

Symmetries in Sorting

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Abstract

Sorting algorithms are fundamental to computer science, and their correctness criteria are well understood as rearranging elements of a list according to a specified total order on the underlying set of elements. As mathematical functions, they are functions on lists that perform combinatorial operations on the representation of the input list. In this paper, we study sorting algorithms conceptually as abstract sorting functions.

There is a canonical surjection from the free monoid on a set (lists of elements) to the free commutative monoid on the same set (multisets of elements). We show that sorting functions determine a section (right inverse) to this surjection satisfying two axioms, that do not presuppose a total order on the underlying set. Then, we establish an equivalence between (decidable) total orders on the underlying set and correct sorting functions.

The first part of the paper develops concepts from universal algebra from the point of view of functorial signatures, and gives constructions of free monoids and free commutative monoids in (univalent) type theory. Using these constructions, the second part of the paper develops the axiomatisation of sorting functions. The paper uses informal mathematical language, and comes with an accompanying formalisation in Cubical Agda.

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1 Introduction

Consider a puzzle about sorting, inspired by Dijkstra’s Dutch National Flag problem [15, Ch.14]. Suppose there are balls of three colours, corresponding to the colours of the Dutch flag: red, white, and blue. Given an unordered list (bag) of such balls, how many ways can



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you sort them into the Dutch flag?

[red, red, blue, white, blue, red, white, blue]

Obviously there is only one way, decided by the order the colours appear in the Dutch flag: red < white < blue.

[red, red, red, white, white, blue, blue, blue]

What if we are avid enjoyers of vexillology who also want to consider other flags? How many ways can we sort our unordered list? To answer this, we distinguish between a sorting *algorithm* (a computational procedure) and a sorting *function* (a mathematical mapping). Since we sort the elements of a carrier set A directly (so stability and metadata are not concerns), a sorting function must produce a unique, deterministic output list for each input bag. Thus, “how many ways to sort” means how many extensionally distinct sorting functions can be defined. Because there are $3! = 6$ permutations of {red, white, blue}, there are 6 possible orderings, corresponding to 6 categories of tricolour flags (see Wikipedia):



For a carrier set with 3 elements, the 6 possible total orderings yield exactly 6 *extensionally correct* sorting functions. Intuitively, a *correct* sorting function takes a bag and returns a unique, sorted list of the same elements. Extensional correctness therefore requires the output list to be a permutation of the input elements, and sorted according to a fixed total order. Formally, there is a bijection between total orderings on a carrier set A and correct sorting functions on lists over A . Our main contribution is showing how a sorting function can be correctly axiomatised and formalised from this point of view.

Outline and Contributions

The paper is organised as follows:

- In § 2, we describe a formalisation of universal algebra developed from the point of view of functorial signatures, the definition and universal property of free algebras, and equational systems over signatures. These are elementary constructions but necessary groundwork for us, and we give an informal exposition in type theory.
- In § 3, we give constructions of free monoids, and proofs of their universal property. Following this, in § 4, we add commutativity to each representation of free monoids, and extend the proofs of universal property from free monoids to free commutative monoids. These constructions are standard, and we formalise by using our general framework.
- In § 5, we build on the previous constructions and study sorting functions. The main result connects total orders, sorting, and commutativity, by proving an equivalence between decidable total orders on a carrier set A , and correct sorting functions defined using free monoids and free commutative monoids over A .
- Finally, § 6 discusses aspects of the formalisation, related and future work.

These sections are largely independent. Readers interested in universal algebra can start at § 2; those focused on free (commutative) monoids can skip to §§ 3 and 4; and those interested primarily in sorting can proceed directly to § 5. All results are formalised in Cubical Agda, available online at [10]. We use informal mathematical language in the paper to keep it accessible. An extended version with additional details is on arXiv [11].

No prior exposure to Homotopy Type Theory (HoTT) or Univalence is required. We assume only basic familiarity with dependent type theory and functional programming (such as Agda, Coq, or Haskell). While the formalisation is written in Cubical Agda and uses higher inductive types and univalence for quotient constructions, the paper’s text is presented in informal mathematical language.

2 Universal Algebra

We develop some basic notions from universal algebra and equational logic [6], which gives us a vocabulary and framework to express our results in. The point of view we take is the standard category-theoretic approach to universal algebra [32], which is an alternative perspective to the Lawvere theory or abstract clone point of view [23, 19]. We keep a running example of monoids in mind, while explaining and defining the abstract concepts.

2.1 Algebras

► **Definition 1** (⚙️ Signature). A signature, denoted σ , is a (dependent) pair consisting of a set of operations, $\text{op} : \text{Set}$, and an arity function for each operation, $\text{ar} : \text{op} \rightarrow \text{Set}$.

► **Example 2.** A premonoid is a set with an identity element (a nullary operation), and a binary multiplication operation, with signature $\sigma_{\text{Mon}} := (\text{Fin}_2, \lambda\{0 \mapsto \text{Fin}_0; 1 \mapsto \text{Fin}_2\})$. Informally, we denote the two operations as a tuple $(e, \bullet)$.

Every signature σ induces a signature functor F_σ on Set .

► **Definition 3** (⚙️ Signature functor).

$$X \mapsto (o : \text{op}) \times X^{\text{ar}(o)} \quad X \xrightarrow{f} Y \mapsto (o : \text{op}) \times X^{\text{ar}(o)} \xrightarrow{(o, - \circ f)} (o : \text{op}) \times Y^{\text{ar}(o)}$$

► **Example 4.** The signature functor for monoids, $F_{\sigma_{\text{Mon}}}$, assigns to a carrier set X , the set of inputs for each operation. Expanding the dependent product on Fin_2 , we obtain a coproduct of sets: $F_{\sigma_{\text{Mon}}}(X) \simeq X^{\text{Fin}_0} + X^{\text{Fin}_2} \simeq \mathbf{1} + X \times X$.

A σ -structure is given by a carrier set, with functions interpreting each operation symbol. The signature functor applied to a carrier set gives the inputs to each operation, and the output is simply a map back to the carrier set. These two pieces of data constitute an algebra for the F_σ functor. We write $\mathfrak{X}$ for a σ -structure with carrier set X , following the model-theoretic notational convention.

► **Definition 5** (⚙️ Structure). A σ -structure $\mathfrak{X}$ is an F_σ -algebra, that is, a pair consisting of a carrier set X , and an algebra map $\alpha_X : F_\sigma(X) \rightarrow X$.

► **Example 6.** Concretely, an $F_{\sigma_{\text{Mon}}}$ -algebra has the type

$$\alpha_X : F_{\sigma_{\text{Mon}}}(X) \rightarrow X \quad \simeq \quad (\mathbf{1} + X \times X) \rightarrow X \quad \simeq \quad (\mathbf{1} \rightarrow X) \times (X \times X \rightarrow X)$$

which is the pair of functions interpreting the two operations. Natural numbers $\mathbb{N}$ with $(0, +)$ or $(1, \times)$ are examples of monoid structures.

► **Definition 7** (⚙️ σ -Homomorphism). A homomorphism between σ -structures $\mathfrak{X}$ and $\mathfrak{Y}$ is a morphism of F_σ -algebras, that is, a map $f : X \rightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} F_\sigma(X) & \xrightarrow{\alpha_X} & X \\ F_\sigma(f) \downarrow & & \downarrow f \\ F_\sigma(Y) & \xrightarrow{\alpha_Y} & Y \end{array}$$

► **Example 8.** Given two monoid structures $\mathfrak{X}$ and $\mathfrak{Y}$, the top half denotes:

$1 + (X \times X) \xrightarrow{\alpha_X} X \xrightarrow{f} Y$, which applies f to the output of each operation, and the bottom half denotes the map: $1 + (X \times X) \xrightarrow{F_{\sigma\text{Mon}}(f)} 1 + (Y \times Y) \xrightarrow{\alpha_Y} Y$. In other words, a homomorphism between $\mathfrak{X}$ and $\mathfrak{Y}$ is a map f on the carrier sets that commutes with the interpretation of the monoid operations, or simply, preserves the monoid structure.

For a fixed signature σ , the category of F_σ -algebras and their morphisms form a category of algebras, written $F_\sigma\text{-Alg}$, or simply, $\sigma\text{-Alg}$, given by the obvious definitions of identity and composition for the underlying functions.

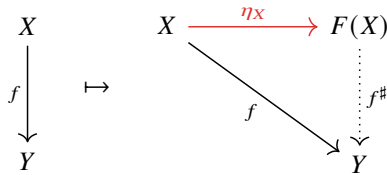
2.2 Free Algebras

The concrete category $\sigma\text{-Alg}$ of structured sets and structure-preserving maps admits a forgetful functor $U: \sigma\text{-Alg} \rightarrow \text{Set}$. In our notation, $U(\mathfrak{X})$ is simply X , a fact we exploit to simplify our notation and formalisation. The left adjoint to this forgetful functor is the free algebra construction, also known as the term algebra or the *absolutely free* algebra without equations [27]. We rephrase this in more concrete terms.

- **Definition 9** (⚙️ Free Algebras). *A free σ -algebra construction consists of the following:*
- a set $F(X)$, for every set X ,
 - a σ -structure on $F(X)$, written as $\mathfrak{F}(X)$,
 - a universal map $\eta_X: X \rightarrow F(X)$, for every X , such that,
 - for any σ -algebra $\mathfrak{Y}$, the operation assigning to each homomorphism $f: \mathfrak{F}(X) \rightarrow \mathfrak{Y}$, the map $f \circ \eta_X: X \rightarrow Y$ (or, pre-composition with η_X), is an equivalence.

