2-morphisms.

These notions of algebra give rise to different forms of recursion equations: For Lambek algebras, it has the form $h \circ in = \alpha \circ f(h)$ asking – for fixed (X, α) – for a unique solution h; this is expressed diagrammatically on the right below as the unique morphism from the initial Lambek f-algebra $(\mu f, in)$ (whenever it exists) to (X, α).

For Mendler algebras we again consider the unique morphism from the initial Mendler algebra $(\mu f, in)$ to some Mendler algebra (X, ϕ), and thus obtain a priori a family of sets of equations (for each A, one for each $g : A \to \mu f$), but by choosing the component μf (diagram on the left) and evaluating in the identity on μf (diagram in the middle), we obtain a canonical single equation where $h : \mu f \to X$ is the unique solution of the recursion equation $h \circ in_{\mu f}(id) = \phi_{\mu f}(h)$.

$$
\begin{tikzcd}
{[\mu f, \mu f]} \arrow[r, "{[\mu f, h]}"] \arrow[d, "in_{\mu f}"] & {[\mu f, X]} \arrow[d, "\phi_{\mu f}"] \\
{[f(\mu f), \mu f]} \arrow[r, "{[f(\mu f), h]}"] & {[f(\mu f), X]}
\end{tikzcd}
\qquad
\begin{tikzcd}
f(\mu f) \arrow[r, "in_{\mu f}(id)"] \arrow[dr, "\phi_{\mu f}(h)"'] & \mu f \arrow[d, "h"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
f(\mu f) \arrow[r, "in"] \arrow[d, "f(h)"] & \mu f \arrow[d, "h"] \\
f(X) \arrow[r, "\alpha"] & X
\end{tikzcd}
$$

In the parlance of natural-deduction calculi, commutation of the middle, respectively of the right, diagram corresponds to the *computation rule*. In particular we see that in case of Lambek algebras the signature f needs to be functorial (at least in the morphism h it is applied to) while in the Mendler case this is not a requirement since f is not applied to any morphism. This shows an advantage of the Mendler style at least if one is working with a syntactic deduction calculus since in such a calculus there is a priori no rule asserting functoriality of the signature, and for higher-order signatures such as in System F_ω there can be fundamental obstacles to establish functoriality at all (see "syntactic restriction" and "monotonicity witness" in [2, §3.4, p.19; §5, p.31]). While in category theory, the Yoneda lemma[2] tells us that the categories of Mendler f-algebras and that of Lambek f-algebras are isomorphic, it was recognized in [48] that one can adapt the definition of Mendler algebra to difunctors $f : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$. A Mendler algebra for such an "endo-difunctor" is now a pair (X, ϕ) where again $X : \mathcal{C}$, but $\phi : [Idd(-, =), X] \to [f(-, =), X]$ is a *dinatural transformation* [9] (see also Definition 1), and $Idd : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is the difunctor ignoring the first argument and acting as the identity on the second. This mixed-variant notion of Mendler algebra for $f : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is not in general equivalent to that of a Lambek algebra[3] for an endofunctor on $\mathcal{C}$. In [48], a sufficient criterion was also given for when Mendler algebras of the variety considered there can be reduced to algebras for an endofunctor on $\mathcal{C}$.

In Section 2 we give a motivation for the notion of Mendler algebra in terms of set-theoretic induction, illustrative of the difference between the two styles of inductive definition. In Section 3 we recall necessary definitions and give examples of other types of Mendler algebras. In Section 4 we define a category of *diindexed dinatural transformations (ddts)* and show that it is the total category of a two-sided fibration; this structure will be the key for the reduction

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[2] There does not seem to be an original textual reference for this "arguably [...] most important result in category theory" [41], but according to [31] it was proved by Nobuo Yoneda in 1954; the obituary [30] relates the story how it became known. The lemma and its variants (see Example 4) play an important rôle in the theory of Mendler algebras.

[3] Although there is a notion of *Freyd dialgebra* [14] (see also Definition 23) for f that (under some conditions) can be reduced to a Lambek algebra for an endofunctor on $\mathcal{C}^{op} \times \mathcal{C}$.

of *Mendler dialgebras* (see Definition 16) – which will by definition lie in the "diagonal" of the category of ddts – to Lambek algebras as we will explain in Section 5. In Section 6 we will give some more examples of Mendler dialgebras – in particular for modelling higher-order abstract syntax – and for reduction to Lambek algebras. In Section 7 we point to some differences regarding initial algebra semantics for Mendler dialgebras compared to Lambek algebras. In Section 8 we make concluding remarks and formulate some ideas for future work related to Mendler dialgebras.

The contributions of our paper are: We introduce the notion of Mendler dialgebra, give a general categorical semantics of recursion schemes of mixed variance in its terms, and establish a correspondence between Mendler dialgebras and Lambek algebras under a certain sufficient criterion. From the mathematical point of view, we observe that the same arguments more generally establish a relation between categories of Lambek algebras and "diagonals" of two-sided fibrations with fibered initial object.

2 Background: Relation To Induction In Set Theory

Uustalu and Vene [50] ask: "What are the origins of the Mendler style and why the name? It seems that the technique was invented by N. P. Mendler, in type theory."

The origin of the idea might appear less mysterious when realizing that the difference between a Lambek style and a Mendler style is present in set-theoretic induction as well:

An *inductive definition* (see e.g. [36], [40, p.83–84]) is defined in the following way with respect to a fixed set A.

1. A *clause* is a pair (X, x) where $X \subseteq A$, and $x \in A$.
2. A signature σ is a set of clauses.
3. For a signature σ we define a predicate $cl(\sigma, -)$ on the powerset of A by (for $Y \subseteq A$)

   $$cl(\sigma, Y) :\Leftrightarrow (\forall X.(X \subseteq Y) \to (\{x \in A \mid (X, x) \in \sigma\} \subseteq Y))$$

   In words: 'Y is closed under σ', and we write $\Sigma(X) := \{x \mid (X, x) \in \sigma\}$.
4. The *inductive set* defined by the signature σ is defined by $Ind(\sigma) := \bigcap\{Y \mid Y \subseteq A,\ cl(\sigma, Y)\}$.

If we assume proof terms, this is to say that $\tilde{\sigma} : cl(\sigma, Y)$ is a proof that Y is closed under σ. The notation

$$\tilde{\sigma} : \forall X.(X \subseteq Y) \to (\Sigma(X) \subseteq Y)$$

expresses then essentially (in particular if "$\subseteq$" is generalized to any preorder or morphism "$\to$" in a category) that the pair $(Y, \tilde{\sigma})$ is a *Mendler algebra* with *carrier* Y, and $Ind(\sigma)$ is (the carrier of) the initial such Mendler algebra.

It is then usually shown that a signature σ can equivalently be expressed as a monotone (with respect to inclusion) endofunction ϕ_σ on the powerset of A, namely $\phi_\sigma : Y \mapsto \{x \mid x \in A,\ \exists X \subseteq Y.\ x \in \Sigma(X)\}$ and its least fixpoint as the inductive set it determines; conversely for any monotone endofunction $\phi : P(A) \to P(A)$ of the powerset of A, we obtain a signature as the set $\sigma_\phi := \{(X, y) \mid X \subseteq A,\ y \in \phi(X)\}$.

A comparison in the situation of a deduction calculus can be found in [48].

3 Preliminaries

We assume the reader has some familiarity with category theory even though we include those definitions from categorical logic [21] essential for the main argument. We let throughout $\mathcal{V}$ denote the category of sets and (total) functions. For hom-sets we use angular brackets $[,]$ if no confusion can arise. We call a functor of the form $f : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ for some categories $\mathcal{C}$ and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{D})$-*bifunctor.* We sometimes denote the identity morphism on an object A by A itself.

▶ **Definition 1** (Dinatural transformation [9]). *For bifunctors $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, a* dinatural transformation *from F to G consists of a $\mathcal{C}$-indexed family τ of morphisms in $\mathcal{D}$ such that for all $h : b \to a$ diagrams of the following form commute.*

$$\begin{tikzcd}
 & F(a,a) \arrow[r, "\tau_a"] & G(a,a) \arrow[dr, "G(h,a)"] & \\
F(a,b) \arrow[ur, "F(a,h)"] \arrow[dr, "F(h,b)"'] & & & G(b,a) \\
 & F(b,b) \arrow[r, "\tau_b"] & G(b,b) \arrow[ur, "G(b,h)"'] &
\end{tikzcd}$$

Natural transformations are examples of dinatural transformations between bifunctors that both ignore the contravariant variable. But unlike natural transformations which are morphisms in functor categories, dinatural transformations cannot in general constitute the class of morphisms of any category[4] since they cannot (generally) be composed – a "defect" that is rectified by the following more restrictive definition due to Barr [38].

▶ **Definition 2** (Strong dinatural transformation). *For bifunctors $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, a* strong dinatural transformation *from F to G consists of a $\mathcal{C}$-indexed family σ of morphisms in $\mathcal{D}$ satisfying the following condition: for every $a : \mathcal{C}$ it holds: for every $h : b \to a$, and every triple (Q, p_1, p_2), if the left square commutes, so does the hexagon.*

$$\begin{tikzcd}
 & F(a,a) \arrow[rr, "\sigma_a"] \arrow[dr, "F(h,a)"] & & G(a,a) \arrow[dr, "G(h,a)"] & \\
Q \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & F(b,a) & & G(b,a) \\
 & F(b,b) \arrow[rr, "\sigma_b"] \arrow[ur, "F(b,h)"] & & G(b,b) \arrow[ur, "G(b,h)"'] &
\end{tikzcd}$$

In case $\mathcal{D}$ has pullbacks, for a given $h : b \to a$, the premise in the defining condition, that for all triples (Q, p_1, p_2) the left square commutes, can of course be replaced by the premise that the left square is a pullback. Every strong dinatural transformation is also a dinatural transformation.

