Compression for Coinductive Rewriting and the Cut-Elimination of Non-Wellfounded Proofs

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Abstract

We introduce a generic presentation of "syntactic objects built by mixed induction and coinduction" encompassing all standard kinds of infinitary terms, as well as derivation trees in non-wellfounded proof systems. We then define a coinductive notion of infinitary rewriting of such objects, which is equivalent to the original presentation of infinitary rewriting relying on metric convergence and ordinal-indexed sequences of rewriting steps. This provides a unified coinductive presentation of *e.g.* first-order infinitary rewriting, infinitary λ-calculi, and cut-elimination in non-wellfounded proofs.

We then formulate and study the coinductive counterpart of compression, *i.e.* the property of an infinitary rewriting system such that all rewriting sequences of any ordinal length can be "compressed" to equivalent sequences of length at most ω (which ensures that they can be finitely approximated). We characterise compression in our generic setting for coinductive rewriting, "factorising" the part of the proof that can be performed at this level of generality. Our proof is fully coinductive, avoiding any detour via rewriting sequences.

Finally we focus on the non-wellfounded proof system $\mu\mathsf{MALL}^{\infty}$ for multiplicative-additive linear logic with fixed points, and we put our results to work in order to prove that compression holds for cut-elimination in this setting, which is a key lemma of several extensions of cut-elimination to similar systems.

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1 Introduction

Infinite objects and processes pervade the study of computing. It is indeed well-known that considering non-terminating computations potentially resulting in undefined results is fundamental in the development of any universal model of computation. Non-terminating computations are also crucial in the study of productive processes, for instance for reactive programs: in such a setting, programs may run forever but one requires either a good behaviour with respect to the environment (*fairness*) or that arbitrary approximations of a result are computed in finite time (*productivity*).

In the presence of non-terminating behaviours, it is natural to consider ideal objects representing what is computed at the limit of the process (typically built *via* a completion of the set of finite processes with respect to some algebraic or topological structure). Standard examples include infinite streams obtained by completing finite words, or Böhm trees resulting from the ideal completion of λ-terms extended with an undefined value [5], that can be seen as a "syntactic description of the semantics" of a program. Once such limit objects enter the picture, it makes sense to give them a first-class status and to allow computation to be performed directly on infinite objects; in particular in a functional setting one wants to ensure compositionality, *i.e.* that the (possibly infinite) result of a computation should in turn be applicable to some arguments, giving rise to a new (possibly infinite) computation.

This motivated the introduction and the study of *infinitary rewriting* in the 1990s: in an infinitary rewriting system the terms may be infinite (they result from the metric completion of usual, finite terms) and the sequences of rewriting steps may be indexed by arbitrary ordinals. This idea was successfully applied to first-order rewriting [13, 18], to the λ-calculus [19], to higher-order rewriting [21], etc. Therein, the limit objects of interest are no longer floating somewhere between syntax and semantics: they become plain syntactic objects (typically the normal forms of the infinitary rewriting).

Infinitary rewriting for non-wellfounded proof theory. On the other side of the Curry-Howard correspondence between programs and proofs, infinite objects and their dynamics are the subject of a blooming line of work around non-wellfounded proof systems (following [25, 29]). In such systems, proof derivations are potentially infinite trees subject to a validity criterion ensuring the productivity of the proof (and hence its correctness). Such systems are used in particular for modal logics (*e.g.* an infinite proof can account for the unfolding of a "forever" temporal modality) or for logics extended with fixed point operators (here an infinite proof typically has a greatest fixed point as a conclusion, or a least fixed point as a hypothesis). In practice such systems (particularly their *circular* or *cyclic* fragments, whose proof derivations are regular) are very well-suited to proof search [31].

As usual in proof theory, good properties of a non-wellfounded proof system (like correctness, or more practically the good behaviour of proof search procedures) are ensured by *cut-elimination*, *i.e.* the fact that the cut rule is admissible [1, 3, 12, 16, 26]. A standard way to prove this property is to show that each occurrence of the cut rule can be "moved upwards" in the proof derivation: repeated applications of this fact produce finite approximations of a limit cut-free derivation. This can be presented in the form of an infinite sequence of rewriting steps (acting on the proof derivations); unfortunately this rewriting of non-wellfounded proofs does not really fit the usual presentation of infinitary rewriting, which had to be partially adapted specifically for this purpose [3, 26]. This calls for more general definitions encompassing the infinitary rewriting of both infinitary terms and non-wellfounded proofs.

The compression property. The "traditional" line of work on infinitary rewriting is based on ordinal-indexed strongly Cauchy convergent rewriting sequences (*i.e.* rewriting sequences that do not only converge in the topological sense, but such that in addition the computation steps occur deeper and deeper in the rewritten objects). A desirable property of such rewriting systems is *compression*, *i.e.* the fact that rewriting sequences of arbitrary ordinal length can be compressed to sequences of length ω. For example consider the first-order rewriting rules $a \longrightarrow_1 f(g(a))$ and $g(f(x)) \longrightarrow_2 f(x)$, then the (strongly converging) rewriting sequence

$$a \longrightarrow_1 f(g(a)) \longrightarrow_1^{\infty} f(g(f(g(\dots)))) \longrightarrow_2 f(f(g(\dots))) \longrightarrow_2^{\infty} f(f(f(\dots))) \tag{1}$$

is of ordinal length $\omega \cdot 2$ but can be compressed by interleaving the rewriting steps:

$$a \longrightarrow_1 f(g(a)) \longrightarrow_1 f(g(f(g(a)))) \longrightarrow_2 f(f(g(a))) \longrightarrow^{\infty}_{1,2} f(f(f(\dots))). \tag{2}$$

This example illustrates the key benefit of compression: in a strongly converging rewriting sequence of length ω, finite approximations of the limit are computed in finite time, which is clearly not the case with sequences of bigger ordinal length (in the example above, it takes $\omega + 1$ steps to produce the two outermost f). This "approximation" or "continuity" property enjoyed by rewriting sequences of length ω is at the heart of most practical motivations for the use of infinitary rewriting. For example, in infinitary λ-calculus, it is the reason why infinitary rewriting allows to provide an easy proof of the continuous approximation theorem [8], a result that is of paramount importance for the classical study of the λ-calculus [6].

To the best of our knowledge, compression lemmas have been proved for three kinds of term rewriting systems: left-linear first-order rewriting systems [18, 20, 23, 14], infinitary λ-calculi [19, 4], fully-extended, left-linear higher-order rewriting systems [21]. The compression property also plays a significant role in non-wellfounded proof theory: although direct proofs of cut-elimination typically produce a rewriting sequence of length ω, some other cut-elimination results involve rewriting sequences of arbitrary ordinal length as they rely on translations of formulæ and proof derivations from one logic to another, where single cut-elimination steps are translated into sequences of ordinal length [26]. In these cases the result crucially relies on a compression lemma ensuring that finite approximations of a cut-free derivation are produced in finitely many steps. Again, this variety of compression lemmas all enjoying more or less the same proof suggests a generic, uniform treatment.

The coinductive turn. As an alternative to the traditional definition of infinite objects *via* ideal or metric completion, a more recent and very fruitful line of work is based on the use of *coinduction*: the metric completion of any algebraic type (*e.g.* a type of terms) can indeed be described as the corresponding coalgebraic type [7]. This coinductive reformulation has been in particular extensively conducted in the setting of infinitary λ-calculi [17, 15, 11, 8] and has noticeably allowed for a clean presentation of α-equivalence for infinitary λ-terms [22, 9].

Although this coinductive redefinition of infinitary terms is now very standard, finding a coinductive counterpart to strongly converging rewriting sequences turns out to be less easy. A first answer has been proposed for an infinitary λ-calculus [15], but in a presentation corresponding solely to sequences of length ω, and a generic presentation of coinductive infinitary rewriting was designed only recently by Endrullis, Hansen, Hendriks, Polonsky and Silva [14] but is limited to first-order term rewriting.

Once more, this calls for a generic presentation of coinductive rewriting unifying and completing the aforementioned lines of work. In particular expressing the infinitary rewriting induced by cut-elimination on non-wellfounded proofs coinductively seems particularly natural, and would pave the way for fully coinductive cut-elimination proofs.

Contributions and organisation of the paper. In Section 2, we propose a framework for describing arbitrary non-wellfounded objects and rewriting systems acting on such objects, which encompasses all the use cases mentioned so far. We extend the correspondence between topology-based and coinduction-based infinitary rewriting to this generic framework. In Section 3, we formulate a characterisation of the compression property "factorising" the part of the proof that can be performed at this level of generality, hence identifying the key features a compressible infinitary rewriting system should enjoy; to our knowledge this proof is new. We then apply it to first-order coinductive rewriting and to the λ-calculus. Finally,

in Section 4 we recall the non-wellfounded proof system $\mu\mathsf{MALL}^\infty$ for multiplicative-additive linear logic with fixed points, and we prove that compression also holds for the rewriting induced by cut-elimination on this proof system. The choice of $\mu\mathsf{MALL}^\infty$ is justified by the fact that compression in this setting is a cornerstone argument of the proof of cut-elimination for $\mu\mathsf{LL}^\infty$ (the non-wellfounded proof system for linear logic with fixed points), in which non-wellfounded proof systems for a whole range of logics (intuitionistic, classical, temporal, etc.) can be embedded. Finally, we recap and suggest directions for future work in Section 5.

2 Infinitary rewriting: From strong convergence to coinduction

In this section, we first recall how infinitary first-order rewriting based on strong Cauchy convergence [18] can be reformulated using coinduction, as exposed in [14], and we similarly connect the presentations of infinitary λ-calculi based on convergence [19] and on coinduction [15]. Finally we provide a generic framework for infinitary rewriting of non-wellfounded objects, extending again the correspondence between the two views on infinitary rewriting.