In other words, we ask for a bijection between the set of homomorphisms out of the free algebra to any other algebra, and the set of functions from the carrier set of the free algebra to the carrier set of the other algebra. There should be no more data in homomorphisms out of the free algebra than there is in functions out of the carrier set, which is the property of *freeness*. The inverse operation to post-composing with η_X extends a function to a homomorphism.

- **Definition 10** (⚙️ Universal extension). *The universal extension of a function $f: X \rightarrow Y$ to a homomorphism out of the free σ -algebra on X is written as $f^\sharp: \mathfrak{F}(X) \rightarrow \mathfrak{Y}$. It satisfies the identities: $f^\sharp \circ \eta_X = f$, $\eta_X^\sharp = \text{id}_{\mathfrak{F}(X)}$, and $(g^\sharp \circ f)^\sharp = g^\sharp \circ f^\sharp$. The universal morphism η_X is highlighted in red.*



Free algebra constructions are canonically equivalent.

- **Proposition 11** (⚙️). *If $\mathfrak{F}(X)$ and $\mathfrak{G}(X)$ are free σ -algebras on X , then $\mathfrak{F}(X) \simeq \mathfrak{G}(X)$ in $\sigma\text{-Alg}$, naturally in X , via the canonical extensions.*

We have discussed specifications of free algebras, but not actually given a construction. In type theory, *free* constructions are often given by inductive types, where the constructors are the pieces of data that freely generate the structure, and the type-theoretic induction principle enforces the category-theoretic universal property.

► **Definition 12** (⚙️ Construction of Free σ -algebras). *The free σ -algebra on a type X is given by the inductive type $\text{Tree}(X)$, generated by two constructors:*

$$\begin{aligned} \text{leaf} &: X \rightarrow \text{Tree}(X) \\ \text{node} &: F_\sigma(\text{Tree}(X)) \rightarrow \text{Tree}(X) \end{aligned}$$

The constructors `leaf` and `node` are the generators for the universal map, and the algebra map, respectively. Together, they describe the type of abstract syntax trees for terms in the signature σ – the leaves are the free variables, and the nodes are the branching operations of the tree, marked by the operations in σ .

► **Proposition 13** (⚙️). *$(\text{Tree}(X), \text{leaf})$ is the free σ -algebra on X .*

2.3 Equations

The algebraic framework described so far only captures operations, not equations. These algebras are *lawless* (or *wild* or *absolutely free*) – $F_{\sigma_{\text{Mon}}}$ -algebras are premonoids rather than monoids, and $\mathfrak{F}_{\sigma_{\text{Mon}}}$ -algebras are free premonoids, not free monoids, since they are missing the unit and associativity laws. To impose equations on the generated abstract syntax trees, we adopt the point of view of equational logic.

► **Definition 14** (⚙️ Term Arity). *A term arity, denoted ε , is a (dependent) pair consisting of: a set of names for equations, $\text{eq} : \text{Set}$, and an arity of free variables for each equation, $\text{fv} : \text{eq} \rightarrow \text{Set}$.*

This will be used to define a *system of equations* (Definition 17).

► **Example 15.** The term arity for monoids ε_{Mon} is: $(\text{Fin}_3, \lambda\{0 \mapsto \text{Fin}_1; 1 \mapsto \text{Fin}_1; 2 \mapsto \text{Fin}_3\})$. The three equations are the left and right unit laws, and the associativity law – a 3-element set of names $\{\text{unitl}, \text{unitr}, \text{assoc}\}$. The two unit laws use one free variable, and the associativity law uses three free variables.

Similar to the term signature functor in Definition 3, this produces a *term arity functor* on Set .

► **Definition 16** (⚙️ Term Arity Functor).

$$X \mapsto (e : \text{eq}) \times X^{\text{fv}(e)} \quad X \xrightarrow{f} Y \mapsto (e : \text{eq}) \times X^{\text{fv}(e)} \xrightarrow{(e, - \circ f)} (e : \text{eq}) \times Y^{\text{fv}(e)}$$

To build equations out of this, we use the σ -operations and construct trees for the left and right-hand sides of each equation using the free variables available.

► **Definition 17** (⚙️ System of Equations). *A system of equations over a signature (σ, ε) is a pair of natural transformations:*

$$\ell, \nu : F_\varepsilon \Rightarrow \mathfrak{F}_\sigma.$$

For any set (of variables) V , this gives a pair of functions $\ell_V, \nu_V : F_\varepsilon(V) \rightarrow \mathfrak{F}_\sigma(V)$, and naturality ensures correctness of renaming.

Finally, we say how a given σ -structure $\mathfrak{X}$ satisfies the system of equations $T_{(\sigma, \varepsilon)}$. We need to assign a value to each free variable in the equation in the carrier set, using a valuation function $\rho : V \rightarrow X$. Given such an assignment, we evaluate the left and right trees of the equation, by extending ρ with Definition 10, that is by construction, a homomorphism from $\mathfrak{F}(V)$ to $\mathfrak{X}$. To satisfy an equation, these two evaluations should agree.

► **Definition 18** (⚙️ Satisfaction $\mathfrak{X} \models T$). A σ -structure $\mathfrak{X}$ satisfies the system of equations $T_{(\sigma, \varepsilon)}$ if for every set V , and every assignment $\rho: V \rightarrow X$, $\rho^\#$ is a (co)fork:

$$F_\varepsilon(V) \begin{array}{c} \xrightarrow{\ell_V} \\ \xrightarrow{r_V} \end{array} \mathfrak{F}(V) \xrightarrow{\rho^\#} \mathfrak{X}$$

Given a signature (σ, ε) with a system of equations $T_{(\sigma, \varepsilon)}$, the σ -algebras satisfying $T_{(\sigma, \varepsilon)}$, or varieties, form a full subcategory of $\sigma\text{-Alg}$. While free algebras exist for any signature σ as in Definition 12, constructions of free varieties for arbitrary systems of equations require non-constructive principles [7, § 7, pg.142], in particular, the arity sets need to be projective – we do not pursue this matter further. We have a rudimentary framework for universal algebra and equational logic, which gives us enough tools to develop the next sections.

3 Constructions of Free Monoids

In this section, we consider various constructions of free monoids in type theory, with proofs of their universal property. Since each construction satisfies the same categorical universal property, by Proposition 11, these are canonically equivalent (hence equal, by univalence) as types (and as monoids), allowing us to transport proofs between them. There are countably many such representations of free monoids, allowing us to choose the most convenient one for a given task. Using the universal property allows us to define and prove our programs correct in one go, which we use in § 5.

3.1 Lists

Cons-lists are simple inductive datatypes, well-known to functional programmers, and are a common representation of free monoids in programming languages. More generally, they correspond to list objects [12, 25] and algebraically-free monoids [21] in category theory.

► **Definition 19** (⚙️ Lists). $\text{List}(A)$ is generated by the following constructors:

$$\begin{aligned} \text{nil} &: \text{List}(A) \\ - :: - &: A \rightarrow \text{List}(A) \rightarrow \text{List}(A) \end{aligned}$$

The (universal) generators map is the singleton: $\eta_A(a) := [a] \equiv a :: \text{nil}$, the identity element is the empty list nil , and the monoid multiplication $\bullet$ is given by list concatenation.

► **Proposition 20** (⚙️ Universal property for List). $(\text{List}(A), \eta_A)$ is the free monoid on A .

3.2 Arrays

An alternate (non-inductive) representation of the free monoid on a carrier set, or alphabet A , is A^* , the set of all finite strings or sequences of characters *drawn* from A , as in [16]. We call this an *array*, which is a pair of a natural number n , denoting the length of the array, and a lookup (or index) function $\text{Fin}_n \rightarrow A$, mapping indices to elements of A . In type theory, this is also understood as an extension of a container [1], where the type of shapes is a $n: \mathbb{N}$, and the type of positions for each shape is Fin_n , where the array elements are stored.

► **Definition 21** (⚙️ Arrays).

$$\text{Array}(A) := (n: \mathbb{N}) \times (\text{Fin}_n \rightarrow A)$$

For example, $(3, \lambda\{0 \mapsto 3, 1 \mapsto 1, 2 \mapsto 2\})$ represents the same list as $[3, 1, 2]$. The (universal) generators map is the singleton: $\eta_A(a) = (1, \lambda\{0 \mapsto a\})$, the identity element is $(0, \lambda\{\})$ and the monoid operation $\bullet$ is given by array concatenation.

► **Definition 22** (⚙️ Concatenation). *The concatenation operation $\bullet$, is defined below, where $\oplus: (\text{Fin}_n \rightarrow A) \rightarrow (\text{Fin}_m \rightarrow A) \rightarrow (\text{Fin}_{n+m} \rightarrow A)$ combines the two lookup functions.*

$$(n, f) \bullet (m, g) = (n + m, f \oplus g) \quad (f \oplus g)(k) = \begin{cases} f(k) & \text{if } k < n \\ g(k - n) & \text{otherwise} \end{cases}$$

We need two auxiliary lemmas about array concatenation to do induction on arrays.