Adjacent strong dinatural transformations can always be composed and since the ensuing category is interesting and important, we introduce a notation for it:

▶ **Definition 3** ($sd_{(\mathcal{C},\mathcal{D})}$). *For categories $\mathcal{C}$, $\mathcal{D}$, the category $sd_{(\mathcal{C},\mathcal{D})}$ of strong dinatural transformations has $(\mathcal{C}, \mathcal{D})$-bifunctors as objects, strong dinatural transformations as morphisms, and the identity dinatural transformation – which is strong – as identity. The composition operation is given by pointwise composition of the families.*

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[4] Bifunctors and dinatural transformations form, however, a paracategory in the sense of [19, 20].

Compositionality of dinatural transformations (see [42] for a comprehensive discussion) is e.g. of interest when bifunctors serve as models for types in a type theory and terms are modelled by dinatural transformations (see [8]). To have a category of (certain) dinatural transformations will also simplify some of our arguments later on (see Proposition 12). Beyond their virtue of compositionality, strong dinatural transformations are interesting for a variety of other reasons. Interesting for us is the connection to (initial) Lambek algebras, for, if the domain F is of the form $[G-,=]$ for some endofunctor G, the premise in the condition for strong dinaturality expresses exactly that the morphisms $h : b \to a$ figuring in the dinaturality condition (see Definition 1) are homomorphisms of G-Lambek algebras; strong dinaturality has also been used to define a notion of *strongly weighted limit* [46] to prove a generalization of Tarski's "non-constructive" fixpoint theorem. We have the following Yoneda-like statements:

▶ **Example 4** ([47]). For $\mathcal{V}$-enriched $\mathcal{C}$ we have:
1. $sd_{(\mathcal{C},\mathcal{V})}([F-,=],[C,=]) \simeq Cone_{U_F}(C)$ for the set of cones with vertex C of the forgetful functor U_F from the category of Lambek algebras for $F : \mathcal{C} \to \mathcal{C}$.
2. $sd_{(\mathcal{C},\mathcal{V})}([H-,=],K=) \simeq K(\mu H)$ for $H : \mathcal{C} \to \mathcal{C}$, $K : \mathcal{C} \to \mathcal{V}$.
3. $sd_{(\mathcal{C},\mathcal{V})}([-,H=],K-) \simeq K(\nu H)$ for $H : \mathcal{C} \to \mathcal{C}$, $K : \mathcal{C}^{op} \to \mathcal{V}$.
4. $sd_{(\mathcal{C},\mathcal{V})}([F-,=],Idd(-,=)) \simeq \mu F$[5] for $F : \mathcal{C} \to \mathcal{C}$.

In [47] the question was asked whether the statements in the previous example had already been observed elsewhere. We find the following:

▶ Remark 5 (Kelly's strong Yoneda lemma). Let $K : \mathcal{C} \to \mathcal{V}$ and define $\widetilde{K} : Alg(T)^{op} \times \mathcal{C} \to \mathcal{V}$ by $\widetilde{K}((X,a),Y) := K(Y)$. Then Kelly's formula [25, (2.38)], $\int_{B:\mathcal{B}} K(B,JB) \simeq [\mathcal{B}^{op} \times \mathcal{A},\mathcal{V}](\mathcal{A}(J-,=),K(-,=))$, can be instantiated by taking $\mathcal{B} := Alg(T)$, $\mathcal{A} := \mathcal{C}$, $J := U : Alg(T) \to \mathcal{C}$ the functor forgetting the algebra structure, and $K := \widetilde{K}$. With this choice, we obtain

$$K(\mu T) \simeq \int_{(X,a):Alg(T)} K(X) \simeq [Alg(T)^{op} \times \mathcal{C},\mathcal{V}]([U-,=],\widetilde{K}(-,=)).$$

Indeed, Kelly's formula gives the right-hand isomorphism, while the left-hand isomorphism is the composite

$$K(\mu T) \simeq [Alg(T),\mathcal{V}](\Delta_1, K \circ U) \simeq \int_{(X,a):Alg(T)} K(X),$$

where $\Delta_1 : Alg(T) \to \mathcal{V}$ is the constant functor at the singleton set.

Another justification for the significance of the definition of strong dinatural transformations is that the Church numerals, i.e. the set of dinatural transformations from a hom functor to itself sending an endomorphism to its n-fold iteration for some natural number n are the only strong dinatural transformations between these functors (see [17]).

We officially give the central definition of Mendler (co)algebras and their morphisms that we already mentioned in the introduction.

▶ **Definition 6** (Classical (covariant) Mendler (co)algebra [35]). *A Mendler algebra for an endofunctor f on a category $\mathcal{C}$ is a pair (X,ϕ) where X is an object of $\mathcal{C}$, and $\phi : [-,X] \to [f-,X]$ is a natural transformation. A morphism between such Mendler f-algebras from*

[5] Where here again Idd is the identity functor supplemented by an ignored contravariant argument.

(X, ϕ) to (Y, ψ) consists of a morphism $m : X \to Y$ such that for all A squares of the form on the left commute. Dually, a Mendler coalgebra is a pair (X, ζ) where $X : ob(\mathcal{C})$ is an object, and $\zeta : [X, -] \to [X, f-]$ is a natural transformation; a morphism $(X, \zeta) \to (Y, \gamma)$ consists of a morphism $m : X \to Y$ such that for all A the squares of the form on the right commute.

$$\begin{tikzcd} {[A,X]} \arrow[r, "{[A,m]}"] \arrow[d, "\phi_A"] & {[A,Y]} \arrow[d, "\psi_A"] \\ {[fA,X]} \arrow[r, "{[fA,m]}"] & {[fA,Y]} \end{tikzcd} \qquad \begin{tikzcd} {[Y,A]} \arrow[r, "{[m,A]}"] \arrow[d, "\gamma_A"] & {[X,A]} \arrow[d, "\zeta_A"] \\ {[Y,fA]} \arrow[r, "{[m,fA]}"] & {[X,fA]} \end{tikzcd}$$

Even though, as mentioned in the introduction, the categories of classical Mendler f-(co)algebras are isomorphic to the categories of Lambek f-algebras, their syntactic difference matters in practice as one could discuss by writing out an appropriate natural deduction calculus with fixpoint operator.

In [48] it was remarked that the definition of Mendler algebra also makes sense for endo-bifunctors, i.e. functors of mixed variance of the form $f : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ and Mendler f-algebras of the form $(X, \phi : [Idd(-,=), X] \to [f(-,=), X])$ where ϕ is a dinatural transformation. In this case it is however not true anymore that the categories of Mendler f-algebras are always equivalent to the category of Lambek f-algebras – not even for Lambek g-algebras for a possibly different endofunctor g. Nevertheless it was given – under the condition that certain initial objects exist – a construction of an endofunctor g on $\mathcal{C}$ such that the category of Mendler f-algebras is isomorphic to the category of Lambek g-algebras.

In this paper we will generalize the results of [48] to Mendler-style algebras of a more general form that we call *Mendler dialgebras*, and characterize the reduction of Mendler dialgebras to Lambek algebras in terms of universal properties of a category of *diindexed dinatural transformations* which contains the category of Mendler dialgebras as a subcategory (the "diagonal"). Our main tool for this reduction will be (Grothendieck) (op)fibrations.

▶ **Definition 7** ((Grothendieck) (op)fibration [44][21]). *Let $p : \mathcal{A} \to \mathcal{I}$ be a functor.*

1. a. *A morphism t is* cartesian *over $p(t)$ if for all $j : A \to C$ for which there exists a morphism $lift_{p(t)}p(j)$ such that $p(t) \circ lift_{p(t)}p(j) = p(j)$ (i.e. a lift through its image), there exists a unique $Lift_t j$ such that $t \circ Lift_t j = j$, and $p(Lift_t j) = lift_{p(t)}p(j)$*

$$\begin{tikzcd} & B \arrow[d, "t"] \\ A \arrow[ur, dotted, "Lift_t j"] \arrow[r, "j"] & C \end{tikzcd} \qquad \begin{tikzcd} & p(B) \arrow[d, "p(t)"] \\ p(A) \arrow[ur, dotted, "lift_{p(t)}p(j)"] \arrow[r, "p(j)"] & p(C) \end{tikzcd}$$

b. *p is called a* fibration *if every morphism of the form $u : J \to p(C)$ has a* cartesian lift*, i.e. there is a cartesian morphism t such that $p(t) = u$. A choice, for every morphism $u : J \to p(C)$, of a cartesian morphism t with codomain C and $p(t) = u$ is called a cleavage, and a fibration is called* split *if this choice preserves identities and composition.*

2. a. *A morphism t is* opcartesian *over $p(t)$ if for all j for which there exists some $ext_{p(t)}p(j) : p(B) \to p(A)$ with $ext_{p(t)}p(j) \circ p(t) = p(j)$, there exists a unique $Ext_t j$ such that $p(Ext_t j) = ext_{p(t)}p(j)$, and $Ext_t j \circ t = j$.*

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "t"'] & A \\ B \arrow[ur, dotted, "Ext_t j"'] & \end{tikzcd} \qquad \begin{tikzcd} p(C) \arrow[r, "p(j)"] \arrow[d, "p(t)"'] & p(A) \\ p(B) \arrow[ur, dotted, "ext_{p(t)}p(j)"'] & \end{tikzcd}$$

b. *p is called an* opfibration *if every morphism of the form $u : p(C) \to I$ has an* opcartesian lift*, i.e. there is an opcartesian morphism t such that $p(t) = u$. A choice, for every morphism $u : p(C) \to I$, of an opcartesian morphism t with domain C and $p(t) = u$ is called an opcleavage, and an opfibration is called* opsplit *if this choice preserves identities and composition.*

A morphism is called p-vertical (for an (op)fibration p) if it is sent to an identity by p.

