Bilateralism with Incompatible Proofs and Refutations

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Abstract

Logical bilateralism challenges traditional concepts of logic by treating assertion and denial as independent yet opposed acts. While initially devised to justify classical logic, its constructive variants show that both acts admit intuitionistic interpretations. This paper presents a bilateral system where a formula cannot be both provable and refutable without contradiction, offering a framework for modelling mathematical proofs and refutations that exclude inconsistency. We formalise the logic via a bilateral natural deduction system with the desirable proof-theoretic properties of normalisation, subformula property and consistency, together with a base-extension semantics grounded in explicit proofs and refutations. Finally, refutation is shown to coincide with Nelson's constructive falsity, extending intuitionistic logic for constructive epistemic reasoning.

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1 Introduction

A central issue in constructive logic concerns the nature of refutation and how falsity should be represented within a constructive framework. In this context, David Nelson [24] was the first to highlight a subtle but important distinction between two kinds of refutative results that can be established constructively for statements of the form $\forall x \phi$.

About falsity. The first, *weak falsity*, is demonstrated by exhibiting a procedure that transforms any construction of the mathematical object into a construction of a contradiction. This is the notion of falsity traditionally adopted in constructive logics, as witnessed by the fact that $\neg\phi$ is often defined as $\phi \to \bot$. In contrast, *strong falsity* (represented by Nelson's strong negation $\sim\phi$) is proven by explicitly constructing witnesses to the falsity of the statement itself.

For example, let $\varphi =$ "there exists an odd perfect number" and define $\phi(0) = \varphi, \phi(1) = \neg\varphi$. To establish the weak falsity of a statement such as $\forall n \in \{0,1\}\phi(n)$, one assumes it and derives a contradiction. On the other hand, to establish its strong falsity, one must instead *construct* one object n for which $\phi(n)$ does not hold. The significance of this difference resides in the fact that, while from the existence of a counterexample to a claim we can trivially obtain a procedure reducing purported constructions of its object to a proof of absurdity, it is not possible in general to obtain a particular counterexample to a claim from a procedure reducing it to a contradiction. This is exactly why $\neg\forall x\phi(x)$ does not imply $\exists x\neg\phi(x)$ in, *e.g.*, intuitionistic or minimal logic.

Nelson argues in [24] that this novel conception of strong negation naturally entails a revision of the principles of intuitionistic logic, thereby explaining why these systems are often regarded as *alternatives* to intuitionism rather than mere extensions of it. In fact, although both weak and strong falsity may reasonably be regarded as constructive, the contents of a proof of strong negation are substantially stronger.

This also leads to a surprising (constructive) recovery of some classical equalities. In Nelson's original definitions, from a proof of $\sim\forall x\phi(x)$ it follows that the existence of a particular a for which $\sim\phi(a)$ holds, hence also $\exists x \sim\phi(x)$. Conversely, since a proof of $\exists x \sim\phi(x)$ requires production of some object a for which $\sim\phi(a)$ holds, if such a proof is available a itself is an effective counterexample to the claim that $\forall x\phi(x)$, hence it follows that $\sim\forall x\phi(x)$. This means that, just as in classical logic, $\exists x\sim\phi(x)$ is equivalent to $\sim\forall x\phi(x)$, but *for entirely different reasons*! In classical logic the equivalence arises because the conditions for proving an existential statement are *weaker*, whereas in Nelson's logic it arises because the conditions for proving a negation are *stronger*.

Falsity and bilateralism. Constructively proving a formula ϕ requires providing a *constructive proof* of ϕ. Dually, proving the negation $\sim\phi$ requires a *constructive disproof* – a constructive method that demonstrates that ϕ cannot hold. Hence Nelson's constructive falsity can be interpreted as putting the contents of positive and negative statements on equal footing. This is essentially what is also done in *logical bilateralism*, an approach to logic in which positive and negative statements are not taken as mutually interdefinable. Bilateralism was originally presented by Rumfitt [30] to provide a proof-theoretic justification of classical logic, but it was later repurposed for intuitionism and its falsificationalist dual [36] through Wansing's logic 2Int [41]. In short, 2Int is a bilateral bi-intuitionistic logic [29, 12], combining the verificationist account of constructivism expressed through intuitionistic logic and its falsificationalist dual-intuitionistic counterpart in a single, unified framework.

Stronger strong falsity. One can associate the *disprovability* of ϕ with the constructive strong negation $\sim\phi$: ϕ is disprovable exactly when $\sim\phi$ is provable, and $\sim\phi$ is disprovable exactly when ϕ is provable. The main formalisms of this idea, proposed by Nelson, are the logical systems N3 and N4 [24, 27]. The system N3, essentially intuitionistic logic extended with strong negation, was designed to provide a formal account of falsity in mathematics. By contrast, N4 is obtained from N3 by removing the *explosion principle* $\phi \to (\sim\phi \to \psi)$, which allows arbitrary conclusions to be derived from a contradiction. As a result, N4 is *paraconsistent*, in the sense that contradictions do not automatically trivialise the system.

Semantically, this distinction has a clear consequence: N4 allows for *simultaneous* proofs and refutations of the same formulas, whereas N3 does not [41, p. 447][1]. This contrast suggests that N4 and N3 capture *distinct notions* of strong falsity. For instance, if one aims to

[1] The model theory of both N3 and N4 have also been extensively studied (see, *e.g.*, [14, 15]). A concise overview of Nelson's logics is provided in [23], which also introduces a semantics for Nelson's lesser-known but conceptually rich system S, presented as a possible alternative to both N3 and intuitionistic logic [25].

model the set of *beliefs* of an agent, it is reasonable to allow both truth and falsity to coexist without collapse, since agents may hold inconsistent beliefs without accepting everything. Wansing's 2Int captures well this N4 perspective. By contrast, when modelling falsity in *mathematics*, such coexistence is problematic: a proof of a statement cannot coherently stand alongside its refutation. The present work is devoted to this N3 view of strong falsity[2].

Contributions and challenges. We investigate the proof-theoretic structure of falsity by introducing BPR, a natural deduction system for a logic of proofs and refutations (Section 2). Unlike existing proof systems for N3, BPR is *pure*, in the sense that each rule operates on a single connective. This design yields key proof-theoretic properties, including the subformula property. We also relate BPR to Nelson's logic, highlighting both the conceptual and technical consequences of our treatment of refutation.

We then show that BPR smoothly extends Wansing's natural deduction proof system N2Int for 2Int [41]. However, unlike N2Int, a *direct proof* of (weak) normalisation for BPR is presented, and *consistency*[3] is shown for both proofs and refutations (Section 3).

The semantics for BPR (Section 4), formulated via base extension semantics (BeS) [31, 35, 1, 5, 43], makes explicit the idea of counterexample construction through atomic refutational rules, ensuring that no formula is simultaneously provable and refutable within the same atomic base. Furthermore, it retains all properties established in [2, 1], including the possibility of semantic embeddings via bilateral semantic harmony. Soundness and completeness results are shown, guaranteeing the correctness of the framework (Section 5).

We end this introduction by noting that there are extensive and well-developed discussions in the literature on proof systems for Nelson's strong negation (see, *e.g.*, [13, 22, 21, 16]) and bilateralism (see, *e.g.*, [20, 8, 11, 9, 38, 37]) – we highlight some in Section 2.1. This paper aims at offering a new perspective that bridges these two strands.

2 Bilateral Proof and Refutation (BPR)

As noted in [42], inferential purity does not hold in some other natural deduction calculi for Nelson's logics with strong constructive negation [13, 21, 16], in the sense that they display two connectives at a time in definitions (they also fail the subformula property), *e.g.*:

$$\frac{\phi \quad {\sim}\psi}{{\sim}(\phi \to \psi)} \qquad\qquad \frac{{\sim}(\phi \land \psi) \quad \begin{array}{c}[{\sim}\phi]\\ \gamma\end{array} \quad \begin{array}{c}[{\sim}\psi]\\ \gamma\end{array}}{\gamma}$$

Wansing's solution consisted of replacing the truth-preserving transitions to/from negated formulas by falsity preserving transitions to/from non-negated formulas, so the above rules are replaced by:

$$\dfrac{\overline{\phi} \quad \overline{\overline{\psi}}}{\overline{\overline{\phi \to \psi}}} \qquad\qquad \frac{\overline{\overline{\phi \land \psi}} \quad \begin{array}{c}[\![\phi]\!]\\ \gamma\end{array} \quad \begin{array}{c}[\![\psi]\!]\\ \gamma\end{array}}{\gamma}$$

where the strong negation operator is substituted by double inference/discharge bars/brackets. The natural deduction system N2Int [42] (Fig 1a) formalises a conservative extension of propositional intuitionistic logic over a set of atomic variables At, with connectives $\to$,

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[2] For additional sources containing in-depth defences of N3 and strong negation from a conceptual perspective, the reader is referred to [17, 19].

