Algorithmic Information Bounds for Distances and Orthogonal Projections

Peter Cholak
University of Notre Dame, IN, USA

Marianna Csörnyei
University of Chicago, IL, USA

Neil Lutz
University of Notre Dame, IN, USA

Patrick Lutz
University of Michigan, Ann Arbor, MI, USA

Elvira Mayordomo
University of Zaragoza, Spain

D. M. Stull
University of Chicago, IL, USA

Abstract

We introduce a new technique for proving bounds on the Kolmogorov complexity of geometric objects in Euclidean space, such as points and lines. We apply this technique to prove two theorems on algorithmic information theory, both of which have consequences for well-known problems in geometric measure theory. First, we show that for any point x in the plane and any other point y sufficiently independent of x, the distance between x and y retains at least half the complexity of the original point x. By the point-to-set principle of J. Lutz and N. Lutz, this yields an improved lower bound on the Hausdorff dimension of pinned distance sets, a topic closely related to Falconer's distance set conjecture. Second, we prove an analogous result for orthogonal projections: for any point x in the plane and any line through the origin which is sufficiently independent of x, the projection of x onto that line retains at least half the complexity of x. As a consequence, we obtain a generalization of a theorem of Bourgain on exceptional sets for orthogonal projections.

2012 ACM Subject Classification Theory of computation → Computability

Keywords and phrases effective fractal dimensions, algorithmic randomness, Kolmogorov complexity

Digital Object Identifier 10.4230/LIPIcs.MFCS.2026.13

Related Version *Full Version*: `https://arxiv.org/abs/2509.05211`

Funding The project was begun at and was made possible by a SQuaRE at the American Institute of Mathematics. The authors thank AIM for providing a supportive and mathematically rich environment. We also want to acknowledge support by Notre Dame Global via an International Research Travel Grant.
Peter Cholak: Partially funded by NSF-DMS-2502292.
Neil Lutz: This research was conducted while Neil Lutz was at Swarthmore College.
Patrick Lutz: Research partially supported by NSF Grant DMS-2203072.
Elvira Mayordomo: Partially funded by Spain-PID2022-138703OB-I00, Aragon-T64_20R (COSMOS), and EU-HORIZON-MSCA-2024-101236610 (NFFC).

1 Introduction

This paper is a contribution to a growing area of research which connects algorithmic information theory and geometric measure theory and which is based on a close correspondence between Kolmogorov complexity and Hausdorff dimension. This correspondence has two facets. First, there is a conceptual analogy between Kolmogorov complexity and Hausdorff dimension. For example, using Kolmogorov complexity, one can define a notion of effective dimension for objects such as infinite binary sequences and real numbers which shares many features with Hausdorff dimension [23, 2, 12]. Second, there is a specific technical result, proved by J. Lutz and N. Lutz and known as the *point-to-set principle*, which allows one to use facts about Kolmogorov complexity and effective dimension to prove theorems about Hausdorff dimension [13].

In recent years, the point-to-set principle has been applied to make progress on a few long-standing questions in geometric measure theory. For example, N. Lutz and Stull used it to improve the best then-known lower bound for the Furstenberg set conjecture (a generalization of the Kakeya conjecture) [14] and Stull used it to improve the best known lower bound for Falconer's distance set conjecture [25], an important open question in geometric measure theory [9].

The main goal of this paper is to introduce a new technique for proving lower bounds on the Kolmogorov complexity of geometric objects in Euclidean space. One of our primary motivations for introducing this new technique is that, when used in conjunction with the point-to-set principle, it seems especially useful for proving results in geometric measure theory which had previously appeared difficult to approach using the methods of algorithmic information theory. However, even ignoring these applications to geometric measure theory, we believe that the theorems in algorithmic information theory which can be proved using our technique are interesting in their own right.

In this paper, we will give two example applications of our technique. In each case, we will first use our technique to prove a theorem on algorithmic information theory and then use the point-to-set principle to transfer this theorem to the setting of geometric measure theory. We have included these applications not just for their own sake, but also as a demonstration of the utility of our technique, which we believe likely has further applications within geometric measure theory. In fact, in unpublished work, Fiedler [6] has already used our technique to prove a new theorem on the Hausdorff dimension of planar extensions of subsets of Euclidean space, a topic which is closely connected to the Kakeya Conjecture.

Our first application concerns the following question: suppose that x and y are points in $\mathbb{R}^2$ and that x is relatively complex (in the sense of Kolmogorov complexity). Does this imply anything about the complexity of $|x - y|$, the distance between x and y? Of course, if we make no assumptions whatsoever about y then the answer is no. However, we will show that if x and y are reasonably independent then $|x - y|$ must retain at least half the complexity of x.

Slightly more precisely, let $K_r(x)$ denote the Kolmogorov complexity of an r-bit approximation of x (i.e. of the first r bits of both coordinates of x) and let $K_r^y(z)$ denote the Kolmogorov complexity of an r-bit approximation of z relative to an oracle for y. We will show, in Theorem 3, that if x and y satisfy a mild independence condition then

$$K_r^y(|x - y|) \geq \frac{K_r(x)}{2} - o(r). \tag{1}$$

We will then use this theorem to prove a new lower bound on the Hausdorff dimension of pinned distance sets. Given a set $E \subseteq \mathbb{R}^2$, the *distance set* of E is the set $\Delta E = \{|x - y| \mid x, y \in E\}$ of pairwise distances between elements of E. Given a set $E \subseteq \mathbb{R}^2$ and a point

$x \in \mathbb{R}^2$, the *pinned distance set* is the set $\Delta_x E = \{|x-y| \mid y \in E\}$ of distances between x and elements of E. Falconer's distance set conjecture states that if $\dim_H(E) > 1$ then $\dim_H(\Delta E) = 1$. Most progress on this conjecture has come from estimating the dimension of pinned distance sets. Using our Theorem 3, we establish in Theorem 6 that for every analytic set $E \subseteq \mathbb{R}^2$ satisfying $s := \dim_H(E) \leq 1$,

$$\sup_{x\in E} \dim_H(\Delta_x E) \geq \frac{3s}{4}.$$

This improves on the best previously known bounds, due to Shmerkin and Wang [22] and Fiedler and Stull [7], for all $0 < s < 1$.

Our second application concerns the complexity of orthogonal projections of points in the plane. In particular, for any point $x \in \mathbb{R}^2$ and direction $e \in S^1$, let $p_e x$ denote the orthogonal projection of x onto the line in direction e. In Theorem 4, we show that if e is sufficiently independent of x then

$$K_r^e(p_e x) \geq \frac{K_r(x)}{2} - o(r). \tag{2}$$

In other words, the projection of x retains at least half the complexity of x.