Examples of Mendler Algebras

The main part of this paper will be dedicated to the reduction of Mendler-style algebras to Lambek (co)algebras and properties of this reduction. To give the reader nevertheless some more sense of the utility of the concept of Mendler algebra, we review some well-known examples of Mendler algebras; we will see some more examples later in Section 6. For now, a Mendler-style algebra is a pair of a carrier and a dinatural transformation; all these variant Mendler algebras will be instances of *Mendler dialgebras* that we will define later on.

▶ **Example 8** (Primitive (co)recursion [34])**.** A Mendler-style algebra for *primitive recursion* has algebras of the form (X, ϕ), where the natural transformation is of the form $\phi : [-, X] \times [-, \mu f] \to [f-, X]$ for a covariant endofunctor f and the carrier μf of its initial (Lambek or conventional Mendler) algebra. The corresponding scheme for primitive corecursion is obtained by switching the arguments of the hom-functors and taking the carrier νf of the terminal f-coalgebra instead of μf.

An example illustrating that Mendler algebras give some legitimacy to the "hate-love" relationship between algebra and coalgebra [23, p.x] by subsuming both aspects is *course-of-value iteration.*

▶ **Example 9** (Mendler-style course-of-value iteration (aka histomorphism)[49, 50, 51])**.** A Mendler-style algebra for *course-of-value iteration* with carrier X is a pair $(X, \phi : [-, X \times F =] \to [F-, X])$ where ϕ is required to be strong in [50][6] for reasons that will be explained shortly.

One can read off the terminal sequence $1 \leftarrow X \times F(1) \leftarrow X \times F(X \times F(1)) \leftarrow X \times F(X \times F(X \times F(1))) \leftarrow \dots$ (whenever it exists) of the functor $F^X(A) := X \times F(A)$ how all "earlier" arguments – the "course of values of F" – are made available for reference in the recursion defined by the initial Mendler algebra of this kind.

To a course of values $\zeta_C : F(\nu F^C) \to C$ corresponds via Example 4.3, a strong dinatural transformation with components $\eta_A^{\zeta_C} : (h : A \to C \times FA) \mapsto \mathcal{C}(F([\![h]\!]), C)(\zeta_C)$ where $[\![h]\!]$ is the coiterative function defined by the F^C-coalgebra h. The cofree comonad on F has the underlying functor $C \mapsto \nu F^C$. Denoting its counit by ϵ, we obtain a course of values as the composite $\zeta_{\mu F} : F(\nu F^{\mu F}) \xrightarrow{F(\epsilon_{\mu F})} F(\mu F) \xrightarrow{in} \mu F$ on μF (where in is the unique isomorphism provided by Lambek's lemma [27]) which we can thus regard as canonical. Hence, there is a unique morphism (induced by an $i_{(\mu F, \eta^{\zeta_{\mu F}})} : C_0 \to \mu F$) of course-of-value Mendler algebras $(C_0, In) \to (\mu F, \eta^{\zeta_{\mu F}})$, from the initial such Mendler algebra. We can furthermore for any course of values $\zeta_C : F(\mu F^C) \to C$ derive recursion equations by asking for unique morphisms $(\mu F, \eta^{\zeta_{\mu F}}) \to (C, \eta^{\zeta_C})$.

[6] It is not explicitly required to be strong in [3] but there is an implicit assumption of parametricity, which implies (under conditions) strong dinaturality (see [10, 11]); for the implication to dinaturality see [39].

Examples of recursive functions definable as histomorphisms on the natural numbers are Fibonacci numbers, and the even-number predicate where the “next” value can be computed from the previous two, respectively the immediately preceding, value.

There is a wealth of other examples of Mendler-style algebras (some of which we will discuss later) and in all these cases it is of interest whether they can be reduced to Lambek algebras. Thus, the main concern of this paper is firstly to determine a general form of Mendler-style algebra, and secondly to give a sufficient criterion for a reduction of Mendler-style algebras to Lambek algebras, as well as to clarify the mathematics underlying this reduction.

4 The Two-Sided Fibration Of Diindexed Dinatural Transformations

The results in this section will be used in the next section to provide a sufficient criterion under which Mendler dialgebras (defined in the next section) can be reduced to Lambek algebras. This criterion will be formulated in terms of certain (Grothendieck) (op)fibrations. We begin with the definition of the appropriate total category which will contain the category of Mendler dialgebras as a subcategory.

▶ **Definition 10** (Diindexed (strong) dinatural transformation). *For functors $F, G : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{I} \to \mathcal{V}$, we define the category $\mathcal{W}$ of* diindexed dinatural transformations (ddts)*:*

1. *An object of $\mathcal{W}$ is a pair $((X, Y), \delta)$ where $X, Y : \mathcal{I}$, and $\delta : F(-, =, X) \to G(-, =, Y)$ is a dinatural transformation.*
2. *A morphism $h : ((X, Y), \delta) \to ((X', Y'), \delta')$ is a pair of morphisms $h_0 : X \to X'$, and $h_1 : Y \to Y'$ such that for each $a : \mathcal{C}$ the following diagram commutes*

$$\begin{tikzcd} F(a,a,X) \arrow[r, "F(a,a,h_0)"] \arrow[d, "\delta_a"'] & F(a,a,X') \arrow[d, "\delta'_a"] \\ G(a,a,Y) \arrow[r, "G(a,a,h_1)"] & G(a,a,Y') \end{tikzcd}$$

The definition of the category $\mathcal{W}^s$ of diindexed strong dinatural transformations *is as above except that we additionally require the participating dinatural transformations to be strong.*

We could make the above definition more general by allowing for functors of more than two variables (see [32]) and/or use polycategories (see [7]) to cover models of sequent calculi with fixpoint operators, but the examples we presently seek to cover do not imply the necessity to do so.

The following generalization of Grothendieck fibrations is appropriate for our situation of mixed variance. Two-sided fibrations are also called fibered spans in [43] [21]. Compared to Definition 7 we use from now on the usual notations “$f^*(e)$” for domains of cartesian lifts, and “$g_!(e)$” for codomains of opcartesian ones.

▶ **Definition 11** (Two-sided fibration [43][29]). *A span $\mathcal{A} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{B}$ in the 2-category Cat of categories (or any 2-category) is a* two-sided fibration *if:*

1. *Any morphism of the form $f : b \to p(e)$ has a cartesian lift with codomain e that is q-vertical.*
2. *Any morphism of the form $g : q(e) \to a$ has an opcartesian lift with domain e that is p-vertical.*

3. *Given a cartesian lift $f^*(e) \to e$ of f, and an opcartesian lift $e \to g_!(e)$ of g, it follows from the previous two conditions that the composite $f^*(e) \to e \to g_!(e)$ lies over f via p, and over g via q. In this situation, the induced morphism $g_! f^* e \to f^* g_! e$ is required to be an isomorphism.*

▶ **Proposition 12.** *Let $\mathcal{W}$ be the category defined in Definition 10, $p((X,Y),\delta) := X$, $p(h_0,h_1) := h_0$, and $q((X,Y),\delta) := Y$, $q(h_0,h_1) := h_1$. Then $\mathcal{I} \xleftarrow{q} \mathcal{W} \xrightarrow{p} \mathcal{I}$ is a two-sided fibration. p as well as q are split.*

Proof. By [43] (see also [53, Example 2.1][29]) every lax pullback in the category of (small) categories carries the structure of a fibered span and there is an obvious way to exhibit $\mathcal{W}$ as a lax pullback (every such fibered span is a discrete fibered span but we will not need to make use of this additional fact). In Appendix A we also spelled out an explicit proof. ◀

We denote the p-fiber, respectively the q-fiber, over an object X by $\mathcal{W}_{X-}$, respectively $\mathcal{W}_{-X}$; the simultaneous (p,q)-fiber over X and Y we denote by $\mathcal{W}_{XY}$, and analogous definitions for $\mathcal{W}^s$.

We move on to the last ingredient we still need for the reduction of Mendler dialgebras to Lambek algebras.

(Op)fibered initial (terminal) objects

Provided what we have seen so far, a sufficient criterion for the reduction to Lambek (co)algebras will be that p have a fibered initial object (respectively that q has an opfibered terminal object). The following characterization will be useful for us (for the terminal case, it can be found in [21, Lemma 1.8.8, p.85]).