[3] [40, 32] study systems for nonmonotonic inference based on expansions of Nelson's logics $\mathsf{N4}^{\perp}$, $\mathsf{N3}^{\perp}$ [24, 39, 27] with Gabbay's consistency operator M [10]. The connection with our approach is marginal since, in our system, consistency is guaranteed purely within the proof system, without adding extra operators.

$\wedge, \vee, \top, \bot$, obtained by adding co-implication ($\mapsfrom$), a connective dual to implication. In N2Int, co-implication internalizes the preservation of *disproof* from the conclusion of a valid inference (understood as logical consequence) to its premises. Hence the implication plays in *verificationism* a role dual to that played by co-implication in *falsificationism*.

The system N2Int* [2] is a variant of N2Int. The key difference lies in the addition of the rules $E \vee (+)$, $E \wedge (-)$, $\bot(+)$, and $\top(-)$ (Fig. 1b), without which a direct proof of normalisation is not possible. The bilateral natural deduction system BPR (Fig. 1) then builds on N2Int* by introducing the rule PR (Fig. 1c), which coordinates *proofs and refutations*. This single addition allows one to move from a paraconsistent to a consistent system, but it also introduces significant proof-theoretic challenges, as discussed in the following sections.

BPR features two kinds of derivations: those concluding with a formula under a single line (proofs) and those concluding under a double line (refutations). A proof (resp. a refutation) may contain refutations (resp. proofs) as subderivations. Dotted lines serve as placeholders for either single or double lines, with the proviso that in any application of $\vee(+)$ or $\wedge(-)$, all dotted lines must be replaced uniformly, either all by single lines or all by double lines.

The premises of a derivation in BPR are partitioned into two classes: assumptions, representing proofs, and contra-assumptions, representing refutations. Accordingly, the conclusion of a derivation is indexed by an ordered pair $(\Gamma; \Delta)$, where Γ and Δ are finite sets of assumptions and contra-assumptions, respectively. Intuitively, Γ collects hypotheses taken to be proved, whereas Δ collects those taken to be refuted. Finally, single (resp. double) square brackets $[\cdot]$ (resp. $[\![\cdot]\!]$) indicate the discharge of assumptions (resp. contra-assumptions).

We denote the existence of deductions through indexed turnstiles: $\Gamma; \Delta \vdash^{+} \phi$ holds if and only if there is a deduction in BPR which has conclusion ϕ, depends on the premises $(\Gamma'; \Delta')$ for $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ and ends with an application of a proof rule, and $\Gamma; \Delta \vdash^{-} \phi$ holds if and only if there is such a deduction ending with an application of a refutation rule instead.

At this point, one may reasonably ask: how do these systems capture Nelson's strong negation, if the operator $\sim$ is not part of the grammar? The next result, presented in [26], shows that extending N2Int (and hence BPR) with strong negation does not increase its expressive power.

▶ **Proposition 1.** *Strong negation* $\sim\phi$ *is definable in* N2Int/BPR *as* $(\phi \wedge (\phi \rightarrow (\phi \mapsfrom \phi))) \vee ((\phi \rightarrow \phi) \mapsfrom \phi)$*. Moreover, the following rules are derivable in* N2Int/BPR*:*

$$\begin{array}{c} \Gamma;\Delta \\ \Pi \\ \hline\hline \phi \\ \hline \sim\phi \end{array}\ {\scriptstyle I\sim(+)} \qquad \begin{array}{c} \Gamma;\Delta \\ \Pi \\ \hline \sim\phi \\ \hline\hline \phi \end{array}\ {\scriptstyle E\sim(+)} \qquad \begin{array}{c} \Gamma;\Delta \\ \Pi \\ \hline \phi \\ \hline\hline \sim\phi \end{array}\ {\scriptstyle I\sim(-)} \qquad \begin{array}{c} \Gamma;\Delta \\ \Pi \\ \hline\hline \sim\phi \\ \hline \phi \end{array}\ {\scriptstyle E\sim(-)}$$

This is particularly interesting for a number of reasons. First, proofs of strong negation are equivalent to refutations and refutations of strong negation are equivalent to proofs, in the sense that from the proof in [26] plus soundness and completeness of 2Int it follows that $(\Vdash^{+}_{\mathcal{B}} {\sim}\phi$ iff $\Vdash^{-}_{\mathcal{B}} \phi)$ and $(\Vdash^{-}_{\mathcal{B}} {\sim}\phi$ iff $\Vdash^{+}_{\mathcal{B}} \phi)$. This allows Nelson's negation to function as a sort of *structural* bilateral operator which allows direct passage from proofs to refutations.

In fact, the use of a framework in which we have both a weak falsity operator $\neg\phi$ and a concept of refutation equivalent to the strong falsity operator ${\sim}\phi$ also allows the study of their interactions in more detail. In BPR we can easily conclude that from a refutation of ϕ follows a constructive proof of the negation $\phi \rightarrow \bot$, as well as that from a proof of ϕ follows a refutation of the co-negation $\top \mapsfrom \phi$:

$$\dfrac{\dfrac{\overline{\phi}^{\,1} \qquad \overline{\overline{\phi}}}{\bot}\,\mathrm{PR}(+)}{\phi \rightarrow \bot}\,1 \qquad\qquad \begin{array}{c} \overline{\phi} \qquad \overline{\overline{\phi}}^{\,1} \\ \hline\hline \top \\ \hline\hline \top \mapsfrom \phi \end{array}\ \begin{array}{l} \mathrm{PR(-)} \\ \\ 1 \end{array}$$

$$\begin{array}{c}\begin{array}{c}\Gamma,[\phi];\Delta\\ \Pi\\ \hline \psi\end{array}\\ \hline \phi\to\psi\end{array}\ I\to(+)\qquad \begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline \phi\to\psi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2\\ \Pi_2\\ \hline \phi\end{array}\end{array}\\ \hline \psi\end{array}\ E\to(+)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline \phi\end{array}\\ \hline \phi\vee\psi\end{array}\ I_1\vee(+)$$

$$\begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline \psi\end{array}\\ \hline \phi\vee\psi\end{array}\ I_2\vee(+)\qquad \begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline \phi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2\\ \Pi_2\\ \hline \psi\end{array}\end{array}\\ \hline \phi\wedge\psi\end{array}\ I\wedge(+)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline \phi\wedge\psi\end{array}\\ \hline \phi\end{array}\ E_1\wedge(+)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline \phi\wedge\psi\end{array}\\ \hline \psi\end{array}\ E_2\wedge(+)$$

$$\begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline \phi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2\\ \Pi_2\\ \hline\hline \psi\end{array}\end{array}\\ \hline \phi\mapsfrom\psi\end{array}\ I\mapsfrom(+)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline \phi\mapsfrom\psi\end{array}\\ \hline \phi\end{array}\ E_1\mapsfrom(+)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline \phi\mapsfrom\psi\end{array}\\ \hline\hline \psi\end{array}\ E_2\mapsfrom(+)$$

$$\begin{array}{c}\begin{array}{c}\Gamma;\Delta,[\![\psi]\!]\\ \Pi\\ \hline\hline \phi\end{array}\\ \hline\hline \phi\mapsfrom\psi\end{array}\ I\mapsfrom(-)\qquad \begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline\hline \phi\mapsfrom\psi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2\\ \Pi_2\\ \hline\hline \psi\end{array}\end{array}\\ \hline\hline \phi\end{array}\ E\mapsfrom(-)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline\hline \phi\end{array}\\ \hline\hline \phi\wedge\psi\end{array}\ I_1\wedge(-)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline\hline \psi\end{array}\\ \hline\hline \phi\wedge\psi\end{array}\ I_2\wedge(-)$$

$$\begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline\hline \phi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2\\ \Pi_2\\ \hline\hline \psi\end{array}\end{array}\\ \hline\hline \phi\vee\psi\end{array}\ I\vee(-)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline\hline \phi\vee\psi\end{array}\\ \hline\hline \phi\end{array}\ E_1\vee(-)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline\hline \phi\vee\psi\end{array}\\ \hline\hline \psi\end{array}\ E_2\vee(-)$$

$$\begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline \phi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2\\ \Pi_2\\ \hline\hline \psi\end{array}\end{array}\\ \hline\hline \phi\to\psi\end{array}\ I\to(-)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline\hline \phi\to\psi\end{array}\\ \hline\hline \phi\end{array}\ E_1\to(-)\qquad \begin{array}{c}\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline\hline \phi\to\psi\end{array}\\ \hline\hline \psi\end{array}\ E_2\to(-)$$

(a) System N2Int.

