Finite-State Dimension and the Davenport-Erdős Theorem

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Abstract

A 1952 result of Davenport and Erdős states that if p is an integer-valued polynomial, then the real number $0.p(1)p(2)p(3)\dots$ is Borel normal in base ten. A later result of Nakai and Shiokawa extends this result to polynomials with arbitrary real coefficients and all bases $b \geq 2$. It is well-known that finite-state dimension, a finite-state effectivization of the classical Hausdorff dimension, characterizes the Borel normal sequences as precisely those sequences of finite-state dimension 1. For an infinite set A of natural numbers, and a base $b \geq 2$, the base-b Copeland-Erdős sequence of A, $\mathrm{CE}_b(A)$, is the infinite sequence obtained by concatenating the base-b expansions of the numbers in A in increasing order. In this work we investigate the possible relationships between the finite-state dimensions of $\mathrm{CE}_b(A)$ and $\mathrm{CE}_b(p(A))$ where p is a polynomial. We show that, if the polynomial is permitted to have arbitrary real coefficients, then for any s, s' in the unit interval, there is a set A of natural numbers and a linear polynomial p so that the finite-state dimensions of $\mathrm{CE}_b(A)$ and $\mathrm{CE}_b(p(A))$ are s and s' respectively. The corresponding result for strong finite-state dimension is also shown. We demonstrate that linear polynomials with rational coefficients do not change the finite-state dimension of any Copeland-Erdős sequence, but there exist polynomials with rational coefficients of every larger integer degree that change the finite-state dimension of some sequence. We also prove the surprising fact that there exist sets A and integer-valued monomials p such that $\mathrm{CE}_b(A)$ is normal, but $\mathrm{CE}_b(p(A))$ has finite-state dimension strictly less than one.

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1 Introduction

A real number x is said to be Borel normal in base b if the base-b expansion of x contains, for every positive integer n, each string of length n over the b-ary alphabet with limiting density b^{-n}. While it has long been known that almost all real numbers are normal [15], and it is conjectured [5] that every irrational algebraic number is absolutely normal (i.e. normal to every positive integer base), all known explicit examples of normal numbers have arisen through artificial construction [2]. The first such example, the base ten Champernowne constant,

$$0.12345678910\dots$$

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obtained by concatenating the base ten representations of the positive integers was shown to be normal in 1933 [7]. A later work by Besicovitch [3] demonstrated that the same result holds for the concatenation of the square numbers in base ten. Copeland and Erdős [9] further demonstrated this for the primes and derived a simple sufficient condition in terms of natural density for the concatenation, in increasing order, of an infinite set of natural numbers to be a normal sequence in any given positive integer base.

The present work has its origins in a 1952 construction of Davenport and Erdős. In [11], they proved that if $p(x)$ is a non-constant polynomial taking only positive integer values on the positive integers, then the real number

$$0.p(1)p(2)p(3)p(4)\dots$$

is normal in base ten. Their result was extended by Nakai and Shiokawa [22, 23, 28] who showed that for any function of the form

$$f(x) = \alpha_n x^{\beta_n} + \alpha_{n-1} x^{\beta_{n-1}} + \cdots + \alpha_1 x^{\beta_1}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $\beta_n > \beta_{n-1} > \cdots > \beta_1 \geq 0$ and $f(x) > 0$ for $x > 0$, the concatenation

$$0.[f(1)]_b[f(2)]_b[f(3)]_b[f(4)]_b\dots$$

where $[x]_b$ represents the integer part of x in base b is normal in base b.

Finite-state dimension, first introduced by Dai, Lathrop, Lutz and Mayordomo [10] is an effectivized version of the classical Hausdorff dimension. The finite-state dimension of an infinite sequence S (or real number), $\dim_{\mathrm{FS}}(S)$, quantifies the lower asymptotic density of finite-state information contained in S. The dual notion of strong finite-state dimension, $\mathrm{Dim}_{\mathrm{FS}}(S)$, is a finite-state effectivization of the classical packing dimension and correspondingly quantifies the upper asymptotic density of finite-state information in S. It is well established [4, 27] that a sequence S is normal if and only if $\dim_{\mathrm{FS}}(S) = 1$. In general, however, it is the case that $0 \leq \dim_{\mathrm{FS}}(S) \leq \mathrm{Dim}_{\mathrm{FS}}(S) \leq 1$ and these two quantities may differ.

A direction for research first proposed by Jack Lutz in 2005 involves investigating the claim: Every theorem about Borel normality is the dimension-1 special case of a more interesting theorem about finite-state dimension. This claim, now known as the "Normality/Dimension Thesis" has inspired a number of subsequent works. For example, the aforementioned results of Copeland and Erdős [9] were extended in [16], and the 1949 theorem of Wall [30] asserting the normality of $q + x$ and qx for normal $x \in \mathbb{R}$ and $q \in \mathbb{Q} \setminus \{0\}$ was shown to hold for finite-state dimension in [14]. In this work, we seek to address the Davenport-Erdős theorem and its subsequent generalization by Nakai and Shiokawa through the lens of this thesis. Existing literature concerning constructions like that of Davenport and Erdős has sought to produce a normal number (and therefore a dimension-1 sequence) by applying a function to $\mathbb{N}$ or the set of primes. As both of these sets themselves give normal numbers when concatenated, we seek in the present work to investigate the finite-state dimension of the sequence $p(n_1)p(n_2)p(n_3)\dots$ when the sequence $n_1n_2n_3\dots$ is permitted to have a finite-state dimension other than 1, and p is a polynomial.

