Towards Fuzzy Constructive Type Theories

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Abstract

In this paper, we explore combinations of Fuzzy Logic and Constructive Higher Order Type Theory. Although Fuzzy Logic is more than 60 years old, and Fuzzy Type Theories have been studied in the classical case of Church's Theory of Types, there is yet no satisfactory understanding of how to deal with judgements of the shape $\Gamma \vdash M :_d A$, where d is a fuzzy degree of confidence. Addressing this issue is important in view of the growing interest in quantitative and non-idempotent type theories. To this end, we introduce *fuzzy type assignment systems* in order to investigate how minimal fuzzy formulæ behave as types, according to some *fuzzy proposition-as-types* paradigm. Moreover, we study a *fuzzy version* of *intersection types*. In both systems assumptions are multisets and d, in $\Gamma \vdash M :_d A$, is a formula of minimal propositional Fuzzy Logic. Evaluating d in a residuated lattice [0,1], endowed with a left-continuous T-norm, we can analyse how the *degree of confidence* propagates from the assumptions to the conclusion, thereby yielding new information on the derivation. The former system sheds light on the connection between minimal Fuzzy Logics and *affine λ-calculus* and permits to show that the tautologies in the **BCK**-logic amount to the simple types which are inhabited by **BCK**-combinators. The latter system keeps track of the number of times a given fuzzy assumption is made. We discuss the standard suite of metatheorems (inversion lemmata and subject-conversion) for the systems, and point to possible uses of the fuzzy formula attached to the membership construct.

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1 Introduction

Fuzzy Logics [16, 12, 14] are more than 60 years old. They are a family of many-valued logics which arise when multiple and linearly ordered *degrees of truth* are considered, namely when the truth value of a propositional formula may be taken to be any real number between 0 and 1. Fuzzy Logics have been extensively used in control systems, in artificial intelligence, and recently in digital health and life-enhancing technologies. Significant work has already appeared on Fuzzy Type Theory in the classical setting of Church's Theory of Types [13], and as far as first order theories [6]. However, to the best of our knowledge, there is yet no complete and satisfactory understanding of how the basic idea behind Fuzzy Logic combines with Higher Order *Constructive* Type Theory. We think that this issue needs to be addressed, since decorating a typing judgement with a fuzzy coefficient adds a new expressive dimension to the invariants which types can capture, thus leading to *fuzzy constructive type theory.* This should be particularly relevant in view of the growing interest, in the theoretical computer science community, towards systems supporting a finer analysis of programs based on quantitative and non-contractive reasoning, *e.g.* [10, 5, 4, 11]. One of the characteristics of Fuzzy Logic, which lies at the root of its explanation of the Sorites Paradox [15], is that each application of *modus ponens* further decreases the confidence value of a formula.

In this paper, we introduce and investigate various notions of *fuzzy type assignment systems* for pure λ-calculus, thus providing possible answers to the following questions.

- How do intuitionistic fragments of Fuzzy Logic behave as types in a possible *Fuzzy Proposition-as-Types* and *Proofs as λ-terms* paradigm?
- What is a fuzzy version of *Intersection Type Assignment Systems*?

Answering these two questions should pave the way for addressing more general questions such as: What are "fuzzy twins" of other type assignment systems *à la* Curry? What is the appropriate notion of *Fuzzy Constructive Type Theory* for λ-terms *à la* Church? Moreover, since Fuzzy Logic has in general a non-idempotent semantics, how do fuzzy type assignment systems relate to existing quantitative and non-idempotent non-contractive type theories and program logics?

In particular, in the present paper, we introduce three fuzzy type assignment systems. The first two formal systems $\mathcal{F}$ and $\mathcal{Q}$ deal with judgements of the of shape $\Gamma \vdash M : d$, where the type d is a formula of negation-free Fuzzy Logic. They provide answers to the first question. These systems differ in that $\mathcal{F}$ is contractive, namely contexts are taken as sets of assumptions, while $\mathcal{Q}$ is non-contractive, *i.e.* contexts are multisets of assumptions. Evaluating the semantics of the formula d in the lattice [0,1], enriched with a left-continuous T-norm (see Definition 32), permits us to relate the notion of *inhabited* fuzzy formula, in $\mathcal{F}$ and $\mathcal{Q}$, to its degree of truth. The value of d depends on the value of the degree of confidence assigned to the assumptions. If d is a tautology, the value of the formula is clearly 1, but if the confidence degree in the assumption is not 1, its value in general is smaller, thereby providing information on the number of occurrences a variable has been used. This permits us to recover, along a different route (see Corollary 11), the connection between the closed *affine* λ-terms, namely the **BCK**-combinators, and the theory of **BCK**-logic, which has been extensively studied in the context of Fuzzy Logic, see *e.g.* [14]. We recall that **BCK**-logic, namely the sound and complete logic of **BCK**-algebras, is included in all Fuzzy-Logics. Moreover, we show that taking contexts as multisets preserves the interpretation of fuzzy types under β-conversion.

Building on $\mathcal{Q}$, we introduce the *Fuzzy Intersection Type Assignment System* $\mathcal{Q}^\cap$, which is a "fuzzification" of the Intersection Type Assignment System [2, 3], for establishing judgements of the shape $\Gamma \vdash_d M : A$, where d is a formula of Fuzzy Logic and A is an intersection-type. Intuitively we may say that the term M has *fuzzy degree d of membership* to the type A. Incidentally, we remark that the former systems $\mathcal{F}$ and $\mathcal{Q}$ amount to studying the fuzzy membership to a unique reflexive type A, such that $A \sim A \to A$, in a suitable Intersection Type Theory, see Remark 25. The system $\mathcal{Q}^\cap$ is non-contractive, hence *relevant*, where fuzzy coefficients index the membership relation and contexts are *fuzzy multisets*, rather than sets. All the standard suite of results which hold in the *crisp* version, *i.e.* the original Intersection Types system, also hold for $\mathcal{Q}^{\cap}$. The term "crisp", which belongs to the terminology in Fuzzy Logic, denotes the system where the truth value of propositions is just 0 or 1. Namely, we have: inversion lemmata, subject expansion, and subject reduction. The benefit of this system, besides providing a theory of how the degree of confidence propagates via applications and abstractions, permits us to keep track of the number of occurrences of a variable in a term. We hint to some applications. We chose to focus on Intersection Types because they generalise simple types and, moreover, they can capture the class of strongly normalising terms, as well as many other classes of λ-terms. We are confident that the same "fuzzification" can be similarly carried out for other types assignment systems for λ-terms *à la* Curry, such as higher order impredicative types.

For all notations and notions which are not explicitly spelled out in this paper, we refer to [3], for typed λ-calculi, and to *e.g.* [12, 14], for Fuzzy Logic.

Synopsis. In Section 2 we give basic definitions on Fuzzy Logic and Type Assignment Systems. In Section 3 we introduce the type assignment systems $\mathcal{F}$ and $\mathcal{Q}$ for assigning a formula of Fuzzy Logic, and ultimately a *confidence degree* to a λ-term, by evaluating the fuzzy formula, and study their main properties. Full Intersection Type Theory is considered in Section 4. Final remarks on Constructive Type Theories with a *fuzzy typing relation*, comparison with related work and directions for future work appear in Section 5. The Appendix A contains additional standard results and definitions in Fuzzy Logic.

2 Preliminaries

2.1 Fuzzy Logic

The language of *minimal Fuzzy Logic* is given by the following grammar:

$$d, e, f ::= X \mid d \wedge e \mid d \otimes e \mid d \Rightarrow e \mid 0 \mid 1.$$

The semantics of d, which is denoted by $[\![d]\!]^{T}_{\rho}$, is given by interpreting propositional variables X, according to an assignment ρ, as values in the interval [0,1] endowed with a *left-continuous T-norm*, thus making the interval [0,1] a residuated lattice, see Appendix A. The constants 0 and 1 denote falsity and truth. By abuse of notation, when clear from the context, we will use values in $[0, 1]$ as constants in the language. The T-norm allows us to interpret the connectives, *viz.* $\otimes$ by the T-norm itself, $\Rightarrow$ by its residuum, and $\wedge$ by the minimum. We assume that $\otimes$ has higher priority than $\Rightarrow$, the latter being right-associative, *e.g.* $d \Rightarrow e \Rightarrow f \otimes f'$ means $d \Rightarrow (e \Rightarrow (f \otimes f'))$. The three prominent examples of left-continuous T-norms on [0,1], namely Gödel's T-norm, Goguen's product T-norm, and Łukasiewicz's T-norm, appear in Appendix A, together with some of the main properties of T-norms. We refer to [12, 14] for more details and basic definitions. A T-norm is said to be *idempotent* if $d \otimes d = d$. Gödel's T-norm is the only idempotent left-continuous T-norm. By abuse of notation, when clear from the context, we denote the norm and its residuum, by $\otimes$ and $\Rightarrow$.

2.2 Lambda terms and Intersection Type Assignment Systems

In this subsection, we introduce the untyped λ-calculus and the set of types that can be assigned to it. We focus on Intersection Type systems [2, 3, 8].

▶ **Definition 1** (λ-terms).

1. *The set* Λ *of (untyped)* λ*-terms is defined as usual:*

$$M, N ::= x \mid \lambda x.M \mid MN$$

where x *is a variable,* $\lambda x.M$ *is an* abstraction*, and* MN *is a term* application.