► **Lemma 23** (⚙️ Array cons). *Any array $(S(n), f)$ is equal to $\eta_A(f(0)) \bullet (n, f \circ S)$.*

► **Lemma 24** (⚙️ Array split). *For any array $(S(n), f)$ and (m, g) ,*

$$(n + m, (f \oplus g) \circ S) = (n, f \circ S) \bullet (m, g) .$$

► **Definition 25** (⚙️ Universal extension). *Given a monoid $\mathfrak{X}$, and a map $f: A \rightarrow X$, we define $f^\# : \text{Array}(A) \rightarrow X$, by induction on the length of the array:*

$$\begin{aligned} f^\#(0, g) &= e \\ f^\#(S(n), g) &= f(g(0)) \bullet f^\#(n, g \circ S) \end{aligned}$$

► **Proposition 26** (⚙️ Universal property for Array). *$(\text{Array}(A), \eta_A)$ is the free monoid on A .*

Proof sketch. We need to show that $(-)^{\#}$ is an inverse to $(-) \circ \eta_A$. $f^\# \circ \eta_A = f$ for all set functions $f: A \rightarrow X$ holds trivially. To show $(f \circ \eta_A)^\# = f$ for all homomorphisms $f: \text{Array}(A) \rightarrow \mathfrak{X}$, we need to show $\forall xs. (f \circ \eta_A)^\#(xs) = f(xs)$. We use Lemmas 23 and 24 to do induction on xs , similar to doing induction on the inductive datatype $\text{List}(A)$. ◀

An alternative proof of the universal property for Array can be given by directly constructing an equivalence (of types, and monoid structures) between $\text{Array}(A)$ and $\text{List}(A)$ (using `tabulate` and `lookup`), and then applying univalence and transport (see formalisation).

4 Constructions of Free Commutative Monoids

The next step is to add commutativity to the monoid multiplication in each construction of free monoids. Informally, adding commutativity to free monoids turns “ordered lists” to “unordered lists”, where the ordering is the one imposed by the position or index of the elements in the list. This is crucial to our goal of studying sorting, as we will study sorting as a function mapping back unordered lists to ordered lists, later in § 5.3.

It is known that finite multisets are (free) commutative monoids, under the operation of multiset union: $xs \cup ys = ys \cup xs$. The order is “forgotten” in the sense that it doesn’t matter how two multisets are union-ed together, such as $\{a, a, b, c\}$ and $\{b, a, c, a\}$ are equal as finite multisets (justifying the bag notation). Formally, $\{x, y, \dots\}$ denotes $\eta_A(x) \bullet \eta_A(y) \bullet \dots : \mathcal{M}(A)$.

This is unlike free monoids, where $[a, a, b, c] \neq [b, a, c, a]$ (justifying the list notation). Formally, $[x, y, \dots]$ denotes $\eta_A(x) \bullet \eta_A(y) \bullet \dots : \mathcal{L}(A)$, and $x :: xs$ denotes $\eta_A(x) \bullet xs : \mathcal{L}(A)$.

4.1 Free monoids quotiented by permutation relations

Instead of constructing free commutative monoids directly, the first construction we study is to take *any* free monoid and quotient by *symmetries*. The constructions in § 4.2 and § 4.4 are specific instances of this recipe.

From the perspective of universal algebra and equational logic developed in § 2, we work in the signature of monoids, but extend the equational theory of monoids with commutativity. If $(\mathfrak{F}(A), \eta)$ is a free monoid construction satisfying its universal property, then we quotient $F(A)$ by an *appropriate* symmetry relation $\approx$. This is precisely a *permutation relation*.

► **Definition 27** (⚙️ Permutation relation). *A binary relation on a free monoid $(F(A), e, \bullet)$ is a permutation relation iff it is:*

- *reflexive, symmetric, transitive (an equivalence),*
- *a congruence w.r.t $\bullet$: $\forall a, b, c, d, a \approx b \rightarrow c \approx d \rightarrow a \bullet c \approx b \bullet d$,*
- *commutative: $\forall a, b, a \bullet b \approx b \bullet a$, and*
- *respects $(-)^{\#}$: $\forall a, b, f, a \approx b \rightarrow f^{\#}(a) = f^{\#}(b)$.*

Note that being a permutation relation is a proposition. We write $F(A) // \approx$ for the quotient of $F(A)$ by $\approx$, with the inclusion map $q: F(A) \rightarrow F(A) // \approx$. The generators map into $F(A) // \approx$ is given by $q \circ \eta_A$, the identity element is $\hat{e} = q(e)$, and the multiplication operation $\bullet$ is lifted to $\hat{\bullet}$ in the quotient by congruence.

► **Proposition 28** (⚙️). *$(F(A) // \approx, \hat{e}, \hat{\bullet})$ is a commutative monoid.*

The monoid extension operation of $F(A)$, denoted $(-)^{\#}$, lifts to an extension operation for commutative monoids, for $F(A) // \approx$, which we denote by $(-)^{\hat{\#}}$.

► **Definition 29** (⚙️). *Given a commutative monoid $\mathfrak{X}$ and a map $f: A \rightarrow X$, we define the extension operation as follows: we first obtain $f^{\#}: \mathfrak{F}(A) \rightarrow \mathfrak{X}$ by the universal property of F , then lift it to $f^{\hat{\#}}: F(A) // \approx \rightarrow \mathfrak{X}$, by using that $\approx$ respects $(-)^{\#}$, as in Definition 27.*

► **Proposition 30** (⚙️ Universal property for $F(A) // \approx$). *$(F(A) // \approx, \eta_A: A \xrightarrow{\eta_A} \mathfrak{F}(A) \xrightarrow{q} F(A) // \approx)$ with the extension operation $(-)^{\hat{\#}}$ forms the free commutative monoid on A .*

4.2 Lists quotiented by permutation relations

A specific instance of this construction is List quotiented by a concrete permutation relation to get commutativity. Of course, there are many permutation relations in the literature, we consider a simple one which swaps any two adjacent elements anywhere in the middle of the list. This PList construction has also been considered in [20], who prove that PList is equivalent to the free commutative monoid (constructed as a HIT). We give a direct proof of its universal property using our generalisation.

► **Definition 31** (⚙️ Perm). *The inductive family Perm is generated by two constructors and defines a permutation relation on List:*

perm-refl: $\forall xs, \text{Perm } xs \ xs$

perm-swap: $\forall x, y, xs, ys, zs, \text{Perm } (xs \bullet x :: y :: ys) \ zs \rightarrow \text{Perm } (xs \bullet y :: x :: ys) \ zs$

where $\bullet$ is the usual list concatenation operation. PList(A) is the quotient of List(A) by Perm:

PList(A) := List(A) // Perm .

By § 4.1, to obtain that $\text{PList}(A)$ is the free commutative monoid on A , it suffices to prove the property of $\text{Perm}(A)$ being a permutation relation.

► **Proposition 32** (⚙️ Perm is a permutation relation). *The relation Perm is a permutation relation on $\text{List}(A)$.*

► **Proposition 33** (⚙️). *Let $\mathfrak{X}$ be a commutative monoid, and $f: A \rightarrow X$. For $x, y: A$ and $xs, ys: \text{List}(A)$, $f^\#(xs \bullet x :: y :: ys) = f^\#(xs \bullet y :: x :: ys)$. Hence, Perm respects $(-)^\#$.*

PList being a quotient of List makes it easy to lift functions and properties defined on List along the quotient map. The inductive nature of Perm makes it easy to define algorithms that inductively rearrange elements in a list, such as insertion sort. This property also makes it possible to normalize inductively constructed list elements in PList to their simplest forms, e.g. $q([x]) \bullet q([y, z])$ normalizes to $q([x, y, z])$ by definition, saving the efforts of defining auxiliary lemmas to prove their equality.

This inductive nature, however, makes it difficult to define efficient operations on PList . Consider a divide-and-conquer algorithm such as merge sort, which involves partitioning a PList of length $n + m$ into two smaller PList of length n and length m . We must iterate all n elements before we can make such a partition, thus making PList unintuitive to work with when we want to reason with operations that involve arbitrary partitioning.

4.3 Swap Lists

Informally, quotients are defined by generating all the points, then quotienting out into equivalence classes by the congruence relation. Alternately, HITs use generators (points) and higher generators (paths) (and higher higher generators and so on...). We can also define free commutative monoids using HITs where adjacent swaps generate all symmetries, for example swap-lists taken from [8], and in the Cubical library [29].

► **Definition 34** (⚙️ SList). *The higher inductive type $\text{SList}(A)$ is generated by:*

```

nil: SList(A)
- :: -: A → SList(A) → SList(A)
swap: ∀x, y, xs, x :: y :: xs = y :: x :: xs
trunc: ∀x, y, (p, q: x = y) → p = q

```

We refer the reader to [8] or the Cubical Agda library [29] for further details. Much like List , SList is inductively defined, therefore making it intuitive to reason with when defining inductive operations or proofs on SList , but difficult to reason with when defining operations that involve arbitrary partitioning, for reasons similar to those given in § 4.2.

4.4 Bag

Alternatively, we can also quotient the Array type by symmetries to get commutativity. This construction has been considered in [4] and [24], partially considered in [8], and in [20], who give a similar construction, where only the index function is quotiented, instead of the entire array. [14] also considered Bag as a setoid relation on List in an intensional MLTT setting. [20] prove that their version of Bag is the free commutative monoid by equivalence to the other HIT constructions. We give a direct proof of its universal property instead, using our general recipe.