The following section introduces our notational conventions and definitions. In Section 3 we address the case of polynomials with arbitrary real coefficients, in Section 4 we consider the case of polynomials with rational coefficients, and in section 5 we describe the behavior of a related quantity, zeta-dimension, under polynomial transformations.

2 Preliminaries

Let Σ denote a finite alphabet, and for $b \geq 2$ let Σ_b denote the b-ary alphabet $\{0, \ldots, b-1\}$. The set of all finite strings over Σ is denoted Σ^*. The set of strings in Σ^* of length l is denoted Σ^l, and Σ^∞ denotes the set of infinite sequences over Σ. For any $w \in \Sigma^* \cup \Sigma^\infty$, $w[i \ldots j]$ is the substring of w beginning at the index i and terminating at the index j where $0 \leq i \leq j < |w|$ and $|w|$ is the length of w.

Following the notation in [16], for $A \subseteq \mathbb{N}$ and $b \geq 2$, the base-b Copeland-Erdős sequence $\mathrm{CE}_b(A)$ is the infinite b-ary sequence obtained by concatenating the b-ary expansions of the numbers in A in increasing order of their magnitude. For example,

$$\mathrm{CE}_2(\mathbb{N}) = 011011100101\ldots \text{ and } \mathrm{CE}_{10}(\mathrm{PRIMES}) = 235711131719\ldots.$$

For any real number $x \in \mathbb{R}$, denote the base-b expansion of x by $\sigma_b(x)$. For $f : \mathbb{N} \to \mathbb{R}$ and $A \subseteq \mathbb{N}$, define

$$\mathrm{CE}_b(f(A)) = \sigma_b\left([f(n_1)]\right)\sigma_b\left([f(n_2)]\right)\ldots$$

where $[\ ]$ denotes the unsigned integer part and $n_1 < n_2 < \ldots$ are the elements of A.

Finite-state dimension admits many equivalent characterizations including finite-state gamblers and finite-state compressors [10], finite-state log-loss predictors [17], automatic Kolmogorov complexity [19], and a generalized Weyl criterion [20]. In this work, we utilize a characterization in terms of sliding block entropy. Disjoint block entropy was shown to be a characterization by Bourke, Hitchcock, and Vinodchandran [4], and Kozachinskiy and Shen [18] showed that the following sliding block version is equivalent.

▶ **Definition 1** ([4]). *Let $w, s \in \Sigma^*$.*

- *The number of sliding block occurences of w in s is the quantity*

$$N(w, s) = \left|\{i \mid s[i \ldots i + |w| - 1] = w\}\right|.$$

- *The sliding count probability of w in s is*

$$P(w, s) = \frac{1}{|s| - |w| + 1} N(w, s).$$

- *For $l \in \mathbb{N}^+$ the l^{th} sliding block entropy of $x \in \Sigma^*$ is*

$$H_l(x) = \frac{1}{l} \sum_{w \in \Sigma^l} P(w, x) \log(P(w, x)^{-1}).$$

▶ **Definition 2** ([18],[10]). *The finite-state dimension and finite-state strong dimension of a sequence $S \in \Sigma^\infty$ are given by*

$$\dim_{\mathrm{FS}}(S) = \inf_l \liminf_{n \to \infty} H_l(S[0 \ldots n])$$

and

$$\mathrm{Dim}_{\mathrm{FS}}(S) = \inf_l \limsup_{n \to \infty} H_l(S[0 \ldots n]).$$

A sequence $S \in \Sigma_b^\infty$ (or equivalently, a real number) is said to be Borel normal in base b if, for all $w \in \Sigma_b^*$,

$$\lim_{n\to\infty} \frac{N(w, S[0\ldots n])}{n} = \frac{1}{b^{|w|}}.$$

It is well-known [4, 27] that the set of normal sequences is precisely the set of sequences of finite-state dimension 1, and that sequences of all possible dimensions and strong dimensions exist. In particular, it has been shown [16] that for any $s, t \in [0,1]$ with $s < t$, there exists $A \subseteq \mathbb{N}$ such that

$$\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = s \text{ and } \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = t.$$

We will make repeated use of the following result, for which we provide a short proof.

▶ **Proposition 3.** *Let $S, T \in \Sigma^\infty$. If T can be obtained by inserting or deleting a symbol in S at a set of indices of asymptotic density zero*[1]*, then* $\dim_{\mathrm{FS}}(S) = \dim_{\mathrm{FS}}(T)$ *and* $\mathrm{Dim}_{\mathrm{FS}}(S) = \mathrm{Dim}_{\mathrm{FS}}(T)$.

Proof. Let $l \in \mathbb{N}$. For any $w \in \Sigma^*$, as $n \to \infty$ we have that

$$|P(w, S[0\ldots n]) - P(w, T[0\ldots n])| \to 0$$

and thus also

$$|H_l(S[0\ldots n]) - H_l(T[0\ldots n])| \to 0.$$

Therefore $\dim_{\mathrm{FS}}(S) = \dim_{\mathrm{FS}}(T)$ and $\mathrm{Dim}_{\mathrm{FS}}(S) = \mathrm{Dim}_{\mathrm{FS}}(T)$. ◀

3 Polynomials with Arbitrary Real Coefficients

In this section, we investigate the finite-state dimensions of $\mathrm{CE}_b(A)$ and $\mathrm{CE}_b(p(A))$ where p is a polynomial with arbitrary real coefficients. We will use the following result based on the concavity of H_l block entropies. Let P_1 and P_2 be two probability distributions over Σ^l and $0 \le \lambda \le 1$. Then

$$H_l(\lambda P_1 + (1-\lambda)P_2) \ge \lambda H_l(P_1) + (1-\lambda)H_l(P_2)$$

where $H_l(P)$ is defined in the natural way for a probability distribution instead of a string. The proof is analogous to the proof for Shannon entropy. In particular, we have the following.