2. Affine λ-terms *are* λ*-terms, where each abstracted variable occurs* at most once. Linear λ-terms *are* λ*-terms, where each abstracted variable occurs* exactly once. *Closed terms are called* combinators.

We refer to [1, 3] for standard notions and results concerning λ-calculus.

We briefly recall the basic definitions concerning simple and intersection types.

▶ **Definition 2.**

1. *Given a set of constants* $\mathbb{A}$ *and a distinguished constant* U*, the set* $\mathbb{T}_{\mathbb{A}}$ *of* intersection types *over* $\mathbb{A}$ *are defined by the grammar:*

$$A, B ::= c \mid \mathsf{U} \mid A \to B \mid A \cap B$$

where $c \in \mathbb{A}$*;* simple *types are those types which do not contain the constant* U *and the* $\cap$ *constructor.*

2. A subtyping relation $\leq$ *is a binary relation on* $\mathbb{T}_{\mathbb{A}}$ *closed under the following axioms and rules:*

$$A \leq A \quad (\mathit{Refl}) \qquad A \cap B \leq A \quad (\mathit{incL}) \qquad B \cap A \leq A \quad (\mathit{incR})$$

$$B \to A \cap A' \leq (B \to A) \cap (B \to A') \quad (\to \cap)$$

$$A \leq \mathsf{U} \quad (\mathit{Top}) \qquad \mathsf{U} \leq A \to \mathsf{U} \quad (\to \mathsf{U})$$

$$\frac{A \leq B \quad A \leq B'}{A \leq B \cap B'}\ (\mathit{Glb}) \qquad \frac{A \leq B \quad B \leq B'}{A \leq B'}\ (\mathit{Trans})$$

$$\frac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}\ (\to)$$

3. *An* intersection type theory (itt) $\mathcal{T}$ *is determined by a set of type constants* $\mathbb{A}$ *and a subtyping relation on the set* $\mathbb{T}_{\mathbb{A}}$ *which includes those in (ii),* i.e. $\mathcal{T} = \langle \mathbb{A}, \leq_{\mathcal{T}} \rangle$.

We assume that $\cap$ has higher priority than $\to$, the latter being right-associative e.g. $A \to B \to C \cap\ C'$ means $A \to (B \to (C \cap C'))$. We write $A \sim B$ as short for $A \leq B$ and $B \leq A$. The above rules permit us to show that $\sim$ is a congruence w.r.t. to the signature of intersection types, given by the $\cap$ and $\to$ constructors, and that $\mathbb{T}/{\sim}$ is a meet semilattice. In this paper we will discuss various type assignment systems, here we just recall the general definition of Intersection Type Assignment system [3, 8]. We also recall that a type is said to be *inhabited* when it is the type of a closed λ-term.

▶ **Definition 3** (Intersection Type Assignment System). *The* intersection type assignment system *induced by an itt* $\mathcal{T} = \langle \mathbb{A}, \leq_{\mathcal{T}} \rangle$ *is a formal system deriving judgements of the shape* $\Gamma \vdash_{\mathcal{T}} M : A$*, where* $A \in \mathbb{T}_{\mathbb{A}}$ *and a* basis Γ *is a finite mapping from term variables to types in* $\mathbb{T}_{\mathbb{A}}$*:*

$$\Gamma ::= \emptyset \mid \Gamma, x : A.$$

The axioms and rules of the type system are the following:

$$\frac{}{\Gamma, x : A \vdash x : A}\ (Ax) \qquad \frac{}{\Gamma \vdash M : \mathsf{U}}\ (\mathsf{U})$$

$$\frac{\Gamma, x : B \vdash M : A}{\Gamma \vdash \lambda x.M : B \to A}\ (\to) \qquad \frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}\ (\to E)$$

$$\frac{\Gamma \vdash M : B \quad \Gamma \vdash M : A}{\Gamma \vdash M : B \cap A}\ (\cap I) \qquad \frac{\Gamma \vdash M : B \quad B \leq_{\mathcal{T}} A}{\Gamma \vdash M : A}\ (\leq)$$

3 Fuzzy Propositions-as-types

In this section we define type assignment systems for untyped λ-terms, where types are formulæ in minimal Fuzzy Logic.The systems can be viewed according to two different perspectives. On the one hand, viewing fuzzy formulæ as types, λ-terms can be viewed as proofs for this fragment of Fuzzy Logic. Thus we can characterise the sets of fuzzy formulæ which are inhabited, thereby providing an alternate route (see Corollary 11) to showing that **BCK**-logic is included in all Fuzzy Logics, see [14]. On the other hand, evaluating the semantics of the formulæ appearing as types (by means of an assignment to fuzzy propositional variables, ρ, into [0,1], endowed with a residuated T-norm), these systems can be also viewed as systems for assigning a *confidence degree* to λ-terms. In Remark 25, we will see that this degree can be viewed as the confidence in their membership to a universal type A such that $A \sim A \to A$.

In particular, we discuss two type assignment systems $\mathcal{F}$ and $\mathcal{Q}$, which differ only in that, the first is *contractive* and hence assumptions are taken to be sets, while the second is *non-contractive* and assumptions are taken to be multisets.

3.1 Minimal Fuzzy Logic - Qualitative Case

We introduce a fuzzy assignment system, *i.e.* a system for deriving judgements of the shape $\Gamma \vdash M : d$, where Γ is a context, M is a λ-term, and d is a formula of minimal Fuzzy Logic.

▶ **Definition 4** (Fuzzy Assumptions, Fuzzy Qualitative Contexts).

- *A* fuzzy assumption *is a pair of the form* (x, d)*, usually written as* $x : d$*, where* x *is a variable and* d *is a fuzzy formula. The fuzzy assumption,* (x, d)*, means that the variable* x *has* confidence *degree* $[\![d]\!]^{\mathcal{T}}_{\rho}$*, once we evaluate the formula* d *according to some assignment* ρ *in [0,1] endowed with a residuated T-norm.*
- *A* fuzzy qualitative context Γ *is a finite set of fuzzy assumptions. Note that* Γ *may contain two or more* different *assumptions for the same variable, with possibly different confidence degrees.*

▶ **Definition 5** (The type assignment system $\mathcal{F}$). *The* qualitative Fuzzy Assignment System*,* $\mathcal{F}$*, assigns fuzzy formulas to pure* λ*-terms according to the following rules:*

$$\frac{}{\Gamma, x : d \vdash_{\mathcal{F}} x : d}\ (\mathrm{Ax})$$

$$\frac{\Gamma, x : d_1, \ldots, x : d_n \vdash_{\mathcal{F}} M : d \qquad x \notin \Gamma}{\Gamma \vdash_{\mathcal{F}} \lambda x.M : d_1 \otimes \cdots \otimes d_n \Rightarrow d}\ (\Rightarrow\mathrm{I})$$

$$\frac{\Gamma \vdash_{\mathcal{F}} M : d_1 \qquad \Gamma \vdash_{\mathcal{F}} N : d_2}{\Gamma \vdash_{\mathcal{F}} MN : d_1 \otimes d_2}\ (\Rightarrow\mathrm{E})$$

▶ Remark 6. As we pointed out already, given an interpretation of fuzzy propositional variables ρ in [0,1], endowed with a residuated T-norm, we can evaluate a fuzzy formula d. Thus $\mathcal{F}$ can be viewed also as a system for assigning degrees of confidence to λ-terms, once we view the judgements of $\mathcal{F}$ as meaning

$$x_1 : [\![d_1]\!]^T_\rho, \dots x_n : [\![d_n]\!]^T_\rho \vdash_{\mathcal{F}} M : [\![d]\!]^T_\rho.$$

Hence, when convenient and clear from the context, we may conflate d with its value $[\![d]\!]^T_\rho \in [0,1]$, given ρ and T, and blur the distinction between the two interpretations of the judgement in $\mathcal{F}$.

The following comments on the rules of the system $\mathcal{F}$ are in order.

- In Rule ($\Rightarrow$I), we use the residuum, which corresponds to fuzzy implication.
- Since system $\vdash_{\mathcal{F}}$ satisfies the contraction rule, then we can always assume that any fresh variable x such that $x \notin M$ appears as the subject of a fuzzy assumption in Γ. Thus from $\Gamma \vdash_{\mathcal{F}} M : d$ by Rule ($\Rightarrow$I) we can conclude $\Gamma \vdash_{\mathcal{F}} \lambda x.M : e \Rightarrow d$, for any e.
- When [0,1] is endowed with a T-norm, we always have that $x \otimes y \leq x$ and $x \leq y \Rightarrow x$, see Proposition 35. Hence when we evaluate the Fuzzy Logic formulæ involved in Rule ($\Rightarrow$E), by means of any ρ in [0,1], we see that ($\Rightarrow$E) *decreases* confidence, while ($\Rightarrow$I) *increases* confidence. Thus, the system exhibits the feature which permits Fuzzy Logic to explain the so-called *Sorites* Paradox [15], namely that each application of *Modus Ponens* decreases the confidence degree. In the context of types this permits to detect the number of applications which have occurred.
- When dealing with multiple assumptions, we use the T-norm, *i.e.* weak conjunction, in the antecedent of $\Rightarrow$. Alternatively, we could have used the $\wedge$ operator corresponding to fuzzy strong conjunction [14], namely the *minimum.* We do not make this choice because it would make the system more difficult to deal with, since the T-norm is well-behaved with respect to its residuum, see Proposition 35.8, and moreover the feature of Fuzzy Logic of potentially registering the number of well-typed applications, mentioned above, would be lost. This makes our system more resource-sensitive. Of course, if we have in mind Gödel-Dummet Logic, there is no difference.