2.1 A first example: First-order rewriting

First-order terms. Fix a countable set $\mathcal{V}$ of *variables*. A *first-order signature* is given by a countable set Σ of symbols endowed with an arity function $\mathrm{ar} : \Sigma \to \mathbf{N}$; we fix such a signature. The set T_Σ of all *finite first-order terms* on the signature Σ is defined by the following set of inductive rules:

$$\frac{}{x \in \mathrm{T}_\Sigma} \qquad \frac{s_1 \in \mathrm{T}_\Sigma \quad \dots \quad s_{\mathrm{ar}(\mathsf{c})} \in \mathrm{T}_\Sigma}{\mathsf{c}(s_1, \dots, s_{\mathrm{ar}(\mathsf{c})}) \in \mathrm{T}_\Sigma}\,. \tag{3}$$

(Whenever we give such a set of rules where a rule features a variable x, a constructor c, or more generally an element of an external set, we assume that there is one rule for each element of this set.) The *truncation* of a term $s \in \mathrm{T}_\Sigma$ at depth $d \in \mathbf{N}$ is defined inductively by $\lfloor s \rfloor_0 := *$, $\lfloor x \rfloor_{d+1} := x$, and $\lfloor \mathsf{c}(s_1, \dots, s_k) \rfloor_{d+1} := \mathsf{c}(\lfloor s_1 \rfloor_d, \dots, \lfloor s_k \rfloor_d)$, where $*$ is a fresh nullary symbol. The set T_Σ is equipped with a metric $\mathbf{d}$ defined by $\mathbf{d}(s,t) := \inf \left\{ 2^{-d} \;\middle|\; \lfloor s \rfloor_d = \lfloor t \rfloor_d \right\}$. The set T_Σ^∞ of all *(infinitary) first-order terms* on the signature Σ is the metric completion of T_Σ wrt. $\mathbf{d}$. As a consequence of [7, Proposition 3.1], T_Σ^∞ can be equivalently presented as the set defined by treating the rules of Equation (3) coinductively. (In short, $\mathrm{T}_\Sigma := \mu X.FX$ and $\mathrm{T}_\Sigma^\infty := \nu X.FX$ where $FX := \mathcal{V} + \coprod_{\mathsf{c} \in \Sigma} X^{\mathrm{ar}(\mathsf{c})}$ and the metric completion is carried by the canonical (co)algebra morphism $\mathrm{T}_\Sigma \to \mathrm{T}_\Sigma^\infty$.)

First-order rewriting. A *substitution* is any function $\mathcal{V} \to \mathrm{T}_\Sigma^\infty$. Given a substitution σ and a term t, we denote by $\sigma \cdot t$ the term defined coinductively by $\sigma \cdot x := \sigma(x)$ and $\sigma \cdot \mathsf{c}(s_1, \dots, s_k) := \mathsf{c}(\sigma \cdot s_1, \dots, \sigma \cdot s_k)$. A (first-order) *rewrite rule* is a pair $(l, r) \in \mathrm{T}_\Sigma^\infty \times \mathrm{T}_\Sigma^\infty$ such that **(i)** l is not a variable, **(ii)** all variables occurring in r also occur in l. An *infinitary (first-order) term rewriting system*, ITRS in short, is a set of rewrite rules. An ITRS $\mathcal{R}$ defines a rewriting relation $\longrightarrow$ on T_Σ^∞ by the following set of inductive rules:

$$\frac{(l, r) \in \mathcal{R} \qquad \sigma : \mathcal{V} \to \mathrm{T}_\Sigma^\infty}{\sigma \cdot l \longrightarrow_0 \sigma \cdot r} \qquad \frac{s_i \longrightarrow_d s_i' \qquad 1 \leq i \leq \mathrm{ar}(\mathsf{c})}{\mathsf{c}(s_1, \dots, s_{\mathrm{ar}(\mathsf{c})}) \longrightarrow_{d+1} \mathsf{c}(s_1, \dots, s_i', \dots, s_{\mathrm{ar}(\mathsf{c})})} \tag{4}$$

where, as syntactic sugar, we sometimes annotate $\longrightarrow$ with integers indicating the depth at which the rewriting step occurs. From now on we fix an ITRS $\mathcal{R}$.

▶ **Definition 1.** *Given an ordinal γ, a* rewriting sequence *of length γ from s_0 to s_γ is the data of terms $(s_\delta)_{\delta\leq\gamma}$ together with rewriting steps $(s_\delta \longrightarrow_{d_\delta} s_{\delta+1})_{\delta<\gamma}$. It is* strongly converging *if for every limit ordinal $\gamma' \leq \gamma$,* **(i)** $\lim_{\delta\to\gamma'} s_\delta = s_{\gamma'}$ *and* **(ii)** $\lim_{\delta\to\gamma'} d_\delta = \infty$. *We write $s \longrightarrow^\infty t$ whenever there is such a strongly converging rewriting sequence (of any ordinal length γ) such that $s_0 = s$ and $s_\gamma = t$.*

We now introduce an alternative definition using coinduction. In general, the sets of rules we introduce to do so contain both inductive and coinductive rules; to distinguish them, we draw double rules for the latter. In such a system, non-wellfounded derivations are allowed provided any infinite branch crosses infinitely many coinductive rules.

▶ **Definition 2.** *For all ordinals $\gamma < \omega_1$, relations $\underset{\gamma}{\longrightarrow}^\infty$ and $\underset{\gamma}{\rightarrow}^\infty$ on T^∞_Σ are mutually defined by the rules:*

$$\frac{s \overset{\gamma,m}{\rightsquigarrow} s' \qquad s' \underset{\gamma}{\rightarrow}^\infty t}{s \underset{\gamma}{\longrightarrow}^\infty t}\ (\mathsf{split}) \qquad \frac{}{x \underset{\gamma}{\rightarrow}^\infty x}\ (\mathsf{lift}_x) \qquad \frac{s_1 \underset{\gamma}{\longrightarrow}^\infty s'_1 \quad \cdots \quad s_{\mathsf{ar(c)}} \underset{\gamma}{\longrightarrow}^\infty s'_{\mathsf{ar(c)}}}{\mathsf{c}(s_1,\dots,s_{\mathsf{ar(c)}}) \underset{\gamma}{\rightarrow}^\infty \mathsf{c}(s'_1,\dots,s'_{\mathsf{ar(c)}})}\ (\mathsf{lift}_{\mathsf{c}})$$

where the notation $s \overset{\gamma,m}{\rightsquigarrow} s'$ stands for any sequence $s \longrightarrow^* s'_1 \underset{\delta_1}{\rightarrow}^\infty t_1 \longrightarrow^* s'_2 \underset{\delta_2}{\rightarrow}^\infty \dots \underset{\delta_m}{\rightarrow}^\infty t_m \longrightarrow^* s'$ such that $\forall 1 \leq i \leq m,\ \delta_i < \gamma$.

Observe that $\underset{\gamma}{\rightarrow}^\infty$ is just $\underset{\gamma}{\longrightarrow}^\infty$ under a constructor, which enforces strong convergence. This Definition and the following Theorem are instantiated on the example rewriting sequence from Equation (1) in Section 3.4, Equation (6).

▶ **Theorem 3** ([14, Theorem 5.2])**.** *$\longrightarrow^\infty$ is the union of all relations $\underset{\gamma}{\longrightarrow}^\infty$ (over $\gamma < \omega_1$).*

Let us mention that we marginally depart from the original definition [14, Definition 4.2] by annotating the relations with ordinals and ensuring that these annotating ordinals decrease along certain branches of the derivations, whereas the original authors explicitly add the (clearly equivalent) constraint that no branch should cross infinitely many $\rightsquigarrow$. Relaxing this constraint would give rise to "bi-infinite" rewriting, corresponding to rewriting sequences that may also be transfinite "to the left", *i.e.* towards their source. In particular our ordinal annotations *do not correspond* to the ordinal length of a corresponding rewriting sequence! (We can only say that any rewriting sequence of length γ can be turned into $\underset{\gamma}{\longrightarrow}^\infty$, see Theorem 10.) We also replace their rule (id), which adds $s \underset{\gamma}{\rightarrow}^\infty s$ as an axiom, by its version (lift_x) restricted to variables. This is enough for the defined relations to be reflexive (see Lemma 13(2)).

2.2 A second example: Infinitary λ-calculi

λ-terms. The construction of infinitary λ-terms follows the same path, let us summarise it. The set Λ of *finite λ-terms* is defined inductively by: $s,t,\dots \coloneqq x \mid \lambda x.s \mid st$ (where $x \in \mathcal{V}$). Fix $a,b,c \in \{0,1\}^3$. The *abc*-truncation of $s \in \Lambda$ at depth $d \in \mathbf{N}$ is defined inductively by $\lfloor s \rfloor^{abc}_0 \coloneqq *$, and by $\lfloor x \rfloor^{abc}_{d+1} \coloneqq x$, $\lfloor \lambda x.s \rfloor^{abc}_{d+1} \coloneqq \lambda x.\lfloor s \rfloor^{abc}_{d+1-a}$ and $\lfloor st \rfloor^{abc}_{d+1} \coloneqq \lfloor s \rfloor^{abc}_{d+1-b} \lfloor t \rfloor^{abc}_{d+1-c}$. For instance, taking $abc = 001$ means that we consider that one goes "deeper" in a λ-term only when entering the argument of an application. We equip Λ with a metric $\mathbf{d}^{abc}$ defined by $\mathbf{d}^{abc}(s,t) \coloneqq \inf\left\{2^{-d} \;\middle|\; \lfloor s \rfloor^{abc}_d = \lfloor t \rfloor^{abc}_d\right\}$. The set Λ^{abc} of *abc-infinitary λ-terms* is defined as the metric completion of Λ wrt. $\mathbf{d}^{abc}$.