▶ **Proposition 4.** *Let $u, v \in \Sigma^*$ and $w = uv$. Then*

$$H_l(w) \ge \frac{|u|}{|w|} H_l(u) + \frac{|v|}{|w|} H_l(v) + o(1)$$

as $|w|$ goes to infinity.

Proof. Let $\lambda = \frac{|u|}{|w|}$ and P_1, P_2 be the probability distributions corresponding to the frequencies in the string u and v respectively. The statement would hold without the error term, except there are $O(l)$ strings that cross the boundary of u and v altering the probability distribution on w from being a true convex combination. As $|w|$ goes to infinity, this difference goes to zero yielding the result by the continuity of entropy. ◀

[1] The asymptotic density of $A \subseteq \mathbb{N}$ is given by $\lim_{n\to\infty} |A \cap \{1, \ldots, n\}|/n$.

In our constructions we will concatenate prefixes of a given sequence. The following result allows us to do this without changing the dimension for sufficiently fast growing prefixes.

▶ **Proposition 5.** *Let $S \in \Sigma^\infty$ with $\dim_{\mathrm{FS}}(S) = s$. Then there exists $n_1 < n_2 < n_3 < \dots \in \mathbb{N}$ such that $T = S[0 \dots n_1]S[0 \dots n_2]S[0 \dots n_3] \dots$ has $\dim_{\mathrm{FS}}(T) = s$.*

Proof. Let $(a_n)_{n\in\mathbb{N}} = (1, 1, 2, 1, 2, 3, \dots)$. Define n_i inductively as follows at stage i:

- Let $S_{i-1} = S[0 \dots n_1] \dots S[0 \dots n_{i-1}]$.
- For $l < i$ let s_l be such that $H_l(S[0 \dots k]) \geq \liminf_{k\to\infty} H_l(S[0 \dots k]) - 2^{-i}$ for all $k > s_l$. Let t_l be such that

  $$H_l(S_{i-1}S[0 \dots k]) \geq \liminf_{k\to\infty} H_l(S[0 \dots k]) - 2^{-i-1}$$

  for all $k > t_l$.
- Let $k_l = (i)(t_l)(s_l)$.
- Pick $n_i > \max\{n_{i-1}, k_l | l \leq i\}$ with

  $$H_{a_i}(S_{i-1}S[0 \dots n_i]) \leq \liminf_{k\to\infty} H_{a_i}(S[0 \dots k]) + 2^{-i}.$$

By the last criterion, for each l there are infinitely many k such that

$$H_l(T[0 \dots k]) \leq \liminf_{k\to\infty} H_l(S[0 \dots k]) + 2^{-i}$$

for every i, so

$$\liminf_{k\to\infty} H_l(T[0 \dots k]) \leq \liminf_{k\to\infty} H_l(S[0 \dots k]).$$

By Proposition 4, for each i

$$H_l(S_{i-1}S[0 \dots k]) \geq \frac{|S_{i-1}|}{|S_{i-1}| + k} H_l(S_{i-1}) + \frac{k}{|S_{i-1}| + k} H_l(S[0 \dots k]) + o(1).$$

For $k < s_l$ we have

$$H_l(S_{i-1}S[0 \dots k]) \geq \frac{i-1}{i}(\liminf_{k\to\infty} H_l(S[0 \dots k]) - 2^{-i}) + o(1)$$

and for $k > s_l$,

$$H_l(S_{i-1}S[0 \dots k]) \geq \min\{H_l(S_{i-1}), H_l(S[0 \dots k])\} \geq \liminf_{k\to\infty} H_l(S[0 \dots k]) - 2^{-i} + o(1).$$

So for all k,

$$H_l(S_{i-1}S[0 \dots k]) \geq \frac{i-1}{i}(\liminf_{k\to\infty} H_l(S[0 \dots k]) - 2^{-i}) + o(1).$$

Thus,

$$\liminf_{k\to\infty} H_l(T[0 \dots k]) \geq \liminf_{k\to\infty} H_l(S[0 \dots k])$$

and hence,

$$\liminf_{k\to\infty} H_l(T[0 \dots k]) = \liminf_{k\to\infty} H_l(S[0 \dots k]).$$ ◀

The above result also holds for strong finite-state dimension, however our proof utilizes a different characterization of finite-state dimension than what is given by definition 2. The alternative characterization is given in terms of betting on sequences using a finite-state machine called a *finite-state gambler*. We refer the reader to [1] and [10] for the relevant definitions regarding finite-state gamblers.

▶ **Proposition 6.** *Let $S \in \Sigma^\infty$ with $\mathrm{Dim}_{\mathrm{FS}}(S) = t$. Then there exists $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$ such that $T = S[0 \ldots n_1]S[0 \ldots n_2]S[0 \ldots n_3] \ldots$ has $\mathrm{Dim}_{\mathrm{FS}}(T) = t$.*

Proof. Let $(a_n)_{n\in\mathbb{N}} = (1, 1, 2, 1, 2, 3, \ldots)$ and let $(t_k)_{k=0}^{\infty}$ be a sequence of rationals approaching t from above. Since each $t_k > \mathrm{Dim}_{\mathrm{FS}}(S)$, there is a finite-state gambler d_k that t_k-strongly succeeds on S. We inductively define n_i as follows for $i \geq 1$.