When viewing system $\mathcal{F}$ as a system for assigning confidence degrees (given an assignment ρ in [0,1] endowed with a T-norm) a few more comments are in order:

- it is convenient to assume that ρ does not assign the trivial value 0 to any propositional fuzzy variable;
- it might be convenient to add in Rule ($\Rightarrow$E) the side condition $[\![d_1 \otimes d_2]\!] > 0$, if we want to get rid of trivial judgement where the type evaluates to 0.

The system $\mathcal{F}$ is very permissive and allows all λ-terms to be typed with a minimal fuzzy formula, albeit not a tautology, as shown by the following example.

▶ **Example 7** (Self-application)**.**

$$\dfrac{\dfrac{x : d \vdash_{\mathcal{F}} x : d \qquad x : d \vdash_{\mathcal{F}} x : d}{x : d \vdash_{\mathcal{F}} xx : d \otimes d}}{\vdash_{\mathcal{F}} \lambda x.xx : d \Rightarrow d \otimes d.}$$

Clearly for all T-norms which are non-idempotent and for all ρ's such that $[\![d]\!]^T_\rho \neq 1$, we have $[\![d \Rightarrow d \otimes d]\!]^T_\rho \neq 1$.

The following Theorem 8, which has the form of a deduction theorem, and Proposition 9 characterise which proofs, *i.e.* λ-terms, yield valid fuzzy tautologies in system $\mathcal{F}$, when the predicates in the typing judgements are evaluated according to a left-continuous T-norm.

▶ **Theorem 8.**

1. *If M is affine, then for all assignments ρ to propositional fuzzy variables of values in [0,1], endowed with a left-continuous T-norm, we have:*

   $$x_1 : d_1, \dots x_n : d_n \vdash_{\mathcal{F}} M : d$$

   implies

   $$[\![d_1 \Rightarrow \dots \Rightarrow d_n \Rightarrow d]\!]^T_\rho = 1.$$

2. *If the T-norm is idempotent,* i.e. *T is Gödel's T-norm (see Definition 34), then*

   $$x_1 : d_1^1, \dots x_1 : d_1^{m_1}, \dots, x_n : d_n^1, \dots x_n : d_n^{m_n} \vdash_{\mathcal{F}} M : d$$

   implies

   $$[\![d_1^1 \otimes \dots \otimes d_1^{m_1} \Rightarrow \dots \Rightarrow d_n^1 \otimes \dots \otimes d_n^{m_n} \Rightarrow d]\!]^T_\rho = 1.$$

Proof. If $\Gamma = x_1 : d_1, \dots x_n : d_n$ we use $\bigotimes \Gamma$ to denote $d_1 \otimes \dots \otimes d_n$.

1. Let Γ_M be the set of the assumptions in Γ for the variables which occur in M. By induction on the derivation in $\mathcal{F}$, we prove that if M is affine and $\Gamma \vdash_{\mathcal{F}} M : d$, then $[\![\bigotimes \Gamma_M \Rightarrow d]\!]^T_\rho = 1$, or equivalently, by Proposition 35 in Appendix A, that $[\![\bigotimes \Gamma_M]\!]^T_\rho \leq [\![d]\!]^T_\rho$.
   (Ax). Immediate, since $[\![\bigotimes \Gamma \otimes d]\!]^T_\rho \leq [\![d]\!]^T_\rho$ by Definition 33.
   ($\Rightarrow$I). By induction hypothesis and Proposition 35.7.
   ($\Rightarrow$E). By affinity we know that $\Gamma_M \cup \Gamma_N = \Gamma_{MN}$, hence we apply the monotonicity of the norm to the induction hypotheses.
   Finally using again Propositions 35.6 and 35.7 we get the result.
2. By induction on the derivation in $\mathcal{F}$, we prove that if $\Gamma \vdash_{\mathcal{F}} M : d$, then $[\![\bigotimes \Gamma \Rightarrow d]\!]^T_\rho = 1$, or equivalently by Definition 34 in Appendix A, that $[\![\bigotimes \Gamma]\!]^T_\rho \leq [\![d]\!]^T_\rho$.
   (Ax). Immediate, since $[\![\bigotimes \Gamma \otimes d]\!]^T_\rho \leq [\![d]\!]^T_\rho$ by Definition 33.
   ($\Rightarrow$I). By induction hypothesis and Proposition 35.7.
   ($\Rightarrow$E). We use the fact that if $[\![d \Rightarrow e_1]\!]^T_\rho = 1$ and $[\![d \Rightarrow e_2]\!]^T_\rho = 1$, then we also have $[\![d \Rightarrow e_1 \otimes e_2]\!]^T_\rho = 1$ since $e_1 \otimes e_2 = \min\{e_1, e_2\}$. Then we conclude as above. ◀

We now give a sort of converse to Theorem 8, namely that using a non-idempotent norm, non-affine λ-terms do not inhabit tautologies.

▶ **Proposition 9.** *Let M be a λ-term where a variable, say x, occurs more than once in M, then we can derive a judgement $\Gamma \vdash_{\mathcal{F}} M : d$ where, for some T-norm, and assignment ρ, $[\![d]\!]^T_\rho$ does not evaluate to 1, and hence, d is not a fuzzy tautology.*

Proof. We take Goguen's *product* T-norm, see Definition 34 in the Appendix. We define a derivation $\mathcal{D} : \Gamma \vdash_{\mathcal{F}} M : d$ as follows. Fuzzy assumptions in $\mathcal{D}$ assign propositional variables to λ-variables. The assignment ρ assigns the value 1 to propositional variables typing λ-variables which are vacuously abstracted and the value $\frac{1}{2}$ to propositional variables typing λ-variables which do occur in M. In the applications of Rule ($\Rightarrow$I) we have $n = 1$. We can show that for every judgement $\Gamma' \vdash N : d'$ which appears in $\mathcal{D}$:

$$[\![d']\!]^T_\rho = \frac{2^{\#_\lambda(N)}}{2^{\#_{var}(N)}}$$

where $\#_\lambda(N)$ denotes the number of non vacuous abstractions in N, and $\#_{var}(N)$ denotes the number of variable occurrences in N. Clearly we have that $\#_\lambda(N) \leq \#_{var}(N)$, the two being equal if and only if the term is affine.

The proof is by induction on $\mathcal{D}$. The only interesting case is Rule ($\Rightarrow$I) when x occurs in M':

$$\frac{\Gamma, x : X \vdash_{\mathcal{F}} M' : d' \qquad x \notin \Gamma}{\Gamma \vdash_{\mathcal{F}} \lambda x.M' : X \Rightarrow d'}$$

By induction hypothesis $[\![d']\!]^T_\rho = \dfrac{2^{\#_\lambda(M')}}{2^{\#_{var}(M')}}$ where $\#_\lambda(M') < \#_{var}(M')$ since x is free in M'. If $\#_\lambda(M') < \#_{var}(M') - 1$, then $[\![X \Rightarrow d']\!]^T_\rho = \dfrac{2^{\#_\lambda(M')}}{2^{\#_{var}(M')}} \times 2 = \dfrac{2^{\#_\lambda(\lambda x.M')}}{2^{\#_{var}(\lambda x.M')}}$. If $\#_\lambda(M') = \#_{var}(M') - 1$, then $[\![X \Rightarrow d']\!]^T_\rho = 1 = \dfrac{2^{\#_\lambda(\lambda x.M')}}{2^{\#_{var}(\lambda x.M')}}$.
Since the term is non-affine there exists at least one judgement in the derivation $\mathcal{D}$ such that for all subsequent judgements $[\![d']\!]^T_\rho < 1$. ◀

We illustrate the above results by the following example, which deals with an affine combinator using Theorem 8.1.

▶ **Example 10.** Proposition 9 implies that if $\vdash_{\mathcal{F}} \mathbf{S} \equiv \lambda xyz.xz(yz) : d$, then it is not the case that $d = 1$ for every left-continuous T-norm, and assignment ρ. However, for a linear variant of $\mathbf{S}$, namely $\lambda xyzw.xz(yw)$ we have that if $\vdash_{\mathcal{F}} \lambda xyzw.xz(yw) : d$, then the evaluation of d in any left-continuous T-norm and assignment ρ is 1, by Theorem 8.1.

The following corollary provides the natural connection between simple types of affine λ-terms and formulæ in **BCK**-logic, see [14].