We then apply this to generalize a theorem of Bourgain [1] on the Hausdorff dimension of exceptional sets for orthogonal projections. Our theorem extends Bourgain's from analytic sets to a more general class of sets which admit *optimal Hausdorff oracles* [24]. This result is similar in spirit to Shmerkin's [21] generalization of Bourgain's theorem in the context of distance sets.

We now outline the main idea of our technique. We will illustrate the idea in the context of our Theorem 3, discussed above. Suppose that the conclusion of that theorem fails and that in general, $K_r^y(|x-y|)$ may be much less than $K_r(x)/2$, even when x and y are fairly independent. By standard reasoning, we may assume that it actually fails for two different values of y, say y_1 and y_2 (with respect to the same value of x).

It is now tempting to reason as follows: if we know y_1 and y_2, as well as the distances $|x-y_1|$ and $|x-y_2|$ then we can easily recover x with very little additional information – the main point is that the circles with centers y_1 and y_2 and radii $|x-y_1|$ and $|x-y_2|$ should in general intersect in only two points. Thus the total information content of x should be no larger than the combined information content of $y_1, y_2, |x-y_1|$, and $|x-y_2|$. Furthermore, since we are assuming that y_1 and y_2 are independent of x, we can ignore the contribution of y_1 and y_2 and conclude that the information content of x should be no larger than the combined information content of $|x-y_1|$ and $|x-y_2|$. Thus it would seem that at least one of $|x-y_1|$ and $|x-y_2|$ must have at least half the information content of x, which is exactly what we want to conclude.

However, there is an important problem with this argument. Whenever we talk about the information content of geometric objects, like points in $\mathbb{R}^2$, what we really mean is the Kolmogorov complexity of finite approximations of these objects. When we assumed that the theorem fails for x, y_1 and for x, y_2, what we actually meant was that for infinitely many r, we have $K_r^{y_1}(|x-y_1|) < K_r(x)/2$ and for infinitely many r, $K_r^{y_2}(|x-y_2|) < K_r(x)/2$. But the values of r for which these inequalities hold may be different for y_1 and y_2. In other words, it may be that at every level of precision, either the finite approximation to $|x-y_1|$ or the finite approximation to $|x-y_2|$ at that precision has low complexity, but they happen to never both have low complexity at the same time.

What we really need in order for the argument sketched above to work is not just some y_1 and y_2 for which the theorem fails with respect to the same x, but y_1 and y_2 for which the theorem fails with respect to the same x *and the same level of precision* r. At the core of our technique is a method for finding such y_1, y_2 and r.

The key idea is that we replace x with a different point with similar properties, which we will refer to as a *surrogate point* for x. In order to do so, we first show that if the theorem fails then for each r, there are many pairs x, y for which it fails. We then use a combinatorial lemma (Lemma 1, proved by a Cauchy-Schwarz double counting argument) to show that it fails for so many pairs that we can find two pairs x_1, y_1 and x_2, y_2 with $x_1 = x_2$.

Of course, this is only a rough description of the idea; the full details will be given in the proofs of Theorems 3 and 4 below. Some proofs are omitted from this conference version of the paper.

2 Preliminaries

2.1 Kolmogorov Complexity in Discrete Domains

We now briefly state the basic definitions and essential properties of (prefix) Kolmogorov complexity, which will be used repeatedly in our proofs. For a detailed treatment of this topic, see Chapter 3 of [11].

Let U be an oracle universal prefix Turing machine, which will remain fixed throughout this paper. For all binary strings $\sigma, \tau \in \{0,1\}^*$, the *conditional (prefix) Kolmogorov complexity* of σ given τ is

$$K(\sigma \mid \tau) = \min\left\{|\pi| \;\middle|\; \pi \in \{0,1\}^* \text{ and } U^{\emptyset}(\pi, \tau) = \sigma\right\},$$

and the *(prefix) Kolmogorov complexity* of σ is

$$K(\sigma) = K(\sigma \mid \lambda),$$

where λ denotes the empty binary string. Standard binary encoding readily extends these definitions to other countable domains, including natural numbers, rationals, dyadic rationals, and vectors of these.

Prefix Kolmogorov complexity obeys the *symmetry of information*: For all $\sigma, \tau \in \{0,1\}^*$,

$$K(\sigma, \tau) = K(\sigma \mid \tau, K(\tau)) + K(\tau) + O(1). \tag{3}$$

The prefix Kolmogorov complexity of a string is never significantly longer than the string's length. In particular, for all $n \in \mathbb{N}$,

$$K(n) \leq \log n + 2 \log \log n + O(1). \tag{4}$$

A family $\{E_r \mid r \in \mathbb{N}\}$ of sets is *uniformly computably enumerable* if the set $\{\langle x, r\rangle \mid x \in E_r\}$ is computably enumerable. Where the dependence on r is clear, we will say that an individual set E_r in such a family as uniformly computably enumerable. We will often show that a set is large by showing that it is uniformly computably enumerable and contains objects of high complexity. In particular, if $\sigma, \tau \in \{0,1\}^*$ and E is a finite set such that $\sigma \in E$ and E is uniformly computably enumerable given τ, then

$$K(\sigma \mid \tau) \leq \log |E| + 2 \log \log |E| + O(1). \tag{5}$$

Intuitively, this is true because, given τ, we could algorithmically specify $\sigma \in E$ by giving its position in the enumeration of E, and this position is at most $|E|$.

2.2 Kolmogorov Complexity in Euclidean Spaces

To apply Kolmogorov complexity in Euclidean spaces, we consider the complexity of finite-precision rational approximations.

For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, let $\mathcal{B}_\varepsilon(x)$ denote the open ball of radius ε about x. For all $\sigma \in \{0,1\}^*$, $E, F \subseteq \mathbb{R}^n$, $r, s \in \mathbb{N}$, and $x, y \in \mathbb{R}^n$, define

$$\begin{aligned}
K(E \mid \sigma) &= \min_{q \in E \cap \mathbb{Q}^n} K(q \mid \sigma) \\
K(E) &= K(E \mid \lambda) \\
K(E \mid F) &= \max_{q \in F \cap \mathbb{Q}^n} K(E \mid q) \\
K(\sigma \mid E) &= \max_{q \in E \cap \mathbb{Q}^n} K(\sigma \mid q) \\
K_r(x) &= K(\mathcal{B}_{2^{-r}}(x)) \\
K_{r,s}(x \mid y) &= K(\mathcal{B}_{2^{-r}}(x) \mid \mathcal{B}_{2^{-s}}(y)).
\end{aligned}$$

While $K_r(x)$ is defined as the minimum complexity of a rational point that is 2^{-r}-close to x, it is often more convenient to analyze the complexity of a specific rational point that is close to x, namely, the point obtained by truncating, r bits to the right of the binary point, the binary expansions of each of x's coordinates. Fortunately, these two complexities can only differ by $O(\log r)$, and we will frequently switch between the two types of approximations in our proofs.