Equivalently, the following coinductive definition of Λ^{abc} has been proposed [8]:

$$\frac{}{x \in \Lambda^{abc}} \qquad \frac{\triangleright_a\ s \in \Lambda^{abc}}{\lambda x.s \in \Lambda^{abc}} \qquad \frac{\triangleright_b\ s \in \Lambda^{abc} \qquad \triangleright_c\ t \in \Lambda^{abc}}{st \in \Lambda^{abc}} \qquad \frac{S}{\triangleright_0\ S}\ (\triangleright_0) \qquad \frac{S}{\triangleright_1\ S}\ (\triangleright_1) \tag{5}$$

where S stands for any statement (in this case, of the form "$s \in \Lambda^{abc}$"). While $\triangleright_0$ is an "inductive guard" (hence could as well be omitted), $\triangleright_1$ introduces a coinductive guard (it is the "later" modality from [24, 2]): $a = 1$ means that λ-abstraction is coinductive, while $a = 0$ means that it is inductive – and similarly for b and c for the left and right sides of application. (In short[1], using fixed points again, $\Lambda \coloneqq \mu X.F^{abc}(X, X)$ and $\Lambda^{abc} \coloneqq \nu X_1.\mu X_0.F^{abc}(X_0, X_1)$ where $F^{abc}(X_0, X_1) \coloneqq \mathcal{V} + \mathcal{V} \times X_a + X_b \times X_c$.)

β-reduction. We denote by $s[t/x]$ the λ-term obtained by substituting all free occurrences of x with t in s, in a capture-avoiding manner. *β-reduction* is the relation on Λ^{abc} defined inductively by:

$$\frac{}{(\lambda x.s)t \longrightarrow_0 s[t/x]} \qquad \frac{u \longrightarrow_d u'}{\lambda x.u \longrightarrow_{d+a} \lambda x.u'} \qquad \frac{u \longrightarrow_d u'}{uv \longrightarrow_{d+b} u'v} \qquad \frac{v \longrightarrow_d v'}{uv \longrightarrow_{d+c} uv'}$$

where again we sometimes annotate $\longrightarrow$ with the depth at which the rewriting step occurs. As in Definition 1, we write $s \longrightarrow^\infty t$ whenever there is a strongly converging β-reduction sequence from s to t. The following counterpart of Theorem 3 states that this topological presentation can be turned into a coinductive one. This is another particular case of Theorem 10, to be stated in the next section.

▶ **Theorem 4.** *$\longrightarrow^\infty$ is the union of all relations $\xrightarrow[\gamma]{}{}^\infty$ (over ordinals $\gamma < \omega_1$) defined by the rules* (split) *and* (lift_x) *from Definition 2, the rules* ($\triangleright_0$) *and* ($\triangleright_1$) *from Equation* (5)*, as well as the rules* $\dfrac{\triangleright_a u \xrightarrow[\gamma]{}{}^\infty u'}{\lambda x.u \xrightarrow[\gamma]{}{}^\infty \lambda x.u'}\ (\mathsf{lift}_\lambda)$ *and* $\dfrac{\triangleright_b u \xrightarrow[\gamma]{}{}^\infty u' \quad \triangleright_c v \xrightarrow[\gamma]{}{}^\infty v'}{uv \xrightarrow[\gamma]{}{}^\infty u'v'.}\ (\mathsf{lift}_@)$.

2.3 Infinitary rewriting of many-sorted infinitary terms

We will now introduce a description of rewriting for arbitrary non-wellfounded objects, in the form of derivation trees that we call "many-sorted terms". We want in particular to mix not only inductive constructors and coinductive constructors, but to feature constructors with mixed inductive and coinductive inputs, as displayed by the example of *abc*-infinitary λ-calculi; and instead of manipulating the rather heavy $\triangleright_0$ and $\triangleright_1$ modalities and the corresponding rules, we introduce the following kind of mixed rules:

$$\frac{}{x \in \Lambda^{001}} \qquad \frac{s \in \Lambda^{001}}{\lambda x.s \in \Lambda^{001}} \qquad \frac{s \in \Lambda^{001} \quad t \in \Lambda^{001}}{st \in \Lambda^{001}}$$

that are equivalent (here for defining Λ^{001}) to the system from Equation (5).

In general, we first fix a set $\mathcal{S}$ of *sorts*. It is typically an inductively defined set using the singleton type, one or more alphabets, as well as tuples, lists, multisets, etc. For instance one could consider the set of two-sided sequents in a given logic, encoded as the set of pairs of lists of formulæ.

▶ **Definition 5.** *A* construction rule (r) *is given by* **(i)** *its arity* $\mathrm{ar}(\mathsf{r}) \in \mathbf{N}$*,* **(ii)** *a partial function* $\mathsf{r} : \mathcal{S}^{\mathrm{ar}(\mathsf{r})} \rightharpoonup \mathcal{S}$ *mapping its* premises *to its* conclusion*,* **(iii)** *a map* $\mathrm{coind}_\mathsf{r} : [1, \mathrm{ar}(\mathsf{r})] \to \{0, 1\}$ *indicating the (co)inductive character of each premise. It is represented by:* $\dfrac{\underline{\quad S_1 \quad} \quad \cdots \quad \underline{\quad S_{\mathrm{ar}(\mathsf{r})} \quad}}{\mathsf{r}(S_1, \ldots, S_{\mathrm{ar}(\mathsf{r})})}\ (\mathsf{r})$

[1] In this summary we carefully omitted to mention any quotient by α-equivalence (*i.e.* by renaming of bound variables). A rigorous construction can be found in [22, 9]; it consists in working in the category of nominal sets and defining $F^{abc}(X_0, X_1) \coloneqq \mathcal{V} + [\mathcal{V}]X_a + X_b \times X_c$, where $[\mathcal{V}]$ is a functor encoding "nameless" abstraction. We do not detail these technicalities, as all the following work could be straightforwardly transported from the naive presentation above to the rigorous one.

where the dashed line under S_i means a full line whenever $\mathrm{coind}_{\mathsf{r}}(i) = 1$*, and an absence of a line otherwise. (When we apply such a rule* r *to some arguments we implicitly suppose that they belong to its domain of definition.) We denote by* $\mathrm{DT}^{\infty}_{\mathcal{D}}$ *the set of all (non-wellfounded) derivation trees generated by a family* $\mathcal{D}$ *of construction rules such that all infinite branches cross infinitely many double lines (*i.e. *coinductive premises), that we call* many-sorted terms.

▶ **Examples 6.**

1. Pre-proofs in your favourite non-wellfounded proof system, *e.g.* for some logic with fixed points, can be presented as an instance of this system (see Section 4.1 for a detailed presentation of the non-wellfounded system $\mu\mathsf{MALL}^{\infty}$ for multiplicative-additive linear logics with fixed points).
2. The set $\mathrm{T}^{\infty}_{\Sigma}$ of first-order terms on the signature Σ can be seen as the many-sorted terms generated by $\mathcal{S} \coloneqq \{\bullet\}$ and $\mathcal{D}_{\Sigma} \coloneqq \left\{\mathrm{Var}_x : \mathcal{S}^0 \to \mathcal{S} \;\middle|\; x \in \mathcal{V}\right\} \cup \left\{\mathrm{Cons}_{\mathsf{c}} : \mathcal{S}^{\mathrm{ar}(\mathsf{c})} \to \mathcal{S} \;\middle|\; \mathsf{c} \in \Sigma\right\}$, together with $\mathrm{coind}_{\mathrm{Cons}_{\mathsf{c}}}(i) \coloneqq 1$ for all c and i.
3. Similarly, *abc*-infinitary λ-terms can be seen as the many-sorted terms generated by $\mathcal{S} \coloneqq \{\bullet\}$ and $\mathcal{D}^{abc}_{\Lambda} \coloneqq \left\{\mathrm{Var}_x : \mathcal{S}^0 \to \mathcal{S} \;\middle|\; x \in \mathcal{V}\right\} \cup \{\mathrm{Abs}_x : \mathcal{S} \to \mathcal{S} \mid x \in \mathcal{V}\} \cup \left\{\mathrm{App} : \mathcal{S}^2 \to \mathcal{S}\right\}$, together with $\mathrm{coind}_{\mathrm{Abs}_x}(1) \coloneqq a$, $\mathrm{coind}_{\mathrm{App}}(1) \coloneqq b$ and $\mathrm{coind}_{\mathrm{App}}(2) \coloneqq c$.

By abuse of notation, we write $s = \mathsf{r}(s_1, \dots, s_k)$ to express that **(i)** the last construction rule of the many-sorted term s is (r), so that there are sorts $S_1, \dots, S_k \in \mathcal{S}$ such that s has conclusion $\mathsf{r}(S_1, \dots, S_k)$, **(ii)** each derivation s_i has conclusion S_i and is the subtree rooted at the ith premise of the concluding (r). Using this notation, a definition of $\mathrm{DT}^{\infty}_{\mathcal{D}}$ using fixed points is given by $\mathrm{DT}^{\infty}_{\mathcal{D}} \coloneqq \nu X_1.\mu X_0.\coprod_{\mathsf{r}\in\mathcal{D}} \mathsf{r}\left(X_{\mathrm{coind}_{\mathsf{r}}(1)}, \dots, X_{\mathrm{coind}_{\mathsf{r}}(\mathrm{ar}(\mathsf{r}))}\right)$. This construction can also be performed in richer categories than the category of sets, *e.g.* the category of nominal sets, as already suggested in Footnote 1. (Concentrating on such a "nu-mu" data type is what allows to express all the notions of rewriting we have in mind; one could however extend our work to any alternation of fixed points.) In the following, we fix a set $\mathcal{D}$ as in Definition 5.