► **Definition 35** (⚙ Bag). *Bags are defined as arrays quotiented by bag equivalence $\approx$:*

$$(n, f) \approx (m, g) := (\sigma: \text{Fin}_n \xrightarrow{\sim} \text{Fin}_m) \times (f = g \circ \sigma)$$

$$\text{Bag}(A) := \text{Array}(A) // \approx$$

Note that by a pigeonhole argument, σ may only be constructed when $n = m$. In other words, we are quotienting by an automorphism on the indices, and its action on the elements. We have already shown Array to be the free monoid in § 3.2. By Proposition 30, to obtain that $\text{Bag}(A)$ is the free commutative monoid on A , it suffices to prove the property of $\approx$ being a permutation relation.

► **Proposition 36** (⚙ Bag equivalence is a permutation relation). *The relation $\approx$ is a permutation relation on $\text{Array}(A)$.*

► **Proposition 37** (⚙). *$\approx$ is congruent with respect to $\bullet$.*

Proof. Given $(n, f) \approx (m, g)$ by σ and $(u, p) \approx (v, q)$ by ϕ , we want to show $(n, f) \bullet (u, p) \approx (m, g) \bullet (v, q)$ by some τ . We construct τ as follows:

$$\tau := \text{Fin}_{n+u} \xrightarrow{\sim} \text{Fin}_n + \text{Fin}_u \xrightarrow{\sigma+\phi} \text{Fin}_m + \text{Fin}_v \xrightarrow{\sim} \text{Fin}_{m+v}$$

which operationally performs:

$$\begin{array}{c} \{0, 1, \dots, n-1, n, n+1, \dots, n+u-1\} \\ \downarrow \sigma+\phi \\ \{\sigma(0), \sigma(1), \dots, \sigma(n-1), \phi(0), \phi(1), \dots, \phi(u-1)\} \end{array} \quad \blacktriangleleft$$

► **Proposition 38** (⚙). *$\approx$ respects commutativity.*

Proof. We want to show for any arrays (n, f) and (m, g) , $(n, f) \bullet (m, g) \approx (m, g) \bullet (n, f)$ by some ϕ . We use formal combinators (see [9]) to define ϕ :

$$\phi := \text{Fin}_{n+m} \xrightarrow{\sim} \text{Fin}_n + \text{Fin}_m \xrightarrow{\text{swap}_+} \text{Fin}_m + \text{Fin}_n \xrightarrow{\sim} \text{Fin}_{m+n}$$

which operationally performs:

$$\begin{array}{c} \{0, 1, \dots, n-1, n, n+1, \dots, n+m-1\} \\ \downarrow \phi \\ \{n, n+1, \dots, n+m-1, 0, 1, \dots, n-1\} \end{array} \quad \blacktriangleleft$$

To show that $f^\#$ is invariant under permutation: for all $\phi: \text{Fin}_n \xrightarrow{\sim} \text{Fin}_n$, $f^\#(n, i) = f^\#(n, i \circ \phi)$, we need some formal combinators for *punching in* and *punching out* indices. These operations are borrowed from [26] and developed further in [9] for studying permutation codes.

► **Lemma 39** (⚙). *Given $\phi: \text{Fin}_{S(n)} \xrightarrow{\sim} \text{Fin}_{S(n)}$, there is a permutation $\tau: \text{Fin}_{S(n)} \xrightarrow{\sim} \text{Fin}_{S(n)}$ such that $\tau(0) = 0$, and $f^\#(S(n), i \circ \phi) = f^\#(S(n), i \circ \tau)$.*

Proof. Let k be $\phi^{-1}(0)$, and $k + j = S(n)$, we construct τ :

$$\begin{array}{ccc} \tau := \text{Fin}_{S(n)} \xrightarrow{\phi} \text{Fin}_{S(n)} \xrightarrow{\sim} \text{Fin}_{k+j} \xrightarrow{\sim} \text{Fin}_k + \text{Fin}_j \xrightarrow{\text{swap}_+} \text{Fin}_j + \text{Fin}_k \xrightarrow{\sim} \text{Fin}_{j+k} \xrightarrow{\sim} \text{Fin}_{S(n)} & & \\ \{0, 1, 2, \dots, k, k+1, k+2, \dots\} & & \{0, 1, 2, \dots, k, k+1, k+2, \dots\} \\ \downarrow \phi & & \downarrow \tau \\ \{x, y, z, \dots, 0, u, v, \dots\} & & \{0, u, v, \dots, x, y, z, \dots\} \end{array}$$

It is trivial to show $f^\#(S(n), i \circ \phi) = f^\#(S(n), i \circ \tau)$, since the only operation on indices in τ is swap_+ . It suffices to show $(S(n), i \circ \phi)$ can be decomposed into two arrays such that $(S(n), i \circ \phi) = (k, g) \bullet (j, h)$ for some g and h . Since the image of $f^\#$ is a commutative monoid, and $f^\#$ is a homomorphism, $f^\#((k, g) \bullet (j, h)) = f^\#(k, g) \bullet f^\#(j, h) = f^\#(j, h) \bullet f^\#(k, g) = f^\#((j, h) \bullet (k, g))$, thereby proving $f^\#(S(n), i \circ \phi) = f^\#(S(n), i \circ \tau)$. $\blacktriangleleft$

► **Lemma 40** (⚙️). *Given $\tau: \text{Fin}_{S(n)} \xrightarrow{\sim} \text{Fin}_{S(n)}$ where $\tau(0) = 0$, there is a $\psi: \text{Fin}_n \xrightarrow{\sim} \text{Fin}_n$ such that $\tau \circ S = S \circ \psi$.*

Proof. We construct ψ as $\psi(x) = \tau(S(x)) - 1$. Since τ maps only 0 to 0 by assumption, $\forall x. \tau(S(x)) > 0$, therefore the (-1) is well defined. This corresponds to the special case for $k = 0$ of the standard punch-in and punch-out operations from the Agda standard library [13], as used for Lehmer codes in [9].

$$\begin{array}{ccc} \{0, 1, 2, 3, \dots\} & & \{0, 1, 2, \dots\} \\ \downarrow \tau & & \downarrow \psi \\ \{0, x, y, z, \dots\} & & \{x-1, y-1, z-1, \dots\} \end{array} \quad \blacktriangleleft$$

► **Theorem 41** (⚙️ Permutation invariance). *For all $\phi: \text{Fin}_n \xrightarrow{\sim} \text{Fin}_n$, $f^\#(n, i) = f^\#(n, i \circ \phi)$.*

Proof. By induction on n .

- At $n = 0$, $f^\#(0, i) = f^\#(0, i \circ \phi) = e$.
- At $n = S(m)$,

$$\begin{aligned} & f^\#(S(m), i \circ \phi) \\ &= f^\#(S(m), i \circ \tau) && \text{by Lemma 39} \\ &= f(i(\tau(0))) \bullet f^\#(m, i \circ \tau \circ S) && \text{by definition of } (-)^\# \\ &= f(i(0)) \bullet f^\#(m, i \circ \tau \circ S) && \text{by construction of } \tau \\ &= f(i(0)) \bullet f^\#(m, i \circ S \circ \psi) && \text{by Lemma 40} \\ &= f(i(0)) \bullet f^\#(m, i \circ S) && \text{induction} \\ &= f^\#(S(m), i) && \text{by definition of } (-)^\# \end{aligned} \quad \blacktriangleleft$$

Unlike `PList` and `SList`, `Bag` and its underlying construction `Array` are not inductively defined, making it difficult to do induction on them. For example, in the proof of Proposition 26, both Lemmas 23 and 24 are needed to do induction on `Array`, as opposed to `List` and its quotients, where we can do induction simply by pattern matching. Much like `PList`, when defining functions on `Bag`, we need to show they respect $\approx$, i.e. $as \approx bs \rightarrow f(as) = f(bs)$. Notably, this is much more difficult than `PList` or `SList` – with `PList` and `SList` we only need to consider swapping adjacent elements, while with `Bag` we need to consider all possible permutations. For example, in the proof of Theorem 41, we need to first construct a τ which satisfies $\tau(0) = 0$ and prove $f^\#(n, i \circ \sigma) = f^\#(n, i \circ \tau)$ before we can apply induction.

5 Sorting Functions

We will now put to work the universal properties of our types of (ordered) lists and unordered lists, to define operations on them systematically, which are mathematically sound, and then reason about them. First, we explore definitions of various operations on both free monoids and free commutative monoids. By univalence (and the structure identity principle [2, 30]),

everything henceforth holds for any presentation of free monoids and free commutative monoids. We use $\mathcal{F}(A)$ to denote the free monoid or free commutative monoid on A , $\mathcal{L}(A)$ to exclusively denote the free monoid construction, and $\mathcal{M}(A)$ to exclusively denote the free commutative monoid construction.

For example `length` is a common operation defined inductively for `List`, but usually, properties about `length`, such as, $\text{length}(xs \bullet ys) = \text{length}(xs) + \text{length}(ys)$, are proven separately after defining it. In our framework of free algebras, where the $(-)^{\#}$ operation is a correct-by-construction homomorphism, we can define operations like `length` directly by universal extension, which also gives us a proof that they are homomorphisms for free. A further application of the universal property is to prove that two different types (such as lists and arrays) are equal, by showing they both satisfy the same universal property as in Proposition 11, which is desirable especially when proving a direct equivalence between the two types turns out to be a difficult exercise in combinatorics.

5.1 Prelude

Any presentation of free monoids or free commutative monoids has a $\text{length}: \mathcal{F}(A) \rightarrow \mathbb{N}$ function, where $\mathbb{N}$ carries the additive monoid structure $(0, +)$, which is also a commutative monoid structure since addition is commutative.

