

Some Recent Developments in Space Complexity

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Abstract

Given a function, what is the minimal memory necessary to compute it? We will describe some old [4, 5, 3] and new algorithmic developments that give surprisingly low-space solutions to this question in many cases. We will survey the recent proof [6] that $\text{TIME}[t]$ is contained in $\text{SPACE}[\sqrt{t} \log t]$ for the multitape Turing machine model, and the engine that makes the proof possible: the amazing Cook-Mertz algorithm [1] for a problem called Tree Evaluation [2]. We will also briefly outline some new developments and generalizations that we've recently proved, pushing the low-space frontier beyond multitape Turing machines. The latter is joint work with Danil Sibgatullin (to appear).

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