▶ **Definition 7.** *For each* $S \in \mathcal{S}$*, we define a nullary construction rule* $\mathsf{trunc}_S : \mathcal{S}^0 \to \mathcal{S}$*,* $() \mapsto S$*,* i.e. *a rule adding* S *as an axiom. The* truncation *at depth* $d \in \mathbf{N}$ *of a many-sorted term* $s \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ *with conclusion* $S \in \mathcal{S}$ *is defined inductively by* $\lfloor s \rfloor_0 \coloneqq \mathsf{trunc}_S()$ *and* $\lfloor \mathsf{r}(s_1, \dots, s_k) \rfloor_{d+1} \coloneqq \mathsf{r}(\lfloor s_1 \rfloor_{d+1-\mathrm{coind}_{\mathsf{r}}(1)}, \dots, \lfloor s_k \rfloor_{d+1-\mathrm{coind}_{\mathsf{r}}(k)})$*. The set* $\mathrm{DT}^{\infty}_{\mathcal{D}}$ *is equipped with a metric* $\mathbf{d}$ *defined by* $\mathbf{d}(s,t) \coloneqq \inf\left\{2^{-d} \;\middle|\; \lfloor s \rfloor_d \simeq \lfloor t \rfloor_d\right\}$*, where* $\simeq$ *is the equivalence relation inductively generated on truncated many-sorted terms by equating all rules* trunc_S*.*

▶ **Definition 8.** *A set* $\longrightarrow \subseteq \mathrm{DT}^{\infty}_{\mathcal{D}} \times \mathrm{DT}^{\infty}_{\mathcal{D}}$ *of* root steps *generates a relation* $\longrightarrow$ *by the following set of inductive rules:*

$$\frac{s \longrightarrow t}{s \longrightarrow_0 t} \qquad \frac{s_i \longrightarrow_d s_i' \quad 1 \le i \le \mathrm{ar}(\mathsf{r})}{\mathsf{r}(s_1, \dots, s_i, \dots, s_{\mathrm{ar}(\mathsf{r})}) \longrightarrow_{d+\mathrm{coind}_{\mathsf{r}}(i)} \mathsf{r}(s_1, \dots, s_i', \dots, s_{\mathrm{ar}(\mathsf{r})})}$$

where, as in Equation (4)*, we sometimes annotate* $\longrightarrow$ *with the depth at which the rewriting step occurs. As in Definition 1, we write* $s \longrightarrow^{\infty} t$ *whenever there is a strongly converging reduction sequence from* s *to* t*.*

The constructions of first-order rewriting in a given ITRS (Section 2.1) and of β-reduction (Section 2.2) are particular cases of this definition applied to the derivations generated by the sets of construction rules $\mathcal{D}_{\Sigma}$ and $\mathcal{D}^{abc}_{\Lambda}$ from Examples 6, respectively. (In particular the root steps for a given ITRS are given by the first rule of Equation (4), namely by the closure of all rewrite rules under substitutions.) Theorems 3 and 4, which provided a coinductive presentation of strongly converging rewriting sequences in these particular settings, can also be extended to the general framework we introduced.

▶ **Definition 9.** *For all ordinals $\gamma < \omega_1$, a relation $\xrightarrow[\gamma]{}{}^{\infty}$ on $\mathrm{DT}^{\infty}_{\mathcal{D}}$ is defined by the following rules:*

$$\frac{s \overset{}{\rightsquigarrow}_{\gamma,m} s' \qquad s' \xrightarrow[\gamma]{}{}^{\infty} t}{s \xrightarrow[\gamma]{}{}^{\infty} t}\ (\mathsf{split}) \qquad \frac{s_1 \xrightarrow[\gamma]{}{}^{\infty} s'_1 \quad \cdots \quad s_{\mathrm{ar}(\mathsf{r})} \xrightarrow[\gamma]{}{}^{\infty} s'_{\mathrm{ar}(\mathsf{r})}}{\mathsf{r}(s_1,\ldots,s_{\mathrm{ar}(\mathsf{r})}) \xrightarrow[\gamma]{}{}^{\infty} \mathsf{r}(s'_1,\ldots,s'_{\mathrm{ar}(\mathsf{r})})}\ (\mathsf{lift}_{\mathsf{r}})$$

where a rule $(\mathsf{lift}_{\mathsf{r}})$ *is defined for each* $\mathsf{r} \in \mathcal{D}$*, and its ith premise is coinductive iff* $\mathrm{coind}_{\mathsf{r}}(i) = 1$.

▶ **Theorem 10.** *$\longrightarrow^{\infty}$ is the union of all relations $\xrightarrow[\gamma]{}{}^{\infty}$ (over ordinals $\gamma < \omega_1$).*

Proof sketch. The proof is as in [14, Thm. 5.2]. If there is a strongly convergent sequence of length γ from s to t (denote it by $s \overset{\gamma}{\Longrightarrow}^{\infty} t$), we build a derivation $s \xrightarrow[\gamma]{}{}^{\infty} t$ by (co)induction, using the fact that only finitely many rewriting steps in the sequence are not performed above a coinductive premise of a construction rule, so that the sequence can be written: $s \longrightarrow^{*} s'_1 \overset{\delta_1}{\Longrightarrow}^{\infty} t_1 \longrightarrow^{*} s'_2 \overset{\delta_2}{\Longrightarrow}^{\infty} \cdots \overset{\delta_{m+1}}{\Longrightarrow}^{\infty} t$ with $\forall 1 \leq i \leq m,\ \delta_i < \gamma$ and $\delta_{m+1} \leq \gamma$, which exactly gives the premises of the rule (split).

Conversely, the proof proceeds by well-founded induction on γ such that $s \xrightarrow[\gamma]{}{}^{\infty} t$. The induction hypothesis implies that $\overset{}{\rightsquigarrow}_{\gamma,m} \subseteq \Longrightarrow^{\infty}$, for all m, hence we obtain $s \Longrightarrow^{\infty} s_1 \xrightarrow[\gamma]{}{}^{\infty} t$. By iterating in the inductive premises of $s_1 \xrightarrow[\gamma]{}{}^{\infty} t$, we obtain $s_1 \Longrightarrow^{\infty} s'_1 \xrightarrow[\gamma]{}{}^{\infty} t$. Then we iterate in the coinductive premises of $s'_1 \xrightarrow[\gamma]{}{}^{\infty} t$, building $s'_1 \Longrightarrow^{\infty} s'_2 \Longrightarrow^{\infty} \ldots$. Since this sequence does not contain any $\longrightarrow_0$ any more, we can prove strong convergence. ◀

3 A generic compression lemma

As recalled in the introduction, the compression lemma is a result about several kinds of infinitary rewriting systems stating that strongly convergent rewriting sequences of arbitrary ordinal length can be "compressed" into sequences of length ω with the same source and target. The goal of this section is to characterise this compression property in the general coinductive framework we just introduced. To do so, we first provide a coinductive counterpart to strongly convergent sequences of length ω, then we prove the main result of the paper (Theorem 14), a characterisation identifying "just what needs to be adapted to each system" for compression to hold. We conclude by applying the theorem to our running examples, first-order rewriting and λ-calculus. (Let us also already mention that Section 3.4 details the proof for the concrete example we gave in Equations (1) and (2) in the introduction.)

Whereas Theorem 10 above was an extension of [14] to our newly introduced "many-sorted terms", our characterisation of the compression property is completely new to our knowledge. In the unique coinductive proof of compression in the literature, namely the Coq/Rocq proof of [14] (which is limited to first-order rewriting, hence only corresponds to our Theorem 20) the slightly different definition of $\longrightarrow^{\infty}$ featuring least and greatest fixed points to constrain the use of coinduction (where we preferred ordinal annotations) results in a completely different treatment of the inductive part of the proof. All other existing proofs of compression properties are formulated in a topology-based setting (usually using the same transfinite induction as in [18], which cannot be straightforwardly translated coinductively since "the ordinal length of a rewriting sequence" has no clear coinductive counterpart); they do of course yield an indirect proof in the coinductive setting, thanks to our Theorem 10. Thus, besides its generality, another advantage of our approach with respect to all those is to provide a direct, explicit coinductive procedure for compressing derivations of infinitary rewritings.

In the following we again fix sets $\mathcal{S}$ and $\mathcal{D}$ as in Definition 5, and a rewriting relation $\longrightarrow$ on $\mathrm{DT}^{\infty}_{\mathcal{D}}$ as in Definition 8.

3.1 Compressed rewriting sequences coinductively

We first need to give a coinductive presentation of strongly converging rewriting sequences of length at most ω. This is quite straightforward, and is an adaption of [14, Equation 9.3].

▶ **Definition 11.** *The relation* $\longrightarrow^{\omega}$ *of* compressed infinitary rewriting *is defined on* $\mathrm{DT}^{\infty}_{\mathcal{D}}$ *by* $\longrightarrow^{\omega} := \underset{0}{\longrightarrow}^{\infty}$, i.e. *it is defined by the following rules* $(\mathsf{split}^{\omega})$ *and, for each* $\mathsf{r} \in \mathcal{D}$, $(\mathsf{lift}^{\omega}_{\mathsf{r}})$*:*

$$\frac{s \longrightarrow^{*} s' \qquad s' \underset{\gamma}{\longrightarrow}^{\omega} t}{s \longrightarrow^{\omega} t}\ (\mathsf{split}^{\omega}) \qquad \frac{s_1 \longrightarrow^{\omega} s'_1 \quad \dots \quad s_{\mathrm{ar}(\mathsf{r})} \longrightarrow^{\omega} s'_{\mathrm{ar}(\mathsf{r})}}{\mathsf{r}(s_1, \dots, s_{\mathrm{ar}(\mathsf{r})}) \longrightarrow^{\omega} \mathsf{r}(s'_1, \dots, s'_{\mathrm{ar}(\mathsf{r})})}\ (\mathsf{lift}^{\omega}_{\mathsf{r}})$$

where, as in Definition 5, the i*th premise of* $(\mathsf{lift}^{\omega}_{\mathsf{r}})$ *is coinductive iff* $\mathrm{coind}_{\mathsf{r}}(i) = 1$.