► **Definition 42** (⚙️ `length`). *The length homomorphism is defined as $(\lambda x. 1)^{\#}: \mathcal{F}(A) \rightarrow \mathbb{N}$.*

Further, any presentation of free monoids or free commutative monoids has an element membership predicate $- \in -: A \rightarrow \mathcal{F}(A) \rightarrow \mathbf{hProp}$, for any set A . Here, we use the fact that $\mathbf{hProp}$ forms a (commutative) monoid under disjunction and falsehood $(\perp, \vee)$.

► **Definition 43** (⚙️ `Membership` $\in$). *The membership predicate on a set A for any element $x: A$ is $x \in - := \mathfrak{L}_A(x)^{\#}: \mathcal{F}(A) \rightarrow \mathbf{hProp}$, where we define $\mathfrak{L}_A(x) := \lambda y. x = y: A \rightarrow \mathbf{hProp}$.*

$\mathfrak{L}$ is formally the Yoneda map under the “types are groupoids” correspondence, where $x: A$ is being sent to its representable in the Hom-groupoid (formed by the identity type), of type $\mathbf{hProp}$. Note that the proofs of (commutative) monoid laws for $\mathbf{hProp}$ use equality, which requires the use of univalence (or at least, propositional extensionality). By construction, this membership predicate satisfies its homomorphic properties, which are colloquially the properties of inductively defined de Bruijn indices.

We note that $\mathbf{hProp}$ is actually one type level higher than A . To make the type levels explicit, A is of type level ℓ , and since $\mathbf{hProp}_{\ell}$ is the type of all types $X: \mathbf{Set}_{\ell}$ that are mere propositions, $\mathbf{hProp}_{\ell}$ has type level $\ell + 1$. We do not assume any propositional resizing axioms [31], and instead use Agda’s universe polymorphism in our formalisation to accommodate this.

Any presentation of free (commutative) monoids $\mathcal{F}(A)$ also supports the `Any` and `All` predicates, which allow lifting a predicate $A \rightarrow \mathbf{hProp}$ (on A), to *any* or *all* elements of $xs: \mathcal{F}(A)$, respectively. We note that $\mathbf{hProp}$ forms a (commutative) monoid in two different ways: $(\perp, \vee)$ and $(\top, \wedge)$ (disjunction and conjunction), which are the two different ways of getting `Any` and `All`, respectively, by extension.

► **Definition 44** (`Any` and `All`).

$$\text{Any}(P) := P^{\#}: \mathcal{F}(A) \rightarrow (\mathbf{hProp}, \perp, \vee) \quad \text{All}(P) := P^{\#}: \mathcal{F}(A) \rightarrow (\mathbf{hProp}, \top, \wedge)$$

Note that Cubical Agda has problems with indexing over HITs [28, 3, § 8], hence it is preferable to program with our universal properties, such as when defining `Any` and `All`, because the indexed-inductive definitions of these predicates get stuck on `transp` terms.

There is a **head** function on lists, which is a function that returns the first element of a non-empty list. Formally, this is a monoid homomorphism from $\mathcal{L}(A)$ to $1 + A$.

► **Definition 45** (⚙️ **head**). *The head homomorphism is defined as $\text{head} := \text{inr}^\# : \mathcal{L}(A) \rightarrow 1 + A$, where the monoid structure on $1 + A$ has unit $e := \text{inl}(\star) : 1 + A$, and multiplication picks the leftmost element that is defined.*

$$\begin{aligned} \text{inl}(\star) \oplus b &:= b \\ \text{inr}(a) \oplus b &:= \text{inr}(a) \end{aligned}$$

This monoid operation $\oplus$ is not commutative, since swapping the input arguments to $\oplus$ would return the leftmost or rightmost element. To make it commutative would require a canonical way to pick between a choice of two elements – this leads us to the next section.

5.2 Total orders

First, we recall the axioms of a total order or linear order $\leq$ on a set A .

► **Definition 46** (⚙️ **Total order**). *A total order on a set A is a relation $\leq : A \rightarrow A \rightarrow \text{hProp}$ that satisfies:*

- *reflexivity: $x \leq x$,*
- *transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$,*
- *antisymmetry: if $x \leq y$ and $y \leq x$, then $x = y$,*
- *totality: $\forall x, y$, either $x \leq y$ or $y \leq x$.*

A decidable total order requires the $\leq$ relation to be decidable:

- *decidable totality: $\forall x, y$, we have $x \leq y + \neg(x \leq y)$.*

Note that *either-or* means a (truncated) logical disjunction. In the context of this paper, we want to make a distinction between “decidable total order” and “total order”. The decidability axiom strengthens the totality axiom, where we have either $x \leq y$ or $y \leq x$ merely as a proposition, but decidability allows us to produce a witness if $x \leq y$ is true.

► **Proposition 47** (⚙️). *In a decidable total order, it holds that $\forall x, y, (x \leq y) + (y \leq x)$. Further, this makes A discrete, that is $\forall x, y, (x = y) + (x \neq y)$.*

Proof. We decide if $x \leq y$ and $y \leq x$, and then by cases:

- if $x \leq y$ and $y \leq x$: by antisymmetry, $x = y$.
- if $\neg(x \leq y)$ and $y \leq x$: assuming $x = y$ leads to a contradiction, hence $x \neq y$.
- if $x \leq y$ and $\neg(y \leq x)$: similar to the previous case.
- if $\neg(x \leq y)$ and $\neg(y \leq x)$: by totality, either $x \leq y$ or $y \leq x$, which leads to a contradiction. ◀

We also recall the axioms of a *strict* total order $<$ on A .

► **Definition 48** (⚙️ **Strict total order**). *A strict total order on a set A is a relation $< : A \rightarrow A \rightarrow \text{hProp}$ that satisfies:*

- *irreflexivity: $\neg(x < x)$,*
- *transitivity: if $x < y$ and $y < z$, then $x < z$,*
- *asymmetry: if $x < y$, then $\neg(y < x)$,*
- *cotransitivity: $\forall x, y, z$, if $x < z$, then either $x < y$ or $y < z$.*
- *connectedness: $\forall x, y$, if $\neg(x < y)$ and $\neg(y < x)$, then $x = y$.*

A decidable strict total order requires the $<$ relation to be decidable:

- *decidability: $\forall x, y$, we have $x < y + \neg(x < y)$.*

► **Proposition 49** (⚙️). *In a decidable strict total order, it holds that $\forall x, y, (x < y) + (y < x)$. Further, this makes A discrete, that is $\forall x, y, (x = y) + (x \neq y)$.*

Proof. We decide if $x < y$ and $y < x$, and by cases:

- if $\neg(x < y)$ and $\neg(y < x)$: by connectedness, $x = y$.
- if $x < y$ or $y < x$: by irreflexivity, $x \neq y$. ◀

We can show that giving a decidable *strict* total order structure on A is equivalent to giving a decidable total order structure on A . In practice, however, this equivalence is thoroughly awful to work with in a proof assistant, leading to horrible goals when formalising proofs that shift between these representations.

► **Proposition 50** (⚙️). *The set of decidable strict total orders on A and the set of decidable total orders on A are equivalent.*

Proof sketch. Given a decidable total order $\leq$ on A , by Proposition 47, we map it to a decidable strict total order $<$ on A by $x < y := (x \leq y) \times (x \neq y)$. Vice versa, by Proposition 49, given a decidable strict total order $<$ on A , we map it to a decidable total order $\leq$ on A by $x \leq y := (x < y) + (x = y)$. These maps are checked to be inverses of each other. ◀

An equivalent way of defining a total order is using a binary meet operation, without starting from an ordering relation.

► **Definition 51** (⚙️ Meet semi-lattice). *A meet semi-lattice is a set A with a binary operation $\sqcap : A \rightarrow A \rightarrow A$ that is:*

- *idempotent:* $x \sqcap x = x$,
- *associative:* $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$,
- *commutative:* $x \sqcap y = y \sqcap x$.

A total meet semi-lattice further satisfies:

- *total:* $\forall x, y$, either $x \sqcap y = x$ or $x \sqcap y = y$.

A decidable total meet semi-lattice strengthens this to:

- *decidable totality:* $\forall x, y, (x \sqcap y = x) + (x \sqcap y = y)$.

► **Proposition 52** (⚙️). *If A has decidable equality, all total meet semi-lattice structures on A are decidable.*

Proof. We decide if $x \sqcap y = x$ and $x \sqcap y = y$ to compute whether $x \sqcap y = x$ or $x \sqcap y = y$. By totality, one of them must hold. ◀

► **Proposition 53** (⚙️). *A total order structure on a set A is equivalent to a total meet semi-lattice structure on A . Further, a decidable total order on A induces a decidable total meet semi-lattice structure on A .*

Proof sketch. Given a (mere) total order $\leq$ on a set A , we define $x \sqcap y := \text{if } x \leq y \text{ then } x \text{ else } y$. Crucially, this map is *locally-constant*, allowing us to eliminate from an $\mathbf{hProp}$ to an $\mathbf{hSet}$. Meets satisfy the universal property of products, that is, $c \leq a \sqcap b \Leftrightarrow c \leq a \wedge c \leq b$, and the axioms follow by calculation using $\mathbb{J}$ -arguments. Conversely, given a meet semi-lattice, we define $x \leq y := x \sqcap y = x$, which defines an $\mathbf{hProp}$ -valued total ordering relation. If the total order is decidable, by Proposition 47 and Proposition 52, the mapped meet operation is decidable. The converse does not hold however, as a decidable total meet semi-lattice structure does not imply discreteness of A . ◀

$$\mathcal{L}(A) \begin{array}{c} \xrightarrow{\mathbf{q}} \\ \xleftarrow{\mathbf{s}} \end{array} \mathcal{M}(A)$$

■ **Figure 1** Relationship of $\mathcal{L}(A)$ and $\mathcal{M}(A)$.