- If $i = 1$ let $N_1 = 0$. Otherwise, for $k \leq i$ let $r_{i,k} \in \mathbb{N}$ be the least length such that the state of d_k after processing $S[0 \ldots n_{i-1}]$ is the same as after processing $S[0 \ldots r_{i,k}]$. Define m_k such that $d_k(S[0 \ldots r_{i,k}]) \leq 2d_k(S[0 \ldots m_k])$ for all $m \geq m_k$. Let $N_1 = \max\{m_k \mid k \leq i\}$.
- Let N_2 be such that
$$H_{a_i}(S[0 \ldots n_1] \ldots S[0 \ldots n_{i-1}]S[0 \ldots N_2]) \geq \limsup_{k\to\infty} H_{a_i}(S[0 \ldots k]) - 2^{-i}.$$
- Choose $n_i = \max\{N_1, N_2, n_{i-1} + 1\}$.

Note that the values of N_2 above ensure that $\mathrm{Dim}_{\mathrm{FS}}(T) \geq \mathrm{Dim}_{\mathrm{FS}}(S) = t$. Therefore it suffices to prove that $\mathrm{Dim}_{\mathrm{FS}}(T) \leq t$. To do this we will show that $\mathrm{Dim}_{\mathrm{FS}}(T) \leq t_k$ for every t_k. Note that there are finitely many states in d_k and thus there is a common bound for all the $r_{i,k}$ values defined above. Hence for the sequence

$$T' = S[0 \ldots n_1]S[r_{1,k} \ldots n_2]S[r_{2,k} \ldots n_3] \ldots$$

we get that $\mathrm{Dim}_{\mathrm{FS}}(T') = \mathrm{Dim}_{\mathrm{FS}}(T)$ by Proposition 3. Therefore the proof follows if we show that the gambler d_k t_k-strongly succeeds on T'.

To see this let $q = d_k(S[0 \ldots n_1] \ldots S[r_{k-2,k} \ldots n_{k-1}]) > 0$. The above definition of m_k guarantees that d_k will at least double its value after processing every added prefix of S after this point. After m prefixes have been added we then have

$$d_k(S[0 \ldots n_1]S[r_{k+m-1,k} \ldots n_{k+m}]S[r_{k+m,k} \ldots c]) \geq q2^m \frac{d_k(S[0 \ldots c])}{d_k(S[0 \ldots r_{k+m,k}])}$$

for all $c \geq r_{k+m,k}$. Since there is a bound on the denominator for all m and d_k t_k-strongly succeeds on S, it also t_k-strongly succeeds on T'. ◀

▶ **Corollary 7.** *Given finitely many sequences $S_1, S_2, \ldots S_n$, a single sequence of naturals $n_1 < n_2 < n_3 \cdots \in \mathbb{N}$ exists such that Propositions 5, 6 hold simultaneously for each S_i.*

Proof. Use the same constructions as the proof of Propositions 5, 6 and choose n_i to be the max of those obtained for each sequence at each step. ◀

It is natural to wonder what happens for concatenations of prefixes which are not of fast-growing length. The following result gives an answer to this by looking at concatenating every single prefix. In particular, note that for any normal sequence S the sequence T defined below is normal. Thus, the following proposition generalizes the result of Pellegrino [24] to arbitrary finite state dimension in support of the Normality/Dimension Thesis.

▶ **Proposition 8.** [2] *Let* $S \in \Sigma^\infty$ *and* $T = S[0]S[0\dots1]S[0\dots2]\dots$ *Then* $\dim_{\mathrm{FS}}(S) \le \dim_{\mathrm{FS}}(T)$.

▶ Remark 9. It is straightforward to construct, for each $b \ge 2$, an $S \in \Sigma_b^\infty$ and $n_1 < n_2 < n_3 < \dots \in \mathbb{N}$ so that

$$\dim_{\mathrm{FS}}(S) \lneqq \dim_{\mathrm{FS}}(S[0\dots n_0]S[0\dots n_1]S[0\dots n_2]\dots)$$

by dilution of a normal sequence ([10], construction 6.4) and careful choice of prefix lengths.

▶ **Proposition 10.** *For any* $b \ge 2$ *and* $s, s', t, t' \in [0,1]$ *with* $s \le t$ *and* $s' \le t'$ *there exists an infinite set* $A \subseteq \mathbb{N}$ *and a linear polynomial* $p(x)$ *so that*

$$\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = s, \ \ \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = t$$

and

$$\dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A)) = s', \ \ \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A)) = t'.$$

Proof. Let $\alpha, \beta \in \mathbb{R}^+$ be such that $\dim_{\mathrm{FS}}(\sigma_b(\alpha)) = s$, $\mathrm{Dim}_{\mathrm{FS}}(\sigma_b(\alpha)) = t$, $\dim_{\mathrm{FS}}(\sigma_b(\beta)) = s'$, and $\mathrm{Dim}_{\mathrm{FS}}(\sigma_b(\beta)) = t'$. Take $c = \frac{\beta}{\alpha}$. Note that such a choice is possible by [16]. By Corollary 7, we can choose $n_1 < n_2 < \dots \in \mathbb{N}$ so that $p(x) = cx$ and $A = \{\sigma_b(\alpha)[0\dots n_i] | i \in \mathbb{N}\}$ yield the result. As the set of indices at which the base-b representations of each product cn for $n \in A$ disagree with the corresponding prefix of $c\alpha$ form a set of asymptotic density zero, we have $\dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A)) = \dim_{\mathrm{FS}}(c\alpha)$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A)) = \mathrm{Dim}_{\mathrm{FS}}(c\alpha)$. ◀

The above result relies critically on our ability to encode information into the irrational coefficient c in the linear polynomial. In the following section, we investigate the behavior of polynomials with rational coefficients.