▶ **Lemma 12.** *For* $s, t \in \mathrm{DT}^{\infty}_{\mathcal{D}}$, $s \longrightarrow^{\omega} t$ *iff there is a strongly converging sequence of length at most* ω *from* s *to* t.

We say that $\longrightarrow$ has the *compression property* if $\longrightarrow^{\infty} = \longrightarrow^{\omega}$.

3.2 A characterisation of the compression property

We reach the core technical content of this paper, where we provide a characterisation of compression for any rewriting system presented in the generic framework introduced in the previous section. We state this result just after the following useful observations (which are all straightforward consequences of Definition 9).

▶ **Lemma 13.**

1. *If* $s \underset{\gamma}{\longrightarrow}^{\infty} t$ *and* $\delta \geq \gamma$, *then* $s \underset{\delta}{\longrightarrow}^{\infty} t$.
2. *The relation* $\underset{\gamma}{\longrightarrow}^{\infty}$ *is reflexive.*
3. *If* $s \underset{\gamma}{\longrightarrow}^{\infty} t \underset{\delta}{\longrightarrow}^{\infty} u$, *then* $s \underset{\epsilon}{\longrightarrow}^{\infty} u$ *for* $\epsilon := \max(\gamma + 1, \delta)$.
4. *Items (1)–(3) also hold with* $\underset{\gamma}{\longrightarrow}^{\infty}$, $\underset{\delta}{\longrightarrow}^{\infty}$ *instead of* $\underset{\gamma}{\longrightarrow}^{\infty}$, $\underset{\delta}{\longrightarrow}^{\infty}$.
5. *As a consequence, the relation* $\longrightarrow^{\infty}$ *is reflexive and transitive.*
6. *The inclusions* $\underset{\delta}{\longrightarrow}^{\infty} \subset \underset{\delta}{\longrightarrow}^{\infty}$ *hold for every ordinal* δ, *hence also* $\longrightarrow^{\infty} \subset \longrightarrow^{\infty}$.

▶ **Theorem 14.** *We define the following properties of the rewriting relation* $\longrightarrow$*:*

$$\mathfrak{P}_{\delta}: \quad \forall n \in \mathbf{N},\ \forall s, s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ s \underset{\delta, n}{\rightsquigarrow} s' \ \Rightarrow\ \exists s'' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \exists \epsilon < \delta,\ \ s \longrightarrow^{*} s'' \underset{\epsilon}{\longrightarrow}^{\infty *} s',$$

$$\mathfrak{Q}: \quad \forall \delta,\ \forall s, t, t' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ \mathfrak{P}_{\delta}\ \wedge\ s \underset{\delta}{\longrightarrow}^{\infty} t \longrightarrow t' \ \Rightarrow\ \exists s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ s \longrightarrow^{*} s' \underset{\delta}{\longrightarrow}^{\infty} t'.$$

Then $\longrightarrow$ *has the compression property iff the property* $\mathfrak{Q}$ *holds.*

The property $\mathfrak{Q}$ seems quite convoluted at first sight, but is in fact not very surprising: it essentially means that given a certain induction hypothesis (namely $\mathfrak{P}_{\delta}$), a rewriting sequence of ordinal length $\delta + 1$ (for an arbitrary infinite ordinal δ) can be turned into an equivalent sequence of length $p + \delta = \delta$ for some $p \in \mathbf{N}$ (recall that Theorem 10 translates rewriting sequences of length δ into $\underset{\delta}{\longrightarrow}^{\infty}$). When one tries to prove a compression lemma in a traditional way, using a transfinite induction and topological arguments, the key case of the proof is exactly the same, namely the case of a successor ordinal [30, § 12.7].

We now give the proof of Theorem 14, which starts with the following key lemmas.

▶ **Lemma 15.** *For all ordinal* δ *such that* $\mathfrak{P}_{\delta}$ *holds and for all many-sorted terms* $s, t \in \mathrm{DT}^{\infty}_{\mathcal{D}}$, *if* $s \underset{\delta}{\longrightarrow}^{\infty} t$ *then there exists* $s' \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ *such that* $s \longrightarrow^{*} s' \underset{\delta}{\longrightarrow}^{\infty} t$.

Proof. The derivation of $s \xrightarrow[\delta]{}{}^{\infty} t$ ends with the rule (split), with premises $s \overset{\delta,n}{\rightsquigarrow} t'$ and $t' \xrightarrow[\delta]{}{}^{\infty} t$ (for some $n \in \mathbf{N}$ and $t' \in \mathrm{DT}^{\infty}_{\mathcal{D}}$). By $\mathfrak{P}_\delta$ there are $s' \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ and $\epsilon < \delta$ such that $s \longrightarrow^* s' \xrightarrow[\epsilon]{}{}^{\infty *} t'$, and we can conclude by Lemmas 13(1) and 13(3). ◀

▶ **Lemma 16.** *If $\mathfrak{Q}$ holds then $\mathfrak{P}_\gamma$ holds for any ordinal γ.*

Proof. Instead of $\mathfrak{Q}$, we suppose the following property $\mathfrak{Q}'$:

$$\forall \delta,\ \forall s,t,t' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ \mathfrak{P}_\delta\ \wedge\ s \xrightarrow[\delta]{}{}^{\infty} t \longrightarrow t'\ \Rightarrow\ \exists s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ s \longrightarrow^* s' \xrightarrow[\delta]{}{}^{\infty} t'.$$

The only difference with the definition of $\mathfrak{Q}$ in Theorem 14 is that the two occurrences of $\xrightarrow[\delta]{}{}^{\infty}$ have been replaced with $\xrightarrow[\delta]{}{}^{\infty}$. This is straightforwardly weaker by Lemma 13(6) (for the first occurrence) and Lemma 15 (for the second one).

Then we proceed by well-founded induction on γ, *i.e.* we suppose that $\mathfrak{P}_\delta$ holds for any ordinal $\delta < \gamma$ and we prove that $\mathfrak{P}_\gamma$ holds, that is to say:

$$\forall m \in \mathbf{N},\ \forall s,s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ s \overset{\gamma,m}{\rightsquigarrow} s'\ \Rightarrow\ \exists s'' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \exists \delta < \gamma,\ \ s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s'.$$

We do this by induction on $m \in \mathbf{N}$. Take $s,s' \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ such that $s \overset{\gamma,m}{\rightsquigarrow} s'$. We want to build $s'' \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ such that $\exists \delta < \gamma,\ s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s'$. If $m = 0$ the result is immediate: observe that $s \overset{\gamma,0}{\rightsquigarrow} s'$ means that $s \longrightarrow^* s'$ and use Lemma 13(2). Otherwise, $s \overset{\gamma,m}{\rightsquigarrow} s'$ can be decomposed as follows: $s \overset{\gamma,m-1}{\rightsquigarrow} s'_m \xrightarrow[\delta_m]{}{}^{\infty} t'_m \longrightarrow^* s'$, with $\delta_m < \gamma$. By iterated applications of $\mathfrak{Q}'$, using the induction hypothesis on δ_m (*i.e.* the fact that $\mathfrak{P}_{\delta_m}$ holds), there is an $s''_m \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ such that: $s \overset{\gamma,m-1}{\rightsquigarrow} s'_m \longrightarrow^* s''_m \xrightarrow[\delta_m]{}{}^{\infty *} s'$. Notice that $s \overset{\gamma,m-1}{\rightsquigarrow} s'_m \longrightarrow^* s''_m$ can be reformulated as $s \overset{\gamma,m-1}{\rightsquigarrow} s''_m$, whence we can apply the induction hypothesis on $m-1$ and obtain a term s'' and an ordinal $\epsilon < \gamma$ such that: $s \longrightarrow^* s'' \xrightarrow[\epsilon]{}{}^{\infty *} s''_m \xrightarrow[\delta_m]{}{}^{\infty *} s'$, which can be simplified as $s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s'$ with $\delta \coloneqq \max(\epsilon, \delta_m) < \gamma$, thanks to Lemma 13(1). ◀

Proof of Theorem 14. Assume $\mathfrak{Q}$ holds. We want to design a procedure using a derivation of $s \longrightarrow^{\infty} t$ to coinductively produce a derivation of $s \longrightarrow^{\omega} t$. To formalise this, let us add the following rule to Definition 11: $\dfrac{s \xrightarrow[\gamma]{}{}^{\infty} t}{s \longrightarrow^{\omega} t}\ (\mathsf{comp})$, and show that any derivation of $s \longrightarrow^{\omega} t$ can be coinductively turned into a derivation of $s \longrightarrow^{\omega} t$ that does not use this rule (comp), *i.e.* that the rule is admissible. Observe that (comp) can occur at most once in any branch of a derivation, so that its occurrences delimit a compressed prefix and subtrees remaining to be compressed; we have to prove **(i)** that in this situation the compressed prefix can be extended wherever the rule (comp) occurs (which we do by analysing the remaining subtrees), so that iterating the process results in an infinitary compressed derivation (as (comp) will be pushed to infinity, see Section 3.4 for an illustration), and **(ii)** that this infinitary derivation is valid, *i.e.* each of its infinite branches crosses infinitely often a coinductive premise of a rule ($\mathsf{lift}^{\omega}_{\mathsf{r}}$).