Finally, tying this back to Definition 45, we have the following for free commutative monoids.

► **Definition 54** (⚙️ head). *Assume a total order $\leq$ on a set A . We define a commutative monoid structure on $1 + A$, with unit $e := \text{inl}(\star) : 1 + A$, and multiplication defined as:*

$$\begin{aligned} \text{inl}(\star) \oplus b &:= b \\ \text{inr}(a) \oplus \text{inl}(\star) &:= \text{inr}(a) \\ \text{inr}(a) \oplus \text{inr}(b) &:= \text{inr}(a \sqcap b) . \end{aligned}$$

This gives a homomorphism $\text{head} := \text{inr}^\# : \mathcal{M}(A) \rightarrow 1 + A$, which picks out the least element of the free commutative monoid.

5.3 Sorting functions

The free commutative monoid is also a monoid, hence, there is a canonical monoid homomorphism $\mathbf{q} : \mathcal{L}(A) \rightarrow \mathcal{M}(A)$, which is given by $\eta_A^\#$, the extension of the unit $\eta_A : A \rightarrow \mathcal{M}(A)$. Since $\mathcal{M}(A)$ is (upto equivalence), a quotient of $\mathcal{L}(A)$ by symmetries (or a permutation relation), it is a surjection (in particular, a regular epimorphism in **Set** as constructed in type theory), the canonical inclusion into the quotient. Concretely, $\mathbf{q}$ simply includes the elements of $\mathcal{L}(A)$ into equivalence classes of lists in $\mathcal{M}(A)$, which “forgets” the order that was imposed by the indexing of the list.

Classically, assuming the Axiom of Choice would allow us to construct a section (right-inverse, in **Set**) to the surjection $\mathbf{q}$, that is, a function $\mathbf{s} : \mathcal{M}(A) \rightarrow \mathcal{L}(A)$ such that $\forall x. \mathbf{q}(\mathbf{s}(x)) = x$. Or in informal terms, given the surjective inclusion into the quotient, a section (uniformly) picks out a canonical representative for each equivalence class. The core question we want to study is the existence of $\mathbf{s}$ in a constructive setting, or equivalently, whether the order factored out by the symmetry quotient can be constructively recovered. Viewing the quotienting relation as a permutation relation (from § 4.1), a section $\mathbf{s}$ to $\mathbf{q}$ has to pick out canonical representatives of equivalence classes generated by permutations. Using **SList** as an example, $\mathbf{s}(x :: y :: xs) = \mathbf{s}(y :: x :: xs)$ for any $x, y : A$ and $xs : \mathbf{SList}(A)$, by **swap**. Since $\forall xs. \mathbf{q}(\mathbf{s}(xs)) = xs$, $\mathbf{s}$ must preserve all the elements of xs . It cannot be a trivial function such as $\lambda xs. []$ – it must produce a permutation of the elements of xs . But to place these elements side-by-side in the list, $\mathbf{s}$ must somehow impose an order on A (invariant under permutation), turning unordered lists of A into ordered lists of A . Axiom of Choice (AC) giving us a section $\mathbf{s}$ to $\mathbf{q}$ “for free” is analagous to how AC implies the well-ordering principle, which states every set can be well-ordered. Thus, if AC was assumed, we could easily recover an order on A from the section $\mathbf{s}$. Instead we study how to constructively define such a section, and in fact, that is exactly the extensional view of a sorting algorithm, and the implications of its existence is that A can be ordered, or sorted.

5.3.1 Section from Order

► **Proposition 55** (⚙️). *Assume a decidable total order on A . There is a sort function $\mathbf{s} : \mathcal{M}(A) \rightarrow \mathcal{L}(A)$ which constructs a section to $\mathbf{q} : \mathcal{L}(A) \rightarrow \mathcal{M}(A)$*

Proof sketch. We may construct such a sort function by implementing any sorting algorithm. In our formalisation we chose insertion sort, because it can be defined easily using the inductive structure of $\text{SList}(A)$ and $\text{List}(A)$. To implement other sorting algorithms like mergesort, other representations such as **Bag** and **Array** would be preferable, as explained in § 4.4. To see how this proposition holds: $\mathbf{q}(\mathbf{s}(xs))$ first orders an unordered list xs by $\mathbf{s}$, then forgets the order again by $\mathbf{q}$ – imposing and then forgetting an order on xs simply *permutes* its elements, which proves $\mathbf{q} \circ \mathbf{s} = \text{id}$. ◀

5.3.2 Order from Section

The previous section allowed us to construct a section, but an arbitrary section may not be a sorting function. To show a section is indeed a sort function, we need to show the section imposes some total order on A which it sorts by. Indeed, just by the virtue of $\mathbf{s}$ being a section, we can *almost* construct a total-order on the carrier set.

► **Definition 56** (⚙️ *least*). Given a section $\mathbf{s}$, we define a relation $\leq_{\mathbf{s}}$ parametrised by $\mathbf{s}$:

$$x \leq_{\mathbf{s}} y := \text{head}(\mathbf{s}(\wr x, y)) = \text{inr}(x) .$$

Recall that the head homomorphism $\text{head} := \text{inr}^\# : \mathcal{L}(A) \rightarrow 1 + A$ (as in Definition 45) is defined by equipping $1 + A$ with a monoid structure with unit $e := \text{inl}(\star)$ and multiplication defined as $\text{inl}(\star) \oplus b := b$ and $\text{inr}(a) \oplus b := \text{inr}(a)$, which picks out the first (leftmost) element of a list. Here, we take the two-element bag $\wr x, y$, “sort” it by $\mathbf{s}$, and test if the **head** element is x . Note, this is equivalent to $x \leq_{\mathbf{s}} y := \mathbf{s} \wr x, y = [x, y]$, because $\mathbf{s}$ preserves length, and the second element is forced to be y .

► **Proposition 57** (⚙️). $\leq_{\mathbf{s}}$ is reflexive, antisymmetric, and total.

Proof. For all x , $\text{least}(\wr x, x)$ must be $\text{inr}(x)$, therefore $x \leq_{\mathbf{s}} x$, giving reflexivity. For all x and y , given $x \leq_{\mathbf{s}} y$ and $y \leq_{\mathbf{s}} x$, we have $\text{least}(\wr x, y) = \text{inr}(x)$ and $\text{least}(\wr y, x) = \text{inr}(y)$. Since $\wr x, y = \wr y, x$, $\text{least}(\wr x, y) = \text{least}(\wr y, x)$, therefore we have $x = y$, giving antisymmetry. For all x and y , $\text{least}(\wr x, y)$ is merely either $\text{inr}(x)$ or $\text{inr}(y)$, therefore we have merely either $x \leq_{\mathbf{s}} y$ or $y \leq_{\mathbf{s}} x$, giving totality. ◀

A crucial observation is that $\mathbf{s}$ correctly orders 2-element bags, but it does not necessarily sort bags with 3 or more elements.

► **Proposition 58** (⚙️). $\leq_{\mathbf{s}}$ is not necessarily transitive.

Proof. A sorting section $\mathbf{s}$ would indeed give a $\leq_{\mathbf{s}}$ that satisfies transitivity. However, we can turn a sorting section into a non-sorting misbehaving section that gives a non-transitive $\leq_{\mathbf{s}}$. Take the sorting section $\text{sort} : \text{SList}(\mathbb{N}) \rightarrow \text{List}(\mathbb{N})$ which sorts $\text{SList}(\mathbb{N})$ ascendingly, we can use sort to construct a counterexample $\mathbf{s}$ as follows:

$$\mathbf{s}(xs) := \begin{cases} [3, 1] & \text{if } xs = \wr 1, 3 \\ \text{sort}(xs) & \text{otherwise} \end{cases}$$

Indeed, we have $1 \leq_{\mathbf{s}} 2$ and $2 \leq_{\mathbf{s}} 3$ as expected, but $\text{least}(\wr 1, 3) = \text{inr}(3)$, therefore $1 \not\leq_{\mathbf{s}} 3$, thus violating transitivity. ◀

To make sure that $\mathbf{s}$ is a well-behaved section that recovers a total order on A , we will enforce additional constraints on the *image* of $\mathbf{s}$.

► **Definition 59** (⚙️ $- \in \text{im}(\mathfrak{s})$). The fiber of $\mathfrak{s}$ at a point in the codomain $xs: \mathcal{L}(A)$ is given by $\text{fib}_{\mathfrak{s}}(xs) := (ys: \mathcal{M}(A)) \times (\mathfrak{s}(ys) = xs)$. The image of $\mathfrak{s}$ is given by $\text{im}(\mathfrak{s}) := (xs: \mathcal{L}(A)) \times \|\text{fib}_{\mathfrak{s}}(xs)\|_{-1}$. Simplifying, we say that $xs: \mathcal{L}(A)$ is “in the image of $\mathfrak{s}$ ”, or, $xs \in \text{im}(\mathfrak{s})$, if there merely exists a $ys: \mathcal{M}(A)$ such that $\mathfrak{s}(ys) = xs$.