▶ **Corollary 11.** *All simple types inhabited by an affine λ-term are tautologies in all Fuzzy Logics, once we read $\to$ as $\Rightarrow$.*

Proof. Let $\mathcal{D} : \vdash M : A$ be a derivation of the simple type A for the closed affine λ-term M. We can, inductively, parallel the derivation $\mathcal{D}$ in the the system $\vdash_{\mathcal{F}}$, appropriately replacing arrows. The main difference in the new derivation is that the application of the *modus ponens* rule in the simple types assignment system, namely $\dfrac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$ is replaced by $\dfrac{\Gamma' \vdash M : A' \Rightarrow B' \qquad \Gamma' \vdash N : A'}{\Gamma' \vdash MN : (A' \Rightarrow B') \otimes A'}$, where A', B' are the corresponding types. Proceeding top-down in the derivation consider the first use of the application rule in a branch of the derivation. By Theorem 8.1 we have that $\otimes\Gamma' \Rightarrow (A' \Rightarrow B') \otimes A'$ is a tautology, but since $[\![(A' \Rightarrow B') \otimes A']\!]^T_\rho \leq [\![B']\!]^T_\rho$ by Proposition 35.8 we have also that $[\![\otimes\Gamma' \Rightarrow B']\!]^T_\rho = 1$. Recall that each application of modus ponens decreases the confidence value. Proceeding along the derivation we can then replace all occurrences of $(A' \Rightarrow B') \otimes A'$ by B' still preserving the fact that its value is 1. Hence all simple types inhabited by an affine λ-term are fuzzy tautologies. ◀

It is folklore that the *simple* types inhabited by affine λ-terms are the simple types which can be assigned to applications of the combinators $\mathbf{B} \equiv \lambda xyz.x(yz)$, $\mathbf{C} \equiv \lambda xyz.xzy$, and $\mathbf{K} \equiv \lambda xy.x$. These are the tautologies of **BCK**-logic. By the above Corollary 11 therefore, we recover along a different route the fact that all such implicational formulæ hold in the Minimal T-norm Fuzzy Logic MTL, see [14], and hence in all its extensions, which include all left-continuous T-norm Fuzzy Logics.

Similarly from Theorem 8.2, we derive that all minimal intuitionistic tautologies are valid formulæ for the idempotent T-norm, namely Göde's T-norm. These are the purely implicational, *i.e.* minimal, intuitionistic tautologies, which are all valid in Gödel-Dummet Fuzzy Logic [12]. Clearly every minimal intuitionistic tautology is inhabited by a simply typed λ-term. Less obvious is that extending the inhabitants to all λ-terms does not increase the class of inhabited types.

3.2 Minimal Fuzzy Logic - Quantitative Case

In this subsection we modify system $\mathcal{F}$ by taking contexts to be *multisets*, namely we allow them to contain two or more *equal*, or different, fuzzy assumptions, in the sense of Definition 4, for the same variable. In fact, if we interpret the formulæ via a norm and an assignment these are indeed fuzzy multisets. We will see that by doing so all inhabited Fuzzy Logic formulæ necessarily have value 1, in any left-continuous T-norm and assignment ρ.

▶ **Definition 12** (Fuzzy Quantitative Context)**.** *A* fuzzy quantitative context Γ *is a finite multiset of fuzzy assumptions.*

The above definition is just the *multiset* version of Definition 4, which, interpreting d's in [0,1], is the appropriate definition of *fuzzy multiset*.

▶ **Definition 13** (Quantitative Fuzzy Assignment, $\mathcal{Q}$)**.** *The rules of the* quantitative Fuzzy Assignment System*,* $\mathcal{Q}$*, are the following:*

$$\frac{}{x : d \vdash_{\mathcal{Q}} x : d}\ (\mathrm{Ax})$$

$$\frac{\Gamma, x : d_1, \ldots, x : d_n \vdash_{\mathcal{Q}} M : d \qquad x \notin \Gamma}{\Gamma \vdash_{\mathcal{Q}} \lambda x.M : d_1 \otimes \cdots \otimes d_n \Rightarrow d}\ (\Rightarrow\mathrm{I})$$

$$\frac{\Gamma \vdash_{\mathcal{Q}} M : d \qquad x \notin \Gamma}{\Gamma \vdash_{\mathcal{Q}} \lambda x.M : 1 \Rightarrow d}\ (\Rightarrow 0)$$

$$\frac{\Gamma_1 \vdash_{\mathcal{Q}} M : d_1 \qquad \Gamma_2 \vdash_{\mathcal{Q}} N : d_2}{\Gamma_1 :: \Gamma_2 \vdash_{\mathcal{Q}} MN : d_1 \otimes d_2}\ (\Rightarrow\mathrm{E})$$

where :: denotes multiset union.

Notice that in (Ax), above, there is no extra-context; this enforces the *relevance* of this quantitative system. Therefore, we need to add Rule ($\Rightarrow$0) to take care explicitly of *weakening*.

Theorem 15, below, shows that the inhabited fuzzy formulæ in $\mathcal{Q}$ evaluate to a confidence degree equal to 1 for any left-continuous T-norm, and any assignment ρ. First, we need a lemma.

▶ **Lemma 14** ($\mathcal{Q}$-Invariance)**.** *Let* $\bigotimes \Gamma$ *denote the tensor product of all confidence degrees assigned to variables in the context* Γ*. If* $\Gamma \vdash_{\mathcal{Q}} M : d$*, then* $[\![\bigotimes \Gamma]\!]^T_\rho \leq [\![d]\!]^T_\rho$*, for any T-norm, and any assignment* ρ*.*

Proof.
By induction on the derivation in $\mathcal{Q}$:
(Ax). Immediate;
($\Rightarrow$I) and ($\Rightarrow$0). Immediate by induction hypothesis and Definition 33;
($\Rightarrow$E). By monotonicity of the T-norm, see Definition 32. ◀

▶ **Theorem 15.**
If the T-norm is left-continuous, then

$$x_1 : d_1^1, \ldots, x_1 : d_1^{m_1}, \ldots, x_n : d_n^1, \ldots, x_n : d_n^{m_n} \vdash_{\mathcal{Q}} M : d$$

implies

$$[\![d_1^1 \otimes \cdots \otimes d_1^{m_1} \Rightarrow \ldots \Rightarrow d_n^1 \otimes \cdots \otimes d_n^{m_n} \Rightarrow d]\!]^T_\rho = 1$$

for all assignments ρ*.*

Proof. By Lemma 14 we see that if $\Gamma \vdash_{\mathcal{Q}} M : d$, then $[\![\bigotimes \Gamma]\!]^T_\rho \leq [\![d]\!]^T_\rho$, hence using Proposition 35.5 in Appendix A we get the result. ◀

It is natural, now, to ask the following question: what are the inhabited types in System $\mathcal{Q}$? The simple answer is given in the following proposition.

▶ **Proposition 16.** *The class of inhabited types in $\mathcal{Q}$ is the same as that of the types inhabited in $\mathcal{F}$ by affine terms.*

Proof. Taking different variables for different uses of the Axiom (Ax), any derivation in $\mathcal{Q}$ can be turned in a derivation in $\mathcal{F}$ for an affine term. ◀

We end this subsection remarking that both in $\mathcal{F}$ and $\mathcal{Q}$ all λ-terms can receive infinitely many degrees of confidence depending on which (quantitative) fuzzy contexts we start from. For instance, using the trivial assignment, all terms can receive degree of confidence 1, even unsolvable terms, and hence also unsolvable terms can play the role of inhabitants. Fuzzy Intersection Type theories allow us to harness the class of derivations, as we will see in Section 4.

3.3 Subject Conversion

For all type assignment systems, it is natural to ask whether types are preserved under β-conversion, namely reduction/expansion, of the term. Since both $\mathcal{F}$ and $\mathcal{Q}$ are sensitive to the number of times an assumption appears in the proof, albeit in different ways, preservation in reduction or expansion of the subject are problematic, both as far as the fuzzy formula itself as well as its evaluation. The following counterexamples and proposition illustrate the situation.

▶ **Counterexample 17** (Subject Conversion Fails in $\mathcal{F}$ for $\mathbf{\Delta I} \to_\beta \mathbf{II}$)**.** *We recall that $\mathbf{\Delta} \equiv \lambda x.xx$ and $\mathbf{I} \equiv \lambda x.x$. Consider the following reduction $(\lambda x.xx)(\lambda x.x) \to_\beta (\lambda x.x)(\lambda x.x)$, we have the derivation:*

$$\dfrac{\dfrac{\dfrac{x : d \vdash_{\mathcal{F}} x : d \qquad x : d \vdash_{\mathcal{F}} x : d}{x : d \vdash_{\mathcal{F}} xx : d \otimes d}}{\emptyset \vdash_{\mathcal{F}} \lambda x.xx : d \Rightarrow d \otimes d} \qquad \dfrac{x : d \vdash_{\mathcal{F}} x : d}{\emptyset \vdash_{\mathcal{F}} \lambda x.x : d \Rightarrow d}}{\emptyset \vdash_{\mathcal{F}} (\lambda x.xx)(\lambda x.x) : (d \Rightarrow d \otimes d) \otimes (d \Rightarrow d)}$$

and the derivation:

$$\dfrac{\dfrac{x : d \vdash_{\mathcal{F}} x : d}{\emptyset \vdash_{\mathcal{F}} \lambda x.x : d \Rightarrow d} \qquad \dfrac{x : d \vdash_{\mathcal{F}} x : d}{\emptyset \vdash_{\mathcal{F}} \lambda x.x : d \Rightarrow d}}{\emptyset \vdash_{\mathcal{F}} (\lambda x.x)(\lambda x.x) : (d \Rightarrow d) \otimes (d \Rightarrow d).}$$

But in Goguen's product norm, for a ρ such that $[\![d]\!]^T_\rho \lneq 1$ we have $[\![d \Rightarrow d]\!]^T_\rho = 1$, but $[\![(d \Rightarrow d \otimes d) \otimes (d \Rightarrow d)]\!]^T_\rho \lneq [\![(d \Rightarrow d) \otimes (d \Rightarrow d)]\!]^T_\rho = 1$, in general. Hence in $\mathcal{F}$ the reduct can have a higher confidence degree than the redex.