We start from a derivation whose conclusion $s \longrightarrow^{\omega} t$ is obtained by rule (comp), with hypothesis $s \xrightarrow[\gamma]{}{}^{\infty} t$. The latter can only be obtained through the rule (split), hence there are $s' \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ and $m \in \mathbf{N}$ such that $s \overset{\gamma,m}{\rightsquigarrow} s' \xrightarrow[\gamma]{}{}^{\infty} t$. Then by Lemma 16, $\mathfrak{P}_\gamma$ holds, hence $s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s' \xrightarrow[\gamma]{}{}^{\infty} t$ for some $s'' \in \mathrm{DT}^{\infty}_{\mathcal{D}}$ and some ordinal $\delta < \gamma$. We apply Lemma 13(3) and obtain $s \longrightarrow^* s'' \xrightarrow[\gamma]{}{}^{\infty} t$. In particular we have the derivation below left for some $\mathsf{r} \in \mathcal{D}$, hence we can form the derivation below right:

$$\dfrac{\begin{array}{ccc} \vdots & & \vdots \\ s''_1 \xrightarrow[\gamma]{}{}^{\infty} t_1 & \cdots & s''_k \xrightarrow[\gamma]{}{}^{\infty} t_k \end{array}}{s'' = \mathsf{r}(s_1,\dots,s_{\mathrm{ar}(\mathsf{r})}) \xrightarrow[\gamma]{}{}^{\infty} \mathsf{r}(s'_1,\dots,s'_{\mathrm{ar}(\mathsf{r})}) = t}\ (\mathsf{lift}_{\mathsf{r}}) \qquad \dfrac{s \longrightarrow^* s'' \qquad \dfrac{\begin{array}{ccc} \vdots & & \vdots \\ s''_1 \xrightarrow[\gamma]{}{}^{\infty} t_1 & & s''_k \xrightarrow[\gamma]{}{}^{\infty} t_k \\ \sim\sim\sim\ (\mathsf{comp}) & & \sim\sim\sim\ (\mathsf{comp}) \\ s''_1 \longrightarrow^{\omega} t_1 & \cdots & s''_k \longrightarrow^{\omega} t_k \end{array}}{s'' \longrightarrow^{\omega} t}\ (\mathsf{lift}^{\omega}_{\mathsf{r}})}{s \longrightarrow^{\omega} t}\ (\mathsf{split}^{\omega})$$

and we iterate the process coinductively in each subtree rooted at $s_i'' \longrightarrow^{\omega} t_i$. In addition the derivation we produce is correct (*i.e.* every of its infinite branches crosses infinitely many coinductive premises) because its structure exactly follows the one of the target t: each $(\mathsf{split}^{\omega})$ followed by $(\mathsf{lift}^{\omega}_{\mathsf{r}})$ in the obtained derivation corresponds to an (r) in the derivation t, hence the validity of the former is ensured by the validity of the latter.

Conversely, if the compression property holds then any reduction $s \xrightarrow[\delta]{}{}^{\infty} t \longrightarrow t'$ can be compressed to $s \longrightarrow^{\omega} t'$. This is equivalent to $s \xrightarrow[0]{}{}^{\infty} t'$, hence $s \xrightarrow[\delta]{}{}^{\infty} t'$ by Lemma 13(1). Finally, $s \longrightarrow^{*} s \xrightarrow[\delta]{}{}^{\infty} t'$. ◀

3.3 Compression for left-linear (coinductive) first-order rewriting

In this subsection, we fix an ITRS $\mathcal{R}$ as in Section 2.1, and we prove the compression lemma for first-order rewriting: if $\mathcal{R}$ is left-linear and left-finite then it has the compression property, as proved in [18] in the topological setting. Our proof relies on Theorem 14 and on two lemmas: the first one isolates the finite portion of an infinite rewriting that is necessary to produce a given finite prefix of the output (*pattern extraction*), the second one performs the remaining infinitary rewriting on the branches starting from this prefix (*pattern filling*). These lemmas form the base case of a straightforward induction proving the property $\mathfrak{Q}$. A concrete illustration of the whole compression procedure is given in Section 3.4.

▶ **Definition 17.** *A term $l \in \mathrm{T}_{\Sigma}$ is* linear *if no variable occurs twice (or more) in l. A rewrite rule (l, r) is* left-linear *(resp.* left-finite*) if l is linear (resp. $l \in \mathrm{T}_{\Sigma}$). An* ITRS *$\mathcal{R}$ is* left-linear *(resp.* left-finite*) if each rule $(l, r) \in \mathcal{R}$ is so.*

▶ **Lemma 18** (pattern extraction)**.** *Consider an ordinal δ such that $\mathfrak{P}_{\delta}$ holds, and $s \in \mathrm{T}^{\infty}_{\Sigma}$, $\sigma : \mathcal{V} \to \mathrm{T}^{\infty}_{\Sigma}$ and a linear $l \in \mathrm{T}_{\Sigma}$ such that $s \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l$. Then there is a substitution $\tau : \mathcal{V} \to \mathrm{T}^{\infty}_{\Sigma}$ such that $s \longrightarrow^{*} \tau \cdot l$ and $\forall x \in \mathcal{V},\ \tau(x) \xrightarrow[\delta]{}{}^{\infty} \sigma(x)$.*

Proof. By induction over l. If $l = x$, then we define $\tau(x) \coloneqq s$ and $\tau(y) \coloneqq \sigma(y)$ otherwise. If $l = \mathsf{c}(l_1, \dots, l_k)$ then there is derivation as follows:

$$\dfrac{s \overset{\delta,n}{\rightsquigarrow} \mathsf{c}(t_1, \dots, t_k) \qquad \dfrac{t_1 \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_1 \quad \dots \quad t_k \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_k}{\mathsf{c}(t_1, \dots, t_k) \xrightarrow[\delta]{}{}^{\infty} \mathsf{c}(\sigma \cdot l_1, \dots, \sigma \cdot l_k)}\ (\mathsf{lift}_{\mathsf{c}})}{s \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l = \mathsf{c}(\sigma \cdot l_1, \dots, \sigma \cdot l_k)}\ (\mathsf{split})$$

By $\mathfrak{P}_{\delta}$ applied to the left premise of (split), there are an ordinal $\epsilon < \delta$ and $s_1, \dots, s_k \in \mathrm{T}^{\infty}_{\Sigma}$ such that $s \longrightarrow^{*} \mathsf{c}(s_1, \dots, s_k) \xrightarrow[\epsilon]{}{}^{\infty *} \mathsf{c}(t_1, \dots, t_k)$. As a consequence, for all $1 \leq i \leq k$ there are reductions $s_i \xrightarrow[\epsilon]{}{}^{\infty *} t_i \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_i$, therefore by Lemma 13(3) $s_i \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_i$. By induction there exist substitutions τ_i such that $s_i \longrightarrow^{*} \tau_i \cdot l_i$ and $\forall x \in \mathcal{V},\ \tau_i(x) \xrightarrow[\delta]{}{}^{\infty} \sigma(x)$. By linearity of l, each variable occurring in l occurs in exactly one of the l_i hence we can define, for all $x \in \mathcal{V}$, $\tau(x) \coloneqq \tau_i(x)$ if x occurs in some l_i and $\tau(x) \coloneqq \sigma(x)$ otherwise. As a consequence, $\forall x \in \mathcal{V},\ \tau(x) \xrightarrow[\delta]{}{}^{\infty} \sigma(x)$. In addition, $s \longrightarrow^{*} \mathsf{c}(s_1, \dots, s_k) \longrightarrow^{*} \mathsf{c}(\tau_1 \cdot l_1, \dots, \tau_k \cdot l_k) = \tau \cdot l$. ◀

▶ **Lemma 19** (pattern filling)**.** *For all $r \in \mathrm{T}^{\infty}_{\Sigma}$ and $\sigma, \tau : \mathcal{V} \to \mathrm{T}^{\infty}_{\Sigma}$, if $\forall x \in \mathcal{V},\ \tau(x) \xrightarrow[\delta]{}{}^{\infty} \sigma(x)$ then $\tau \cdot r \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot r$.*

Proof. By coinduction on $r \in \mathrm{T}^{\infty}_{\Sigma}$. If r is just a variable the result follows by the assumption. Otherwise $r = \mathsf{c}(s_1, \dots, s_{\mathrm{ar}(\mathsf{c})})$. For all $1 \leq i \leq \mathrm{ar}(\mathsf{c})$ we build $\tau \cdot s_i \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot s_i$ coinductively. By applying the rule $(\mathsf{lift}_{\mathsf{c}})$ we obtain $\tau \cdot r \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot r$ and we conclude by Lemma 13(6). ◀

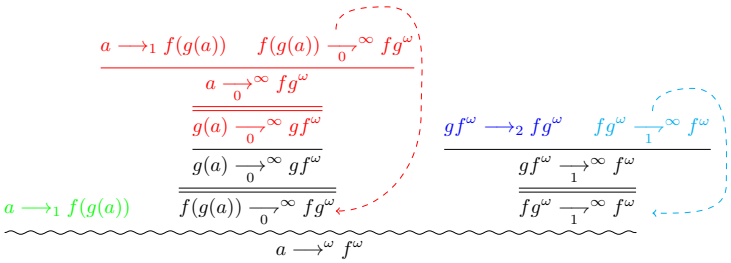
▶ **Theorem 20.** *If $\mathcal{R}$ is left-linear and left-finite then $\longrightarrow$ satisfies the property $\mathfrak{Q}$, and thus has the compression property.*