▶ Remark 13 ((Op)fibered initial (terminal) object). **1.** (Fibered initial object) Let $p : \mathcal{T} \to \mathcal{B}$ denote a fibration, and for $X : \mathcal{B}$ let $0_{\mathcal{T}_X}$ denote the initial object in the p-fiber over X. The following are equivalent:

a. For each X, $0_{\mathcal{T}_X}$ exists and is preserved by reindexing; i.e. for the p-cartesian lifting situation

$$\frac{Cart(h, 0_{\mathcal{T}_{X'}}) : h^*(0_{\mathcal{T}_{X'}}) \to 0_{\mathcal{T}_{X'}}}{h : X \to X'} \downarrow p$$

there is an isomorphism $i : 0_{\mathcal{T}_X} \to h^*(0_{\mathcal{T}_{X'}})$. In particular, composing the isomorphism $i : 0_{\mathcal{T}_X} \to h^*(0_{\mathcal{T}_{X'}})$ with the chosen p-cartesian lift $h^*(0_{\mathcal{T}_{X'}}) \to 0_{\mathcal{T}_{X'}}$ yields a morphism $0_{\mathcal{T}_h} : 0_{\mathcal{T}_X} \to 0_{\mathcal{T}_{X'}}$, which is p-cartesian.

b. There is a fibered left adjoint $\mathbf{0} \dashv p$ from the terminal fibration over $\mathcal{B}$ to p.

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, bend right=10, "t(=p)"'] \arrow[dr, "p"'] & & B \arrow[ll, bend right=10, "\mathbf{0}"'] \arrow[dl, "id"] \\
& \mathcal{B} &
\end{tikzcd}$$

The $\mathbf{0} \dashv p$-counit has as components the canonical morphisms $\iota_w : \mathbf{0}(p(w)) \to w$ and these are p-vertical.

2. (Opfibered terminal object) Let $q : \mathcal{T} \to \mathcal{B}$ denote an opfibration, and for $X : \mathcal{B}$, let $1_{\mathcal{T}_X}$ denote the terminal object in the q-fiber over X. The following are equivalent:

a. For each X, $1_{\mathcal{T}_X}$ exists and is preserved by opreindexing; i.e. for the q-opcartesian lifting situation

$$\frac{opCart(h, 1_{\mathcal{T}_X}) : 1_{\mathcal{T}_X} \to h_!(1_{\mathcal{T}_X})}{h : X \to X'} \downarrow q$$

there is an isomorphism $i : q_!(1_{\mathcal{T}_X}) \to 1_{\mathcal{T}_{X'}}$. In particular, composing the chosen q-opcartesian lift $1_{\mathcal{T}_X} \to h_!(1_{\mathcal{T}_X})$ with the isomorphism $i : q_!(1_{\mathcal{T}_X}) \to 1_{\mathcal{T}_{X'}}$ yields a morphism $1_{\mathcal{T}_h} : 1_{\mathcal{T}_X} \to 1_{\mathcal{T}_{X'}}$, which is q-opcartesian.

b. There is an opfibered right adjoint $q \dashv \mathbf{1}$ from the terminal fibration over $\mathcal{B}$ to q

$$\begin{tikzcd} E \arrow[rr, bend left, "t(=q)"] \arrow[dr, "q"'] & & B \arrow[ll, bend left, "\mathbf{1}"] \arrow[dl, "id"] \\ & B & \end{tikzcd}$$

The $q \dashv \mathbf{1}$-unit has as components the canonical morphisms $\pi : w \to \mathbf{1}(q(w))$ and these are q-vertical.

For simplicity, we sometimes identify $\mathbf{1}(h : X \to X')$ with the opcartesian lift of h with respect to $1_{\mathcal{T}_X}$.

▶ Remark 14. For ease of later reference we summarize the adjunctions $\begin{tikzcd} \mathcal{I} \arrow[r, bend left, "\mathbf{0}"] & \mathcal{W} \arrow[l, "p"] \arrow[r, "q"] & \mathcal{I} \arrow[l, bend left, "\mathbf{1}"] \end{tikzcd}$.

1. For the $\mathbf{0} \dashv p$ adjunction
   a. The unit $\eta : id \to p\mathbf{0}$ is the identity natural transformation.
   b. The counit $\iota : \mathbf{0}p \to id$ is not (generally) the identity, but it is p-vertical.
2. For the $q \dashv \mathbf{1}$ adjunction
   a. The unit $\pi : id \to \mathbf{1}q$ is not (generally) the identity, but it is q-vertical.
   b. The counit $\epsilon : q\mathbf{1} \to id$ is the identity.

Recall that each canonical map ι_w is p-vertical, and each canonical map π_w is q-vertical. (It is neither generally true that ι_w is q-vertical, nor that π_w is p-vertical.)

For later reference, we spell out:

▶ **Lemma 15.**

1. *Assuming p-fibered initial objects, for all $h : ((X, X), \delta) \to ((X', X'), \delta')$ the diagrams of the form on the left below commute.*
2. *Assuming q-opfibered terminal objects, for all $h : ((X, X), \delta) \to ((X', X'), \delta')$ the diagrams of the form on the right below commute.*

$$\begin{tikzcd} q(0_{\mathcal{W}_{X-}}) \arrow[r, "q(\iota_{((X,X),\delta)})"] \arrow[d, "q(0_{\mathcal{W}_{h_0 -}})"] & X \arrow[d, "h_1"] \\ q(0_{\mathcal{W}_{X'-}}) \arrow[r, "q(\iota_{((X',X'),\delta')})"] & X' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "p(\pi_{((X,X),\delta)})"] \arrow[d, "h_0"] & p(1_{\mathcal{W}_{-X}}) \arrow[d, "p(0_{\mathcal{W}_{h_0 -}})"] \\ X' \arrow[r, "p(\pi_{((X',X'),\delta')})"] & p(1_{\mathcal{W}_{-X'}}) \end{tikzcd}$$

Proof. (1) Apply q to

$$\begin{tikzcd} \mathbf{0}(p((X, X), \delta)) \arrow[r, "\iota_{((X,X),\delta)}"] \arrow[d, "\mathbf{0}(p(h))"] & ((X, X), \delta) \arrow[d, "h"] \\ \mathbf{0}(p((X', X'), \delta')) \arrow[r, "\iota_{((X',X'),\delta')}"] & ((X', X'), \delta') \end{tikzcd}$$

The dual statement (2) about counits can be seen analogously. ◀

5 Reduction Of Mendler Dialgebras To Lambek (Co)algebras

We are now ready to explain the reducibility of Mendler dialgebras to Lambek algebras; this generalizes and clarifies results from [48].

▶ **Definition 16** (Mendler Dialgebra). *We denote the "diagonal" of $\mathcal{W}$, i.e. the subcategory of $\mathcal{W}$ with objects w satisfying $p(w) = q(w)$, and morphisms h satisfying $p(h) = q(h)$, by $Alg^{\mathcal{W}}$, and call this the category of* Mendler dialgebras.

For the following to be well-defined, we need to assume that (op)fibered initial (terminal) objects are chosen (as opposed to be merely existent). For the reduction of Mendler dialgebras to

Lambek algebras, we assume that p has a chosen fibered initial object. (5.1)

Lambek coalgebras, we assume that q has a chosen opfibered terminal object. (5.2)

the endofunctors for which Mendler dialgebras (under the given assumptions) will be Lambek algebras are defined as follows:

▶ **Definition 17.**

1. *Provided (5.1), we define a functor*

$$\begin{aligned} &\underline{w} : \mathcal{I} \to \mathcal{I} \\ &\quad \underline{w}(X) := q(0_{\mathcal{W}_{X-}}) \\ &\quad \underline{w}(f : X \to Z) := q(\iota_{f^* 0_{\mathcal{W}_{Z-}}}) \end{aligned}$$

   This is well defined because $q(f^* 0_{\mathcal{W}_{Z-}}) = q(0_{\mathcal{W}_{Z-}})$.

2. *Provided (5.2), we define a functor*

$$\begin{aligned} &\underline{m} : \mathcal{I} \to \mathcal{I} \\ &\quad \underline{m}(X) := p(1_{\mathcal{W}_{-X}}) \\ &\quad \underline{m}(g : X \to Z) := p(\pi_{g_! 1_{\mathcal{W}_{-X}}}) \end{aligned}$$

   This is well defined because $p(g_! 1_{\mathcal{W}_{-X}}) = p(1_{\mathcal{W}_{-X}})$.

The previous definition can be understood in terms of the category-of-elements form of representability, cf. [21, Section 1.8], as follows.

▶ Remark 18 (Fibred initial objects as representability). For fixed $a \in \mathcal{I}$, write $P_a(b) := \mathrm{Dinat}(F(-,=,a), G(-,=,b))$ where on the right dinatural transformations between the respective difunctors are meant, with functoriality in b given, for $h : b \to b'$, by post-composition along $G(-,=,h)$. Then the p-fiber $\mathcal{W}_{a-}$ is the category of elements of P_a; hence $\mathcal{W}_{a-}$ has an initial object iff P_a is representable, i.e. iff there is an object $H(a) \in \mathcal{I}$ and a natural isomorphism $P_a(b) \cong \mathcal{I}(H(a), b)$. In that case, if $0_{\mathcal{W}_{a-}} = ((a, H(a)), \eta_a)$ is the universal element corresponding to $\mathrm{id}_{H(a)}$, then the functor of Definition 17 satisfies $\underline{w}(a) = q(0_{\mathcal{W}_{a-}}) = H(a)$. Equivalently, the $\mathcal{V}$-valued profunctor $P : \mathcal{I}^{op} \times \mathcal{I} \to \mathcal{V}$, defined by $P(a, b) := \mathrm{Dinat}(F(-,=,a), G(-,=,b))$, is represented in its second variable by the endofunctor $\underline{w} : \mathcal{I} \to \mathcal{I}$.

