Functorial Semantics for First-Order Theories

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Abstract

Building on the recent axiomatisation of first-order bicategories, we develop a functorial semantics approach to the model theory of first-order logic. First-order theories $\mathbb{T}$ are captured by free first-order bicategories $\mathcal{F}_{\mathbb{T}}$ and models of $\mathbb{T}$ are structure-preserving functors from $\mathcal{F}_{\mathbb{T}}$ to a first-order bicategory $\mathbf{C}$. Elementary morphisms of models arise as lax natural transformations between such functors, and the classical Tarski–Vaught test and downward Löwenheim–Skolem theorem admit direct diagrammatic proofs. Our results instantiate classically when $\mathbf{C} = \mathbf{Rel}$ and hold uniformly for models valued in $\mathsf{Rel}(\mathbf{D})$ over an arbitrary Boolean geometric category $\mathbf{D}$ in which regular epis split.

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1 Introduction

First-order (fo) logic has a well-understood model theory when its models are sets. Computer science, however, routinely requires fo structures valued elsewhere: e.g. in toposes of sheaves for constructive and topological reasoning, in categories of measurable spaces for probabilistic semantics, or in presheaf categories for concurrency and name-passing. Each such setting has typically required a bespoke development of the basic model-theoretic apparatus (e.g. substitution, elementary embedding, Skolemisation, Löwenheim–Skolem) as witnessed for instance by the model theory of metric structures [53] and topos-theoretic model theory [38]. This paper develops fo model theory via *functorial semantics*, following the spirit of Lawvere's approach to algebraic theories [33], in such a way that the basic apparatus is proved once and instantiates uniformly across a broad class of semantic universes.

In our approach, a fo theory $\mathbb{T}$ is identified with its free *fo bicategory* $\mathcal{F}_{\mathbb{T}}$, a notion recently introduced in [5]. Concretely, $\mathcal{F}_{\mathbb{T}}$ is presented by a finite set of generators and inequational axioms, so that fo formulas are morphisms of $\mathcal{F}_{\mathbb{T}}$ and are rendered as *string diagrams*. This yields a point-free, tacit approach: wires replace variables, composition replaces substitution. Logical connectives and quantifiers arise from the algebraic structure; entailment between

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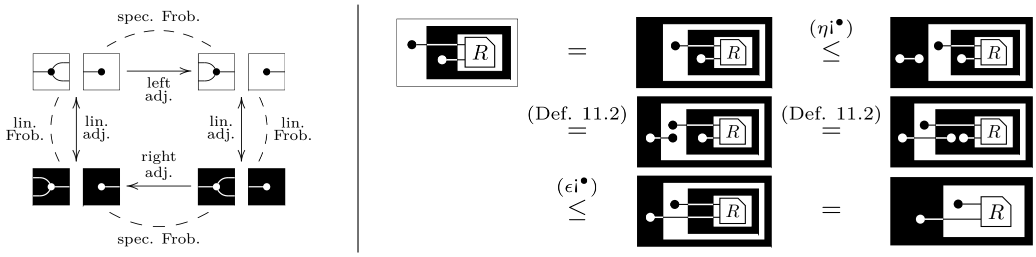


Figure 1 On the left: the Tao of Logic, depicting the laws of fo bicategories, introduced in [5] and recalled in § 3. On the right: diagram rewriting proving $\exists x.\forall y.R(x, y) \Rightarrow \forall y.\exists x.R(x, y)$.

formulas is obtained by diagrammatic rewriting using the (sound and complete) axioms that govern the algebraic structure of fo bicategories, i.e., the laws of (co)cartesian [12] and linear bicategories [15] illustrated in Figure 4 and compactly summarised in Figure 1. Thus, in fo bicategories the *entire logical structure* lives inside a *single* poset-enriched category where arrows are formulas and the order is logical entailment. The reader familiar with hyperdoctrines [34, 41] will recognise a different and complementary architectural choice: hyperdoctrines organise fo logic as an indexed family of Heyting (or Boolean) algebras fibred over a syntactic category, allowing e.g. for an elegant characterisation of quantifiers via adjunctions.