$$\begin{array}{c}\begin{array}{ccc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline \phi\vee\psi\end{array} & \begin{array}{c}\Gamma_2,[\phi];\Delta_2\\ \Pi_2\\ \hdashline \chi\end{array} & \begin{array}{c}\Gamma_3,[\psi];\Delta_3\\ \Pi_3\\ \hdashline \chi\end{array}\end{array}\\ \hdashline \chi\end{array}\ E\vee(+)\qquad \begin{array}{c}\begin{array}{ccc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline\hline \phi\wedge\psi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2,[\![\phi]\!]\\ \Pi_2\\ \hdashline \chi\end{array} & \begin{array}{c}\Gamma_3;\Delta_3,[\![\psi]\!]\\ \Pi_3\\ \hdashline \chi\end{array}\end{array}\\ \hdashline \chi\end{array}\ E\wedge(-)$$

$$\begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline \bot\\ \hdashline \phi\end{array}\ \bot(+)\qquad \begin{array}{c}\Gamma;\Delta\\ \Pi\\ \hline\hline \top\\ \hdashline \phi\end{array}\ \top(-)$$

(b) Rules added to N2Int in order to obtain the System N2Int*.

$$\begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1;\Delta_1\\ \Pi_1\\ \hline \phi\end{array} & \begin{array}{c}\Gamma_2;\Delta_2\\ \Pi_2\\ \hline\hline \phi\end{array}\end{array}\\ \hdashline \psi\end{array}\ PR$$

(c) Proofs and refutations rule.

Figure 1 System BPR.

It follows that there is a (constructive) proof of $\phi \to \bot$ iff there is no proof of ϕ and a (constructive) refutation of $\top \leftarrow \phi$ iff there is no refutation of ϕ. Hence those deductions are essentially showing that *strong falsity* implies *weak falsity* and *strong provability* implies *weak provability* (that is, absence of refutations). Note that these inferences are not valid in N2Int – this is a feature specific to BPR. At the end of Section 5 we prove that the converse does not hold; that is, weak falsity does not imply strong falsity in BPR.

2.1 On BPR and related logics

We finish this section by discussing some parallels between our system, BPR, and other logics in the literature.

Regarding Nelson's N3, we point out a crucial difference, insofar as Nelson's strong negation $\sim$ is involutive by definition: a P-realizer of $\sim \phi$ is exactly an N-realizer of ϕ, and conversely. This collapses proofs of $\sim \phi$ into refutations of ϕ and refutations of $\sim \phi$ into proofs of ϕ. Such duality is essential to N3, but it does not reflect the epistemic notion of constructive falsity we aim to model with BPR. In mathematics one does not "refute a refutation" as refutation is an act of producing a counterexample, not an operator closed under involution. BPR captures this asymmetry directly: proofs and refutations are primitive epistemic acts, governed by coordination rules that enforce incompatibility without relying on an involutive negation.

There are also important comparisons to be made between BPR and other bilateral logics. For instance, BPR can be obtained from the logic $\mathbf{HB_2}$ [6] by adding rules for the units $\top$ and $\bot$ and for co-implication, removing the classical co-ordination rule and using the traditional rules for assertion of disjunction and refutation of conjunction instead of the modified ones[4]. Since $\mathbf{HB_2}$ is equivalent to Rumfitt's calculus [pg. 384][6], BPR can also be interpreted as a intuitionistic version of Rumfitt's logic. $\mathbf{HB_2}$ is itself a variant of the logic $\mathbf{HB_1}$ presented in [18], which is also equivalent to Rumfitt's logic. Those three logics have in common, as do the slightly less similar $C - intelim$ logic in [4], the fact that they aim to provide proof-theoretic characterizations of *classical logic*, a goal we do not share. They also define $\bot$ as a punctuation mark instead of a formula, which prevents them from being able to draw comparisons between weak and strong negation. The difference in the formulation of the rules and the fact that BPR is closer to 2Int and N3 is mostly explained by technical implications of this change in goal.

3 Normalisation and the subformula principle

In this section we prove the normalisation theorem for BPR. The aim is to show that if there exists a proof (respectively, a refutation) of a formula ϕ from a finite set of assumptions Θ and contra-assumptions Δ (that is, a set of premises $\Gamma = \Delta \cup \Theta$), then there exists a derivation of the same kind in *normal form*.

The proof of the normalisation theorem yields a systematic procedure for eliminating detours, that is, redundant segments of derivations that do not contribute to their essential logical structure. In standard natural deduction, such detours arise from consecutive introduction and elimination steps for the same connective. In our setting, however, they also encompass *non-trivial* proof-refutation interactions and other mixed rule combinations, revealing a richer and more intricate notion of redundancy.

[4] The modified rules define such connectives in terms of the *disjunctive syllogism*, essentially defining $\phi \vee \psi$ as $\neg\phi \to \psi$. Since $\phi \vee \phi$ is not equivalent to $\neg\phi \to \psi$ in intuitionistic logic, this modification could not be sensibly used in BPR.

Once normalisation is established, we prove that the subformula property holds: every formula occurring in a normal derivation is a subformula of either an undischarged assumption or the final conclusion. This result is not evident, as it usually does not follow directly from the normalisation result. Indeed, consider the following derivation of $\top$ from $\bot;\emptyset$ in N2Int:

$$\dfrac{\dfrac{\overline{\bot}}{\top \leftarrowtail \top}\ \bot(+)}{\top}\ E_2 \leftarrowtail (+)$$

Since $\top \leftarrowtail \top$ is neither a subformula of any open premise nor of the conclusion, this derivation violates the subformula property, even though a different form of normalisation still holds for N2Int[5].

Finally, as a further consequence, we obtain consistency: $\bot$ is not provable and $\top$ is not refutable in BPR.

▶ **Definition 2.** *The* degree of a formula ϕ*, denoted* $d[\phi]$*, is defined inductively by*

$$\begin{array}{lcll} d[p] & = & 0 & \textit{for } p \in \mathsf{At} \\ d[\phi \circ \psi] & = & d[\phi] + d[\psi] + 1 & \textit{for } \circ \in \{\to, \leftarrowtail, \vee, \wedge\} \end{array}$$

▶ **Definition 3.** *Let Π be a derivation in* BPR*, d be the greatest degree of a formula that occurs as premise or conclusion of an instance of PR, $\bot(+)$ or $\top(-)$ in Π, n the number of occurrences of formulas with degree d as premises of applications of PR, $\bot(+)$ or $\top(-)$ in Π, and m the number of occurrences of formulas with degree d as conclusions of those same rules. The* complexity value *of Π is defined as the pair $\langle d, n+m\rangle$. We say that $\langle d', n'+m'\rangle < \langle d, n+m\rangle$ holds if and only if either $d' < d$ or both $d' = d$ and $n'+m' < n+m$.*

Notice that, by defining the second component of the complexity value as a sum, formula occurrences that are simultaneously a premise and a conclusion of the relevant rules are counted twice – this is essential for proving Lemma 4.

The following lemmas play a crucial role in the normalisation results based on Prawitz's techniques. They establish that certain rule applications can be restricted to the atomic case and that consecutive applications of them can be deleted.

▶ **Lemma 4.** *Any deduction Π with conclusion ϕ and premises Γ can be reduced to a deduction Π' with conclusion ϕ and premises Γ such that all instances of PR, $\bot(+)$ and $\top(-)$ have premises and conclusions in* At*.*

Proof. We prove the result by induction on the complexity value $\langle d, n\rangle$ of Π, showing that, if it is not the case that $d = n = 0$, then Π reduces to a deduction Π' which has complexity value $\langle d', n'\rangle$ for $\langle d', n'\rangle < \langle d, n\rangle$ and the same premises and conclusion. If $d = n = 0$ then all instances of PR, $\bot(+)$ and $\top(-)$ have atomic premises and conclusion.

First take a formula occurrence ϕ in Π which is a premise or conclusion with degree d of an instance of PR, $\bot(+)$ or $\top(-)$, satisfying the requirement that no instance of PR, $\bot(+)$ or $\top(-)$ above the formula occurrence has premises or conclusions with degree d (if ϕ is the conclusion of PR this includes the rule application of which the formula is a conclusion). If the formula is a premise of PR we also require that no such instance occurs above the

[5] In fact, normalisation for N2Int is established indirectly, by embedding its derivations into natural deduction for intuitionistic logic and then appealing to Prawitz's normalisation theorem [28]. On the other hand, in [1] cut-elimination and the subformula property are proved directly for the *sequent system* SC2Int.

formula occurrence side connected with it. We show the reductions for when the application of PR contains a single line, since the cases where it contains a double line instead are always analogous. The reductions are defined based on the shape of ϕ.