For $r \in \mathbb{N}$, let $\mathcal{D}_r = 2^{-r}\mathbb{Z}$ denote the *r-dyadic rationals*, and for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

$$\lfloor x \rfloor_r = (2^{-r}\lfloor 2^r x_1 \rfloor, \ldots, 2^{-r}\lfloor 2^r x_n \rfloor) \in \mathcal{D}_r^n.$$

For $e \in \mathcal{S}^1$, we identify e with the real number $a \in [0,1)$ such that $e = (\cos(2\pi a), \sin(2\pi a))$, so $\lfloor e \rfloor_r$ denotes $\lfloor a \rfloor_r$ in this case.

Let $C \in \mathbb{N}$ be a constant such that, for all $r \in \mathbb{N}$, $\sigma \in \{0,1\}^*$, and $x \in \mathbb{R} \cup \mathbb{R}^2 \cup \mathcal{S}^1$,

$$|K_r(x \mid \sigma) - K(\lfloor x \rfloor_r \mid \sigma)| < C \log r \tag{6}$$

and

$$|K(\sigma \mid \mathcal{B}_{2^{-r}}(x)) - K(\sigma \mid \lfloor x \rfloor_r)| < C \log r. \tag{7}$$

For details on the existence of this constant, see Section 2.3 of [14].

A closely related useful fact is that the complexity of an approximation is only linearly sensitive to changes in the level of precision. In particular, it was shown by [3, 13] that for each $n \in \mathbb{N}$, there is a constant $C_0 \in \mathbb{N}$ such that for all $x, y \in \mathbb{R}^n$ and all $r, s, t \in \mathbb{N}$,

$$K_{r+t,s}(x \mid y) \le K_{r,s}(x \mid y) + nt + C_0 \log(r + s + t) \tag{8}$$

and

$$K_{r,s}(x \mid y) \le K_{r,s+t}(x \mid y) + nt + C_0 \log(r + s + t). \tag{9}$$

2.3 Hausdorff Dimension and Effective Hausdorff Dimension

For $E \subseteq \mathbb{R}^n$ and $s > 0$, the *s-dimensional Hausdorff measure* of E is

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} |U_i|^s \;\middle|\; E \subseteq \bigcup_{i \in \mathbb{N}} U_i \text{ and each } |U_i| < \delta \right\},$$

where $|U_i|$ denotes the diameter of U_i, and the *Hausdorff dimension* of E is

$$\dim_{\mathrm{H}} E = \inf\{s \mid \mathcal{H}^s(E) = 0\}.$$

The statement of Theorem 6, like many classical results on Hausdorff dimension, requires the underlying set to be *analytic*, i.e., a continuous image of a Borel set.

The *(effective Hausdorff) dimension* of an individual point $x \in \mathbb{R}^n$ is defined [12, 19] as

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}.$$

As U was an oracle prefix machine, we can readily define Kolmogorov complexities *relative* to any oracle $A \subseteq \mathbb{N}$. For example, for all $\sigma, \tau \in \{0,1\}^*$,

$$K^A(\sigma \mid \tau) = \min\left\{|\pi| \mid \pi \in \{0,1\}^* \text{ and } U^A(\pi,\tau) = \sigma\right\}.$$

All other Kolmogorov complexity quantities in this section, including $\dim(x)$, are relativized in the same way, always denoted by placing the oracle in the superscript. The bounds (3)–(9) all hold relative to an arbitrary oracle $A \subseteq \mathbb{N}$. Given a point $y \in \mathbb{R}^n$, we will write K^y and $\dim^y$ as shorthand for K^{A_y} and $\dim^{A_y}$, where $A_y \subseteq \mathbb{N}$ is an oracle that encodes $\lfloor y \rfloor_r$ for all $r \in \mathbb{N}$. When multiple oracle sets are placed in the superscript, that indicates relativization to their Turing join.

Our Hausdorff dimension bounds will rely on the *point-to-set principle* [13], which states that, for all $E \subseteq \mathbb{R}^n$,

$$\dim_{\mathrm{H}} E = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \dim^A(x). \tag{10}$$

An oracle A that achieves the minimum in (10) is called a *Hausdorff oracle* for E. We will also use the correspondence principle of Hitchcock [8], which says that, if $E \subseteq \mathbb{R}^n$ is computably compact relative to an oracle A, then A is a Hausdorff oracle for E. That is,

$$\dim_{\mathrm{H}} E = \sup_{x\in E} \dim^A(x). \tag{11}$$

A Hausdorff oracle A for E is an *optimal Hausdorff oracle* for E if, for every auxiliary oracle $B \subseteq \mathbb{N}$ and every $\varepsilon > 0$, there is some $x \in E$ such that

$$\dim^{A,B}(x) \geq \dim_{\mathrm{H}}(E) - \varepsilon$$

and, for almost every $r \in \mathbb{N}$,

$$K_r^{A,B}(x) \geq K_r^A(x) - \varepsilon r.$$

Stull [24] formulated this definition and proved that if E is an analytic set, then there are optimal Hausdorff oracles for E, and that the converse does not hold.

3 Combinatorial Lemma

The proofs of each of our main Kolmogorov complexity bounds will rely on finding objects u, v, and d that satisfy two criteria: First, d is in some sense "suitable" for each of u and v, belonging to certain sets N_u and N_v. Second, u and v are sufficiently dissimilar with respect to d, satisfying $u \not\sim_d v$, where $\sim_d$ is a binary relation that may depend on d. The following lemma says that if every v has many suitable d and, for each such d, there are few u such that $u \sim_d v$, then there is some pair u, v such that many d satisfy these two criteria for that pair.

▶ **Lemma 1.** *Let X and V be finite sets, $\alpha \in (0,1)$, and, for each $d \in X$, $\sim_d$ a binary relation on V. Suppose there is a collection $\{N_v \mid v \in V\}$ of finite subsets of X such that, for all $v \in V$, we have $|N_v| \geq \alpha|X|$ and, for all $d \in N_v$,*

$$|\{u \in V \mid u \sim_d v\}| \leq \frac{\alpha^2}{4}|V|. \tag{12}$$

Then there exist $u, v \in V$ such that $|\{d \in X \mid d \in N_u \cap N_v \text{ and } u \not\sim_d v\}| > \frac{\alpha^2}{2}|X|$. In particular, if there is a single binary relation $\sim$ such that $\sim_d$ is $\sim$ for all $d \in X$, then there exist $u, v \in V$ such that $u \not\sim v$ and $|N_u \cap N_v| > \frac{\alpha^2}{2}|X|$.

4 Kolmogorov Complexity Bound for Distances

The proof of our first main Kolmogorov complexity bound, Theorem 3, relies on triangulating the approximate location of a point d using its approximate distance from two other points u and v. Geometrically, this places d in the intersection of two thin annuli, centered at u and v. We will use the following lemma of Wolff [26] to bound the size of that intersection.