The operations sending Mendler dialgebras to Lambek (co)algebras and vice versa are given by:

▶ **Definition 19.**

1. *Let* $((X,X),\delta) \in Alg^{\mathcal{W}}$.
   a. *Assuming that* p *has a fibered initial object, (5.1), we define a* $\underline{w}$*-algebra by* $\mathbb{A}((X,X),\delta) := (X, q(\iota_{((X,X),\delta)}))$.
   b. *Assuming that* q *has an opfibered terminal object, (5.2), we define a* $\underline{m}$*-coalgebra by* $\mathbb{C}((X,X),\delta) := (X, p(\pi_{((X,X),\delta)}))$.
2. a. *Assuming that* p *has a fibered initial object, (5.1), we define* $\mathbb{M}^{alg}(X, \alpha : \underline{w}(X) \to X) := \alpha_!(0_{\mathcal{W}_{X-}})$.
   b. *Assuming that* q *has an opfibered terminal object, (5.2), we define* $\mathbb{M}^{coalg}(X, \alpha : X \to \underline{m}(X)) := \alpha^*(1_{\mathcal{W}_{-X}})$.

The operations just defined all extend to functors; for 1a and 1b, the action on morphisms is spelled out in Lemma 15.

▶ Remark 20 (Reduction to, and lifting of a bialgebra). Definition 19 1a and 1b taken together, we can define a so-called $(\underline{w}, \underline{m})$-bialgebra

$$\mathbb{D}((X,X),\delta) := \big(X,\ q(\iota_{((X,X),\delta)}),\ p(\pi_{((X,X),\delta)})\big).$$

That is, the carrier X is equipped with a $\underline{w}$-algebra structure $q(\iota_{((X,X),\delta)}) : \underline{w}X \to X$ and a $\underline{m}$-coalgebra structure $p(\pi_{((X,X),\delta)}) : X \to \underline{m}X$. Generally, a bialgebra consists of a carrier that is an algebra for one functor, and a coalgebra for another functor (and as such is a special form of *dialgebra* [28][18]) such that there is a natural transformation – in this context called a *distributive law* [5] – between the composite functors. In the present case, this natural transformation $d : \underline{wm} \to \underline{mw}$ is the composite $d = p\pi\mathbf{0} \circ \eta \circ \eta \circ q\iota\mathbf{1}$ where $\iota, \pi, \epsilon, \eta$ were defined in Remark 14 (in particular ϵ and η are both the identity). Usually, one of the functors is a monad or comonad and the transformation is additionally required to be compatible with this (co)monad structure (see [5], or [22] for a more recent reference). But in our case neither $\underline{w}$ nor $\underline{m}$ generally carry such additional structure. One can also lift a bialgebra to a Mendler dialgebra, see Appendix C.

▶ **Corollary 21.** *The pairs of functors defined in Definition 19 are inverses.*

Proof. See Appendix B. ◀

The previous corollary does not depend on the specifics of the category $\mathcal{W}$ but only on the two-sided fibered structure.

▶ **Theorem 22.** *Let* $\mathcal{I} \xleftarrow{p} \mathcal{T} \xrightarrow{q} \mathcal{I}$ *be any two-sided split fibration where* p *has a fibered initial object, respectively* q *has an opfibered terminal object. Then there is an endofunctor* w *respectively* m *on* $\mathcal{I}$ *such that the "diagonal"*[7] $Alg^{\mathcal{T}}$ *of* $\mathcal{T}$ *is isomorphic to the category of algebras for* w*, respectively to the category of coalgebras for* m*.*

Proof. Inspection of Definition 17, Definition 19, and the proof of Corollary 21. ◀

6 Examples

All recursion schemes from [3] are examples of Mendler dialgebras. Apart from classical (covariant) Mendler algebras (as in Definition 6), the authors list the histomorphism (see Section 3), the so-called "abstract inverse" which we will explain shortly, "higher-order"

[7] Of course defined in analogy to Definition 16 as the subcategory of $\mathcal{T}$ on which p and q coincide.

versions of all systems which are categorically simply obtained by replacing the base category by categories of endofunctors, as well as combinations of these systems which can be obtained by taking products of hom-sets (see Example 27). Since the "abstract inverse", a Mendler variant for defining higher-order abstract syntax, is a paradigmatic example for recursion schemes of mixed variance, we will focus on this example.

Freyd dialgebras and higher-order abstract syntax

The significance of *higher-order abstract syntax*, i.e. the definition of a datatype whose terms are to be interpreted as lambda terms of some object language is to use meta-language variables in place of object-language variables which simplifies certain routines such as capture-avoiding substitution [52, p.1]. A motivating example is the untyped lambda calculus where the signature functor $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is of mixed variance and defined by $T(X, Y) = (X \to Y) + (Y \times Y) + V$ for some fixed set of variables V. Since it is not possible to define a category of Lambek algebras of the form $(X, \alpha : T(X, X) \to X)$ as there is no appropriate notion of morphism, Freyd [14] (see also [13]) defined the following notion which served as a basis for further developments in encoding higher-order abstract syntax in [33], [52] and [3].

▶ **Definition 23** (Freyd dialgebra). *A* Freyd dialgebra *for a difunctor $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is a pair $(X, \alpha : TXX \to X, \alpha' : X \to TXX)$. A* homomorphism of Freyd dialgebras *is a pair $(m : X \to Y, m' : Y \to X) : (X, \alpha, \alpha') \to (Y, \beta : TYY \to Y, \beta' : Y \to TYY)$ such that $(\beta \circ Tm'm = m \circ \alpha) \wedge (Tmm' \circ \beta' = \alpha' \circ m')$.*

▶ Remark 24 (Freyd Dialgebras vs. Mendler Dialgebras). Freyd dialgebras (α, α') give rise to (strong) Mendler dialgebras of the form $[\alpha', -] \times [-, \alpha] : [T(X, X), -] \times [-, T(X, X)] \to [X, -] \times [-, X]$, and morphisms of Mendler dialgebras induced by a morphism $(m, m') : (X, X) \to (Y, Y)$ in $\mathcal{C}^{op} \times \mathcal{C}$ are exactly those pairs (m, m') constituting morphisms of Freyd dialgebras. We display the correspondence for morphisms in the following diagrams showing how the morphism (m, m') gives rise to a family of maps of the form $(\psi, \psi') \mapsto (\psi \circ T(m, m'), T(m', m) \circ \psi'$, $(h, h') \mapsto (h \circ m', m \circ h')$ natural in A constituting the vertical maps in the upper square where $\delta_A := [\alpha', -] \times [-, \alpha]$ and $\gamma_A := [\beta', -] \times [-, \beta]$ (and (h, h') which is not in the picture factors for verification of commutation as $(\psi \circ \alpha', \alpha \circ \psi')$).

$$\begin{tikzcd}
{[TXX, A] \times [A, TXX]} \arrow[r, "\delta_A"] \arrow[d] & {[X, A] \times [A, X]} \arrow[d] \\
{[TYY, A] \times [A, TYY]} \arrow[r, "\gamma_A"] & {[Y, A] \times [A, Y]}
\end{tikzcd}$$

$$\forall A \forall \psi'. \quad \begin{tikzcd}
TXX \arrow[d, "\alpha"] & A \arrow[l, "\psi'"'] \arrow[r, "\psi'"] & TXX \arrow[d, "Tm'm"] \\
X \arrow[r, "m"] & Y & TYY \arrow[l, "\beta"']
\end{tikzcd} \qquad iff \qquad \begin{tikzcd}
TXX \arrow[r, "Tm'm"] \arrow[d, "\alpha"'] & TYY \arrow[d, "\beta"] \\
X \arrow[r, "m"] & Y
\end{tikzcd}$$

$$\forall A \forall \psi. \quad \begin{tikzcd}
TYY \arrow[d, "Tmm'"'] & Y \arrow[l, "\beta'"'] \arrow[r, "m'"] & X \arrow[d, "\alpha'"] \\
TXX \arrow[r, "\psi"] & A & TXX \arrow[l, "\psi"']
\end{tikzcd} \qquad iff \qquad \begin{tikzcd}
Y \arrow[r, "m'"] \arrow[d, "\beta'"'] & X \arrow[d, "\alpha'"] \\
TYY \arrow[r, "Tmm'"] & TXX
\end{tikzcd}$$

Meijer and Hutton proposed in [33] an implementation of Freyd dialgebras as a realization of higher-order abstract syntax. Given the involutive nature of Freyd dialgebras (see [13]), defining a recursive function out of the "initial Freyd dialgebra" (also called a minimal invariant object) amounts to simultaneously defining a corecursive function into it. In applications to higher-order abstract syntax this is not the desired formulation: the corecursive contribution is intended to be determined by the recursive part.

Fegaras and Sheard [12], and later Washburn and Weirich [52], address this issue by using a form of abstract inverse, relying on parametricity to control the additional structure. Categorically, this should be read against the standard relation between relational parametricity, naturality, and dinaturality: dinaturality follows from relational parametricity in the sense of Plotkin–Abadi [39], while the categorical role of natural and dinatural transformations in polymorphism goes back at least to Bainbridge–Freyd–Scedrov–Scott [4]. For the relation with strong dinaturality, see also [10].

Three issues meet in this example. The first is higher-order abstract syntax itself, which introduces mixed-variance signatures. The second is the use of relational parametricity in the programming-language literature to control the abstract inverse. The third is the categorical distinction between dinatural and ordinary natural transformations. We will not develop the relational parametric semantics here. The point relevant for the present paper is instead that the schemes in [3] are written in a polymorphic naturality-style form, while the basic mixed-variance Mendler algebra is naturally formulated as a dinatural transformation. We therefore recall the Dubuc–Street comparison between dinaturality and ordinary naturality.

▶ Remark 25 (Dinaturality vs. naturality).