▶ **Counterexample 18** (Subject Conversion fails in $\mathcal{F}$ for $\mathbf{2}\ \mathbf{2} \to_\beta \mathbf{4}$)**.** *We recall that $\mathbf{2} \equiv \lambda xy.x(xy)$ and $\mathbf{4} \equiv \lambda xy.x(x(x(xy)))$ are Church encodings of the natural numbers 2 and 4. Consider the following reduction $\mathbf{2}\ \mathbf{2} \to_\beta \mathbf{4}$. We have the derivation:*

$$\dfrac{\mathcal{D} \qquad\qquad \mathcal{D}}{\emptyset \vdash_{\mathcal{F}} \mathbf{2}\ \mathbf{2} : (d \Rightarrow d \Rightarrow d \otimes d \otimes d) \otimes (d \Rightarrow d \Rightarrow d \otimes d \otimes d),}$$

where $\mathcal{D}$ denotes the derivation:

$$\dfrac{x:d\vdash_{\mathcal{F}} x:d \qquad \dfrac{x:d\vdash_{\mathcal{F}} x:d \qquad y:d\vdash_{\mathcal{F}} y:d}{x:d,y:d\vdash_{\mathcal{F}} xy:d\otimes d}}{\dfrac{x:d,y:d\vdash_{\mathcal{F}} x(xy):d\otimes d\otimes d}{\dfrac{x:d\vdash_{\mathcal{F}} \lambda y.x(xy):d\Rightarrow d\otimes d\otimes d}{\emptyset\vdash_{\mathcal{F}} \lambda xy.x(xy):d\Rightarrow d\Rightarrow d\otimes d\otimes d,}}}$$

and also the derivation:

$$\dfrac{x:d\vdash_{\mathcal{F}} x:d \qquad \dfrac{x:d\vdash_{\mathcal{F}} x:d \qquad \dfrac{x:d\vdash_{\mathcal{F}} x:d \qquad \dfrac{x:d\vdash_{\mathcal{F}} x:d \qquad y:d\vdash_{\mathcal{F}} y:d}{x:d,y:d\vdash_{\mathcal{F}} xy:d\otimes d}}{x:d,y:d\vdash_{\mathcal{F}} x(xy):d\otimes d\otimes d}}{x:d,y:d\vdash_{\mathcal{F}} x(x(xy)):d\otimes d\otimes d\otimes d}}{\dfrac{x:d,y:d\vdash_{\mathcal{F}} x(x(x(xy))):d\otimes d\otimes d\otimes d\otimes d}{\dfrac{x:d\vdash_{\mathcal{F}} \lambda y.x(x(x(xy))):d\Rightarrow d\otimes d\otimes d\otimes d\otimes d,}{\emptyset\vdash_{\mathcal{F}} \lambda xy.x(x(x(xy))):d\Rightarrow d\Rightarrow d\otimes d\otimes d\otimes d\otimes d,}}}$$

but in Goguen's product T-norm for a ρ such that $[\![d]\!]^T_\rho \lneq 1$ the value of the confidence degree of the reduct is equal to $[\![d]\!]^T_\rho \otimes [\![d]\!]^T_\rho \otimes [\![d]\!]^T_\rho$, which is strictly smaller than the confidence degree of the redex, which is equal to $[\![d]\!]^T_\rho \otimes [\![d]\!]^T_\rho$.

The counterexamples above show that confidence degree of the reduct can be strictly larger or strictly smaller than that of the redex. So we have the following proposition.

▶ **Proposition 19.** *Subject conversion holds in System $\mathcal{F}$ if and only if the norm is idempotent.*

Proof. The "only if" part follows almost directly from Counterexample 17. Let $[\![d]\!]^T_\rho = a$. Since the T-norm is left continuous we have that $a \Rightarrow a\otimes a = sup\{c \mid a\otimes c \leq a\otimes a\}$. If this were 1, as it would be necessary for subject reduction, then we would have that $a \leq a\otimes a$, and, since the converse always holds, we would have that $a = a\otimes a$, namely the T-norm would be idempotent.
The "if" part follows from the fact that if the T-norm is idempotent by Theorem 8. 2 we have that the value of the confidence degree inhabited by closed λ-terms is always 1. ◀

If we were to consider only affine terms in $\mathcal{F}$, β-conversion would naturally hold. Subject conversion for $\mathcal{Q}$ is satisfactory only for closed terms as the following proposition and counterexamples show.

▶ **Proposition 20.** *Subject Reduction and Subject Expansion hold in System $\mathcal{Q}$ for closed λ-terms, but not for open terms.*

Proof. In system $\mathcal{Q}$, the value of the confidence degree of fully abstracted inhabited types is always 1, thus subject conversion holds for closed terms. ◀

▶ **Counterexample 21** (Subject Conversion Fails in $\mathcal{Q}$ for $\mathbf{\Delta}x \to_\beta xx$). *Consider the following reduction $(\lambda x.xx)x \to_\beta xx$, we have the derivation:*

$$\dfrac{\dfrac{\dfrac{x:d\vdash_{\mathcal{Q}} x:d \qquad x:d\vdash_{\mathcal{Q}} x:d}{x:d,x:d\vdash_{\mathcal{Q}} xx:d\otimes d}}{\emptyset\vdash_{\mathcal{Q}} \lambda x.xx:d\otimes d\Rightarrow d\otimes d} \qquad x:d\vdash_{\mathcal{Q}} x:d}{x:d\vdash_{\mathcal{Q}} (\lambda x.xx)x:(d\otimes d\Rightarrow d\otimes d)\otimes d}$$

and the derivation:

$$\frac{x : d \vdash_{\mathcal{Q}} x : d \qquad x : d \vdash_{\mathcal{Q}} x : d}{x : d, x : d \vdash_{\mathcal{Q}} xx : d \otimes d}$$

As we can see the context of the conclusion of the second derivation has expanded.

▶ **Counterexample 22** (Subject Conversion Fails in $\mathcal{Q}$ for $(\lambda x.y)z \to_\beta y$). *We have the derivation:*

$$\frac{\dfrac{y : d \vdash_{\mathcal{Q}} y : d}{y : d \vdash_{\mathcal{Q}} \lambda x.y : 1 \Rightarrow d} \qquad z : d \vdash_{\mathcal{Q}} z : d}{y : d, z : d \vdash_{\mathcal{Q}} (\lambda x.y)z : (1 \Rightarrow d) \otimes d}$$

which has a longer context than its reduct, namely $y : d \vdash_{\mathcal{Q}} y : d$.

In conclusion the problems with subject conversion come from the interaction between non-idempotent T-norms and duplication or erasure of variables during β-reduction. The remarks above and Proposition 20 show that subject conversion can still work in fuzzy settings, for example, if we assume Gödel's T-norm for affine terms or in system $\mathcal{Q}$ for closed terms.

4 Fuzzy Intersection Types

We are now in the position of fruitfully studying the combination of intersection types, see Definition 2, with fuzzy type-membership. Namely, we introduce a *fuzzy* type assignment system assigning intersection types to pure λ-terms, decorating appropriately the Intersection Type Assignment System, see *e.g.* [2, 3, 8], with minimal fuzzy formulæ.

▶ **Definition 23** (Typing assumption, typing context).

1. *A* typing assumption *is a triple of the form (x, A, d), usually written as $x :_d A$, where x is a variable, A is an intersection type and d is a fuzzy formula, see Section 2.1, or, equivalently, a real number in $[0, 1]$ obtained by its evaluation w.r.t. a T-norm and an assignment ρ. Its intended meaning is that x belongs to the type A with degree of confidence $[\![d]\!]^T_\rho$.*
2. *A* typing context Γ *is a finite multiset of typing assumptions with non-zero degrees.*

Notice that a context Γ may contain two or more assumptions for the same variable, possibly with the same type, and the same confidence degree. Thus, a typing context can be seen as a *fuzzy multiset*, once we evaluate the fuzzy formulæ appearing in the indices, with respect to a norm and an assignment of values to the propositional fuzzy variables.