▶ **Lemma 2** (Wolff [26])**.** *Let $\rho_1, \rho_2 \in [0.99, 1.01]$, $x_1, x_2 \in [0, 0.01]^2$, and $\varepsilon > 0$. Let A_1 and A_2 be the open ε-neighborhoods of the circles with centers x_1 and x_2 and radii ρ_1 and ρ_2, respectively. Then $A_1 \cap A_2$ is contained within $O(1)$ sets of the form*

$$\{y \in A_1 \mid x_1 + \rho_1 e_{x_1,y} \in \gamma\},$$

where each γ is an arc of length

$$\frac{2\pi\rho_1\varepsilon}{\sqrt{\max(|x_1 - x_2| + |\rho_1 - \rho_2|, \varepsilon)\max(||x_1 - x_2| - |\rho_1 - \rho_2||, \varepsilon)}}.$$

For all distinct points $u, v \in \mathbb{R}^2$, let $e_{u,v}$ denote $\frac{v-u}{|v-u|} \in \mathcal{S}^1$, the direction from u to v.

▶ **Theorem 3.** *If $A \subseteq \mathbb{N}$, $x, y \in \mathbb{R}^2$, $r \in \mathbb{N}$, and constants $\delta, \varepsilon \in \mathbb{Q} \cap (0,1)$ satisfy $\delta < \frac{\varepsilon}{6} \cdot \dim^{A,y}(e_{y,x})$,*

$$K_r^{A,y}(x) \geq K_r^A(x) - \delta r, \tag{13}$$

and r is sufficiently large, then for all $s \in \mathbb{N}$,

$$K_{r,s}^{A,y}(|x - y| \mid |x - y|) \geq \frac{K_{r,s}^A(x \mid x)}{2} - \varepsilon r. \tag{14}$$

Before giving the proof, we note that the primary strength of this theorem is the weakness of the assumptions. Namely, the theorem does not require that x, y or $e_{y,x}$ are highly complex. The only assumptions are that the dimension of $e_{y,x}$ is non-zero and that x and y are sufficiently independent. This makes Theorem 3 incomparable to the previous results on the algorithmic complexity of distances, such as Theorem 22 of [25], which all require strong assumptions on the randomness of x, y and $e_{y,x}$.

Proof. Let A, x, y, r, s, δ, and ε satisfy the hypotheses. We suppress the oracle A from the notation below, but the argument is unchanged by working relative to A. There is a computable affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ with computable Lipschitz constants such that $\lfloor f(x) \rfloor_7 = (1, 0)$ and $\lfloor f(y) \rfloor_7 = (0, 0)$. As this transformation is computable and bi-Lipschitz, applying f can only change complexities at precision r by $O(\log r)$ [3, 13], so we assume without loss of generality that $\lfloor x \rfloor_7 = (1, 0)$ and $\lfloor y \rfloor_7 = (0, 0)$; this assumption will make it more convenient to apply Lemma 2 as written, since $2^{-7} < 0.01$.

When $s \geq r - \varepsilon r$, the right-hand side of (14) is negative for all sufficiently large r. Hence, assume $s < r - \varepsilon r$, and assume toward a contradiction that (14) does not hold. We now construct sets X, V, and $\{N_v \mid v \in V\}$, to which we will apply Lemma 1.

Construction. For all $v, d \in \mathbb{R}^2$, denote by $v \rhd d$ the inequality

$$K^v_{r,s}(|d-v| \mid |d-v|) < \frac{K_{r,s}(x \mid x)}{2} - \varepsilon r + 4C_0 \log r,$$

where C_0 is the constant from (8). Intuitively, $v \rhd d$ means that v is a suitable point to use in triangulating the location of the point d. Our goal will be to find two such points u and v such that u, v, and d aren't too close to being collinear.

Define the sets

$$X = \left\{ d \in \mathcal{D}^2_r \;\middle|\; \begin{array}{l} \lfloor d \rfloor_7 = (1,0),\ K(d) \le K_r(x) + C \log r, \\ K(d \mid \lfloor d \rfloor_s) \le K_{r,s}(x \mid x) + 2C \log r \end{array} \right\}$$
$$Y = \left\{ d \in \mathcal{D}^2_r \;\middle|\; \lfloor d \rfloor_7 = (0,0),\ K(d) \le K_r(y) + C \log r \right\},$$

where C is the constant from (6). Let $t = \lfloor \varepsilon r/3 \rfloor$, noting that this is computable given r. Let $z = (r, s, K_r(x), K_{r,s}(x \mid x), K_r(y), K^y_t(e_{y,x}))$, noting that $K(z) = O(\log r)$ by (4), and let $C_3, C_4, C_5 \in \mathbb{N}$ be constants to be defined later.

Let M be an oracle prefix Turing machine that, given v as an oracle and a program for z as an input, internally enumerates all $d \in X$ such that $v \rhd d$. For each $e \in \mathcal{D}_t$, whenever M has internally enumerated exactly

$$\left\lfloor 2^{K_r(x) - K^y_t(e_{y,x}) - \delta r - C_3 \log r} \right\rfloor \tag{15}$$

points d such that $\lfloor e_{v,d} \rfloor_t = e$, the machine labels e "crowded for v" and prints those points.

For each $v \in Y$, let F_v be the set of all directions that are crowded for v. Define the set

$$V_1 = \left\{ d \in Y \;\middle|\; (\exists v \in \mathbb{R}^2) \left[\lfloor v \rfloor_r = d \text{ and } |F_v| > 2^{K^y_t(e_{y,x}) - C_4 \log r} \right] \right\}. \tag{16}$$

For each $d \in V_1$, let $v_d \in \mathbb{R}^2$ be a point satisfying $\lfloor v_d \rfloor_r = d$ and $|F_{v_d}| > 2^{K^y_t(e_{y,x}) - C_4 \log r}$, and define the set $V = \{v_d \mid d \in V_1\}$.

For each $v \in V$, define

$$N_v = \{ d \in X \mid M^v(\pi) \text{ prints } d \text{ and } K^v(\lfloor e_{v,d} \rfloor_t) \ge K^y_t(e_{y,x}) - C_4 \log r - 2 \},$$

where π is a witness to $K(z)$ and $C_4 \in \mathbb{N}$ is the constant from (16).

The hypotheses of Lemma 1 are satisfied by the sets X and V, the collection $\{N_v \mid v \in V\}$, the parameter $\alpha = 2^{-\delta r - (C_3 + C_4) \log r - 3}$, and the collection $\{\sim_d \mid d \in X\}$ of reflexive binary relations defined for $u \ne v$ by $u \sim_d v$ if $|e_{u,d} - e_{v,d}| < 2^{-t}$.