1. We recall the Dubuc–Street comparison between dinaturality and ordinary naturality. Let $H, K : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ be bifunctors, with $\mathcal{C}$ small and with the required ends existing in $\mathcal{V}$. Dubuc and Street associate to K a bifunctor $\overline{K} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ defined by $\overline{K}(A, B) := \int_C \big(\mathcal{C}(B, C) \times \mathcal{C}(C, A)\big) \pitchfork K(C, C)$. In the present paper $\mathcal{V}$ is the category of sets, so the cotensor $\pitchfork$ is simply the function-set construction $(-) \to (-)$. Thus, in our notation,
$$\overline{K}(A, B) = \int_C \big(\mathcal{C}(B, C) \times \mathcal{C}(C, A) \to K(C, C)\big).$$
The Dubuc–Street theorem [9, Theorem 2] gives a natural isomorphism $dn_{(\mathcal{C},\mathcal{V})}(H, K) \simeq n(H, \overline{K})$, where dn denotes dinatural transformations and n ordinary natural transformations. Hence a dinatural transformation into K may equivalently be presented as an ordinary natural transformation into the Dubuc–Street transform $\overline{K}$.
2. The most basic mixed-variance Mendler algebra is obtained by taking $f : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ and, for a carrier $X \in \mathcal{C}$, considering a dinatural transformation $[Idd(-, =), X] \to [f(-, =), X]$, i.e. an element of $dn_{(\mathcal{C},\mathcal{V})}([Idd(-, =), X], [f(-, =), X])$. This is the elementary instance of the Mendler-dialgebra form considered in this paper.
Applying the Dubuc–Street comparison to $K(A, B) := [f(A, B), X]$ rewrites this dinatural datum as an ordinary natural transformation $[Idd(-, =), X] \to \overline{K}$, where
$$\overline{K}(A, B) = \int_C \big(\mathcal{C}(B, C) \times \mathcal{C}(C, A) \to [f(C, C), X]\big).$$
This rewrite into naturality form is to be borne in mind when reading the scheme $\Pi_a\, [X, aX] \times [aX, X] \to [f(aX), X]$ found in [3] (where a is an additional parameter that can act on the carrier). That scheme is not literally the basic mixed-variance Mendler algebra above, but a variant of the basic scheme with an additional parameter. In the

framework of the present paper, such variants are expressed as Mendler dialgebras by separating positive and negative occurrences into the appropriate domain and codomain data. The point of the Dubuc–Street comparison here is only to explain why the dinatural formulation used in this paper may appear in [3] in naturality-style, polymorphic notation.

In context of higher-order abstract syntax one finds the terminology of the "abstract inverse" which can be explained as follows.

▶ **Example 26** (Abstract inverse [3, 12])**.** For $f : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, $F, G : (\mathcal{C} \to \mathcal{C})^{op} \times (\mathcal{C} \to \mathcal{C}) \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ are defined by $F(a, a', X, X') := [X, a'X'] \times [aX, X']$, and $G(a, a', X, X') := [f(a'X', aX), X']$. The action on morphisms is induced by the hom-functors.

Evaluating the initial Mendler dialgebra of this type over the identity we obtain the diagrammatic recursion equation on the left, i.e. the canonical morphism from the initial Mendler dialgebra of this signature to an arbitrary Mendler dialgebra ϕ of this signature (shown is the component over (h', h)).

$$\begin{tikzcd} f(\mu f, \mu f) \arrow[r, "in_{id}(id,id)"] \arrow[d, "f(h',h)"] & \mu f \arrow[d, "h"] \\ f(X, X) \arrow[r, "\phi_{(h\circ h', h\circ h')}"] & X \end{tikzcd} \qquad \begin{tikzcd} Exp \times Exp \arrow[r, "ev \times ev"] \arrow[d, "a"] & Val \times Val \arrow[d, "a'"] \\ Exp \arrow[r, "ev"] & Val \end{tikzcd} \qquad \begin{tikzcd} {[Exp, Exp]} \arrow[r, "{[ev, un]}"] \arrow[d, "l"] & {[Val, Val]} \arrow[d, "l'"] \\ Exp \arrow[r, "ev"] & Val \end{tikzcd}$$

Instantiating with the motivating example, i.e. with the signature $f(X', X) := (X' \to X) + (X \times X)$, we obtain (by taking the coproduct apart) the familiar recursion equations for an interpreter ev of the syntax of untyped lambda calculus with application and abstraction (displayed in the middle and right square above) where we denoted the constructors of Exp by a and l, the algebra structure of Val by a' and l', and ev denotes evaluation and un the corresponding map in the opposite direction, as customary in the context of the abstract inverse.

For the sake of completeness, we mention that we can obtain all schemes in [3] by combining schemes we already discussed by taking products of hom functors; e.g. combining the histomorphism (which, to recall, was the Mendler dialgebra defined by $F(a, b, X) := [a, fb] \times [a, X]$ and $G(a, b, X) := [fa, X]$) with the abstract inverse:

▶ **Example 27** ("Histomorphism $\times$ abstract inverse")**.** For $f : \mathcal{C} \to \mathcal{C}$, $F, G : (\mathcal{C} \to \mathcal{C})^{op} \times (\mathcal{C} \to \mathcal{C}) \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ are defined by $F(a, a', X, X') := [aX, f(a'X)] \times [X, a'X'] \times [aX, X']$, and $G(a, a', X, X') := [f(aX), X']$.

An example for reduction of a scheme

For clarity and emphasis we spell out how the reduction of a scheme looks like in the basic case of a Mendler dialgebra arising from a Freyd dialgebra.

▶ Remark 28 (Freyd dialgebras and the reduction to first-order algebras). We spell out the endofunctor obtained from Definition 17 in the Freyd case considered in Remark 24. Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ be a mixed-variance bifunctor and put $\mathcal{I} := \mathcal{C}^{op} \times \mathcal{C}$. We consider the following representable subcategory of the diindexed dinatural transformations of Definition 10. Its objects are triples $((X, Y), (U, V), \delta)$, where $(X, Y), (U, V) \in \mathcal{I} = \mathcal{C}^{op} \times \mathcal{C}$ and δ is a dinatural transformation

$$\delta : [T(Y, X), -] \times [-, T(X, Y)] \to [U, -] \times [-, V]$$

required to be uniquely of the form $\delta = [s', -] \times [-, s]$, for maps $s' : U \to T(Y, X)$ and $s : T(X, Y) \to V$ in $\mathcal{C}$. A morphism $((X, Y), (U, V), \delta) \to ((X', Y'), (U', V'), \delta')$ is a pair $h = (h_0, h_1)$, with $h_0 : (X, Y) \to (X', Y')$ and $h_1 : (U, V) \to (U', V')$ in $\mathcal{I}$, such that the square of Definition 10 commutes. The projections are $p((X, Y), (U, V), \delta) = (X, Y)$ and $q((X, Y), (U, V), \delta) = (U, V)$. The diagonal objects in this ambient category are the Mendler dialgebras of Definition 16; the Freyd dialgebras of Definition 23 give such diagonal objects by Remark 24.

Fix $(X, Y) \in \mathcal{I}$. The object $0_{(X,Y)} = \Big(((X, Y), (T(Y, X), T(X, Y))), \mathrm{id}\Big)$ lies in the ambient p-fiber over (X, Y), since its p-projection is (X, Y). Here id is the identity transformation

$$id : [T(Y, X), -] \times [-, T(X, Y)] \longrightarrow [T(Y, X), -] \times [-, T(X, Y)].$$

This object is not required to be a Mendler dialgebra; it is an object of the ambient p-fiber.

We claim that $0_{(X,Y)}$ is initial in this p-fiber. Let $\big((X, Y), (U, V), \delta\big)$ be another object in the same p-fiber, with $\delta = [s', -] \times [-, s]$, where $s' : U \to T(Y, X)$ and $s : T(X, Y) \to V$ in $\mathcal{C}$. A morphism $0_{(X,Y)} \to \big((X, Y), (U, V), \delta\big)$ in the p-fiber is a morphism in the total category whose p-component is $\mathrm{id}_{(X,Y)}$. Hence it is determined by a morphism $h' = (s', s) : (T(Y, X), T(X, Y)) \to (U, V)$ in $\mathcal{I} = \mathcal{C}^{op} \times \mathcal{C}$, i.e. by maps $s' : U \to T(Y, X)$ and $s : T(X, Y) \to V$ in $\mathcal{C}$, and the square of Definition 10 is $([s', -] \times [-, s]) \circ \mathrm{id} = \delta$. This holds for the displayed h', while uniqueness follows from the assumed unique carrier representation of δ.

Therefore $0_{(X,Y)}$ is initial in the ambient p-fiber over (X, Y). Consequently, the general formula for $\underline{w} : I \to I$, of Definition 17 gives

$$\underline{w}((X, Y)) = q(0_{(X,Y)}) = (T(Y, X), T(X, Y)).$$

The action on morphisms u is given by

$$\underline{w}(u) := q(\iota_{u^*0_{\mathcal{W}(X',Y')-}}) = \big(T(u_2, u_1), T(u_1, u_2)\big) : (T(Y, X), T(X, Y)) \to (T(Y', X'), T(X', Y'))$$

where $u = (u_1, u_2) : (X, Y) \to (X', Y')$ in $\mathcal{I} = \mathcal{C}^{op} \times \mathcal{C}$.

▶ Remark 29 (Freyd's diagonal category and Fiore's doubling trick). The preceding remark shows that, in the Freyd case, the abstract functor $\underline{w}$ of Definition 17 is the endofunctor $\underline{w} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}^{op} \times \mathcal{C}$ given by $\underline{w}((X, Y)) = (T(Y, X), T(X, Y))$. This is the same first-order endofunctor which appears in the classical reductions of mixed-variance recursive types, i.e. types specified by a difunctor of shape $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ to ordinary inductive types, i.e. types defined by an endofunctor.