For the case $\phi = \psi \land \chi$, we rewrite the deduction as follows:

$$\begin{array}{c}\dfrac{\dfrac{\Sigma}{\psi\land\chi}\qquad\dfrac{\Sigma'}{\overline{\psi\land\chi}}}{\sigma}\,PR\\ \Sigma''\end{array}\quad\Longrightarrow\quad\begin{array}{c}\dfrac{\dfrac{\Sigma'}{\overline{\psi\land\chi}}\qquad \dfrac{\overline{\overline{\psi}}^{\,1}\qquad \dfrac{\dfrac{\Sigma}{\psi\land\chi}}{\psi}\,E_1\land(+)}{\sigma}\,PR \qquad \dfrac{\overline{\overline{\chi}}^{\,2}\qquad\dfrac{\dfrac{\Sigma}{\psi\land\chi}}{\chi}\,E_2\land(+)}{\sigma}\,PR}{\sigma}\,E_1\land(-),1,2\\ \Sigma''\end{array}$$

Notice that this reduction replaces the original application of PR by two others and creates a copy of σ, but the inductive measure is set up so that the complexity still decreases. Notice also that, if the formula occurrence σ was also the premise of an application of PR, then it has to be counted twice in the inductive measure, as otherwise the new deduction would have the same inductive value as the original.

Since all applications of PR, $\bot(+)$ and $\top(-)$ in Σ above the original application of PR which are duplicated by this procedure have premises with degree less than d and the two new applications of PR have premises with degree lesser than d, either the occurrences of $\psi \land \chi$ on Π were the last ones appearing on Π as premises or conclusions of PR, $\bot(+)$ or $\top(-)$ which have degree d, in which case $d' < d$, or there are other such applications, in which case $d' = d$ but $n' < n$, thus we have $\langle d', n'\rangle < \langle d, n\rangle$.

The case where ϕ is $\psi \lor \chi$ is analogous to $\psi \land \chi$. The case $\phi = \psi \to \chi$ can be found in [18] and is analogous to the case of co-implication.

If the formula occurrence is a conclusion of PR, $\bot(+)$ or $\top(-)$, the reductions to be applied are uniform. Below is the case for $\phi \land \psi$ being concluded from single line PR:

$$\begin{array}{c}\dfrac{\dfrac{\Sigma}{\chi}\qquad\dfrac{\Sigma'}{\overline{\chi}}}{\phi\land\psi}\,PR\\ \Sigma''\end{array}\quad\Longrightarrow\quad\begin{array}{c}\dfrac{\dfrac{\dfrac{\Sigma}{\chi}\qquad\dfrac{\Sigma'}{\overline{\chi}}}{\phi}\,PR\qquad\dfrac{\dfrac{\Sigma}{\chi}\qquad\dfrac{\Sigma'}{\overline{\chi}}}{\psi}\,PR}{\phi\land\psi}\,I\land(+)\\ \Sigma''\end{array}$$

The restrictions on the formula occurrence we pick guarantee that no formula occurrence which is a premise or conclusion of one of the relevant rules occur in Σ or Σ', as well as that χ has degree lesser than that of the selected formula, so in any case either there are no other relevant formula occurrences with the same degree in the deduction, so it reduces to a Π' with $d < d'$, or there are other occurrences and it reduces to a Π' with $d = d'$ but $n' < n$, so in any case we obtain a deduction with complexity degree $\langle d', n'\rangle$ such that $\langle d', n'\rangle < \langle d, n\rangle$. The cases for $\bot(+)$ are analogous, and the cases for $(\phi \land \psi)$, $(\phi \lor \psi)$, $(\phi \to \psi)$ and $(\phi \leftarrow \psi)$ occurring as a consequence of $\top(-)$ are respectively analogous to those of $(\phi \lor \psi)$, $(\phi \land \psi)$, $(\phi \leftarrow \psi)$ and $(\phi \to \psi)$. ◀

▶ **Lemma 5.** *Any deduction showing $\Gamma; \Delta \vdash^* \phi$ can be reduced to one showing $\Gamma'; \Delta' \vdash^* \phi$ for $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that no formula occurrence is both the conclusion of an application of $\bot(+)$, $\top(-)$ or PR and the premise of an application of $\bot(+)$, $\top(-)$ or PR.*

Proof. We prove the result by induction of the number of formula occurrences which are at the same time conclusion of an application of PR, $\bot(+)$ or $\top(-)$. The argument is straightforward: we delete certain rule applications and, where needed, replace applications of PR using single lines with applications using double lines, and conversely. Some cases in which the first rule applied is PR are given below. The remaining cases are analogous.

$$\dfrac{\dfrac{\dfrac{\Sigma}{\phi} \quad \dfrac{\Sigma'}{\phi}}{\psi}\,PR \qquad \dfrac{\Sigma''}{\psi}}{\dfrac{\chi}{\Sigma'''}}\,PR \quad \Longrightarrow \quad \dfrac{\dfrac{\Sigma}{\phi} \quad \dfrac{\Sigma'}{\phi}}{\dfrac{\chi}{\Sigma'''}}\,PR \qquad\qquad \dfrac{\dfrac{\Sigma}{\phi} \quad \dfrac{\Sigma'}{\phi}}{\dfrac{\dfrac{\bot}{\chi}\,\bot(+)}{\Sigma'''}}\,PR \quad \Longrightarrow \quad \dfrac{\dfrac{\Sigma}{\phi} \quad \dfrac{\Sigma'}{\phi}}{\dfrac{\chi}{\Sigma'''}}\,PR$$

The number of formula occurrences satisfying the original conditions always decreases by 1, which establishes the desired results. ◀

An interesting feature of the proof of Lemma 4 is that it essentially relies on the duality (or harmony) of proof and refutation rules for the logical operators.

A proof-theoretic analysis of this bilateral concept of harmony, which could in principle be used to extend the result to other intuitionistic bilateral logics[6], is carried out in [8]. A semantic counterpart of bilateral harmony can also be found in the weak and strong semantic harmony results proved in [2].

Our normalisation proof adapts Prawitz's strategy for intuitionistic logic [28] to the bilateralist setting. Some modifications are required, however, since the PR rule introduces new complications that may threaten the subformula property. To illustrate this, consider the following deduction:

$$\dfrac{\dfrac{\overline{\phi} \qquad \overline{\psi}}{\phi \wedge \psi}\,\wedge I(+) \qquad \dfrac{\overline{\overline{\phi}}}{\phi \wedge \psi}\,\wedge I(-)}{\chi}\,PR$$

Since $\phi \wedge \psi$ is a subformula of neither the premises nor the conclusion of this deduction, establishing the subformula property requires reductions specific to the bilateralist setting. Such reductions must show either that every deduction can be transformed into one in which no premise of a PR application is the conclusion of an introduction rule, or – following the approach we adopt – that all such premises can be reduced to atomic form.

The main difference in our normalisation proof is that we first apply reductions so as to make the premises and conclusions of any application of $\bot(+)$, $\top(-)$, PR *atomic* in order to obtain deductions in what we call atomic normal form, then reducing the deductions to normal form and simplifying them further.

The following definitions are standard [28].

▶ **Definition 6.** *A* maximal segment *in a deduction* Π *is a sequence* $\phi_1, \ldots, \phi_n$ *with at least one element such that* ϕ_1 *is the conclusion of a* I*-rule,* ϕ_n *is the major premise of an* E*-rule and, for all* $1 \leq i < n$ *(if any),* ϕ_i *is the minor premise of* $\vee(+)$ *or* $\wedge(-)$*.*
A maximal segment α *is said to be* above a formula occurrence β in a deduction *iff all formula occurrences in* α *are above the formula occurrence* β*.*

The degree *of a maximal segment is the degree of the formula* ϕ_1*. The* length *of a maximal segment is the number of formula occurrences in it.*

▶ **Definition 7.** *A deduction is in* atomic form *if all conclusions and premises of applications of* $\bot(+)$*,* $\top(-)$ *and* PR *have degree* 0*.*

▶ **Definition 8.** *The* inductive value *of a deduction* Π *is a pair* $\langle d, s\rangle$*, where* m *is the highest degree of a maximal segment in* Π *and* s *is the sum of the length of all maximal segments with degree* d*. We define* $\langle d, s\rangle < \langle d', s'\rangle$ *as holding whenever* $d < d'$ *or* $d = d'$ *and* $s < s'$*.*
A deduction is in normal form *if it is in atomic form with inductive value* $\langle 0, 0\rangle$*.*

[6] Those rules do not always preserve classical principles, as shown in [7].