Hence, we can apply Lemma 1 to let $u, v \in V$ be such that

$$|\{ d \in X \mid d \in N_u \cap N_v \text{ and } |e_{u,d} - e_{v,d}| \ge 2^{-t} \}| > 2^{K_r(x) - 2\delta r - O(\log r)}.$$

Since this set is large, it contains high-complexity points relative to every oracle; in particular, let $d \in N_u \cap N_v$ such that $|e_{u,d} - e_{v,d}| \ge 2^{-t}$ and

$$K^{u,v}(d) \ge K_r(x) - 2\delta r - O(\log r). \tag{17}$$

Let π_u be a witness to $K^u(\lfloor |d-u| \rfloor_r \mid \lfloor |d-u| \rfloor_s)$ and π_v be a witness to $K^v(\lfloor |d-v| \rfloor_r \mid \lfloor |d-v| \rfloor_s)$, and define the sets $P_u = \{U^u(\pi_u, d_u) \mid d_u \in \mathcal{D}_s\}$ and $P_v = \{U^v(\pi_v, d_v) \mid d_v \in \mathcal{D}_s\}$. Intuitively, each of these sets contains, within each s-dyadic interval, at most one candidate value for an approximation of a distance from d. In particular, $\lfloor |d-u| \rfloor_r \in P_u$ and $\lfloor |d-v| \rfloor_r \in P_v$. Let $D_{u,v}$ be the set of all r-dyadic points that share these properties; that is,

$$D_{u,v} = \left\{ w \in \mathcal{D}^2_r \mid \lfloor |w-u| \rfloor_r \in P_u \text{ and } \lfloor |w-v| \rfloor_r \in P_v \right\}.$$

For any two points $w, w' \in D_{u,v}$ such that $|w - w'| < 2^{1-s}$, we have

$$\big|\lfloor |w-u| \rfloor_r - \lfloor |w'-u| \rfloor_r\big| \leq \big||w-u| - |w'-u|\big| + 2^{1-r} < 2^{2-s},$$

and similarly $\big|\lfloor |w-v| \rfloor_r - \lfloor |w'-v| \rfloor_r\big| < 2^{2-s}$. As each s-dyadic interval contains at most one element of P_u and one element of P_v, this implies that the set

$$P_{u,v} = \{(\lfloor |w-u| \rfloor_r, \lfloor |w-v| \rfloor_r) \mid w \in D_{u,v} \text{ and } \lfloor w \rfloor_s = \lfloor d \rfloor_s\}$$

has cardinality $O(1)$.

Now consider the set

$$\hat{D}_{u,v} = \{w \in \mathcal{D}_r^2 \mid \lfloor w-u \rfloor_r = \lfloor d-u \rfloor_r \text{ and } \lfloor w-v \rfloor_r = \lfloor d-v \rfloor_r\}.$$

This set is contained in the intersection of $O(2^{-r})$-neighborhoods of the circles with centers u and v and radii $\rho_u := |d-u|$ and $\rho_v := |d-v|$, respectively. As $|e_{u,d} - e_{v,d}| \geq 2^{-t}$, for all sufficiently large r we have

$$\max\{|u-v| + |\rho_u - \rho_v|, O(2^{-r})\} \geq 2^{-t-O(1)}$$

and, by a straightforward trigonometric argument (also using $|u-v| < 2^{-8}$ and $\rho_u, \rho_v > 1 - 2^{-7}$),

$$\max\{||u-v| - |\rho_u - \rho_v||, O(2^{-r})\} \geq 2^{-3t-O(1)}.$$

Therefore, by Lemma 2, $\hat{D}_{u,v}$ is covered by $O(1)$ sets, each of diameter

$$\frac{O(2^{-r})}{\sqrt{2^{-4t}}} = O(2^{2t-r}).$$

It follows that some point in $\hat{D}_{u,v}$ is a precision-$(r - 2t - O(1))$ approximation of d that can be computed, relative to u and v, given π_u, π_v, a witness to $K(\lfloor d \rfloor_s)$, $O(1)$ bits to specify $(\lfloor |d-u| \rfloor_r, \lfloor |d-v| \rfloor_r)$ within the set $P_{u,v}$, $O(1)$ bits to specify a specific cover element, and $O(1)$ bits for the description of the Turing machine that performs this computation. That is,

$$K^{u,v}_{r-2t-O(1)}(d) \leq |\pi_u| + |\pi_v| + K(\lfloor d \rfloor_s) + O(1). \tag{18}$$

Therefore,

$$\begin{aligned}
K_{r,s}(x \mid x) &= K(\lfloor x \rfloor_r \mid \lfloor x \rfloor_s) + O(\log r) && \text{(by (6) and (7))}\\
&= K(\lfloor x \rfloor_r) - K(\lfloor x \rfloor_s) + O(\log r) && \text{(by (3))}\\
&= K_r(x) - K(\lfloor x \rfloor_s) + O(\log r) && \text{(by (6))}\\
&\leq K^{u,v}(d) - K_s(x) + 2\delta r + O(\log r) && \text{(by (17))}\\
&= K^{u,v}_r(d) - K(\lfloor d \rfloor_s) + 2\delta r + O(\log r) && \text{(by (6))}\\
&\leq K^{u,v}_{r-2t-O(1)}(d) + 4t - K(\lfloor d \rfloor_s) + 2\delta r + O(\log r) && \text{(by (8))}\\
&\leq |\pi_u| + |\pi_v| + 4t + 2\delta r + O(\log r) && \text{(by (18))}\\
&= K^u_{r,s}(|d-u| \mid |d-u|) + K^v_{r,s}(|d-v| \mid |d-v|) + 4t + 2\delta r + O(\log r)\\
& && \text{(by (6) and (7))}\\
&\leq K_{r,s}(x \mid x) - 2\varepsilon r + 4t + 2\delta r + O(\log r) && (d \in N_u \cap N_v)\\
&< K_{r,s}(x \mid x) - 2\varepsilon r/3 + 2\delta r + O(\log r). && (t \leq \varepsilon r/3)
\end{aligned}$$

As $\delta < \varepsilon \dim^y(e_{y,x})/6 \leq \varepsilon/6$, this implies $\varepsilon r/3 < O(\log r)$, a contradiction for sufficiently large r. We conclude that (14) holds. ◀

5 Kolmogorov Complexity Bound for Orthogonal Projections

▶ **Theorem 4.** *If $A \subseteq \mathbb{N}$, $x \in \mathbb{R}^2$, $e \in \mathcal{S}^1$, $r \in \mathbb{N}$, and constants $\delta, \varepsilon \in \mathbb{Q} \cap (0,1)$ satisfy $\delta < \varepsilon \dim^A(e)/4$,*

$$K_r^{A,e}(x) \geq K_r^A(x) - \delta r, \tag{19}$$

and r is sufficiently large, then for all $s \in \mathbb{N}$,

$$K_{r,s}^{A,e}(p_e x \mid p_e x) \geq \frac{K_{r,s}^A(x \mid x)}{2} - \varepsilon r. \tag{20}$$

Proof. Let A, x, e, r, s, δ, and ε satisfy the hypotheses. We suppress the oracle A from the notation below, but the argument is unchanged by working relative to A. When $s \geq r - \varepsilon r$, the right-hand side of (20) is negative for all sufficiently large r. Hence, assume $s < r - \varepsilon r$, and assume toward a contradiction that (20) does not hold. We now construct sets X and V and a collection $\{N_v \mid v \in V\}$, to which we will apply Lemma 1.