Proof sketch. Take an ordinal δ and terms $s, t, t' \in \mathrm{T}^{\infty}_{\Sigma}$ such that $\mathfrak{P}_{\delta}$ and $s \xrightarrow[\delta]{}{}^{\infty} t \longrightarrow t'$. We want to build a term $s' \in \mathrm{T}^{\infty}_{\Sigma}$ and reductions $s \longrightarrow^{*} s' \xrightarrow[\delta]{}{}^{\infty} t'$. By induction over $t \longrightarrow t'$, **(i)** if there are a rule $(l, r) \in \mathcal{R}$ and a substitution $\sigma : \mathcal{V} \to \mathrm{T}^{\infty}_{\Sigma}$ such that $t = \sigma \cdot l$ and $t' = \sigma \cdot r$, then we apply Lemmas 18 and 19. **(ii)** otherwise we proceed by induction. ◀

3.4 An example in first-order rewriting

In Equations (1) and (2) in the introduction, we gave the example of the ITRS defined on the signature $\Sigma \coloneqq \{a, f, g\}$ with $\mathrm{ar}(a) \coloneqq 0$, $\mathrm{ar}(f) \coloneqq 1$ and $\mathrm{ar}(g) \coloneqq 1$, by the following rules: $a \longrightarrow_1 f(g(a))$ and $g(f(x)) \longrightarrow_2 f(x)$; we presented a strongly converging rewriting sequence $a \longrightarrow^{\infty} f(f(f(\dots)))$ of length $\omega \cdot 2$ as well as an equivalent sequence of length ω. Let us translate the initial sequence into a derivation of $a \longrightarrow^{\infty} f(f(f(\dots)))$ in the coinductive formalism from Definition 2, and describe how the proofs of Theorems 14 and 20 compress it in a derivation of $a \longrightarrow^{\omega} f(f(f(\dots)))$.

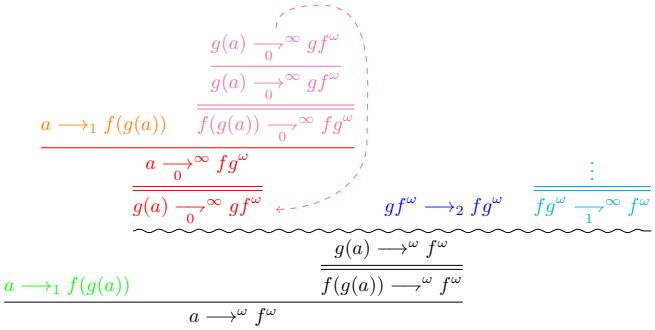
We use the following notations: $f^{\omega} \coloneqq f(f(f(\dots)))$, $fg^{\omega} \coloneqq f(g(f(g(\dots))))$, and $gf^{\omega} \coloneqq g(f(g(f(\dots))))$. The initial sequence $a \longrightarrow^{\infty} f^{\omega}$ of length $\omega \cdot 2$, namely Equation (1), corresponds to the following derivation:



(6)

We omit the label of the rules but they should be clear: simple rules are occurrences of (split) or $(\mathsf{split}^{\omega})$, double rules are occurrences of $(\mathsf{lift}_{\mathsf{c}})$ or $(\mathsf{lift}^{\omega}_{\mathsf{c}})$ for $\mathsf{c} \in \{f, g\}$, and the wavy rule is the rule (comp) from the proof of Theorem 14, delimiting the compressed prefix and branches yet to be compressed.

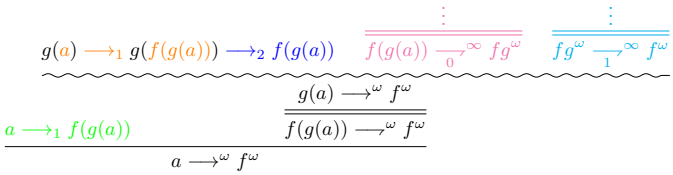
Next, we apply transitivity (Lemma 13(3)). The red, blue and cyan subderivations are reused without modification, and the compressed prefix grows:



(7)

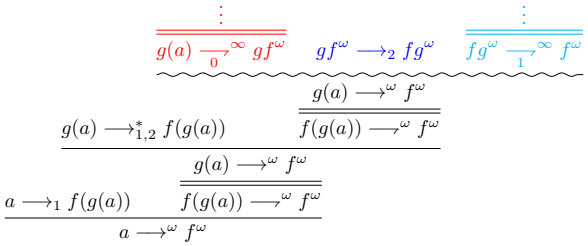
Now we need to permute $g(a) \xrightarrow[0]{}{}^{\infty} gf^{\omega}$ and $gf^{\omega} \longrightarrow_2 fg^{\omega}$. To do so, we do *pattern extraction* (Lemma 18): we start with the necessary steps of the former (in fact only the first step $a \longrightarrow_1 f(g(a))$, performed in the context $g(-)$) so that the latter can be performed. Then we do *pattern filling* (Lemma 19), performing the remainder of the former rewriting.

Concretely:



(8)

Finally, observe that the pink and cyan subderivations are exactly the ones we merged between Equations (6) and (7), hence we can apply again the transitivity Lemma 13(3):



(9)

We obtain again a wider compressed prefix. We continue coinductively above the wavy line, as in Equation (7).

3.5 Compression for (coinductive) infinitary λ-calculi

The very same work can be done for the *abc*-infinitary λ-calculi introduced in Section 2.2:

▶ **Theorem 21.** *The relation* $\longrightarrow$ *on* Λ^{abc} *satisfies the property* $\mathfrak{Q}$*, and thus has the compression property.*

The proof is detailed in the appendices of the long version of this article [10]. A noticeable consequence of the compression property in this setting is that our coinductive presentation of $\longrightarrow^{\infty}$ (as in Theorem 4) is equivalent to its usual one in the literature, where only $\longrightarrow^{\omega}$ is defined [15, 8].

4 Compressing μMALL$^{\infty}$ cut-elimination sequences

In this last section, we prove compression for an example of the other kind of infinitary rewriting appearing in the literature: infinitary cut-elimination in non-wellfounded proof systems. We focus on the multiplicative-additive linear logic with fixed points μMALL and the non-wellfounded proof system μMALL$^{\infty}$ for this logic, as a compression lemma for infinitary cut-elimination in this system can crucially be used to prove cut-elimination for a whole range of non-wellfounded proof systems for (intuitionistic, classical, linear) logics with fixed points [26, Prop. 26].

4.1 The non-wellfounded proof system μMALL$^{\infty}$

We recall the definitions of the formulæ of μMALL and the rules of μMALL$^{\infty}$, from which we build a set $\mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$ of non-wellfounded many-sorted terms. These trees are usually called *pre-proofs* in the literature, where *proofs* are then the subset of pre-proofs satisfying some validity criterion. However compression and validity are quite orthogonal concerns, as **(i)** the following development can be performed entirely at the more general level of pre-proofs,

$$\frac{}{\vdash F, F^\perp}\ (\mathsf{ax}_F) \qquad \frac{\vdash \Gamma, F \quad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta}\ (\mathsf{cut}) \qquad \frac{\vdash F_{1,1}, \dots, F_{1,n_1} \quad \dots \quad \vdash F_{k,1}, \dots, F_{k,n_k}}{\vdash \Gamma}\ (\mathsf{mcut}_{k,\vec{n},\sqcup})$$

$$\frac{\vdash F_{\sigma(1)}, \dots, F_{\sigma(n)}}{\vdash F_1, \dots, F_n}\ (\mathsf{x}_\sigma) \qquad \frac{}{\vdash 1}\ (1) \qquad \frac{}{\vdash \Gamma, \top}\ (\top_\Gamma) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ (\bot) \qquad \frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ (\parr) \qquad \frac{\vdash \Gamma, F \quad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ (\otimes)$$

$$\frac{\vdash \Gamma, F_i}{\vdash \Gamma, F_0 \oplus F_1}\ (\oplus_{i, F_{1-i}}) \qquad \frac{\vdash \Gamma, F \quad \vdash \Gamma, G}{\vdash \Gamma, F \,\&\, G}\ (\&) \qquad \frac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu) \qquad \frac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$$

Figure 1 The derivation rules of $\mu\mathsf{MALL}^\infty$. $F[G/X]$ denotes the formula obtained by substituting the fixed point variable X with G in F, in a capture-avoiding manner.

(ii) the validity of the limit of a compressed cut-elimination sequence is not ensured by the compression lemma, but by the fact that the original transfinite sequence has a limit that is a valid derivation; hence there is no need to discuss or even define validity in this work.

We closely follow the exposition from [27], making some technicalities more precise and showing how this material can be encoded as an instance of the general framework we introduced in Section 2.

Fix a set $\mathcal{A}$ of *atomic formulæ* and a countable set $\mathcal{X}$ of *fixed point variables*. The set $\mathcal{F}_0$ of $\mu\mathsf{MALL}$ *pre-formulæ* is defined by induction by:

$$\begin{aligned} \mathcal{F}_0 \quad \ni \quad F, G, \dots \quad \coloneqq \quad & A \mid A^\perp \mid 0 \mid 1 \mid \top \mid \bot \mid F \parr G \mid F \otimes G \mid F \oplus G \mid F \,\&\, G \\ & \mid X \mid \mu X.F \mid \nu X.F \qquad\qquad (A \in \mathcal{A}, X \in \mathcal{X}). \end{aligned}$$

The set $\mathcal{F}$ of $\mu\mathsf{MALL}$ *formulæ* is the set of all pre-formulæ containing no free fixed point variable (*i.e.* each $X \in \mathcal{X}$ only occurs in the scope of fixed point constructors μX and νX). The set $\mathcal{S}$ of $\mu\mathsf{MALL}$ *sequents* is the set of all finite lists of formulæ, denoted by $\vdash F_1, \dots, F_n$. As usual, portions of sequents are denoted by Γ, Δ, etc. Negation is the involution $(-)^\perp : \mathcal{F} \to \mathcal{F}$ inductively defined by: $(A^\perp)^\perp \coloneqq A$, $0^\perp \coloneqq \top$, $1^\perp \coloneqq \bot$, $(F \parr G)^\perp \coloneqq F^\perp \otimes G^\perp$, $(F \oplus G)^\perp \coloneqq F^\perp \,\&\, G^\perp$ and $(\mu X.F)^\perp \coloneqq \nu X.F^\perp$.