▶ **Definition 24** (Quantitative Fuzzy Intersection Types $\mathcal{Q}^{\cap}$). *The typing rules of the Fuzzy Intersection Types $\mathcal{Q}^{\cap}$ induced by the type theory $\langle \mathbb{A}, \leq_{\mathcal{T}} \rangle$ are the following:*

$$\frac{}{x :_d A \vdash_{\mathcal{Q}^{\cap}} x :_d A}\ (Ax) \qquad \frac{}{\vdash_{\mathcal{Q}^{\cap}} M :_1 \mathsf{U}}\ (\mathsf{U})$$

$$\frac{\Gamma, x :_{d_1} A_1, \ldots, x :_{d_n} A_n \vdash_{\mathcal{Q}^{\cap}} M :_d A \qquad x \notin dom(\Gamma)}{\Gamma \vdash_{\mathcal{Q}^{\cap}} \lambda x.M :_{d_1 \otimes \cdots \otimes d_n \Rightarrow d} A_1 \cap \cdots \cap A_n \to A}\ (\to I)$$

$$\frac{\Gamma \vdash_{\mathcal{Q}^{\cap}} M :_d A \qquad x \notin dom(\Gamma)}{\Gamma \vdash_{\mathcal{Q}^{\cap}} \lambda x.M :_{1 \Rightarrow d} B \to A}\ (\to 0)$$

$$\frac{\Gamma_1 \vdash_{\mathcal{Q}^{\cap}} M :_{d_1} A \to B \qquad \Gamma_2 \vdash_{\mathcal{Q}^{\cap}} N :_{d_2} A}{\Gamma_1 :: \Gamma_2 \vdash_{\mathcal{Q}^{\cap}} MN :_{d_1 \otimes d_2} B}\ (\to E)$$

$$\frac{\Gamma_1 \vdash_{\mathcal{Q}^{\cap}} M :_{d_1} A \qquad \Gamma_2 \vdash_{\mathcal{Q}^{\cap}} M :_{d_2} B}{\Gamma_1 :: \Gamma_2 \vdash_{\mathcal{Q}^{\cap}} M :_{d_1 \otimes d_2} A \cap B}\ (\cap I) \qquad \frac{\Gamma \vdash_{\mathcal{Q}^{\cap}} M :_d A \qquad A \leq_{\mathcal{T}} B}{\Gamma \vdash_{\mathcal{Q}^{\cap}} M :_d B}\ (\leq)$$

where $\Gamma_1 :: \Gamma_2$ denotes multiset union, i.e. *multiplicities of occurrences are not erased.*

Some comments are in order.

- How does the system $\mathcal{Q}^{\cap}$ qualify with respect to *contraction* and *idempotency*? It is contraction-free with respect to the fuzzy formula indexing the "$:_d$" in the judgement $\Gamma \vdash_{\mathcal{Q}^{\cap}} M :_d A$, but although contexts are multisets, the system is nonetheless *idempotent* as far as type intersections are concerned, *e.g.* in Rule $(\to I)$. Hence, since $\otimes$ is not idempotent in general, the judgements store the information on the number of times an assumption is *used*, as if the system were non-contractive, only in a different format. Therefore $\mathcal{Q}^{\cap}$ has a quantitative nature.
- The crisp Intersection Type Assignment System given in Definition 3 can be obtained from $\vdash_{\mathcal{Q}^{\cap}}$ by erasing all fuzzy coefficients from the "$:_d$" symbol, namely $(\Gamma \vdash_{\mathcal{Q}^{\cap}} M :_d A)^{\#}$ yields $\Gamma' \vdash_{\mathcal{T}} M : A$ for $\mathcal{T}$ with the same subtyping, where Γ' is the set obtained from Γ erasing multiple assumptions after having erased the confidence degree in each assumption; we denote by $\Gamma^{\#}$ the crisp context thus obtained.
- Removing the coefficient d in the judgements does not yield the same system that we would get if we take all the d's in assumptions to be the constant 1. Indeed all d's would evaluate to 1, but the indexes still carry information. This is still a fuzzy system, but just with a trivial value. If, on the other hand, we erase the coefficient, we completely forget information and collapse everything to the usual crisp intersection type system where contexts are needlessly multisets.
- In principle, we could introduce a fuzzy intersection type assignment system building on top of $\mathcal{F}$ rather than $\mathcal{Q}$, but then the value of the fuzzy index would not be preserved by reduction or expansion, unless we would focus on Gödel's T-norm, or affine λ-terms.
- Notice, incidentally, that when evaluating the confidence degrees w.r.t. a T-norm and an assignment ρ, the fuzzy constructor $\Rightarrow$ behaves w.r.t. $\leq$ contravariantly on domains and covariantly on codomains, as does the intersection types constructor $\to$, see Definition 2. Namely, if $[\![d]\!]^T_\rho \leq [\![d']\!]^T_\rho$ and $[\![e]\!]^T_\rho \leq [\![e']\!]^T_\rho$, then $[\![d' \Rightarrow e]\!]^T_\rho \leq [\![d \Rightarrow e']\!]^T_\rho$. However $\otimes$ does not behave as $\cap$. For instance Rule $(\to \cap)$ (see Definition 2) fails in Goguen's T-norm.
- In the introduction Rules $(\to I)$ and $(\cap I)$ rather than $d_1 \otimes d_2$, we could consider instead $d_1 \wedge d_2$, which is usually interpreted by the minimum. We adopt $\otimes$ to maintain a uniform propagation of confidence across abstraction and application, thus making no qualitative distinction between uses of the same variable with the same type or uses of the same variable with different types.

The choice between $\otimes$ and $\wedge$ is not immaterial. Using $\wedge$, interpreted as minimum, yields the largest confidence compatible with both premises, namely $\min(d_1, d_2)$. In contrast, $\otimes$ may yield a strictly smaller value and therefore captures quantitatively the repeated use of assumptions. Since $d_1 \otimes d_2 \leq d_1 \wedge d_2$, the system based on $\otimes$ is more sensitive to resource usage, while the system based on $\wedge$ is closer to the set-theoretic interpretation of intersection types. In any case the choice of $\wedge$ would make the theory more involved and we should re-work the theory developed in the earlier sections.

- Rule $(\leq)$ is parametrised on a type theory (see Definition 2). In this rule we could have allowed for a weakening of the degree of confidence in the conclusion, but this would break the preservation of the degree of confidence under conversion for closed λ-terms.
- As we did for the earlier systems, according to the needs of the argument, when clear from the context, we will consider the index in $:_d$ either as a formula of minimal negation-free Fuzzy Logic, or as its value w.r.t. to a given T-norm and assignment ρ.
- The confidence degree in Rule (U) appears to be the natural one.

▶ Remark 25 (On the relation between system $\mathcal{F}$ and $\mathcal{Q}^{\cap}$). As we have pointed out at the outset of Section 3 we can read a fuzzy assumption, $x : d$, in $\mathcal{F}$, see Definition 4, as the typing assumption $x :_d c$, in Definition 23, for an intersection type theory, $\mathcal{T}$, with only one constant c such that $c \sim_{\mathcal{T}} c \to c$. Namely a judgement of the shape $x_1 : d_1, \ldots x_n : d_n \vdash_{\mathcal{Q}} M : d$ can be understood as a judgement $x_1 :_{d_1} c, \ldots x_n :_{d_n} \vdash_{\mathcal{Q}^{\cap}} M :_d c$.

Hence, in studying systems $\mathcal{F}$, and $\mathcal{Q}$, we have clarified the behaviour of fuzzy formulæ as types *per se*, thereby illustrating the non-contractive nature of Fuzzy Logic, which becomes significant once we go beyond purely affine λ-terms. In studying $\mathcal{Q}^{\cap}$ we combine the two systems. The exact shape of the fuzzy minimal formula in the typing judgement is not necessarily preserved, but its evaluation in terms of a T-norm and an appropriate assignment provides interesting invariants under evaluation.

We are now in the position of proving the main theorem.

▶ **Theorem 26.** *Given a left-continuous T-norm and an assignment ρ, if*

$$
\begin{array}{l}
x_1 :_{d^1_{1,1}} A^1_1, \ldots, x_1 :_{d^1_{1,k_1}} A^1_1, \ldots, x_1 :_{d^1_{m_1,1}} A^1_{m_1}, \ldots, x_1 :_{d^1_{m_1,k_{m_1}}} A^1_{m_1}, \ldots \\
\ldots \\
\ldots, x_n :_{d^n_{1,1}} A^n_1, \ldots, x_n :_{d^n_{1,k_1}} A^n_1, \ldots, x_n :_{d^n_{m_n,1}} A^n_{m_n}, \ldots, x_n :_{d^n_{m_n,k_{m_n}}} A^n_{m_n} \vdash_{\mathcal{Q}^{\cap}} M :_d A,
\end{array}
$$

then

$$
\begin{array}{l}
[\![d^1_{1,1} \otimes \cdots \otimes d^1_{1,k_1} \Rightarrow \ldots \Rightarrow d^1_{m_1,1} \otimes \cdots \otimes d^1_{m_1,k_{m_1}} \Rightarrow \ldots \\
\ldots \\
\ldots d^n_{1,1} \otimes \cdots \otimes d^n_{1,k_1} \Rightarrow \ldots \Rightarrow d^n_{m_n,1} \otimes \cdots \otimes d^n_{m_n,k_{m_n}} \Rightarrow d]\!]^T_\rho = 1
\end{array}
$$

and

$$\Gamma^{\#} \vdash_{\mathcal{T}} M : A$$

in the same intersection type theory.

Proof. (sketch) By induction on the structure of derivations we prove that the systems $\mathcal{Q}^{\cap}$ and the intersection types assignment system $\mathcal{T}$ progress in parallel. Theorem 15 ensures that the interpretation of the fuzzy formula indexing the membership relation, once fully abstracted, yields the value 1. ◀

We can now also establish the standard suite of proof theoretical results on intersection types assignment systems.

▶ **Lemma 27** (Inversion Lemma).