If $\mathfrak{s}$ were a sort function, the image of $\mathfrak{s}$ would be the set of $\mathfrak{s}$ –“sorted” lists, therefore $xs \in \text{im}(\mathfrak{s})$ would imply xs is a correctly $\mathfrak{s}$ –“sorted” list. First, we note that the 2-element case is correct.

► **Proposition 60** (⚙️). $x \leq_{\mathfrak{s}} y \iff [x, y] \in \text{im}(\mathfrak{s})$.

Then, we state the first axiom on $\mathfrak{s}$.

► **Definition 61** (⚙️ isHeadLeast). A section $\mathfrak{s}$ satisfies isHeadLeast iff for all x, y, xs :

$$y \in x :: xs \wedge x :: xs \in \text{im}(\mathfrak{s}) \rightarrow [x, y] \in \text{im}(\mathfrak{s}) .$$

There are two different membership symbols in this axiom, where the first membership is list membership from Definition 43, and the second membership is image membership from Definition 59. The $\in$ symbol is intentionally overloaded to make the axiom look like a logical “cut” rule, that pushes the (least) x element to the head of the list. Informally, the head of an $\mathfrak{s}$ –“sorted” list is always the least element of the list.

► **Proposition 62** (⚙️). If A has a total order $\leq$, the insertion sort function defined using $\leq$ satisfies isHeadLeast .

► **Proposition 63** (⚙️). If $\mathfrak{s}$ satisfies isHeadLeast , $\leq_{\mathfrak{s}}$ is transitive.

Proof. Given $x \leq_{\mathfrak{s}} y$ and $y \leq_{\mathfrak{s}} z$, we want to show $x \leq_{\mathfrak{s}} z$. Consider the 3-element bag $\{x, y, z\}: \mathcal{M}(A)$. Let u be $\text{least}(\{x, y, z\})$, by Definition 61 and Proposition 60, we have $u \leq_{\mathfrak{s}} x \wedge u \leq_{\mathfrak{s}} y \wedge u \leq_{\mathfrak{s}} z$. Since $u \in \{x, y, z\}$, u must be one of the elements. If $u = x$ we have $x \leq_{\mathfrak{s}} z$. If $u = y$ we have $y \leq_{\mathfrak{s}} x$, and since $x \leq_{\mathfrak{s}} y$ and $y \leq_{\mathfrak{s}} z$ by assumption, we have $x = y$ by antisymmetry, and then we have $x \leq_{\mathfrak{s}} z$ by substitution. Finally, if $u = z$, we have $z \leq_{\mathfrak{s}} y$, and since $y \leq_{\mathfrak{s}} z$ and $x \leq_{\mathfrak{s}} y$ by assumption, we have $z = y$ by antisymmetry, and then we have $x \leq_{\mathfrak{s}} z$ by substitution. ◀

5.3.3 Embedding orders into sections

Following from Propositions 57 and 63, and Proposition 62, we have shown that a section $\mathfrak{s}$ that satisfies isHeadLeast produces a total order $x \leq_{\mathfrak{s}} y := \text{least}(\{x, y\}) = \text{inr}(x)$, and a total order $\leq$ on the carrier set produces a section satisfying isHeadLeast , constructed by sorting with $\leq$. This constitutes an embedding of decidable total orders into sections satisfying isHeadLeast .

► **Proposition 64** (⚙️). Assume A has a decidable total order $\leq$, we can construct a section $\mathfrak{s}$ that satisfies isHeadLeast , such that $\leq_{\mathfrak{s}}$ constructed from $\mathfrak{s}$ is equivalent to $\leq$.

Proof. By the insertion sort algorithm parameterized by $\leq$, it holds that $[x, y] \in \text{im}(\mathfrak{s})$ iff $x \leq y$. By Proposition 60, we have $x \leq_{\mathfrak{s}} y$ iff $x \leq y$. We now have a total order $x \leq_{\mathfrak{s}} y$ that is equivalent to $x \leq y$. ◀

5.3.4 Equivalence of order and sections

We want to improve the embedding to an isomorphism, and it remains to show that we can turn a section satisfying `isHeadLeast` to a total order $\leq_s$, then reconstruct the *same* section from $\leq_s$. Unfortunately, `isHeadLeast` is not enough to guarantee this.

► **Proposition 65** (⚙️). *There is no equivalence between sections satisfying the `isHeadLeast` axiom and total orders on A .*

Proof. We consider the set $\mathbb{N}$ of natural numbers, and construct a $\mathfrak{s}$ that satisfies `isHeadLeast` but is not a sort function. Let the correct insertion sort function be `sort`: $\mathcal{M}(\mathbb{N}) \rightarrow \mathcal{L}(\mathbb{N})$. We define `reverseTail` which reverses only the tail of a list, and a section $\mathfrak{s}$ that sorts correctly using `sort` then reverses the tail:

$$\begin{aligned} \text{reverseTail}([]) &:= [] \\ \text{reverseTail}(x :: xs) &:= x :: \text{reverse}(xs) \\ \mathfrak{s}(xs) &:= \text{reverseTail}(\text{sort}(xs)) \end{aligned}$$

For example:

$$\begin{aligned} \mathfrak{s}(\lceil 2, 3, 1, 4 \rceil) &= [1, 4, 3, 2] \\ \mathfrak{s}(\lceil 2, 3, 1 \rceil) &= [1, 3, 2] \\ \mathfrak{s}(\lceil 2, 3 \rceil) &= [2, 3] \end{aligned}$$

Note that both `sort` and $\mathfrak{s}$ satisfy `isHeadLeast`, but $\mathfrak{s}$ only sorts 2-element lists correctly, and $\mathfrak{s} \neq \text{sort}$. By Proposition 64 we can use both `sort` and $\mathfrak{s}$ to reconstruct the same $\leq$ on $\mathbb{N}$. Hence, two different sections satisfying `isHeadLeast` map to the same total order. ◀

Therefore, we need to introduce a second axiom of sorting.

► **Definition 66** (⚙️ `isTailSorted`). *A section $\mathfrak{s}$ satisfies `isTailSorted` iff for all x, xs ,*

$$x :: xs \in \text{im}(\mathfrak{s}) \rightarrow xs \in \text{im}(\mathfrak{s})$$

This says that $\mathfrak{s}$ -“sorted” lists are downwards-closed under `cons`-ing, that is, the tail of an $\mathfrak{s}$ -“sorted” list is also $\mathfrak{s}$ -“sorted”. To prove the correctness of our axioms, first we need to show that a section $\mathfrak{s}$ satisfying `isHeadLeast` and `isTailSorted` is equal to insertion sort parameterized by the $\leq_s$ constructed from $\mathfrak{s}$. In fact, the axioms we have introduced are equivalent to the standard inductive characterisation of sorted lists, found in textbooks, such as in [5].

► **Definition 67** (⚙️ `isSorted`). *Given a total order $\leq$ on A , the predicate `isSorted≤` on $\mathcal{L}(A)$ is generated by the following constructors:*

$$\begin{aligned} \text{sorted-nil} &: \text{isSorted}_{\leq}([]) \\ \text{sorted-}\eta &: \forall x. \text{isSorted}_{\leq}([x]) \\ \text{sorted-cons} &: \forall x, y, zs. x \leq y \rightarrow \text{isSorted}_{\leq}(y :: zs) \rightarrow \text{isSorted}_{\leq}(x :: y :: zs) \end{aligned}$$

Note that `isSorted≤`(xs) is a proposition for every xs , and forces the list xs to be permuted in a unique way. This predicate is equivalent to the standard `Sorted` relation in the Agda standard library [13], which is defined pairwise using the more general `Linked` relation (i.e., `Sorted(xs) := Linked(- ≤ -, xs)`). Indeed, many of these definitions are traditional constructions that are completely independent of univalence or cubical features.

► **Lemma 68** (⚙️). *Given a total order $\leq$, for any lists $xs, ys: \mathcal{L}(A)$, if $q(xs) = q(ys)$, $\text{isSorted}_{\leq}(xs)$, and $\text{isSorted}_{\leq}(ys)$, then $xs = ys$.*

Insertion sort by $\leq$ always produces lists that satisfy $\text{isSorted}_{\leq}$. Sections that also produce lists satisfying $\text{isSorted}_{\leq}$ are equal to insertion sort by function extensionality.

► **Proposition 69** (⚙️). *Given a total order $\leq$, if a section $\mathfrak{s}$ always produces a sorted list, that is, $\forall xs. \text{isSorted}_{\leq}(\mathfrak{s}(xs))$, $\mathfrak{s}$ is equal to insertion sort by $\leq$.*

Finally, this gives us the correctness property of our axioms.