A deduction is in simplified normal form *if it is in normal form and no formula occurrence is both the premise of an application of PR, $\bot(+)$ and $\top(-)$ and a conclusion of a rule with one of those shapes.*

We are now ready to establish normalisation for BPR. Our strategy proceeds in three stages. First, we transform the initial deduction into atomic form using the crucial Lemma 4. Next, we obtain a normal deduction by adapting Prawitz's normalisation strategy to the bilateral setting – we illustrate some permutative reductions below.

$$\dfrac{\dfrac{\dfrac{\Sigma}{\phi\vee\psi} \quad \begin{matrix}[\phi]\\ \Sigma'\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \Sigma''\\ \chi\end{matrix}}{\chi}\,{\scriptstyle E\vee(+)} \quad \Xi}{\sigma} \Longrightarrow \dfrac{\dfrac{\Pi_1}{\phi\vee\psi} \quad \dfrac{\begin{matrix}[\phi]\\ \Sigma'\\ \chi\end{matrix} \quad \Xi}{\sigma} \quad \dfrac{\begin{matrix}[\psi]\\ \Sigma''\\ \chi\end{matrix}\quad \Xi}{\sigma}}{\sigma}\,{\scriptstyle E\vee(+)}$$

$$\dfrac{\dfrac{\dfrac{\Sigma}{\phi\vee\psi} \quad \begin{matrix}[\phi]\\ \Sigma'\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \Sigma''\\ \chi\end{matrix}}{\chi}\,{\scriptstyle E\vee(+)} \quad \Xi}{\sigma} \Longrightarrow \dfrac{\dfrac{\Pi_1}{\phi\vee\psi} \quad \dfrac{\begin{matrix}[\phi]\\ \Sigma'\\ \chi\end{matrix} \quad \Xi}{\sigma} \quad \dfrac{\begin{matrix}[\psi]\\ \Sigma''\\ \chi\end{matrix}\quad \Xi}{\sigma}}{\sigma}\,{\scriptstyle E\vee(+)}$$

The remaining proof and refutation proper reductions are standard. Finally, we simplify the resulting deduction by means of Lemma 5. These simplifications are *non trivial* and required in order to secure the subformula property. And this order is *essential*: while normalisation procedures preserve atomic form and simplifications preserve normal form, reductions to atomic form may generate new maximal segments, and normalisation may introduce new formula occurrences that must subsequently be eliminated by simplification.

▶ **Theorem 9.** *Every deduction Π showing $\Gamma;\Delta\vdash^*\phi$ can be reduced to a deduction showing $\Gamma;\Delta\vdash^*\phi$ which is in simplified normal form.*

The subformula property follows immediately from our normalisation proof.

▶ **Corollary 10.** BPR *satisfies the subformula property. Since* BPR *is a conservative extension of* N2Int**, the latter system also satisfies the subformula property.*

The proof of the following result is an adaptation of the one in [3] on atomic normalisation.

▶ **Lemma 11.** *If Π is a normal derivation in* BPR *that does not end in an application of an I-rule, then Π contains at least one undischarged hypothesis.*

Remarkably, we can provide a syntactic proof of consistency for BPR. This result plays a key role in the semantics introduced in the next section, which relies in particular on consistency proofs to establish completeness.

▶ **Corollary 12.** BPR *is consistent.*

Proof. Assume there is a derivation Π showing $\vdash^+\bot$. Since no I-rule concludes $\bot$, Π cannot end with an I-rule. By Lemma 11, and must then contain an undischarged hypothesis, contradicting the fact that it is a derivation of $\vdash^+\bot$. The case of $\vdash^-\top$ is analogous. ◀

4 Base-extension semantics

In this section we provide a base-extension semantics (BeS) for BPR. This choice of semantics is motivated by the fact that BeS enables non-trivial extensions of the results in [2] by combining techniques from [3] with new concepts. These must be integrated with our proof-theoretical results and have no direct counterpart in standard model-theoretic semantics. Moreover, given that this is a paper in proof theory, we find it natural and conceptually appealing to adopt a semantic framework where proof theory is the main star.

Proof-theoretic semantics [33, 34, 35] (PtS) provides an alternative perspective for the meaning of logical operators compared to the viewpoint offered by model-theoretic semantics. In PtS, the concept of *truth* is substituted for that of *proof*, emphasizing the fundamental nature of proofs as a means through which we gain demonstrative knowledge, particularly in mathematical contexts. This makes PtS a superior approach for comprehending reasoning since it ensures that the meaning of logical operators, such as connectives in logics, is defined based on their usage in inferences.

BeS [31] is a strand of PtS where proof-theoretic validity is defined relative to a given collection $\mathcal{B}$ of inference rules defined over basic formulas of the language. Indeed, validity w.r.t. a set $\mathcal{B}$ of atomic rules has the general shape

$$\Vdash_{\mathcal{B}} p \qquad \text{iff} \qquad \vdash_{\mathcal{B}} p$$

where $\vdash_{\mathcal{B}} p$ indicates that p is *derivable* in the proof system determined by $\mathcal{B}$. After defining validity for atoms one can also define validity for logical connectives via semantic clauses that express proof conditions (*e.g.*, $A \wedge B$ is provable from $\mathcal{B}$ if and only if both A and B are provable in $\mathcal{B}$), which results in a framework that evaluates propositions exclusively in terms of proofs of its constituents.

4.1 Basic definitions

We adopt Sandqvist's [31] terminology with the adaptations to the bilateral setting presented in [2]. Our definitions use systems containing natural deduction rules over basic sentences for the semantical analysis, and we allow inference rules to discharge sets of basic hypotheses. On the other hand, we allow the logical constants $\bot$ and $\top$ to be manipulated by the rules as if they were atoms.

The propositional *base language*, denoted by At – abusing the concept of atom – is assumed to have a set $\{p_1, p_2, \ldots\}$ of countably many atomic propositions together with $\bot$ and $\top$. The elements of At are called *basic sentences*, or *atoms*.

▶ **Definition 13.** *A* bilateral atomic system *(a.k.a.* bilateral base*, or* base*)* $\mathcal{B}$ *is a (possibly empty) set of atomic rules of the form*

$$\dfrac{\dfrac{\begin{array}{c}[\Gamma^1_{\mathsf{At}}];[\![\Delta^1_{\mathsf{At}}]\!]\\ \vdots\end{array}}{p_1} \quad \ldots \quad \dfrac{\begin{array}{c}[\Gamma^n_{\mathsf{At}}];[\![\Delta^n_{\mathsf{At}}]\!]\\ \vdots\end{array}}{\overline{p_n}}}{p} \qquad \dfrac{\dfrac{\begin{array}{c}[\Gamma^1_{\mathsf{At}}];[\![\Delta^1_{\mathsf{At}}]\!]\\ \vdots\end{array}}{p_1} \quad \ldots \quad \dfrac{\begin{array}{c}[\Gamma^n_{\mathsf{At}}];[\![\Delta^n_{\mathsf{At}}]\!]\\ \vdots\end{array}}{\overline{p_n}}}{\overline{p}}$$

where $p_i, p \in \mathsf{At}$*,* Γ^i_{At} *is a (possibly empty) set of atomic proof assumptions and* Δ^i_{At} *is a (possibly empty) set of atomic contra-assumptions. The sequence* $\langle p^1, ..., p^n \rangle$ *of premises in a rule can be empty – in this case the rule is called an* atomic axiom*. The sets* $\Gamma^i_{\mathsf{At}}, \Delta^i_{\mathsf{At}}$ *can be omitted when empty. The atom* p *is the rule's* conclusion*.*

▶ **Definition 14.** *An atomic rule is called a* proof rule *if a single line appears above its conclusion, and a* refutation rule *if a double line appear above it instead. Atomic axioms may also be called* proof axioms *if they are proof rules and* refutation axioms *otherwise.*

▶ **Definition 15.** *An atomic system* $\mathcal{C}$ *is an* extension *of an atomic system* $\mathcal{B}$ *(written* $\mathcal{C} \supseteq \mathcal{B}$*) if* $\mathcal{C}$ *results from adding a (possibly empty) set of atomic rules to* $\mathcal{B}$*.*

▶ **Definition 16.** *An* atomic proof *in the base $\mathcal{B}$ is a deduction using only the rules $\mathcal{B}$ which has no undischarged assumptions or contra-assumptions and ends with an application of a proof rule. We denote by $\vdash^+_{\mathcal{B}} p$ the existence of an atomic proof of p in $\mathcal{B}$.*

An atomic refutation *in the base $\mathcal{B}$ is a deduction using only the rules $\mathcal{B}$ which has no undischarged assumptions or contra-assumptions and ends with an application of a refutation rule. We denote by $\vdash^+_{\mathcal{B}} p$ the existence of an atomic proof of p in $\mathcal{B}$.*

Now we define new notions to be used in the new setting.