4 Polynomials with Rational Coefficients

▶ **Proposition 11.** *Let* S *be a normal sequence and* S^0 *be a sequence obtained by inserting strings in* $\{0\}^*$ *such that the number of strings inserted to* S *has asymptotic density zero relative to the number of digits in the prefix. Let* $z(n)$ *be the number of zeroes added into a prefix of* S *when viewed as a prefix of* S^0 *of length* n. *Then*

$$\dim_{\mathrm{FS}}(S^0) = 1 - \limsup_{n\to\infty} \frac{z(n)}{n}$$

and

$$\mathrm{Dim}_{\mathrm{FS}}(S^0) = 1 - \liminf_{n\to\infty} \frac{z(n)}{n}.$$

Proof. Let $l \in \mathbb{N}$. Then the number of sliding blocks over the original parts of S that are impacted by the additions of 0's has asymptotic density zero. Hence, in the limit the probability distribution of length l strings is equivalent to the weighted sum of the probability distribution $P_{0,l}$ of S^0 on the sliding blocks over only the added zero bits and the probability distribution $P_{S,l}$ on the original bits present in S. In particular, letting $P_{S^0,l,n}$ be the probability distribution of length l strings on a prefix of S^0 of length n we have in the limit

$$P_{S^0,l,n}(0^l) = \frac{z(n)}{n} + \frac{n - z(n)}{n} P_{S,l}(0^l)$$

[2] The proof of this proposition has been omitted for space concerns in the present version of the paper.

and

$$P_{S^0,l,n}(w) = \frac{n - z(n)}{n} P_{S,l}(w)$$

for all other $w \in \Sigma^l$ as n goes to infinity. Since S is normal we have

$$P_{S^0,l,n}(0^l) = \frac{z(n)}{n} + \frac{n - z(n)}{n}\frac{1}{b^l}$$

and

$$P_{S^0,l,n}(w) = \frac{n - z(n)}{n}\frac{1}{b^l}$$

as n goes to infinity. Therefore,

$$\liminf_{n\to\infty} H_l(S^0[0\dots n]) = \liminf_{n\to\infty} -\frac{1}{l}\left[\left(\frac{z(n)}{n} + \frac{n-z(n)}{n}\frac{1}{b^l}\right)\log\left(\frac{z(n)}{n} + \frac{n-z(n)}{n}\frac{1}{b^l}\right) + (b^l - 1)\left(\frac{n-z(n)}{n}\frac{1}{b^l}\right)\log\left(\frac{n-z(n)}{n}\frac{1}{b^l}\right)\right].$$

It is routine to verify that this is non-increasing with l and converges to $1 - \limsup_{n\to\infty} \frac{z(n)}{n}$ as l goes to infinity. The lim sup result is analogous. ◀

Note that the l-block entropy in the weighted distribution is minimized when the probability distribution of the inserted strings has l-block entropy zero (e.g. contains only one string of length l). Therefore we get the following as a corollary.

▶ **Corollary 12.** *Let S be a normal sequence and S' be a sequence obtained by inserting arbitrary strings such that the number of strings inserted to S has asymptotic density zero relative to the number of digits in the prefix. Let $z'(n)$ be the number of digits added into a prefix of S when viewed as a prefix of S' of length n. Then*

$$\dim_{\mathrm{FS}}(S') \geq 1 - \limsup_{n\to\infty} \frac{z'(n)}{n}$$

and

$$\mathrm{Dim}_{\mathrm{FS}}(S') \geq 1 - \liminf_{n\to\infty} \frac{z'(n)}{n}.$$

In [14], it is shown that for any rational number q, and real number x, the numbers $x, q + x$, and qx have equal finite-state dimensions and equal finite-state strong dimensions. We now utilize this result to show that linear polynomials with rational coefficients do not change the finite-state dimension nor finite-state strong dimension of Copeland-Erdős sequences.

▶ **Proposition 13.** *Let $p(x) = qx + r$ with $q, r \in \mathbb{Q}$, $q \neq 0$. For all $b \geq 2$ and $A \subseteq \mathbb{N}$,*

$$\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A)))$$

and

$$\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A))).$$

Proof. First note that $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \dim_{\mathrm{FS}}(\mathrm{CE}_b(A+r))$ for any $r \in \mathbb{Q}$ by Proposition 3. We now consider the case that q is an integer and $r = 0$. Let $k \in \mathbb{Z} \setminus \{0\}$ and enumerate $A = \{n_1, n_2, \dots\}$. For each $i \geq 1$, let

$$j_i = \big||\sigma_b(n_i)| - |\sigma_b(kn_i)|\big|.$$

Let $x \in \mathbb{R}$ be such that

$$\sigma_b(x) = 0.0^{j_1}\sigma_b(n_1)0^{j_2}\sigma_b(n_2)\dots$$

so that $\sigma_b(kx) = 0.\mathrm{CE}_b(kA)$ and therefore $\dim_{\mathrm{FS}}(\mathrm{CE}_b(kA)) = \dim_{\mathrm{FS}}(kx)$. By Proposition 3, we have $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \dim_{\mathrm{FS}}(x)$. If $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) \neq \dim_{\mathrm{FS}}(\mathrm{CE}_b(kA))$, then x is a real number such that $\dim_{\mathrm{FS}}(x) \neq \dim_{\mathrm{FS}}(kx)$ contradicting the main theorem of [14].