► **Proposition 70** (⚙️). *Given a section $\mathfrak{s}$ that satisfies isHeadLeast and isTailSorted , and $\leq_{\mathfrak{s}}$ the order derived from $\mathfrak{s}$, then for all $xs: \mathcal{M}(A)$, it holds that $\text{isSorted}_{\leq_{\mathfrak{s}}}(\mathfrak{s}(xs))$. Equivalently, for all lists $xs: \mathcal{L}(A)$, it holds that $xs \in \text{im}(\mathfrak{s})$ iff $\text{isSorted}_{\leq_{\mathfrak{s}}}(xs)$.*

Proof. We inspect the length of $xs: \mathcal{M}(A)$. For lengths 0 and 1, this holds trivially. Otherwise, we proceed by induction: given a $xs: \mathcal{M}(A)$ of length ≥ 2 , let $\mathfrak{s}(xs) = x :: y :: ys$. We need to show $x \leq_{\mathfrak{s}} y \wedge \text{isSorted}_{\leq_{\mathfrak{s}}}(y :: ys)$ to construct $\text{isSorted}_{\leq_{\mathfrak{s}}}(x :: y :: ys)$. By isHeadLeast , we have $x \leq_{\mathfrak{s}} y$, and by isTailSorted , we inductively prove $\text{isSorted}_{\leq_{\mathfrak{s}}}(y :: ys)$. ◀

► **Lemma 71** (⚙️). *Given a decidable total order $\leq$ on A , we can construct a section $t_{\leq}$ satisfying the axioms isHeadLeast and isTailSorted , such that, for the order $\leq_{\mathfrak{s}}$ derived from $\mathfrak{s}$, we have $t_{\leq} = \mathfrak{s}$.*

Proof. From $\mathfrak{s}$ we can construct a decidable total order $\leq_{\mathfrak{s}}$ since $\mathfrak{s}$ satisfies isHeadLeast and A has decidable equality by assumption. We construct $t_{\leq_{\mathfrak{s}}}$ as insertion sort parameterized by $\leq_{\mathfrak{s}}$ constructed from $\mathfrak{s}$. By Proposition 69 and Proposition 70, $s = t_{\leq_{\mathfrak{s}}}$. ◀

We can now state and prove our main theorem.

► **Definition 72** (⚙️ Sorting function). *A sorting function is a section $\mathfrak{s}: \mathcal{M}(A) \rightarrow \mathcal{L}(A)$ to the canonical surjection $q: \mathcal{L}(A) \twoheadrightarrow \mathcal{M}(A)$ satisfying two axioms:*

- $\text{isHeadLeast}: \forall x, y, xs, y \in x :: xs \wedge x :: xs \in \text{im}(\mathfrak{s}) \rightarrow [x, y] \in \text{im}(\mathfrak{s})$,
- $\text{isTailSorted}: \forall x, xs, x :: xs \in \text{im}(\mathfrak{s}) \rightarrow xs \in \text{im}(\mathfrak{s})$.

► **Theorem 73** (⚙️). *Let $\text{DecTotOrd}(A)$ be the set of decidable total orders on A , $\text{Sort}(A)$ be the set of sorting functions with carrier set A , and $\text{Discrete}(A)$ be a predicate which states A has decidable equality. There is a map $o2s: \text{DecTotOrd}(A) \rightarrow \text{Sort}(A) \times \text{Discrete}(A)$, which is an equivalence.*

Proof. $o2s$ is constructed by parameterizing insertion sort with $\leq$. By Proposition 47, A is Discrete . The inverse $s2o(s)$ is constructed by Definition 56, which produces a total order by Propositions 57 and 63, and a decidable total order by $\text{Discrete}(A)$. By Proposition 64 we have $s2o \circ o2s = \text{id}$, and by Lemma 71 we have $o2s \circ s2o = \text{id}$, giving an isomorphism, hence an equivalence. ◀

► **Corollary 74** (⚙️). *Let $\text{DecSTotOrd}(A)$ be the set of decidable strict total orders on A , There is a map $t2s: \text{DecSTotOrd}(A) \rightarrow \text{Sort}(A) \times \text{Discrete}(A)$, which is an equivalence.*

Proof. By Proposition 47, we have an equivalence $\text{DecSTotOrd}(A) \simeq \text{DecTotOrd}(A)$, and the result follows from Theorem 73 by transitivity of equivalences. ◀

► **Corollary 75** (⚙️). *Assume A has decidable equality. Let $\text{TMSLat}(A)$ be the set of total meet semilattices on A , There is a map $l2s: \text{TMSLat}(A) \rightarrow \text{Sort}(A)$, which is an equivalence.*

Proof. By assumption and Proposition 52, every total meet semilattice on A is decidable. Thus, by Proposition 53, we have an equivalence $\text{TMSLat}(A) \simeq \text{DecTotOrd}(A)$, and the result follows from Theorem 73 by transitivity of equivalences. $\blacktriangleleft$

The sorting axioms we have come up with are abstract properties of functions. As a sanity check, we can verify that the colloquial correctness specification of a sorting function (starting from a total order) matches our axioms. We use the correctness criterion developed in [3].

► **Proposition 76 (⚙️).** *Assume a decidable total order $\leq$ on A . A sorting algorithm is a map $\text{sort}: \mathcal{L}(A) \rightarrow \mathcal{OL}(A)$, that turns lists into ordered lists, where $\mathcal{OL}(A)$ is defined as $(xs: \mathcal{L}(A)) \times \text{isSorted}_{\leq}(xs)$, such that:*

$$\begin{array}{ccc} \mathcal{L}(A) & \xrightarrow{\text{sort}} & \mathcal{OL}(A) \\ & \searrow q & \swarrow q \circ \pi_1 \\ & \mathcal{M}(A) & \end{array}$$

Sorting functions (as in Definition 72) give sorting algorithms.

Proof. We construct our section $\mathfrak{s}: \mathcal{M}(A) \rightarrow \mathcal{L}(A)$, and define $\text{sort} := \mathfrak{s} \circ q$, which produces ordered lists by Proposition 70. $\blacktriangleleft$

6 Discussion

We conclude by discussing some high-level observations, related work, and future directions.

Formalisation. The paper uses informal type theoretic language, and is accessible without understanding any details of the formalisation. However, the formalisation is done in Cubical Agda, which has a few differences and shortcomings, that cause certain details to be more verbose than they are in informal type theory.

For simplicity we omitted type levels in the paper, but our formalisation has many verbose uses of universe levels due to Agda’s universe polymorphism. Similarly, h-levels were restricted to sets in the paper, but the formalisation is parameterised in many places for any h-level (to facilitate future generalisations). The free algebra framework currently only works with sets. Due to issues of regularity, certain computations only hold propositionally, and the formalisation requires proving auxiliary β and η computation rules in some places.

Universal Algebra. One of the contributions of our work is a rudimentary framework for universal algebra in a more categorical style, which lends itself to an elegant formalisation in type theory. This framework may be improved and generalised from sets to groupoids, using a system of coherences on top of a system of equations, to talk about 2-algebraic structures on groupoids, such as monoidal groupoids, symmetric monoidal groupoids, and so on.

Free commutative monoids. The construction of finite multisets and free commutative monoids in type theory has a long history, and various authors have different approaches to it. We refer the reader to the discussions in [8, 20] for a detailed survey of these constructions. Our work, in particular, was motivated by the colloquial observation that: “there is no way to represent free commutative monoids using inductive types”. From the categorical point of view, note that the free monoid functor on Set is polynomial, but the free commutative monoid endofunctor is not polynomial, since it only weakly preserves pullbacks. Various

authors have given clever encodings of free commutative monoids using inductive types by adding assumptions on the carrier set – in particular, the assumption of total ordering on the carrier set leads to the construction of “fresh-lists”, by [22], which forces the canonical *sorted* ordering on the elements of the finite multiset.

It is worth noting that in programming practice, it is usually the case that all user-defined types have some sort of total order enforced on them, either because they’re finite, or because they can be canonically enumerated. Under these assumptions, the construction of fresh lists is a very reasonable way to represent free commutative monoids, or finite multisets.

The free monoid and free commutative monoid constructions can be categorified to free monoidal categories and free symmetric monoidal categories, respectively. In type theory, these types can be groupoidified to free monoidal groupoids and free symmetric monoidal groupoids, using the groupoid structure of identity types. This is a natural next step, and its connections to assumptions about total orders on the type of objects would be an important direction to explore.

Correctness of Sorting. Sorting is a classic problem in computer science – the programming point of view of sorting and its correctness has been studied by various authors. The simplest view of sorting is a function $\text{sort}: \mathcal{L}(\mathbb{N}) \rightarrow \mathcal{L}(\mathbb{N})$, which permutes the list and outputs an ordered list, which is studied in [5]. Fundamentally, this is a very extrinsic view of program verification, which is commonly used in program verification and proof assistants, and further, a special case of a more general sorting algorithm.

Henglein [17] studies sorting functions abstractly, without requiring a total order on the underlying set. He considers sorting functions as functions on sequences (lists), and recovers the order by applying a “sorting function” on an n -element list, and looking up the position of the elements to be compared. Unlike our approach, he does not factorize sorting functions through free commutative monoids. Henglein also studies sorting on preorders where antisymmetry may not hold, whereas we have only considered total orders in this paper. We are able to give a more refined axiomatisation of sorting because we consider permutations explicitly, and work in a constructive setting (using explicit assumptions about decidability), and which is an improvement over previous work.

Another intrinsic view of correctness of sorting has been studied in [18] using bialgebras, which was further expanded in [3] with an accompanying formalisation in Cubical Agda, that matches our point of view, as explained in Proposition 76. However, their work is not just about extensional correctness of sorting, but also deriving various sorting algorithms starting from bialgebraic semantics and distributive laws. Our work is complementary, in that we are not concerned with the computational content of sorting, but rather the abstract properties of sorting functions, which are independent of a given ordering. It is unclear whether the abstract property of sorting functions can be combined with the intrinsic complexity of sorting algorithms – and that is a direction for future work.

Although we only talk about sorting lists and bags, the same ideas can be applied to other inductive types (polynomial functors) such as trees. We speculate that this could lead to some interesting connections with sorting (binary) trees, and constructions of (binary) search trees, from classical computer science.

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