Construction. Let $t = \lfloor \varepsilon r/2 \rfloor$, noting that this is computable given r, and define the sets

$$X = \{d \in \mathcal{D}_r^2 \mid K(d) \leq K_r(x) + C\log r \text{ and } K(d \mid \lfloor d \rfloor_s) \leq K_{r,s}(x \mid x) + 2C\log r\}$$
$$S = \{d \in \mathcal{D}_r \mid K(d) \leq K_r(e) + C\log r \text{ and } K_t(d) \leq K_t(e) + 2C\log t\},$$

where C is the constant from (6).

For every direction $v \in \mathcal{S}^1$, define the set

$$N_v = \left\{ d \in X \;\middle|\; K_{r,s}^v(p_v d \mid p_v d) < \frac{K_{r,s}(x \mid x)}{2} - \varepsilon r + 4C_0 \log r \right\},$$

where C_0 is the constant from (8).

Let $z = (r, s, K_r(x), K_{r,s}(x \mid x), K_r(e), K_t(e))$, noting that $K(z) = O(\log r)$ by (4), and let $C_1, C_2 \in \mathbb{N}$ be constants to be defined later. Define the set

$$V_1 = \left\{ d \in S \;\middle|\; (\exists v \in \mathcal{S}^1)\left[\lfloor v \rfloor_r = d \text{ and } |N_v| > 2^{K_r(x) - \delta r - C_1 \log r} \right] \right\}. \tag{21}$$

Let M be a prefix Turing machine that, given any input $\pi \in \{0,1\}^*$ such that $U(\pi) \in \mathbb{N}^6$, does the following. Working under the assumption that $U(\pi) = z$, M internally enumerates V_1. For each $d \in \mathcal{D}_t$, whenever M has internally enumerated exactly $\lfloor 2^{K_r(e) - K_t(e) - C_2 \log r} \rfloor$ directions $d' \in V_1$ such that $\lfloor d' \rfloor_t = d$, the machine labels d "crowded" and prints these $\lfloor 2^{K_r(e) - K_t(e) - C_2 \log r} \rfloor$ directions d'.

Let V_2 be the set of all directions printed by $M(\pi)$, where π is a witness to $K(z)$. As $V_2 \subseteq V_1$, for each $d \in V_2$ there is some $v_d \in \mathcal{S}^1$ satisfying $\lfloor v_d \rfloor_r = d$ and $|N_{v_d}| > 2^{K_r(x) - \delta r - C_1 \log r}$; let $V = \{v_d \mid d \in V_2\}$.

The sets X and V, the collection $\{N_v \mid v \in V\}$, $\alpha = 2^{-\delta r - (C + C_2)\log r - 1}$, and the binary relation $|u - v| < 2^{-t}$ satisfy the hypotheses of Lemma 1. Hence, we can apply Lemma 1 to let $u, v \in V$ be such that $|u - v| \geq 2^{-t}$ and $|N_u \cap N_v| > 2^{K_r(x) - 2\delta r - O(\log r)}$. Since this intersection is large, it contains high-complexity points relative to every oracle; in particular, let $d \in N_u \cap N_v$ such that

$$K^{u,v}(d) \geq K_r(x) - 2\delta r - O(\log r). \tag{22}$$

Let π_u and π_v be witnesses to $K^u(\lfloor p_u d\rfloor_r \mid \lfloor p_u d\rfloor_s)$ and $K^v(\lfloor p_v d\rfloor_r \mid \lfloor p_v d\rfloor_s)$, respectively, and define the sets $P_u = \{U^u(\pi_u, d') : d' \in \mathcal{D}_s\}$ and $P_v = \{U^v(\pi_v, d') : d' \in \mathcal{D}_s\}$. Intuitively, each of these sets contains, within each s-dyadic interval, at most one candidate value for an approximation of a projection of d. In particular, $\lfloor p_u d\rfloor_r \in P_u$ and $\lfloor p_v d\rfloor_r \in P_v$. Let $D_{u,v}$ be the set of all r-dyadic points that share these properties; that is,

$$D_{u,v} = \left\{y \in \mathcal{D}_r^2 \mid \lfloor p_u y\rfloor_r \in P_u \text{ and } \lfloor p_v y\rfloor_r \in P_v\right\}.$$

For any two points $y, y' \in D_{u,v}$ such that $|y - y'| < 2^{1-s}$, we have

$$\begin{aligned}
|\lfloor p_u y\rfloor_r - \lfloor p_u y'\rfloor_r| &\le |\lfloor p_u y\rfloor_r - p_u y| + |p_u y - p_u y'| + |p_u y' - \lfloor p_u y'\rfloor_r| \\
&< 2^{-r} + |y - y'| + 2^{-r} \\
&< 2^{2-s},
\end{aligned}$$

and similarly $|\lfloor p_v y\rfloor_r - \lfloor p_v y'\rfloor_r| < 2^{2-s}$. As each s-dyadic interval contains at most one element of P_u and one element of P_v, this implies that the set

$$P_{u,v} = \{(\lfloor p_u y\rfloor_r, \lfloor p_v y\rfloor_r) \mid y \in D_{u,v} \text{ and } \lfloor y\rfloor_s = \lfloor d\rfloor_s\}$$

has cardinality $O(1)$. Furthermore, since u and v are 2^{-t}-separated, the set

$$\hat{D}_{u,v} = \{y \in \mathcal{D}_r^2 \mid \lfloor p_u y\rfloor_r = \lfloor p_u d\rfloor_r \text{ and } \lfloor p_v y\rfloor_r = \lfloor p_v d\rfloor_r\}$$

is contained within a parallelogram of diameter $O(2^{t-r})$, meaning some point in $\hat{D}_{u,v}$ is a precision-$(r - t - O(1))$ approximation of d that can be computed, relative to u and v, given π_u, π_v, a witness to $K(\lfloor d\rfloor_s)$, $O(1)$ bits to specify $(\lfloor p_u d\rfloor_r, \lfloor p_v d\rfloor_r)$ within the set $P_{u,v}$, and $O(1)$ bits for the description of the Turing machine that performs this computation. That is,

$$K^{u,v}_{r-t-O(1)}(d) \le |\pi_u| + |\pi_v| + K(\lfloor d\rfloor_s) + O(1). \tag{23}$$