When $q = 1/c$, we take $x' \in \mathbb{R}$ to be such that

$$\sigma_b(x') = \sigma_b(n_1')\sigma_b(n_2')\dots$$

where, for each i, n_i' is the nearest multiple of c to n_i. By Proposition 3, we have $\dim_{\mathrm{FS}}(x) = \dim_{\mathrm{FS}}(\mathrm{CE}_b(A))$ and $\dim_{\mathrm{FS}}(qx) = \dim_{\mathrm{FS}}(\mathrm{CE}_b(qA))$. Exactly similar arguments give the corresponding results for finite-state strong dimension. ◀

We will now investigate the case for polynomials of degree at least two. To do so we use the following definition due to Besicovitch [3].

▶ **Definition 14.** *A natural number n is (ϵ, k)-normal in base $b \geq 2$ if*

$$\left|\frac{N(w, \sigma_b(n))}{|\sigma_b(n)|} - \frac{1}{b^k}\right| \leq \epsilon$$

for every string $w \in \Sigma_b^k$.

The following result is well known and appears as Theorem 1 in [25].

▶ **Lemma 15.** *Consider a sequence $\{a_n\}_{n=1}^{\infty}$. Suppose that the lengths of the strings $\sigma_b(a_n)$ are growing on average, but that no one length dominates; more precisely, suppose that as m tends to infinity,*

$$m = o\left(\sum_{n=1}^{m} |\sigma_b(a_n)|\right) \text{ and } m \cdot \max_{1 \leq n \leq m} |\sigma_b(a_n)| = O\left(\sum_{n=1}^{m} |\sigma_b(a_n)|\right).$$

In addition, suppose that for any fixed $\epsilon > 0$ and $k \in \mathbb{N}$, almost all a_n are $(\epsilon, k)-$normal, in the sense that the number of $n \leq m$ for which a_n is not $(\epsilon, k)-$normal is $o(m)$ as m tends to infinity, then $\sigma_b(a_1)\sigma_b(a_2)\sigma_b(a_3)\dots$ is normal.

The following results also appear in [25] with the first part due to Lemma 4.7 in [5].

▶ **Lemma 16.** *Let $\epsilon > 0, k \in \mathbb{N}$, and $b \geq 2$ Then.*

1. *For every $\epsilon > 0$ and $k \in \mathbb{N}$ there is a δ such that the number of integers $n \in [1, m]$ that are not (ϵ, k)-normal is at most $m^{1-\delta}$ for all sufficiently large m.*
2. *The number of integers $n \in [0, m]$ for which n^2 is not (ϵ, k)-normal is $o(m)$ as m goes to infinity.*

We now show that polynomials of degree greater than 1 can change the finite-state dimension, even with rational coefficients.

▶ **Proposition 17.** *For every $b \geq 2$ there exists $A \subseteq \mathbb{N}$ and a polynomial $p \in \mathbb{Q}[x]$ of degree 2 so that*

$$0 < \dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) < \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) < 1.$$

Proof. Note that for any $b \geq 2$, $n \in \mathbb{N}$ and $i \geq |\sigma_b(n)|$ we have

$$\sigma_b\left((b^i + n)\right) = 10^{i-|\sigma_b(n)|}\sigma_b(n).$$

Let $p(x) = x^2$. We have

$$\sigma_b\left((b^i + n)^2\right) = 10^{i-|\sigma_b(2n)|}\sigma_b(2n)0^{i-|\sigma_b(n^2)|}\sigma_b(n^2).$$

Let $\epsilon_i = 2^{-i}$ and $k_i = i$. Then by Lemma 16, for each i there is an $l_i \in \mathbb{N}$ for which there is an $n \in \Sigma_b^l$ with $n, 2n$ and n^2 all (ϵ_i, k_i)-normal for every $l > l_i$.

We construct A in stages as follows starting with $i = 1$ and $l = l_i$.

- Pick an $n \in \Sigma_b^l$ where $n, 2n$ and n^2 are (ϵ_i, k_i)-normal and add $b^{2l} + n$ to A. Set $l = l + 1$.
- Once $l = l_{i+1}$ set $i = i + 1$ and go back to the first step.

Now consider the sequence S formed by stripping the zero block corresponding to each b^{2l} in $\mathrm{CE}_b(A)$. Similarly let T be the sequence obtained by removing the zero block corresponding to the binomial expansion in $\mathrm{CE}_b(p(A))$. By Lemma 15, both S and T are normal. By Proposition 11 we have $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = 0.5$ and $\dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = 0.75$ as $lim_{n\to\infty} \frac{z(n)}{n}$ is 0.5 for $\mathrm{CE}_b(A)$ and 0.25 for $\mathrm{CE}_b(p(A))$. ◀

▶ **Corollary 18.** *For all $d \geq 2$, there exists $A \subseteq \mathbb{N}$ and $p \in \mathbb{Q}[x]$ so*

$$0 < \dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) < \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) < 1.$$

Proof. By the binomial theorem, we have for any $i \geq \binom{d}{\lfloor d/2 \rfloor}$, the string $\sigma_b\left((b^i + n)^d\right)$ is given by

$$10^{i-|\sigma_b(\binom{d}{1}n)|}\sigma_b\left(\binom{d}{1}n\right)0^{i-|\sigma_b(\binom{d}{2}n^2)|}\sigma_b\left(\binom{d}{2}n^2\right)\ldots 0^{i-|\sigma_b(\binom{d}{d}n^d)|}\sigma_b\left(\binom{d}{d}n^d\right).$$

Theorem 2 of the original work of Davenport and Erdős [11] states that for any integer-valued polynomial $p(x)$, the number of numbers $n \leq m$ for which $f(n)$ is not $(\epsilon, k)-$normal in base 10 is $O(m)$ as $m \to \infty$. A simple generalization of the construction used in the preceding proposition to the expression given by the binomial theorem and the observation that the result of Davenport and Erdős holds in every positive integer base $b \geq 2$ proves the corollary. ◀

While the above results show rational polynomials can change the dimension, the following shows that this is not always the case.