1. *If* $\Gamma \vdash_{\mathcal{Q}^\cap} x :_d A$ *and* $A \not\sim \mathsf{U}$, *then*
$$\Gamma = \{x :_{d_1} A_1, \ldots, x :_{d_n} A_n\}$$
for some $d_1, \ldots, d_n, A_1, \ldots, A_n$ *such that* $d = d_1 \otimes \ldots \otimes d_n$ *and* $A_1 \cap \ldots \cap A_n \leq A$;
2. *If* $\Gamma \vdash_{\mathcal{Q}^\cap} MN :_d A$ *and* $A \not\sim \mathsf{U}$, *then there exist* I, B_i, C_i *for* $i \in I$, *such that* $\bigcap_{i\in I} C_i \leq A$ *and* $\Gamma \vdash_{\mathcal{Q}^\cap} M :_{e_i} B_i \to C_i$ *and* $\Gamma \vdash_{\mathcal{Q}^\cap} N :_{f_i} B_i$ *for all* $i \in I$, *and* d *is* $\bigotimes_{i\in I} e_i \otimes f_i$;
3. *If* $\Gamma \vdash_{\mathcal{Q}^\cap} \lambda x.M :_d A$ *and* x *occurs in* M, *then there exist* I *and* B_i, C_i *for* $i \in I$ *such that* $\bigcap_{i\in I}(B_i \to C_i) \leq A$ *and* $\Gamma, x :_{e_i} B_i \vdash_{\mathcal{Q}^\cap} M :_{f_i} C_i$ *for all* $i \in I$, *and* d *is* $\bigotimes_{i\in I}(e_i \Rightarrow f_i)$.
4. *If* $\Gamma \vdash_{\mathcal{Q}^\cap} \lambda x.M :_d A$ *and* x *does not occur in* M, *then there exist* I *and* B_i, C_i *for* $i \in I$ *such that* $\bigcap_{i\in I}(B_i \to C_i) \leq A$ *and* $\Gamma \vdash_{\mathcal{Q}^\cap} M :_{f_i} C_i$ *for all* $i \in I$, *and* d *is* $\bigotimes_{i\in I}(1 \Rightarrow f_i)$.

Proof. (1). A variable can only be typed by Axiom (Ax) and hence the judgements thus obtained can only be premises of the Rules $(\cap I)$ and $(\leq)$. The coefficient of the conclusion of Rule $(\cap I)$ is obtained by applying $\otimes$ to the coefficients of the premises. The coefficient of the conclusion of Rule $(\leq)$ is the coefficient of the premise.

(2). An application can only be typed by Rule $(\to E)$. We conclude as in Item (1).

(3). An abstraction where the abstracted variable occurs in the body can only be typed by Rule $(\to I)$. We conclude as in Item (1).

(4). An abstraction where the abstracted variable does not occur in the body can only be typed by Rule $(\to 0)$. We conclude as in Item (1). ◀

▶ **Theorem 28** (Subject Expansion). *If* $M \to_\beta M'$ *and* $\Gamma \vdash_{\mathcal{Q}^\cap} M' :_d A$ *imply* $\Gamma' \vdash_{\mathcal{Q}^\cap} M :_e A$.

Proof. Paralleling the proof of Subject Expansion [3, Corollary 14.2.5(ii)], taking into account that the context may change, since the system $\mathcal{Q}^\cap$ is quantitative, *i.e.* it accounts for the number of times a variable is effectively used. ◀

Notice that, in the above Theorem 28, the context Γ' may be contained in the context Γ. To see this, consider the analogue of Counterexample 21, where the typing assumptions have as type predicate the constant c, in an an intersection type theory, $\mathcal{T}$, with one constant c such that $c \sim_{\mathcal{T}} c \to c$.

▶ **Theorem 29** (Subject Reduction). *If the background type theory* $\mathcal{T}$ *satisfies the condition that* $\Gamma \vdash_{\mathcal{Q}^\cap} \lambda x.N :_{d'} B \to C$ *implies* $\Gamma, x :_f B \vdash_{\mathcal{Q}^\cap} N :_{f'} C$ *where* d' *id* $f \Rightarrow f'$, *then*

$$M \to M' \text{ and } \Gamma \vdash_{\mathcal{Q}^\cap} M :_d A \text{ imply } \Gamma' \vdash_{\mathcal{Q}^\cap} M' :_e A.$$

Proof. Paralleling the proof of Subject Reduction [3, Proposition 14.2.1(ii)], taking into account that the context may change, since the system $\mathcal{Q}^\cap$ is quantitative, *i.e.* it accounts for the number of times a variable is effectively used. ◀

Notice that, in the above Theorem 29, the context Γ' may contain the context Γ. To see this, consider the analogue of Counterexample 22.

In the above Theorems 28 and 29 one could give a sharper relation on the shapes of the formulæ occurring in the indexes d and e. In the following Subsection 4.1 we elaborate on this issue.

We end this subsection with an example which illustrates the need to consider the possibility of having different contexts in judgements involving converted terms.

▶ **Example 30** (Self-application Revisited)**.**
We have the following derivations

$$\dfrac{\dfrac{\dfrac{x :_d (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_d (A \to B) \cap A}{x :_d (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_d (A \to B)} \quad \dfrac{x :_e (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_e (A \to B) \cap A}{x :_e (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_e A}}{x :_d (A \to B) \cap A, x :_e (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} xx :_{d \otimes e} B}}{\vdash_{\mathcal{Q}^\cap} \lambda x.xx :_{d\otimes e \Rightarrow d \otimes e} (A \to B) \cap A \to B}$$

$$\dfrac{\vdash_{\mathcal{Q}^\cap} \lambda x.xx :_{d\otimes e \Rightarrow d \otimes e} (A \to B) \cap A \to B \qquad x :_d (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_d (A \to B) \cap A}{x :_d (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} (\lambda x.xx)x :_{d \otimes (d\otimes e \Rightarrow d\otimes e)} B}$$

but

$$\dfrac{\dfrac{x :_d (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_d (A \to B) \cap A}{x :_d (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_d (A \to B)} \quad \dfrac{x :_e (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_e (A \to B) \cap A}{x :_e (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} x :_e A}}{x :_d (A \to B) \cap A, x :_e (A \to B) \cap A \vdash_{\mathcal{Q}^\cap} xx :_{d \otimes e} B.}$$

4.1 What sort of information can we extract from the fuzzy degree of a judgement?

If we evaluate the confidence degree of a provable judgement $\Gamma \vdash_{\mathcal{Q}^\cap} M :_d A$ in $[0, 1]$, this gives us a measure on the membership of the term M to the type A. But viewing the confidence degree d as a formula of minimal Fuzzy Logic, we also have a "skeleton" of the derivation which established the inhabitation of the type A by M. The confidence degree encodes a lot of information indeed. Here are some intriguing questions and conjectures, which we shall not address here.

- To what extent can the derivation be reconstructed from a provable judgement in $\mathcal{Q}^\cap$? We conjecture that the *inhabitation problem* $\vdash_{\mathcal{Q}^\cap} ? :_d A$ should be decidable. This conjecture is supported by the decidability of inhabitation for non-idempotent intersection types, see [5].
- The shape of the fuzzy formula should provide an estimate on the length of the longest β-reduction of M, if M is typed without U, that is, if it is strongly normalising.

5 Final remarks

In this paper, we have started an investigation of how Fuzzy Logic fares with Constructive Type Theory in view of the growing interest on quantitative systems in constructive Type Theory. We have proposed a number of fuzzy type assignment systems. These permit us to derive some results on Fuzzy Logic *per se*, through a fuzzy analogue of the formulæ-as-types paradigm, as well as provide a "fuzzification" of intersection type theories for pure λ-calculus. More work needs to be done to answer the open questions left in Section 4.1, as well as to assess the expressive benefits of using $\wedge$ in place of $\otimes$ in Rule $(\cap I)$, this preventing us to treat on a par multiple uses of the same assumption with different types.

We also plan to explore in the future λ-calculi à la Church, such as Constructive Type Theories, where judgements have the shape $\Gamma \vdash_{\mathcal{Q}} M :_d A$, but also have equality judgements, $\Gamma \vdash_{\mathcal{Q}} M_1 = M_2 :_d A$.

A related line of research, which we intend to explore, is how to derive a suitable fuzzy intersection type assignment system starting from a notion of *fuzzy similarity* between types. The natural understanding of fuzzy equality-types in Constructive Type Theory, given above, namely $\Gamma \vdash_{\mathcal{Q}} M_1 =_d M_2 : A$, is a similarity relation, see Definition 36 in the Appendix. More precisely, given basic sets D_i's, assume that a *similarity relation*, S_{D_i}, is given for each. Is there a natural way to extend the similarity to higher types, thereby deriving a fuzzy simple type-assignment system? Moreover, can we take $(D, \cdot)$ to be an applicative structure, and use as similarity the inverse of Hausdorff distance, *i.e.* $S_{\mathcal{P}(D)}(A, B) = \min\{\inf_{x \in A} \sup_{y \in B} S_D(x, y), \inf_{x \in B} \sup_{y \in A} S_D(x, y)\}$, to inductively define similarity between logical relations within D? Appropriate conditions on the applicative behaviour of the points in D need to be enforced, however. This could provide a reading of the results in [11] on λ-quantitative theories, in terms of similarities, rather than of metrics. Yet another line of research is the use of fuzzy sets in some models of untyped λ-calculus such as Engeler's model [9].

Related Work. Some of our results can also be derived from the results on **BCK**-algebras in the theory of Fuzzy Propositional Logics, see *e.g.* [14], but our proofs follow a different route. Vilém Novák and his group have worked considerably on Fuzzy Type theory, but have mainly taken a classical approach [13]. Castelnuovo and Miculan [6] consider first order fuzzy theories in general, whereas we focus on higher order structures. A few authors have considered Fuzzy Type Theory specifically. S. Arya, G. Coraglia, P. North, S. O'Connor, H. Riess, A. Tenório have done some preliminary work on Fuzzy Type Theory in the context of 2-categories (see `https://etagreta.github.io/`).