▶ **Definition 17.** *An atomic system $\mathcal{B}$ is*

- logically consistent *if $\nvdash^+_{\mathcal{B}} \bot$ and $\nvdash^-_{\mathcal{B}} \top$.*
- unit complete *if it contains a proof axiom concluding $\top$ and a refutation axiom concluding $\bot$.*
- epistemically consistent *if it is closed under rules with the following shape for all $p \in \mathsf{At}$[7]:*

$$\frac{\overline{p} \qquad \overline{\overline{p}}}{\bot} \qquad\qquad \frac{\overline{p} \qquad \overline{\overline{p}}}{\overline{\overline{\top}}}$$

- epistemically adequate *if it is logically consistent, unit complete and epistemically consistent.*

In the present paper we assume that bases and their extensions are always epistemically adequate (unless otherwise stated).

4.2 Semantic clauses and epistemic consistency

The semantics considered here are based on the clauses introduced in [2], except that we extend the domain At so as to include both $\bot$ and $\top$.

▶ **Definition 18.** The relations $\Vdash^+_{\mathcal{B}}, \Vdash^-_{\mathcal{B}}, \Vdash^+$ *and* $\Vdash^-$ are defined in Fig 2. An asterisk $*$ may be used as a placeholder for $+$ and $-$ when a result is to be shown for both signs. Whenever an asterisk is used in a statement or proof, all its instances must be uniformly replaced by the same sign.

Unlike the semantics in [2], the epistemic adequacy requirement on bases, specifically, the combination of logical and epistemic consistency ensures that proofs and refutations cannot coexist. This is not the case in $\mathsf{2Int}$, where paraconsistency allows a formula to be both provable and refutable.

The proof of the next result proceeds along the same lines as Lemma 2 in [2], adapted to a framework in which $\top$ and $\bot$ are regarded as atomic formulas.

▶ **Lemma 19.** *If $\Gamma; \Delta \Vdash^*_{\mathcal{B}} \phi$ and $\mathcal{B} \supseteq \mathcal{C}$ then $\Gamma; \Delta \Vdash^*_{\mathcal{C}} \phi$.*

The next result establishes our central claim: BPR does not admit simultaneous proofs and refutations. Notice that, although a similar model-theoretic result for a different language is stated (but not proved) in [39], any proof would look significantly different from the one presented here.

▶ **Theorem 20.** *For any $\mathcal{B}$, $\Vdash^+_{\mathcal{B}} \phi$ implies $\nVdash^-_{\mathcal{B}} \phi$ and $\Vdash^-_{\mathcal{B}} \phi$ implies $\nVdash^+_{\mathcal{B}} \phi$.*

Proof. We prove both directions simultaneously by induction on the degree of ϕ. Only the atomic and conjunction cases are shown as the remaining cases are analogous.

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[7] Technically speaking, our theorem requiring consistency only need bases to be closed under instances of either of the rules pictured above, but we include both in order to give precedence neither to the concept of proof nor of refutation.

(At+) $\Vdash^+_{\mathcal{B}} p$ iff $\vdash^+_{\mathcal{B}} p$, for $p \in \mathsf{At}$;

(At−) $\Vdash^-_{\mathcal{B}} p$ iff $\vdash^-_{\mathcal{B}} p$, for $p \in \mathsf{At}$;

(∧+) $\Vdash^+_{\mathcal{B}} \phi \wedge \psi$ iff $\Vdash^+_{\mathcal{B}} \phi$ and $\Vdash^+_{\mathcal{B}} \psi$

(∧−) $\Vdash^-_{\mathcal{B}} \phi \wedge \psi$ iff $\forall \mathcal{C}(\mathcal{C} \supseteq \mathcal{B})$ and $\forall p(p \in \mathsf{At})$: $(\emptyset; \phi \Vdash^+_{\mathcal{C}} p$ and $\emptyset; \psi \Vdash^+_{\mathcal{C}} p$ implies $\Vdash^+_{\mathcal{C}} p)$ and $(\emptyset; \phi \Vdash^-_{\mathcal{C}} p$ and $\emptyset; \psi \Vdash^-_{\mathcal{C}} p$ implies $\Vdash^-_{\mathcal{C}} p)$;

(∨+) $\Vdash^+_{\mathcal{B}} \phi \vee \psi$ iff $\forall \mathcal{C}(\mathcal{C} \supseteq \mathcal{B})$ and $\forall p(p \in \mathsf{At})$: $(\phi; \emptyset \Vdash^+_{\mathcal{C}} p$ and $\psi; \emptyset \Vdash^+_{\mathcal{C}} p$ implies $\Vdash^+_{\mathcal{C}} p)$ and $(\phi; \emptyset \Vdash^-_{\mathcal{C}} p$ and $\psi; \emptyset \Vdash^-_{\mathcal{C}} p$ implies $\Vdash^-_{\mathcal{C}} p)$;

(∨−) $\Vdash^-_{\mathcal{B}} \phi \vee \psi$ iff $\Vdash^-_{\mathcal{B}} \phi$ and $\Vdash^-_{\mathcal{B}} \psi$;

(→ +) $\Vdash^+_{\mathcal{B}} \phi \to \psi$ iff $\phi; \emptyset \Vdash^+_{\mathcal{B}} \psi$;

(→ −) $\Vdash^-_{\mathcal{B}} \phi \to \psi$ iff $\Vdash^+_{\mathcal{B}} \phi$ and $\Vdash^-_{\mathcal{B}} \psi$;

(↤ +) $\Vdash^+_{\mathcal{B}} \phi \leftarrowtail \psi$ iff $\Vdash^+_{\mathcal{B}} \phi$ and $\Vdash^-_{\mathcal{B}} \psi$;

(↤ −) $\Vdash^-_{\mathcal{B}} \phi \leftarrowtail \psi$ iff $\emptyset; \psi \Vdash^-_{\mathcal{B}} \phi$;

(Inf +) For non-empty finite $\Gamma \cup \Delta$, $\Gamma; \Delta \Vdash^+_{\mathcal{B}} \chi$ iff $\forall \mathcal{C}(\mathcal{C} \supseteq \mathcal{B})$: if $\Vdash^+_{\mathcal{C}} \phi$ for all $\phi \in \Gamma$ and $\Vdash^-_{\mathcal{C}} \psi$ for all $\psi \in \Delta$ then $\Vdash^+_{\mathcal{C}} \chi$.

(Inf −) For non-empty finite $\Gamma \cup \Delta$, $\Gamma; \Delta \Vdash^-_{\mathcal{B}} \chi$ iff $\forall \mathcal{C}(\mathcal{C} \supseteq \mathcal{B})$: if $\Vdash^+_{\mathcal{C}} \phi$ for all $\phi \in \Gamma$ and $\Vdash^-_{\mathcal{C}} \psi$ for all $\psi \in \Delta$ then $\Vdash^-_{\mathcal{C}} \chi$.

(BPR+) For finite $\Gamma \cup \Delta$, $\Gamma; \Delta \Vdash^+ \phi$ iff $\Gamma; \Delta \Vdash^+_{\mathcal{B}} \phi$ for all $\mathcal{B}$;

(BPR−) For finite $\Gamma \cup \Delta$, $\Gamma; \Delta \Vdash^- \phi$ iff $\Gamma; \Delta \Vdash^-_{\mathcal{B}} \phi$ for all $\mathcal{B}$;

Figure 2 BPR semantic clauses.

Base case. Let $p \in \mathsf{At}$ such that $\Vdash^+_{\mathcal{B}} p$. Note that, by the imposed consistency condition, in this case, p can never be $\bot$, as this would imply $\vdash^+_{\mathcal{B}} \bot$. Assume, in order to reach a contradiction, that $\Vdash^-_{\mathcal{B}} p$ holds. If p is $\top$, this would entail $\vdash^-_{\mathcal{B}} \top$, which, once again, contradicts our definition of base. For any other atom p, the two conditions yield that $\vdash^+_{\mathcal{B}} p$ and $\vdash^-_{\mathcal{B}} p$ hold simultaneously, so by epistemic consistency we obtain $\vdash^+_{\mathcal{B}} \bot$ (or $\vdash^-_{\mathcal{B}} \top$). Contradiction. Hence, $\nVdash^-_{\mathcal{B}} p$. The proof that $\Vdash^-_{\mathcal{B}} p$ implies $\nVdash^+_{\mathcal{B}} p$ is analogous.