Freyd's route [14] starts from his custom notion of dialgebra Definition 23. For a bifunctor $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, a T-dialgebra is an object X with maps $f : T(X, X) \to X$ and $f' : X \to T(X, X)$. Freyd calls (E, f, f') free if, for every T-dialgebra (A, a, a'), there is a unique pair $x : E \to A$ and $x' : A \to E$ satisfying the two dialgebra morphism equations $xf = a\,T(x', x)$ and $f'x' = T(x, x')a'$. Thus the universal property is already two-sided at the level of morphisms. Since, however, Freyd's carriers are single objects $X \in \mathcal{C}$, he packages this two-sided morphism data by passing to the diagonal category: it has the same objects as $\mathcal{C}$, but a morphism $X \to Y$ is a pair $\mathcal{C}(X, Y) \times \mathcal{C}(Y, X)$. Equivalently, up to swapping the two components, this is the full subcategory of $\mathcal{C}^{op} \times \mathcal{C}$ on the diagonal objects (X, X). Under this identification, the endofunctor induced by T is the diagonal restriction of the doubled functor, sending (X, X) to $(T(X, X), T(X, X))$. Freyd proves the corresponding di-Lambek lemma and relates regular free dialgebras to minimal invariant objects.

Fiore's route avoids this diagonal category and instead works on the doubled category. An involution on a category is a self-duality $D : \mathcal{D}^{op} \to \mathcal{D}$ with $D^2 = \mathrm{id}$; the example relevant here is $\mathcal{D} = \mathcal{C}^{op} \times \mathcal{C}$, with involution exchanging the two factors. An endofunctor $S : \mathcal{D} \to \mathcal{D}$ is symmetric if it commutes with this involution. The bifunctor $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ determines the symmetric endofunctor $S_T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}^{op} \times \mathcal{C}$ defined by $S_T(X, Y) = (T(Y, X), T(X, Y))$. Fiore's doubling trick is precisely the passage from the mixed-variance bifunctor T, which is not itself an endofunctor, to this symmetric endofunctor S_T on the involutory doubled category [13, Section 6.3].

The point of the construction is that it identifies the recursive fixed-point problem for T with an ordinary initial-algebra problem for S_T. A fixed point of T is an object X with an isomorphism $T(X, X) \cong X$. Fiore calls such a fixed point minimal if it is a retract of every other fixed point of T; this is a categorical minimality condition, not an order-theoretic least-element condition. If $\mathcal{C}$ is parameterised algebraically compact, meaning in particular that initial algebras are free in the sense that their inverses are final coalgebras, then $\mathcal{C}^{op} \times \mathcal{C}$ is again algebraically compact and the initial S_T-algebra yields the canonical minimal fixed point of T [13, Corollary 6.2.7 and Theorem 6.4.1]. Thus Fiore does not define Freyd dialgebras and then compare them with ordinary algebras; rather, he replaces T by the doubled endofunctor S_T, and proves that the initial algebra of S_T supplies the desired canonical/minimal recursive type for T.

Our construction of $\underline{w}$ generalises Freyd's and Fiore's in the sense that it is not limited to signatures of the form $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ and does not require the ambient category $\mathcal{C}$ to be algebraically compact, but still recovers their endofunctor in that special case, namely $\underline{w}((X, Y)) = (T(Y, X), T(X, Y))$.

Example 27 was introduced in [45] to realize a nested inductive type. Generally[8], a nested inductive type is defined in two steps: firstly, a signature functor X_0 for the nested type is itself defined by induction, and secondly the nested type is obtained as the initial algebra of the signature functor X_0. Usually (see [6, 24]), the signature X_0 is obtained as the initial Lambek algebra for a functor H between functor categories (a "higher-order functor"), and the same can equivalently be achieved by instantiating the basic covariant Mendler scheme Definition 6 with H. The following scheme defines X_0 without explicit use of a "higher-order functor"; the signature f for the scheme has the same type as the carriers of the scheme.

▶ **Example 30** (Component-indexed Mendler algebra [45]). For $f : \mathcal{C} \to \mathcal{C}$, $F, G : \mathcal{C}^{op} \times \mathcal{C} \times (\mathcal{C} \to \mathcal{C}) \to \mathcal{V}$ are defined by $F(a, b, X) := [a, Xb]$ and $G(a, b, X) := [fa, Xb]$.

Assuming impredicative quantification, we can now for instance compute the functor $\underline{w}$ induced by this system where the action on objects is given by
$\underline{w}(X) = \Pi_{Y:\mathcal{C}\to\mathcal{C}}\Pi_{\phi:[-,X-]\to[f-,Y-]}Y$. Indeed $(((X, \Pi_{Y:\mathcal{C}\to\mathcal{C}}\Pi_{\phi:[-,X=]\to[f=,Y-]}Y-), in^X), i)$ where $in^X_a(\alpha)(x) = (Y \mapsto \phi_a \mapsto (\phi_a(\alpha)(x)))$ is initial over X: given $((X, Y), \Psi)$, the unique morphism displayed below is induced by $i_{((X,Y),\Psi)} : \eta \mapsto \eta Y \Psi$.

$$\begin{tikzcd}
{[a, Xa]} \arrow[r, "id"] \arrow[d, "in^X_a"] & {[a, Xa]} \arrow[d, "\Psi_a"] \\
{[fa, \Pi_{Y:\mathcal{C}\to\mathcal{C}}\Pi_{\Phi_a:[a,Xa]\to[fa,Ya]}Ya]} \arrow[r, "{[fa, i_{((X,Y),\Psi)}]}"] & {[fa, Ya]}
\end{tikzcd}$$

i.e. $\underline{w}$ is the "higher-order functor" implicit in the system and one can show that Definition 6 instantiated with $\underline{w}$ defines the same nested type as the above system.

[8] There is a spectrum of terminologies in use and there are of course different ways in which a type can be nested: for example the type of "rose trees" and that of ordinal notations [1] fit our nomenclature here, but sometimes also initial algebras of certain higher-order functors themselves, i.e. (certain) inductive families such as "List" or "Tree", are called "nested types" (cf. [2] and [15]).

7 Cones Revisited

This section can be contextualized in the so-called "extended initial-algebra semantics" of [24]. On the level of generality of Mendler dialgebras we can (so far) say little in the direction of constructing initial objects in this category following [26]; nevertheless we can make some observations related to the variant Yoneda lemmas in Example 4. Recall that the "Church encoding" $\Pi_X(F(X) \to X) \to X = \mu F$ of the initial Lambek F-algebra can be interpreted by restriction of the Π-type to strong dinatural[9] transformations $sd_{(\mathcal{C},\mathcal{V})}([F-,=], Idd(-,=)) \simeq \mu F$. Also recall that we generally have:

▶ Remark 31. Let $p : \mathcal{A} \to \mathcal{I}$ be a functor, let $\mathcal{A}$ have an initial object $0_{\mathcal{A}}$, and let i be the assignment, to any object w, of the canonical morphism $i_w : 0_{\mathcal{A}} \to w$. Then $(p(0_{\mathcal{A}}), p(i))$ is a limiting cone for p.

For p being the forgetful functor U_F from a category of Lambek F-algebras, the converse is true as well [16]. This converse fails[10] already for dialgebras. But we still can consider the analog of $sd_{(\mathcal{C},\mathcal{V})}([F-,=],[C,=]) \simeq Cone_{U_F}(C)$, and observe the following for $p : Alg^{\mathcal{W}} \to \mathcal{I}$: Write $C/ : \mathcal{I}^{op} \times \mathcal{I} \to Cat$ for the (degenerate) pseudofunctor defined by $(X,Y) \mapsto [C,Y]$. It follows from Proposition 12 via the Grothendieck construction for two-sided fibrations [29] that $\mathcal{I} \xleftarrow{p} \mathcal{W} \xrightarrow{q} \mathcal{I}$ defines a pseudofunctor $\underline{\mathcal{W}} : \mathcal{I}^{op} \times \mathcal{I} \to Cat$.

▶ **Proposition 32.** *A cone for the functor $p : Alg^{\mathcal{W}} \to \mathcal{I}$ with vertex C is a strong (pseudo[11]) dinatural transformation $\Theta : \underline{\mathcal{W}} \to C/$.*

The proof follows from p being *elementary* (i.e. isomorphism-reflecting and faithful). There is an obvious dual statement for cocones.

For $p, q : \mathcal{W} \to \mathcal{I}$ not restricted to the "diagonal", the above characterization of (co)cones does not hold. An observation in this case is that assuming (op)fibered initial (terminal) objects, we can exchange (co)cones for p with (co)cones for q:

▶ Remark 33.

1. If (L, λ) is a cone for p, then $(q0(L), q(\iota) \circ \lambda)$ is a cone for q.
2. If (R, ρ) is a cocone for q, then $(p1(R), \rho \circ p(\pi))$ is a cocone for p.

8 Conclusion

We introduced the notion of Mendler dialgebra providing an extension of the existing semantics of Mendler-style recursion schemes to cover a variety of examples of schemes of mixed variance using diindexed dinatural transformations. We explained the reduction of Mendler dialgebras to Lambek (co)algebras via universal properties of two-sided fibrations with (op)fibered initial (terminal) objects, and observed that this reduction depends only on these universal properties and not on any other specifics of the total category.