Since, as we already pointed out, Fuzzy Logic has a non-idempotent flavour, our work relates to non-idempotent intersection type theories [10, 7, 5, 4]. The information stored in the type in [5] is included in the information stored in the fuzzy formula that indexes the typing judgement in our system $\mathcal{Q}^{\cap}$. More work needs to be done to compare the two approaches in depth.

We hope that the present paper will stimulate future discussions on the intriguing combination of Fuzzy Logic with Constructive Type Theory, as well as with other formal systems for reasoning about concurrent processes.

References

1 Henk Barendregt. *The Lambda Calculus - its Syntax and Semantics*, volume 103 of *Studies in logic and the foundations of mathematics*. North-Holland, 1985. URL: `https://api.pageplace.de/preview/DT0400.9780080933757_A23543814/preview-9780080933757_A23543814.pdf`.

2 Henk Barendregt, Mario Coppo, and Mariangiola Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *Journal of Symbolic Logic*, 48(4):931–940, 1983. `doi:10.2307/2273659`.

3 Henk Barendregt, Wil Dekkers, and Richard Statman. *Lambda Calculus with Types*. Cambridge University Press, 2013. `doi:10.1017/CBO9781139032636`.

4 Antonio Bucciarelli, Delia Kesner, and Simona Ronchi Della Rocca. Inhabitation for non-idempotent intersection types. *Logical Methods in Computer Science*, 14(3), 2018. `doi:10.23638/LMCS-14(3:7)2018`.

5 Antonio Bucciarelli, Delia Kesner, and Daniel Ventura. Non-idempotent intersection types for the lambda-calculus. *Logic Journal of the IGPL*, 25(4):431–464, 2017. `doi:10.1093/JIGPAL/JZX018`.

6 Davide Castelnovo and Marino Miculan. Fuzzy algebraic theories. In Florin Manea and Alex Simpson, editors, *CSL*, volume 216 of *LIPIcs*, pages 13:1–13:17. Schloss Dagstuhl, 2022. `doi:10.4230/LIPIcs.CSL.2022.13`.

7 Daniel de Carvalho. Execution time of λ-terms via denotational semantics and intersection types. *Mathematical Structures in Computer Science*, 28(7):1169–1203, 2018. `doi:10.1017/S0960129516000396`.

8 Mariangiola Dezani-Ciancaglini, Besik Dundua, Paola Giannini, and Furio Honsell. Sensible intersection type theories. In Thorsten Altenkirch, Franco Barbanera, Ferruccio Damiani, and Ugo de'Liguoro, editors, *Logics and Type Theory: essays dedicated to Stefano Berardi on the occasion of his 1000000th birthday*, volume 441 of *EPTCS*, pages 126–147, 2026. `doi:10.4204/EPTCS.441.8`.

9 Erwin Engeler. Algebras and combinators. *Algebra Universalis*, 13:389–392, 1981. `doi:10.1007/BF02483849`.

10 Philippa Gardner. Discovering needed reductions using type theory. In Masami Hagiya and John C. Mitchell, editors, *TACS*, volume 789 of *LNCS*, pages 555–574. Springer, 1994. `doi:10.1007/3-540-57887-0_115`.

11 Ugo Dal Lago, Furio Honsell, Marina Lenisa, and Paolo Pistone. On quantitative algebraic higher-order theories. In Amy P. Felty, editor, *FSCD*, volume 228 of *LIPIcs*, pages 4:1–4:18. Schloss Dagstuhl, 2022. `doi:10.4230/LIPIcs.FSCD.2022.4`.

12 George Metcalfe. Fundamentals of fuzzy logic, 2005. URL: `https://www.logic.at/tbilisi05/Metcalfe-notes.pdf`.

13 Vilém Novák. On fuzzy type theory. *Fuzzy Sets Syst.*, 149(2):235–273, 2005. `doi:10.1016/J.FSS.2004.03.027`.

14 Rostislav Horčík. Petr Cintula, Petr Hájek. Formal systems of fuzzy logic and their fragments. *Annals of Pure and Applied Logic*, 150:49–72, 2007. `doi:10.1016/j.apal.2007.07.004`.

15 Sorites paradox. URL: `https://plato.stanford.edu/entries/sorites-paradox/`.

16 Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, June 1965. `doi:10.1016/s0019-9958(65)90241-x`.

A Notions in Fuzzy logic

▶ **Definition 31.** *A fuzzy set is a pair (S, μ_S), where S is a set and $\mu_S : S \to [0,1]$ is a function. For all $x \in S$, $\mu_S(x)$ is called the degree of membership of x. We say that:*
- *x is* fully included *in (S, μ_S), if $\mu_S = 1$;*
- *x is* not included *in (S, μ_S), if $\mu_S(x) = 0$;*
- *x is* partially included *in (S, μ_S), if $0 < \mu_S(x) < 1$.*

If all $x \in S$ are fully included in S, the fuzzy set S is said to be crisp.

▶ **Definition 32** (T-norm). *A* T-norm*, see [12], is a binary function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following properties:*
- *commutativity: $T(a,b) = T(b,a)$,*
- *associativity: $T(a, T(b,c)) = T(T(a,b), c)$,*
- *monotonicity: $T(a,b) \leq T(c,d)$ if $a \leq c$ and $b \leq d$,*
- *the number 1 is the identity element: $T(a,1) = a$.*

Usually, T-norms are written in infix notation and the symbol $\otimes$ is used to denote them. In the paper, we consider only left-continuous t-norms.

▶ **Definition 33** (Residuum). *A binary function $\otimes : [0,1]^2 \to [0,1]$ is said to be residuate if and only if there exists a binary function $\Rightarrow : [0,1]^2 \to [0,1]$ called the* residuum of $\otimes$*, such that $\otimes$ and $\Rightarrow$ form an adjoint pair,* i.e.*:*

$$\forall a, b, c \in [0,1]. a \otimes b \leq c \Leftrightarrow a \leq b \Rightarrow c.$$

▶ **Definition 34.** *Prominent examples of left-continuous T-norms are:*

- *Gödel's T-norm: $a \otimes b = \min(a, b)$, whose residuum is*
$$a \Rightarrow b = \begin{cases} b & \textit{if } b \lneq a \\ 1 & \textit{otherwise;} \end{cases}$$
- *Goguen's product T-norm: $a \otimes b = a * b$, whose residuum is*
$$a \Rightarrow b = \begin{cases} b/a & \textit{if } b \lneq a \\ 1 & \textit{otherwise;} \end{cases}$$
- *Łukasiewicz's T-norm: $a \otimes b = \max(0, a + b - 1)$, whose residuum is*
$$a \Rightarrow b = \begin{cases} 1 - a + b & \textit{if } b \lneq a \\ 1 & \textit{otherwise.} \end{cases}$$

▶ **Proposition 35** (Norms with Residuum)**.** *Given a left-continuous T-norm $\otimes$, we have the following standard properties, where $a, b, c \ldots \in [0, 1]$:*

1. *$\otimes$ uniquely determines a* residuum *defined as follows*
$$a \Rightarrow b := \sup\{c \mid a \otimes c \leq b\};$$
2. *$\Rightarrow$ is monotone in the second argument and anti-monotone in its first argument;*
3. *$a \Rightarrow a = a \Rightarrow 1 = 1$, $1 \Rightarrow a = a$;*
4. *$a \leq b \Rightarrow a$;*
5. *$a \leq b$ if and only if $a \Rightarrow b = 1$;*
6. *$a \otimes b \leq a$, and also $a \otimes b \leq \min\{a, b\}$;*
7. *$(a \otimes b) \Rightarrow c = a \Rightarrow (b \Rightarrow c)$;*
8. *$(a \otimes (a \Rightarrow b)) \leq b$, and hence $(a \otimes (a \Rightarrow b)) \leq \min\{a, b\}$.*

Proof.

- Property (1) follows by the adjunction in Definition 33.
- Properties (2)-(5) follow from previous items and the Definition 33. We illustrate (5). From the adjunction we have that $1 \leq a \Rightarrow b \Longleftrightarrow 1 \otimes a \leq b$, but $a = 1 \otimes a$ by (2).
- Property (6) follows from monotonicity of the T-norm and identity.
- Property (7) follows from the adjunction of the T-norm.
- Property (8) follows from the fact that, by adjunction we have $a \otimes (a \Rightarrow b) \leq b \Leftrightarrow (a \Rightarrow b) \leq (a \Rightarrow b)$, and hence $a \otimes (a \Rightarrow b) \leq b$ always holds. ◀

▶ **Definition 36** (Similarity Relation)**.** *A* similarity relation*, S, over a domain D, is a fuzzy subset of $D \times D$, which is reflexive, symmetric, and transitive.*
Namely let $x, y, z \in D$ and let $\mu_S(x, y)$ denote the grade of membership of the ordered pair $(x, y) \in D \times D$. Then S is a similarity relation in D if and only if, for all $x, y, z \in D$,

- *$\mu_S(x, x) = 1$(reflexivity),*
- *$\mu_S(x, y) = 1$ implies $\mu_S(y, x) = 1$ (symmetry),*
- *$\mu_S(x, z) \geq \mu_S(x, y) \otimes \mu_S(y, z)$ (transitivity).*