▶ **Proposition 19.** *For every $s \in [0, 1]$, $d \in \mathbb{N}$ and every $b \geq 2$ there is a set A with*

$$\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = s$$

for a rational polynomial p of degree d.

Proof. Set $p(x) = x^d$ and let $(\frac{c_n}{d_n})_{n\in\mathbb{N}}$ where $c_n, d_n \in \mathbb{N}$ be a sequence of rationals converging to s with the property that $d_{n+1} - d_n \leq a$ for some fixed constant a and $lim_{n\to\infty} d_n = lim_{n\to\infty} c_n = \infty$. Note that we do not require $\frac{c_n}{d_n}$ to be written in lowest terms.

We then construct a set A where ϵ_i, k_i and l_i are defined in the proof of 17. Start with $n = 0$

1. Let $i \in \mathbb{N}$ be the largest satisfying $l_i < c_n$.
2. Pick an $m \in \Sigma_b^{c_n}$ with m and m^d (ϵ_i, k_i)-normal and add $m(b^{(d_n - c_n)})$ to A.
3. Set $n = n + 1$ and go back to step 1.

Since $\lim_{n\to\infty} c_n = \infty$ we can remove the $(d_n - c_n)$ 0's at the end of each integer in $\mathrm{CE}_b(A)$ to get a normal sequence by Lemma 15 and similarly for removing the $d(d_n - c_n)$ 0's from $\mathrm{CE}_b(p(A))$. Since $d_{n+1} - d_n \leq a$ the limit of $z(n)$ behaves nicely for both sequences with

$$\lim_{n\to\infty} \frac{z(n)}{n} = \lim_{n\to\infty} \frac{d_n - c_n}{d_n}$$

and moreover

$$\lim_{n\to\infty} 1 - \frac{d_n - c_n}{d_n} = \frac{c_n}{d_n} = s.$$

Hence, by Proposition 11 we have

$$\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = s. \qquad \blacktriangleleft$$

▶ Remark 20. For $0 \leq s \leq t \leq 1$ it is also possible to create a set A with

$$\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = s \text{ and } \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) = t$$

in the same way using one sequence of rationals corresponding to s and another for t. Then alternating from padding based on the sequence corresponding to s after the frequency of zeros is arbitrarily close to the correct value to padding according to t and vice versa one can get the $\liminf$ and $\limsup$ to correspond to the correct values.

The following natural question arises from [29] (Theorem 5): does the normality of $\mathrm{CE}_b(A)$ imply that $\mathrm{CE}_b(p(A))$ is normal? We now answer this question in the negative by demonstrating the existence of a set A such that $A \subseteq \mathbb{N}$ yields a dimension-1 sequence, but $\mathrm{CE}_b(p(A))$ has dimension strictly less than 1.

▶ **Lemma 21.** [3] *Let $k, b, d \in \mathbb{N}$ be such that $b, d \geq 2$ and $\gcd(d, b) = 1$, and let $\epsilon > 0$. Then there exists $N, c \in \mathbb{N}$ and $p \in \mathbb{Q}[x]$ of degree d such that for any $n \geq N$ there exists $m \in \mathbb{N}$ with*

1. $|\sigma_b(m)| \geq 2n - \log(n) - c$
2. *$\sigma_b(m)$ (ϵ, k)-normal*
3. *$0^{\frac{n}{2}-c}$ is a substring of $\sigma_b(p(m))$.*

▶ **Theorem 22.** *For every $b, d \geq 2$ with $\gcd(d, b) = 1$, there exists $A \subseteq \mathbb{N}$ and $p \in \mathbb{Q}[x]$ of degree d such that*

$$1 = \dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) > \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A)))$$

and

$$1 = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) > \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A))).$$

[3] The proof of this lemma has been omitted for space concerns in the present version of the paper.

Proof. It is straightforward to construct a sequence of numbers $\{a_n\}_{n=1}^{\infty}$ that satisfies Lemma 21 for a sequence of ϵ_i going to 0 and k_i going to infinity in the style of Proposition 17. By Lemma 15, the corresponding set A has $1 = \dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A))$. It is also straightforward to see that the length of the numbers in the sequence can be chosen to grow slowly enough that the blocks of approximately $\frac{n}{2}$ zeros in the outputs $p(m)$ of length approximately $2dn$ cause there resulting sequence to consist of $\frac{1}{4d}$ zero blocks as a fraction of the sequence length in the limit. For large enough l this implies that the l-block entropy will be less than 1 for sufficiently large n as the probability of the string 0^l appearing is larger than $\frac{1}{b^l}$. Thus,

$$1 > \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(p(A))) \geq \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A))).$$ ◀

We conclude this section by demonstrating the existence of a sequence of finite-state dimension zero whose finite-state dimension becomes positive under a rational polynomial transformation.