Case $(\phi = \psi \wedge \chi)$. Assume that $\Vdash^+_{\mathcal{B}} \psi \wedge \chi$. Then $\Vdash^+_{\mathcal{B}} \psi$ and $\Vdash^+_{\mathcal{B}} \chi$. Let $\mathcal{C}$ be any base such that $\mathcal{C} \supseteq \mathcal{B}$ and $\mathcal{D}$ any base such that $\mathcal{D} \supseteq \mathcal{C}$. By transitivity of $\supseteq$ and monotonicity we have $\Vdash^+_{\mathcal{D}} \phi$ and $\Vdash^+_{\mathcal{D}} \chi$, so, by the induction hypothesis, $\nVdash^-_{\mathcal{D}} \psi$ and $\nVdash^-_{\mathcal{D}} \chi$. Furthermore, since $\mathcal{D}$ is an arbitrary extension of $\mathcal{C}$ we conclude that $\emptyset; \phi \Vdash^-_{\mathcal{C}} \top$ and $\emptyset; \chi \Vdash^-_{\mathcal{C}} \top$ hold vacuously but, by logical consistency and the semantic clause for refutations of atoms, $\Vdash^-_{\mathcal{C}} \top$ does not hold. Hence, since $\top \in \mathsf{At}$, $\nVdash^-_{\mathcal{B}} \phi \wedge \chi$. The second case is analogous. ◀

Just like the proof of Lemma 4, the proof of Theorem 20 only works due the harmonic relationship existing between proof and refutation conditions. It also uses both the epistemic and logical consistency constraints in an essential way. If bases were not required to be epistemically consistent[8] we could have a base with both $\Vdash^+_{\mathcal{B}} p$ and $\Vdash^-_{\mathcal{B}} p$, and if instead of using logical consistency we followed [2] and used the traditional BeS definition of proofs of $\bot$ (as well as refutations of $\top$) in terms of atomic explosion (*e.g.* $\Vdash^+_{\mathcal{B}} \bot$ iff $\Vdash^+_{\mathcal{B}} p$ and $\Vdash^-_{\mathcal{B}} p$ for all atoms p) then in any base $\mathcal{B}$ with $\Vdash^+_{\mathcal{B}} p$ and $\Vdash^-_{\mathcal{B}} p$ for all $p \in \mathsf{At}$ we would also have $\Vdash^+_{\mathcal{B}} \phi$

[8] In fact, if bases were only required to be logically consistent and unit complete we would obtain an alternative BeS for 2Int instead of one for BPR.

and $\Vdash^-_{\mathcal{B}} \phi$ for all ϕ. As such, Theorem 20, which shows that the semantics in fact codifies our pre-theoretic intuitions about BPR, can only be obtained by this novel combination of the bilateralist ideas presented in [2] with the treatment of consistency developed in [3].

5 Soundness and Completeness

Having established normalisation and its immediate consequences, we now turn to the semantic adequacy of our system. The aim is to show that BPR is both sound and complete with respect to the BeS proposed in Section 4.2. The proof of Soundness follows closely that of N2Int* in [2], differing only in that it has two additional cases corresponding to the instances of the PR rule. The proof of completeness, however, requires a more detailed analysis, since our semantic framework presupposes that the bases are consistent. In this context, consistency will play the same crucial structural role as it plays in [3].

The soundness proof for BPR follows along the lines of the one in [2]. Since the semantic treatment of $\bot$ and $\top$ here differs from said proof, only in that we regard them as atomic formulas rather than connectives, no new cases arise in the auxiliary lemmas, which therefore have their proofs omitted. The only substantial additions concern the two instances of the co-ordination rule PR, whose soundness must be verified.

▶ **Theorem 21** (Soundness). *If* $\Gamma; \Delta \vdash^*_{\mathsf{BPR}} \phi$ *then* $\Gamma; \Delta \Vdash^* \phi$.

Proof. We prove the result by induction on the length of derivations. If the derivation has length 0, then it is either the single occurrence of a proof assumption ϕ, yielding $\phi; \emptyset \vdash^+_{\mathsf{N2Int^*}} \phi$, or the single occurrence of a refutation assumption ϕ, yielding $\emptyset; \phi \vdash^-_{\mathsf{N2Int^*}} \phi$. In the first case, $\phi; \emptyset \Vdash^+_{\mathcal{B}} \phi$ for every $\mathcal{B}$, hence $\phi; \emptyset \Vdash^+ \phi$. The second case is analogous.

The derivation has length greater than 0. Then the result is proved by considering the last rule applied in it, so we have to show that each rule application preserves semantic validity. Notice, however, that even though a deduction ending with an application of $\top(+)$ or $\bot(-)$ has no open assumptions it can still be considered a deduction showing $\Gamma; \Delta \vdash^+_{\mathsf{N2Int^*}} \top$ or $\Gamma; \Delta \vdash^-_{\mathsf{N2Int^*}} \bot$, respectively. We only show the case of the rule PR in its proof version, as the dual proof case is analogous.

If $\Gamma_1; \Delta_1 \Vdash^+ \phi$ and $\Gamma_2; \Delta_2 \Vdash^- \phi$, then $\Gamma_1, \Gamma_2; \Delta_1, \Delta_2 \Vdash^+ \phi$. Assume $\Gamma_1; \Delta_1 \Vdash^+ \phi$ and $\Gamma_2; \Delta_2 \Vdash^- \phi$ and pick an arbitrary $\mathcal{B}$. Assume, for the sake of proving a contradiction, that there is $\mathcal{C} \supseteq \mathcal{B}$ such that $\Vdash^+_{\mathcal{C}} \gamma$ and $\Vdash^-_{\mathcal{C}} \delta$ for every $\gamma \in \Gamma_1 \cup \Gamma_2$ and $\delta \in \Delta_1 \cup \Delta_2$. Then, by $\Gamma_1; \Delta_1 \Vdash^+ \phi$, we have that $\Vdash^+_{\mathcal{C}} \phi$, and by $\Gamma_2; \Delta_2 \Vdash^- \phi$, we have $\Vdash^-_{\mathcal{C}} \phi$, contradicting theorem 20. Hence there is no extension $\mathcal{C}$ of the arbitrary $\mathcal{B}$ such that $\Vdash^+_{\mathcal{C}} \gamma$ and $\Vdash^-_{\mathcal{C}} \delta$ for every $\gamma \in \Gamma_1 \cup \Gamma_2$ and $\delta \in \Delta_1 \cup \Delta_2$, so $\Gamma_1, \Gamma_2; \Delta_1, \Delta_2 \Vdash^+ \phi$ holds vacuously. ◀

In contrast to soundness, the completeness proof is more involved. The overall strategy follows that of [2], based on Sandqvist's simulation base construction. However, in our case we must additionally ensure that the simulation bases used in the argument satisfy the constraints required in the definition of BeS. To achieve this, we show that the simulation bases are themselves consistent. The key step is to establish atomic normalisation for these bases. Since normalisation has already been proved for BPR, this step follows directly. The resulting normalisation procedure then allows us to extend the consistency argument to the simulation bases, ensuring that they meet the requirements of our semantic framework.

▶ **Definition 22.** *Let* Γ *be a set of formulas and* $\Gamma^{\mathrm{Sub}} = \{\phi : \phi$ *is a subformula of some* $\psi \in \Gamma\}$. *An* atomic mapping α *for a set* Γ^{Sub} *is any injective function* $\alpha : \Gamma^{Sub} \mapsto \mathsf{At}$ *such that, for all* $p \in \Gamma^{Sub}$ *with* $p \in \mathsf{At}$, $\alpha(p) = p$.

For every $\phi \in \Gamma^{Sub}$, we will denote the atom $\alpha(\phi)$ by p^{ϕ}. Note that, since α is injective, this notation is well defined, as no two different formulas may be assigned to the same atom.

Given an atomic mapping α for Γ, a *simulation base* $\mathcal{U}$ is a base which contains rules mimicking the rules of system BPR. The construction of this base is exactly as in [2], with the addition of the following rules for all $\phi \in \Gamma$:

$$\begin{array}{c} \overline{p^{\phi}} \qquad \overline{\overline{p^{\phi}}} \\ \hline q \end{array}\ {\scriptstyle PR(+),\,p} \qquad\qquad \begin{array}{c} \overline{p^{\phi}} \qquad \overline{\overline{p^{\phi}}} \\ \hline\hline q \end{array}\ {\scriptstyle PR(-),\,p}$$

Furthermore, notice that instances of $p^{\bot}$ and $p^{\top}$ are now written as simply $\bot$ and $\top$, as in the present framework those are treated as if they were atoms.

The proof of atomic normalisation follows standard lines. The atomic reductions are analogous to those used in our normalisation proof.

▶ **Theorem 23.** *Let Π be a derivation of ϕ from the set of assumptions Γ and contra-assumptions Δ, in $\mathcal{U}$. Then Π reduces to a normal derivation Π' of ϕ from the set of assumptions Γ and contra-assumptions Δ.*

▶ **Corollary 24.** *$\mathcal{U}$ is consistent.*

The proofs of these results can be obtained through a straightforward adaptation of the proofs of Theorem 9 and Corollary 12 to the atomic setting and shall be omitted.