We discussed the example of higher-order abstract syntax which is paradigmatic for mixed variance recursion schemes. In future we would like to explore how definitions via Mendler dialgebras relate to inductive-inductive definitions [37] which can e.g. define the syntax of type theory by a mutual definition of types and contexts, and whose semantics is given by (certain) dialgebras (aka subequalizers or inserters).

[9] For a syntactic discussion of dinatural terms in second-order lambda calculus, see [8].
[10] Still, one can always obtain a weakly initial object from a free object on this limit.
[11] By "pseudo dinatural transformation" we mean the appropriate notion of dinatural transformation of pseudofunctors.

We also made some observations about the forgetful functor from the category of Mendler dialgebras related to free objects in this category and would like to say more about constructing initial Mendler dialgebras in future.

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A Proof of Proposition 12

After observing what the (op)cartesian lifts are, the proof of Proposition 12 is a simple verification of the conditions; the interest of including it is that some of its details are needed in practice when reducing particular Mendler dialgebras to Lambek algebras (see Example 30).

1. A cleavage $Cart$ is given as follows:
for $f : Z \to p((X,Y),\delta)$, we obtain a lift $Cart(f,((X,Y),\delta)) : f^*((X,Y),\delta) \to ((X,Y),\delta)$ by $(\delta \circ F(-,-,f), id) : ((Z,Y),\delta \circ F(-,-,f)) \to ((X,Y),\delta)$; diagrammatically:

$$\begin{tikzcd}
F(a,b,Z) \arrow[d, "F(a,b,f)"'] \arrow[dr] & & & F(a,b,X) \arrow[dl] \\
F(a,b,X) \arrow[dr] & F(a,a,Z) \arrow[r, "F(a,a,f)"] \arrow[d, "F(a,a,f)"] & F(a,a,X) \arrow[dd, "\delta_a"] & \\
& F(a,a,X) \arrow[d, "\delta_a"] & & \\
& G(a,a,Y) \arrow[r, "id"] \arrow[dl] & G(a,a,Y) \arrow[dr] & \\
G(b,a,Y) & & & G(b,a,Y)
\end{tikzcd}$$

This lift is cartesian because the lifting problem

$$\begin{tikzcd}
p((W,Q),\delta') \arrow[d, dashed, "g"'] \arrow[dr, "p(\eta)"] & \\
Z \arrow[r, "f"] & p((X,Y),\delta)
\end{tikzcd}$$

is (uniquely) solved by

$$\begin{tikzcd}
& F(a,a,W) \arrow[d, "\delta'_a"] \arrow[ddr, "\eta^p_a"] \arrow[ddl, dotted, "F(a,a,g)"'] & \\
& G(a,a,Q) \arrow[dddr, "\eta^q_a"] \arrow[ddl, dotted, "\eta^q_a"'] & \\
F(a,a,Z) \arrow[rr, "F(a,a,f)"] \arrow[d, "\delta_a \circ F(a,a,f)"] & & F(a,a,X) \arrow[d, "\delta_a"] \\
G(a,a,Y) \arrow[rr, "id"] & & G(a,a,Y)
\end{tikzcd}$$

2. Dually, one obtains an opcleavage by lifting $g : q((X,Y),\delta) \to Z$ to

$$(id, G(-,-,g) \circ \delta) : ((X,Y),\delta) \to ((X,Z), G(-,-,g) \circ \delta)$$

3. **a.** The composite morphism in question is

$$\begin{tikzcd}
F(a,a,Z) \arrow[r, "F(a,a,f)"] \arrow[d, "\delta_a \circ F(a,a,f)"] & F(a,a,X) \arrow[r, "id"] \arrow[d, "\delta_a"] & F(a,a,X) \arrow[d, "G(a,a,g) \circ \delta_a"] \\
G(a,a,Y) \arrow[r, "id"] & G(a,a,Y) \arrow[r, "G(a,a,g)"] & G(a,a,Z)
\end{tikzcd}$$

and we readily observe that the verticality conditions are satisfied.

b. The comparison map is even an equality:

$$\begin{aligned} g_!(f^*((X,Y),\delta)) &= ((Z,Z),(G(-,-,g)\circ\delta)\circ F(-,-,f)) \\ &= ((Z,Z),G(-,-,g)\circ(\delta\circ F(-,-,f))) = f^*(g_!(((X,Y),\delta))) \end{aligned}$$

Regarding the splitness, one verifies that
1. $Cart(id,((X,Y),\delta)) = id_{((X,Y),\delta)}$
2. $Cart(f\circ g,(((X,Y),\delta))) = Cart(f,((X,Y),\delta))\circ Cart(g,f^*((X,Y),\delta))$.

The corresponding *op*-ed statement about $OpCart$ is verified analogously.

B Proof of Corollary 21

To prove Corollary 21, we show:
1. $\mathbb{M}^{alg}(\mathbb{A}(-)) \simeq id$
2. $\mathbb{A}(\mathbb{M}^{alg}(-)) \simeq id$
3. $\mathbb{M}^{coalg}(\mathbb{C}(-)) \simeq id$
4. $\mathbb{C}(\mathbb{M}^{coalg}(-)) \simeq id$

1.

$$\begin{aligned} \mathbb{M}^{alg}(\mathbb{A}((X,X),\delta)) &= \mathbb{M}^{alg}(X,q(\iota_{((X,X),\delta)})) \\ &= (q(\iota_{((X,X),\delta)}))_!(0_{\mathcal{W}_{X-}}) \\ &= ((X,X),\delta) \end{aligned}$$

The last identity holds because the lifting problem

$$\begin{tikzcd} & q(q(\iota_{((X,X),\delta)})_!(0_{\mathcal{W}_{X-}})) \\ q(0_{\mathcal{W}_{X-}}) \arrow[ur, "q(\iota_{((X,X),\delta)})"] \arrow[r, "q(\iota_{((X,X),\delta)})"] & q((X,X),\delta) \arrow[u, "id"'] \end{tikzcd}$$

for $q(\iota_{((X,X),\delta)})$ is solved by

$$OpCart(id,((X,X),\delta)) : ((X,X),\delta) \to q(\iota_{((X,X),\delta)})_!(0_{\mathcal{W}_{X-}})$$

which is the identity because q is split.

2.

$$\begin{aligned} \mathbb{A}(\mathbb{M}^{alg}(X,\alpha:\underline{w}(X)\to X)) &= \mathbb{A}(\alpha_!(0_{\mathcal{W}_{X-}})) \\ &= (X,q(\iota_{\alpha_!(0_{\mathcal{W}_{X-}})})) \\ &= (X,\alpha:\underline{w}(X)\to X) \end{aligned}$$

where the last equation holds because $\alpha_!(0_{\mathcal{W}_{X-}}) : \mathcal{W}_{XX}$ gives us an isomorphism, and q is split, whence this isomorphism is the identity.

3.

$$\begin{aligned} \mathbb{M}^{coalg}(\mathbb{C}((X,X),\delta)) &= \mathbb{M}^{coalg}(X,p(\pi_{((X,X),\delta)})) \\ &= (p(\pi_{((X,X),\delta)}))^*(1_{\mathcal{W}_{-X}}) \\ &= ((X,X),\delta) \end{aligned}$$

where the last equation holds by the dual of the corresponding statement about algebras.

4.

$$\begin{aligned}\mathbb{C}(\mathbb{M}^{coalg}(X, \alpha : X \to \underline{m}(X))) &= \mathbb{C}(\alpha^*(1_{\mathcal{W}_{-X}})) \\ &= (X, p(\pi_{\alpha^*(1_{\mathcal{W}_{-X}})})) \\ &= (X, \alpha : X \to \underline{m}(X))\end{aligned}$$

where we use for the last equation that $\alpha^*(1_{\mathcal{W}_{-X}}) : \mathcal{W}_{XX}$ provides an isomorphism, and splitness of p implies that this isomorphism is an identity.

The equations for the actions on morphisms follow from the universal properties involved.

C Lifting Bialgebras

In the opposite direction of Definition 19, i.e. lifting a bialgebra $\underline{w}(X) \xrightarrow{\alpha} X \xrightarrow{\zeta} \underline{m}(X)$ by lifting α, and ζ separately as above, we obtain two objects that we can relate by lifting the composite $\zeta \circ \alpha$ to two morphisms (denoted l and r). l is obtained by lifting through the opfibered terminal object, and r by lifting through the fibered initial object as shown in the following diagram:

$$\begin{tikzcd}
\alpha^*\mathbf{0}X \arrow[d, "Cart(\alpha{,}\mathbf{0}X)"'] \arrow[ddr, bend left, "l"] & \\
\mathbf{0}X \arrow[r, "\iota_X"] \arrow[d, "o"] \arrow[ddr, bend right, "r"'] & \zeta^*\mathbf{1}X \arrow[d, "c"'] \\
\alpha_!\mathbf{0}X \arrow[r, "\pi_X"] & \mathbf{1}X \arrow[d, "OpCart(\zeta{,}\mathbf{1}X)"] \\
& \zeta_!\mathbf{1}X
\end{tikzcd}$$

where $o = \mathrm{OpCart}(\alpha, \mathbf{0}X)$ and $c = \mathrm{Cart}(\zeta, \mathbf{1}X)$.. We have thus: $p(l) = \zeta \circ \alpha$, $q(l) = q(\iota_X)$, and $p(r) = p(\pi_X)$, $q(r) = \zeta \circ \alpha$; in other words, considering "round trips", both possible lifts lie over the bialgebra with which we started and this bialgebra can be recovered by either applying p or q to the lifts.