▶ **Proposition 23.** *For all $b \geq 2$, there exists $A \subseteq \mathbb{N}$ and $p \in \mathbb{R}[x]$ so that $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = 0$ and $\dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A)) > 0$.*

Proof. We will employ the use of the polynomial $p(x) = x^2$. First note that

$$(1 + x + x^2 + \cdots + x^n)^2 = 1 + 2x + 3x + \cdots + (n+1)x^n + nx^{n+1} + \cdots + 2x^{2n-1} + x^{2n}$$

and thus, if $\sigma_b(n) = (10^i)^j 1$ (where exponentiation refers to concatenation), then $\sigma_b(n^2)$ has the form

$$10^{\ell_1}\sigma_b(1)0^{\ell_2}\sigma_b(2)\ldots 0^{\ell_{j+1}}\sigma_b(j+1)0^{\ell_j}\sigma_b(j)\ldots 0^{\ell_2}\sigma_b(2)0^{\ell_1}\sigma_b(1)$$

where $\ell_k = i + 1 - |\sigma_b(k)|$. For example, in base 10,

$$1000010000100001^2 = 1000020000300004000030000200001.$$

Note that if $A = \{n \mid (10^i)^j 1 = \sigma_b(n), n \in S \subseteq \mathbb{N}\}$ for some (infinite) $S \subseteq N$, then $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = 0$. Let A be the set of numbers a_k such that $\sigma_b(a_i)$ has the form $(10^i)^j 1$ and j is such that all numbers with base-b expansions of length i appear in $\sigma_b(a_i)$. For each i, write $\sigma_b(a_i) = x_i^L y_i x_i^R$ where y_i is the portion of $\sigma_b(a_i)$ consisting of numbers whose base-b expansions have length i and which appear in increasing order in $\sigma_b(a_i)$. We have $|\sigma_b(a_i)| = 2i(b^i - 2) + i$ and $|x_i^L| = |x_i^R| = i(b^{i-1} - 1)$. Calculating the relative fraction of the lengths of strings $\sigma_b(a_i)$ consisting of the substrings y_i, we see that

$$\lim_{i\to\infty} \frac{(b-1)b^{i-1}i}{2i(b^i - 2) + i} = \frac{b-1}{2b} > 0.$$

By Corollary 12, the dimension of $\mathrm{CE}_b(p(A))$ is positive. ◀

5 Zeta-Dimension

The *zeta-dimension* of a set $A \subseteq \mathbb{N}$,

$$\mathrm{Dim}_\zeta(A) = \inf\{s | \zeta_A(s) < \infty\}$$

where $\zeta_A(s) = \sum_{n\in A}\frac{1}{n^s}$, is well-known to have the property that $\mathrm{Dim}_\zeta(A) = 1$ implies $\mathrm{CE}_b(A)$ is normal [9, 13, 16]. This notion of dimension admits an additional "entropy characterization" given by

$$\mathrm{Dim}_\zeta(A) = \limsup_{n\to\infty} \frac{\log|A \cap \{1,\ldots,n\}|}{\log n}$$

due to Cahen [6]. As noted in [16], it is natural to define the *lower zeta-dimension*, $\dim_\zeta(A)$ by replacing the limit superior with a limit inferior. In [16], it is shown that $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) \geq \dim_\zeta(A)$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) \geq \mathrm{Dim}_\zeta(A)$, and furthermore that for any $\alpha, \beta, \gamma, \delta$ such that

$$\begin{array}{ccccccc} & & \gamma & \leq & \delta & \leq & 1 \\ & & \vee\!| & & \vee\!| & & \\ 0 & \leq & \alpha & \leq & \beta. & & \end{array}$$

there exists a set $A \subseteq \mathbb{N}$ with $\dim_\zeta(A) = \alpha$, $\mathrm{Dim}_\zeta(A) = \beta$, $\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \gamma$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_b(A)) = \delta$. The following proposition describes the behavior of both zeta-dimensions under polynomial transformations.

▶ **Proposition 24.** *For all $A \subseteq \mathbb{N}$ and $p \in \mathbb{R}[x]$ with $\deg(p) = d$,*

$$\dim_\zeta(p(A)) = \frac{1}{d}\dim_\zeta(A) \quad \textit{and} \quad \mathrm{Dim}_\zeta(p(A)) = \frac{1}{d}\mathrm{Dim}_\zeta(A).$$

Proof. By the series characterization of zeta-dimension, we have

$$\mathrm{Dim}_\zeta(p(A)) = \inf\{s | \zeta_{p(A)}(s) < \infty\}.$$

Noting that, for large n, $p(n) \sim Cn^d$ for some C, we observe that the above infimum is precisely $d^{-1}\mathrm{Dim}_\zeta(A)$. To show the corresponding result for lower zeta-dimension, we utilize the entropy characterization. First note that, without loss of generality,

$$|p(A) \cap \{p(1), \ldots, p(n)\}| = |A \cap \{1, \ldots, n\}|$$

for sufficiently large n. Again noting that $p(x) \sim Cn^d$ for large n, we have

$$\dim_\zeta(A) = \liminf_{n\to\infty} \frac{\log(|A \cap \{1, \ldots, n\}|)}{\log n} = \liminf_{n\to\infty} d\frac{\log(|p(A) \cap \{p(1), \ldots, p(n)\}|)}{\log(Cn^d)}.$$

Now,

$$\limsup_{n\to\infty} \left| \frac{\log(|p(A) \cap \{p(1), \ldots, p(n)\}|)}{\log(Cn^d)} - \frac{\log(|p(A) \cap \{p(1), \ldots, p(n+1) - 1\}|)}{\log(C(n+1)^d)} \right|$$

is zero and thus $\dim_\zeta(p(A)) = d^{-1} \dim_\zeta(A)$ as desired. ◀

6 Future Work

We view this work as a step toward understanding the way in which the Davenport-Erdős theorem is the finite-state dimension-1 special case of a more interesting theorem about finite-state dimension. Remaining open questions include: analysis of bounds on $|\dim_{\mathrm{FS}}(\mathrm{CE}_b(A)) - \dim_{\mathrm{FS}}(\mathrm{CE}_b(p(A)))|$ for polynomials of degree $d \geq 2$, consideration of non-polynomial functions (as in [12, 21, 26, 29]) in our context, and extension to arbitrary pseudo-polynomials (as in [23]).

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