We present the derivation rules of the system $\mu\mathsf{MALL}^\infty$ in Figure 1. Notice that **(i)** instead of the usual binary exchange rule, we define a family of rules (x_σ) parametrised by a permutation $\sigma : [1, n] \to [1, n]$, and acting on sequents of length n; **(ii)** for the rules (ax), $(\top)$ and $(\oplus)$ to be functional, we also need to present them as families of rules parametrised respectively by $F \in \mathcal{F}$, $\Gamma \in \mathcal{S}$, and $(i, F_{i-1}) \in \{0, 1\} \times \mathcal{F}$; **(iii)** the system features an n-ary cut rule, the *multicut rule* [16, 3]. Indeed, cut-elimination theorems for $\mu\mathsf{MALL}^\infty$ rely on moving finitely many consecutive cuts upwards (and not of single cuts): a multicut represents such a tree prefix. The multicut rule is parametrised by $k \in \mathbf{N}$, $\vec{n} \coloneqq (n_1, \dots, n_k) \in \mathbf{N}^k$ and a relation $\sqcup \subset (\mathbf{N}^2)^2$ (a symbol that is reminiscent of cut links in proof nets) relating dual formulæ F and $F^\perp$ that are cut against each other, under some conditions on $\sqcup$ ensuring that this is done sensibly, *e.g.* no formula is cut against two different formulæ; the conclusion of the rule contains all the formulæ that have not been cut. A formal definition is given in the appendices of the long version of this article [10].

$\mu\mathsf{MALL}^\infty$ *pre-proofs* are the elements of $\mathrm{DT}^\infty_{\mu\mathsf{MALL}^\infty}$, the many-sorted terms obtained by the construction rules from Figure 1.

4.2 Compression of infinitary cut-elimination

As explained in the introduction of this article, cut-elimination theorems for non-wellfounded proof systems rely on a rewriting relation "moving the cuts upwards" so that the corresponding infinitary rewriting sequences produce cut-free derivations. For $\mu\mathsf{MALL}^\infty$ this rewriting relation is defined by three kinds of root rewriting steps:

1. steps handling technicalities related to multicuts, *e.g.* merging a cut into a multicut,
2. *principal steps*, corresponding to the situation where dual formulæ are (multi)cut against each other, for example:

$$\dfrac{\vec{Z}\quad \dfrac{\dfrac{\vdash \Gamma, F_0 \quad \vdash \Gamma, F_1}{\vdash \Gamma, F_0 \,\&\, F_1}\,(\&) \qquad \dfrac{\vdash \Delta, F_i^{\perp}}{\vdash \Delta, F_0^{\perp} \oplus F_1^{\perp}}\,(\oplus_{i,F_{i-1}})}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{\vec{Z}\quad \vdash \Gamma, F_i \quad \vdash \Delta, F_i^{\perp}}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup})$$

3. *commutative steps*, corresponding to the situation where a multicut is permuted with the last rule of one of its premises, for example:

$$\dfrac{\vec{Z}\quad \dfrac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\,(\parr)}{\vdash \mathrm{H}, F \parr G}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{\dfrac{\vec{Z}\quad \vdash \Gamma, F, G}{\vdash \mathrm{H}, F, G}\,(\mathsf{mcut}_{k+1,\vec{n}',\sqcup})}{\vdash \mathrm{H}, F \parr G}\,(\parr)$$

Because of space constraints, the description of all root steps is given in the appendices of the long version of this article [10].

▶ **Definition 22.** Cut-elimination *is the relation* $\longrightarrow$ *on* $\mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$ *generated by the root steps given in the appendices of the long version of this article [10], using the construction from Definition 8.*

In the remainder of this section, we prove that compression holds for $\mu\mathsf{MALL}^{\infty}$ cut-elimination. Although the definitions for this case of infinitary rewriting are way heavier than for first-order rewriting or the λ-calculus, the proof of compression happens to be simpler than in those two frameworks thanks to the following observation: all root steps defining $\longrightarrow$ rewrite only *finite (linear) patterns* into *finite patterns*: it is obvious, for instance, for the few examples given above. The proof follows the same steps as the two previous ones.

▶ **Definition 23.** *Given a family* $\mathcal{D}$ *of construction rules as in Definition 5,* derivation patterns *are defined as follows:* **(i)** *for all* $\mathsf{r} \in \mathcal{D}$ *of arity* k, $\mathsf{r}(*_1, \dots, *_k)$ *is a derivation pattern,* **(ii)** *for all* $\mathsf{r} \in \mathcal{D}$ *of arity* k*, for all derivation patterns* $p_i(*_{i,1}, \dots, *_{i,l_i})$ *for* $1 \leq i \leq k$, $\mathsf{r}(p_1(*_{1,1}, \dots, *_{1,l_1}), \dots, p_k(*_{k,1}, \dots, *_{k,l_k}))$ *is a derivation pattern. We denote by* $p(s_1, \dots, s_k)$ *the many-sorted term obtained by substituting a term* s_i *to each symbol* $*_i$ *in the derivation pattern* $p(*_1, \dots, *_k)$*.*

▶ **Lemma 24** (pattern extraction)**.** *Consider an ordinal* δ *such that* $\mathfrak{P}_\delta$ *holds, a derivation pattern* $p(*_1, \dots, *_k)$ *and* $s, t_1, \dots, t_k \in \mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$ *such that* $s \xrightarrow[\delta]{}{}^{\infty} p(t_1, \dots, t_k)$*. Then there exist* $s'_1, \dots, s'_k \in \mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$ *such that* $s \longrightarrow^{*} p(s'_1, \dots, s'_k)$ *and for all* $i \in [1, k]$, $s'_i \xrightarrow[\delta]{}{}^{\infty} t_i$*.*

Proof. The proof is identical as the one for Lemma 18: we proceed by a straightforward induction over p, using $\mathfrak{P}_\delta$ and Lemma 15. ◀

▶ **Lemma 25** (pattern filling)**.** *For all derivation pattern* $q(*_1, \dots, *_k)$ *and for all* $s'_1, \dots, s'_k$, $t_1, \dots, t_k \in \mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$*, if for all* $i \in [1, k]$, $s'_i \xrightarrow[\delta]{}{}^{\infty} t_i$ *then* $q(s'_1, \dots, s'_k) \xrightarrow[\delta]{}{}^{\infty} q(t_1, \dots, t_k)$*.*

Proof. Again, the proof is similar to the one for Lemma 19, but is in fact simpler: as we only consider a finite pattern q, we just need to proceed by induction on q. ◀

▶ **Theorem 26.** $\longrightarrow$ *satisfies the property* $\mathfrak{Q}$*, and thus has the compression property.*

Proof. By induction on arbitrary cut-elimination steps $s \longrightarrow t$ (as defined in Definition 8), one can show that any such step has the shape $p(s_1, \dots, s_k) \longrightarrow q(s_1, \dots, s_k)$ for some derivation prefixes p and q (to be precise, q may not use some of the s_i but this has no

incidence on the proof). Indeed, as explained above it is the case for all cut-elimination root steps (as defined in the appendices of the long version of this article [10]), and the induction case is trivial.

As a consequence, if we take an ordinal δ and $s, t, t' \in \mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$ such that $\mathfrak{P}_{\delta}$ and $s \xrightarrow[\delta]{}^{\infty} t \longrightarrow t'$, we can write $t = p(t_1, \dots, t_n)$ and $t' = q(t_1, \dots, t_n)$. By Lemmas 24 and 25 we obtain $s \longrightarrow^{*} p(s'_1, \dots, s'_k) \xrightarrow[\delta]{}^{\infty} q(t_1, \dots, t_n) = t'$. ◀

5 Conclusion and further work

In the present paper, we did the following:

1. introduce a generic framework for rewriting infinitary objects, and extend to this setting the equivalence between topologically and coinductively presented infinitary rewriting,
2. perform the first generic treatment of the compression property for coinductive infinitary rewriting, by providing a novel characterisation that can be readily applied to all the existing infinitary term rewriting systems (*e.g.*, first-order rewriting and the λ-calculus),
3. take advantage of the previous work to present cut-elimination in a non-wellfounded proof system as a coinductive rewriting procedure, and fully work out the example of the system $\mu\mathsf{MALL}^{\infty}$ for which we provide a simple proof of compression (a property whose significance in this setting has been recalled in the introduction).

The former two achievements unify and complete several threads in the literature and are meant to provide a reasonable level of generality for future investigations into coinductive rewriting. Regarding compression, a natural follow-up would be to investigate the effectiveness of the procedure introduced in the proof of Theorem 14: given a computable derivation witnessing a rewriting $s \longrightarrow^{\infty} t$, is it computable to compress it into $s \longrightarrow^{\omega} t$, as can be conjectured from our proof? The suitable notion of "computable" remains to be clarified.

The latter achievement is also a first step towards developing non-wellfounded proof theory (especially of the system $\mu\mathsf{LL}^{\infty}$ for linear logic with fixed points) in a coinductive setting, which is our longer term goal: a result we typically aim at is a fully coinductive proof of cut-elimination for $\mu\mathsf{LL}^{\infty}$. Similar efforts are currently being conducted in this direction, *e.g.* by Sierra-Miranda *et al.* [28] who present non-wellfounded proofs for Gödel-Löb logic and Grzegorczyk modal logic in a coalgebraic way. A benefit of such a coinductive treatment would be to help in the formalisation of the meta-theory of these systems in proof assistants such as Rocq.

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