▶ **Lemma 25.** *Let $\phi \in \Gamma^{Sub}$, α be an atomic mapping for Γ^{Sub} and $\mathcal{U}$ a simulation base defined with α. Then for all epistemically adequate $\mathcal{B} \supseteq \mathcal{U}$, it holds that $\Vdash^*_{\mathcal{B}} \phi$ iff $\vdash^*_{\mathcal{B}} p^{\phi}$.*

Due to our use of the consistency constraint, a completeness proof uses the strategy developed in [3] instead of the traditional strategy for BeS.

▶ **Theorem 26** (Completeness). *If $\Gamma; \Delta \Vdash^* \phi$ then $\Gamma; \Delta \vdash^*_{\mathsf{BPR}} \phi$.*

Proof. Assume that $\Gamma; \Delta \Vdash^* \phi$. Then, $\Gamma; \Delta \Vdash^*_{\mathcal{U}} \phi$. Let $\Theta = (\Gamma \cup \Delta \cup \{\phi\})^{Sub}$, α an atomic mapping for Θ and $\mathcal{U}$ a simulation base based on α. Consider the sets $\Gamma^{\mathsf{At}} = \{\alpha(\psi) | \psi \in \Gamma\}$ and $\Delta^{\mathsf{At}} = \{\alpha(\chi) | \chi \in \Delta\}$. Furthermore, let $\mathcal{B}$ be a base constructed by adding to $\mathcal{U}$ a proof rule concluding p^{ψ} from empty premises, for every $p^{\psi} \in \Gamma^{\mathsf{At}}$, and a refutation rule concluding p^{χ}, for every $p^{\chi} \in \Delta^{\mathsf{At}}$. We consider two cases:

$\mathcal{B}$ is inconsistent. Then, (i) there is a deduction Π in $\mathcal{B}$ showing $\vdash^+_{\mathcal{B}} \bot$, or (ii) there is a deduction Σ showing $\vdash^-_{\mathcal{B}} \top$. If (i) is the case but every rule in Π is a rule in $\mathcal{U}$, we would have $\vdash^+_{\mathcal{U}} \bot$, contradicting Lemma 24. Hence, there must be some rule applied in Π, which is in $\mathcal{B}$ but not in $\mathcal{U}$. Consider the derivation Π' we obtain from Π by substituting every such rule which is an instance of a proof rule concluding p^{ψ} from empty premises by a pair of assumptions $p^{\psi}; \emptyset$ and every instance of a refutation rule concluding p^{χ} from empty premises by a pair of assumptions $\emptyset; p^{\chi}$. Then, Π' is a derivation showing $\Gamma'; \Delta' \vdash^+_{\mathcal{U}} \bot$ for some $\Gamma' \subseteq \Gamma^{\mathsf{At}}$ and $\Delta' \subseteq \Delta^{\mathsf{At}}$. Consider the derivation Π'' obtained by substituting every instance of a formula p^{φ} by φ in Π'. We can easily show, by induction on the length of the derivations, that Π'' is a derivation showing $\Gamma; \Delta \vdash^+_{\mathsf{BPR}} \bot$. Then, applying $\bot(+)$ at the end of Π'', we obtain a derivation showing $\Gamma; \Delta \vdash^*_{\mathsf{BPR}} \phi$. If (ii) is the case, we follow a similar strategy and obtain a derivation showing $\Gamma; \Delta \vdash^-_{\mathsf{BPR}} \top$, hence, by applying $\top(-)$ at the end, we show that $\Gamma; \Delta \vdash^*_{\mathsf{BPR}} \phi$.

$\mathcal{B}$ is consistent. Then, by construction of this base, we have that $\vdash^+_{\mathcal{B}} p^\psi$ for every $p^\psi \in \Gamma^{\mathsf{At}}$, and $\vdash^-_{\mathcal{B}} p^\chi$ for every $p^\chi \in \Delta^{\mathsf{At}}$. Note that $\mathcal{B}$ is epistemically adequate, $\Gamma, \Delta \subseteq \Theta$, and so, by Lemma 25, $\Vdash^+_{\mathcal{B}} \psi$ for every $\psi \in \Gamma$, and $\Vdash^-_{\mathcal{B}} \chi$ for every $\chi \in \Delta$. Furthermore, as $\mathcal{U} \subseteq \mathcal{B}$ and $\Gamma; \Delta \Vdash^*_{\mathcal{U}} \phi$, we get that $\Vdash^*_{\mathcal{B}} \phi$. Again, by Lemma 25, as $\phi \in \Theta$, we obtain $\vdash^*_{\mathcal{B}} p^\phi$. Let Π^* be a derivation showing $\vdash^*_{\mathcal{B}} p^\phi$ in $\mathcal{B}$. If every rule in Π^* is a rule in $\mathcal{U}$, then Π^* is actually a derivation showing $\vdash^*_{\mathcal{U}} p^\phi$. Otherwise, we follow a similar strategy to that of the previous case, substituting every instance of an application of a proof rule concluding p^ψ from empty premisses by a pair of assumptions $p^\psi; \emptyset$, for every $p^\psi \in \Gamma^{\mathsf{At}}$, and analogously for refutation rules concluding p^χ from empty premises, for $p^\chi \in \Delta^{\mathsf{At}}$. Thus we obtain a derivation in $\mathcal{U}$ showing $\Gamma'; \Delta' \vdash^*_{\mathcal{U}} p^\phi$ for some $\Gamma' \subseteq \Gamma^{\mathsf{At}}$ and $\Delta' \subseteq \Delta^{\mathsf{At}}$. In either case, we have a derivation showing $\Gamma^{\mathsf{At}}; \Delta^{\mathsf{At}} \vdash^*_{\mathcal{U}} p^\phi$. Take this derivation and substitute every instance of an atom p^φ by φ. The atoms which are not the image of some formula under α, such as (possibly) the conclusion of $\bot(-)$, remain unchanged. It is straightforward to show that the obtained derivation is in BPR and shows $\Gamma; \Delta \vdash^*_{\mathsf{BPR}} \phi$. ◀

We can prove now that weak falsity does not imply strong falsity in BPR. In fact, consider a base $\mathcal{B}$ containing only the rule $R1$ and a base $\mathcal{C}$ containing only the rule $R2$ below:

$$\dfrac{\overline{\;p\;}}{\bot}\,R1 \qquad\qquad \dfrac{\overline{\overline{\;p\;}}}{\top}\,R2$$

Since our bases satisfy the consistency constraint there cannot be any $\mathcal{D} \supseteq \mathcal{B}$ with $\Vdash^+_{\mathcal{D}} p$ or any $\mathcal{E} \supseteq \mathcal{C}$ with $\Vdash^-_{\mathcal{E}} p$ since, if there were, we would obtain deductions showing $\vdash^+_{\mathcal{D}} p$ and $\vdash^-_{\mathcal{E}} p$ which could be composed with the rules to show that such bases are inconsistent. So from the semantic clauses $(\to +)$ and $(\leftarrow\!\!\!\!\!- \, -)$ we conclude $\Vdash^+_{\mathcal{D}} p \to \bot$ and $\Vdash^-_{\mathcal{E}} \top \leftarrow\!\!\!\!\!- \, p$. On the other hand, since there are no refutation rules in $\mathcal{B}$ and no proof rules in $\mathcal{C}$ we immediately conclude $\nvdash^-_{\mathcal{B}} p$ and $\nvdash^+_{\mathcal{C}} p$, meaning that in $\mathcal{B}$ we can prove weak falsity without providing a constructive counterexample and in $\mathcal{C}$ we can prove weak provability without providing a constructive proof, showing that the concepts interact as expected.

6 Concluding remarks and future work

In this paper, we develop a form of bilateralism where proofs and refutations are distinct but incompatible assertoric acts, and establish syntactic and semantic results for the logic BPR.

On the syntactic side, we show that our natural deduction system – extending an adapted version of Wansing's 2Int – is weakly normalising. As corollaries, we obtain the subformula property and consistency, ensuring proofs and refutations can all be reduced to analytic form.

On the semantic side, we introduce a base-extension semantics that is sound and complete with respect to the proof system. Because it relies on the explicit construction of proofs and counterexamples via atomic rules, the semantics remains faithful to the mathematical practice of proving and refuting, while also supporting epistemic interpretations.

Finally, we show that BPR internalises constructive falsity via refutation and supports the definition of further operators, including intuitionistic negation. Its relationship to Nelson's N3 mirrors that between 2Int and N4.

There are several directions for future work. A natural next step is to develop alternative proof systems for BPR, in particular a sequent calculus along the lines of [1], enabling systematic comparison with existing frameworks such as [16]. This also opens the door for automatic reasoning. Further directions include first-order extensions, applications to other logics with strong negation, and a deeper analysis of the proof-theoretic and semantic structure of BPR.

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