Therefore,

$$\begin{aligned}
K_{r,s}(x \mid x) &= K(\lfloor x\rfloor_r \mid \lfloor x\rfloor_s) + O(\log r) && \text{(by (6) and (7))}\\
&= K(\lfloor x\rfloor_r) - K(\lfloor x\rfloor_s) + O(\log r) && \text{(by (3))}\\
&= K_r(x) - K(\lfloor x\rfloor_s) + O(\log r) && \text{(by (6))}\\
&\le K^{u,v}(d) - K_s(x) + 2\delta r + O(\log r) && \text{(by (22))}\\
&= K^{u,v}_r(d) - K(\lfloor d\rfloor_s) + 2\delta r + O(\log r) && \text{(by (6))}\\
&\le K^{u,v}_{r-t-O(1)}(d) + 2t - K(\lfloor d\rfloor_s) + 2\delta r + O(\log r) && \text{(by (8))}\\
&\le |\pi_u| + |\pi_v| + 2t + 2\delta r + O(\log r) && \text{(by (23))}\\
&= K^u(\lfloor p_u d\rfloor_r \mid \lfloor p_u d\rfloor_s) + K^v(\lfloor p_v d\rfloor_r \mid \lfloor p_v d\rfloor_s) + 2t + 2\delta r + O(\log r) &&\\
&= K^u_{r,s}(p_u d \mid p_u d) + K^v_{r,s}(p_v d \mid p_v d) + 2t + 2\delta r + O(\log r) && \text{(by (6) and (7))}\\
&\le K_{r,s}(x \mid x) - 2\varepsilon r + 2t + 2\delta r + O(\log r) && (d \in N_u \cap N_v)\\
&< K_{r,s}(x \mid x) - \varepsilon r + 2\delta r + O(\log r). && (t = \lfloor \varepsilon r/2\rfloor)
\end{aligned}$$

As $\delta < \varepsilon \dim(e)/4 \le \varepsilon/4$, this implies $\varepsilon r/2 < O(\log r)$, a contradiction for sufficiently large r. We conclude that (20) holds. ◀

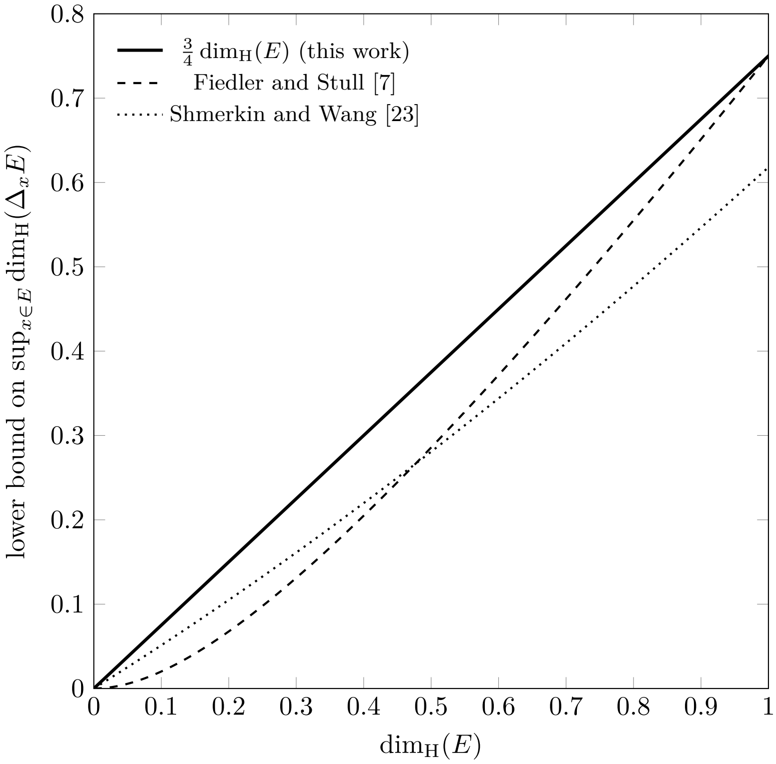

Figure 1 Pinned distance set lower bounds for $\dim_{\mathrm{H}}(E) \in [0,1]$.

6 Hausdorff Dimension Bounds

We now prove our Hausdorff dimension bound for pinned distance sets. For all $s \in (0,1)$, this is a strict improvement over the $\frac{s-2+\sqrt{4+s^2}}{2}$ lower bound of Shmerkin and Wang [22] and the $s\left(1 - \frac{2-s}{2(1+2s-s^2)}\right)$ lower bound of Fiedler and Stull [7], which are the best previously known bounds. For example, where those bounds coincide at $s \approx 0.47671$, they give $\sup_x \dim_{\mathrm{H}}(\Delta_x E) \gtrsim 0.26637$, while our bound gives $\sup_x \dim_{\mathrm{H}}(\Delta_x E) \gtrsim 0.35753$.

Our proof relies on an effective dimension bound, Theorem 5, that is proved using Theorem 3 in conjunction with techniques and results from [7, 25, 15, 5].

▶ **Theorem 5.** *Suppose that $x, y \in \mathbb{R}^2$, $e = \frac{y-x}{|y-x|}$, $0 < \sigma < 1$ and $A \subseteq \mathbb{N}$ satisfy the following conditions.*

(C1) $\dim^A(x), \dim^A(y) > \sigma$

(C2) $K_r^{A,x}(e) > \sigma r - O(\log r)$ *for all r.*

(C3) $K_r^{A,x}(y) \geq K_r^A(y) - O(\log r)$ *for all sufficiently large r.*

(C4) $K_{t,r}^A(e \mid y) > \sigma t - O(\log r)$ *for all sufficiently large r and $t \leq r$.*

Then $\dim^{x,A}(|x-y|) \geq \frac{3\sigma}{4}$.

▶ **Theorem 6.** *Suppose $E \subseteq \mathbb{R}^2$ is an analytic set such that $s = \dim_{\mathrm{H}}(E) \leq 1$. Then,*

$$\sup_{x\in E} \dim_{\mathrm{H}}(\Delta_x E) \geq \frac{3s}{4}.$$

Proof. Let $\sigma < s$. We will use the important fact of geometric measure theory that Hausdorff dimension is inner regular for analytic sets (see, e.g., [17]). That is, for every analytic set G and every $\epsilon > 0$, there is a compact subset K such that $\dim_{\mathrm{H}} K > \dim_{\mathrm{H}} G - \epsilon$. In particular, since E is analytic, there is a compact subset $F \subseteq E$ such that $\dim_{\mathrm{H}}(F) > \sigma$. Let A be a Hausdorff oracle for F relative to which F is computably compact. Using Proposition 7 of [25], $\Delta_x F$ is computably compact relative to (A, x) for every $x \in \mathbb{R}^2$. Hence, by Hitchcock's point-to-set principle, (11), (A, x) is a Hausdorff oracle for $\Delta_x F$, for every $x \in \mathbb{R}^2$. Therefore, since σ can be arbitrarily close to s, it suffices to show that

$$\sup_{x \in F} \sup_{y \in F} \dim^{A,x}(|x-y|) \geq \frac{3\sigma}{4}.$$

As shown in [7], Lemmas 33 and 34, there are points $x, y \in F$ and $e = \frac{y-x}{|x-y|}$ which satisfy the conditions of Theorem 5. We may therefore apply Theorem 5, to conclude that, for any such x, y, we have $\dim^{A,x}(|x-y|) \geq 3\sigma/4$, and the conclusion follows. ◀

Combined with the point-to-set principle, Theorem 4 allows us to generalize, from analytic sets to sets with optimal Hausdorff oracles, Bourgain's theorem on the Hausdorff dimension of *exceptional sets* for orthogonal projections. Marstrand [16] proved that, for every analytic $E \subseteq \mathbb{R}^2$ and almost every $e \in \mathcal{S}^1$, $\dim_{\mathrm{H}}(p_e E) = \min\{\dim_{\mathrm{H}}(E), 1\}$. A direction e is thus "exceptional" for E if $\dim_{\mathrm{H}}(p_e E) < s$ for some $s \leq \min\{\dim_{\mathrm{H}}(E), 1\}$; lesser values of s correspond to "more exceptional" directions. There is a long and active line of research in bounding the Hausdorff dimension of sets of such directions [10, 1, 18, 20, 4]. We prove the following theorem.

▶ **Theorem 7** (generalizing Bourgain [1]). *Let $E \subseteq \mathbb{R}^2$, and suppose that optimal Hausdorff oracles for E exist. Then*

$$\dim_{\mathrm{H}} \left\{ e \in \mathcal{S}^1 \;\middle|\; \dim_{\mathrm{H}}(p_e E) < \frac{\dim_{\mathrm{H}} E}{2} \right\} = 0.$$

References

1. Jean Bourgain. The discretized sum-product and projection theorems. *J. Anal. Math.*, 112:193–236, 2010. doi:10.1007/s11854-010-0028-x.
2. Jin-Yi Cai and Juris Hartmanis. On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line. *J. Comput. System Sci.*, 49(3):605–619, 1994. doi:10.1016/S0022-0000(05)80073-X.
3. Adam Case and Jack H. Lutz. Mutual dimension. *ACM Transactions on Computation Theory*, 7(3):12, 2015. doi:10.1145/2786566.
4. Peter Cholak, Marianna Csörnyei, Neil Lutz, Patrick Lutz, Elvira Mayordomo, and D. M. Stull. Bounding the dimension of exceptional sets for orthogonal projections. *Proceedings of the American Mathematical Society*, to appear.
5. Marianna Csornyei and D. M. Stull. Improved bounds for radial projections in the plane, 2025. arXiv:2508.18228.
6. Jacob B. Fiedler. The information content of points on lines and k-plane extensions. Preprint, arXiv:2510.11645 [math.CA] (2025), 2025. arXiv:2510.11645.
7. Jacob B. Fiedler and D. M. Stull. Pinned distances of planar sets with low dimension, 2024. arXiv:2408.00889.
8. John M. Hitchcock. Correspondence principles for effective dimensions. *Theory of Computing Systems*, 38(5):559–571, 2005. doi:10.1007/S00224-004-1122-1.
9. Alex Iosevich. What is... Falconer's conjecture? *Notices of the American Mathematical Society*, 66(4):552–555, 2019. doi:10.1090/noti1843.
10. Robert Kaufman. On Hausdorff dimension of projections. *Mathematika*, 15:153–155, 1968. doi:10.1112/S0025579300002503.

11 Ming Li and Paul M. B. Vitányi. *An Introduction to Kolmogorov Complexity and its Applications*. Springer-Verlag, Berlin, 2019. Fourth Edition. doi:10.1007/978-3-030-11298-1.

12 Jack H. Lutz. The dimensions of individual strings and sequences. *Inf. Comput.*, 187(1):49–79, 2003. doi:10.1016/S0890-5401(03)00187-1.

13 Jack H. Lutz and Neil Lutz. Algorithmic information, plane Kakeya sets, and conditional dimension. *ACM Transactions on Computation Theory*, 10(2):7:1–7:22, 2018. doi:10.1145/3201783.

14 Neil Lutz and D. M. Stull. Bounding the dimension of points on a line. *Information and Computation*, 275:104601, 2020. doi:10.1016/j.ic.2020.104601.

15 Neil Lutz and D. M. Stull. Projection theorems using effective dimension. *Information and Computation*, 297:105137, 2024. doi:10.1016/j.ic.2024.105137.

16 J. M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)*, 4:257–302, 1954. doi:10.1112/plms/s3-4.1.257.

17 Pertti Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Cambridge University Press, 1999. doi:10.1017/CBO9780511623813.

18 Pertti Mattila. Hausdorff dimension, projections, intersections, and Besicovitch sets. In *New Trends in Applied Harmonic Analysis, Volume 2*, pages 129–157. Springer, 2019. doi:10.1007/978-3-030-32353-0_6.

19 E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Information Processing Letters*, 84(1):1–3, 2002. doi:10.1016/S0020-0190(02)00343-5.

20 Kevin Ren and Hong Wang. Furstenberg sets estimate in the plane, 2023. arXiv:2308.08819.

21 Pablo Shmerkin. A non-linear version of Bourgain's projection theorem: Dedicated to the memory of Jean Bourgain. *Journal of the European Mathematical Society (EMS Publishing)*, 25(10), 2023. doi:10.4171/JEMS/1283.

22 Pablo Shmerkin and Hong Wang. On the distance sets spanned by sets of dimension $d/2$ in $\mathbb{R}^d$. *Geometric and Functional Analysis*, 35(1):283–358, 2025. doi:10.1007/s00039-024-00696-5.

23 Ludwig Staiger. Kolmogorov complexity and Hausdorff dimension. *Inform. and Comput.*, 103(2):159–194, 1993. doi:10.1006/inco.1993.1017.

24 D. M. Stull. Optimal oracles for point-to-set principles. *Journal of Mathematical Logic*, to appear.

25 D. M. Stull. Pinned distance sets using effective dimension. *Israel Journal of Mathematics*, to appear.

26 Thomas Wolff. A Kakeya-type problem for circles. *American Journal of Mathematics*, 119(5):985–1026, 1997. doi:10.1353/ajm.1997.0034.