Following the functorial semantics tradition, a model of $\mathbb{T}$ in a fo bicategory $\mathbf{C}$ is a structure-preserving functor $\mathcal{F}_{\mathbb{T}} \to \mathbf{C}$ [5]. Here we begin the study of basic features of model theory in this setting and observe that an *elementary morphism* of models [51] is a lax natural transformation between such functors. We then exploit the complete axiomatisation of fo logic from [5] to obtain direct diagrammatic proofs of classical model theoretic results, notably Skolemisation (Lemma 24) and the downward Löwenheim–Skolem theorem (Theorem 27). As a consequence of the algebraic setting, the proofs of these results unfold as sequences of diagram equalities justified by named axioms, rather than quantifier manipulations punctuated by freshness side-conditions.

This matters structurally. The overhead of fo reasoning in its traditional presentation – α-equivalence, capture-avoiding substitution, de Bruijn indices, context weakening, freshness side-conditions etc. – is an artefact of variable-based syntax, not of the logic itself. String diagrams dispense with this overhead: *wires* are the only mechanism for variable-passing, and there is nothing to α-rename because there are no names. Equality of formulas is equality of morphisms in a freely generated category. While we do not pursue mechanised formalisation in the present paper, we regard the diagrammatic presentation of fo model theory developed here as a natural starting point: proof assistants handle equational reasoning over a fixed algebraic signature considerably more comfortably than they handle the metatheoretic apparatus of variable-based fo logic. Moreover, free fo bicategories ought to admit a combinatorial characterisation as hypergraphs, allowing the implementation of inference as a string diagrammatic rewriting system, generalising the approach of [4].

Against this background, the contributions of this paper are:

- A structural characterisation of elementary morphisms of models as lax natural transformations (Definition 16), together with a proof that such morphisms are necessarily injective maps that preserve truth of sentences (Lemma 17, Theorem 18).
- An inductive proof of Tarski–Vaught test (Lemma 19), providing a principle for verifying elementarity.
- A diagrammatic Skolemisation lemma (Lemma 24), valid in any cartesian bicategory with tabulations and choice, converting existential witnesses into single-valued arrows.

- A downward Löwenheim–Skolem theorem (Theorem 27) for models valued in $\mathrm{Rel}(\mathbf{C})$ for $\mathbf{C}$ a Boolean geometric category in which regular epimorphisms split, proved uniformly by iteration of the diagrammatic Skolem construction.

Related work. Categorical semantics of fo logic has a long history, principally via hyperdoctrines and their relatives [34, 41, 39]. The direct predecessor of this work is the functorial semantics programme for relational theories proposed in [7] and continued in a series of papers that expanded the logical expressivity [9, 19, 3]. The extension to full classical fo logic was a challenge [24] that resulted in fo bicategories [5]. The correspondence between the latter and (Boolean) hyperdoctrines is well-understood [6] as an adjunction that restricts to an equivalence for well-behaved hyperdoctrines – namely, those satisfying extensional equality [25]. This discrepancy stems from a fundamental difference in how the two formalisms treat terms and formulas: hyperdoctrines, following the traditional presentation of fo logic, keep them as distinct entities, whereas in fo bicategories terms are simply formulas satisfying additional equations (see Remark 15).

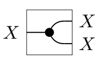
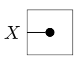
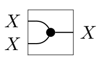
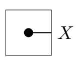
Other categorical treatments of fo logic include allegories [23], whose tight correspondence with cartesian bicategories, i.e. the basis for fo bicategories, is also well-understood, see e.g. [22, 18]. To the best of our knowledge, the literature on model theory in these categorical frameworks remains sparse. It is worth mentioning [52], where Priestley duals of hyperdoctrines are used to prove some model-theoretic results for coherent logic, including Craig's interpolation and Robinson's consistency theorem.

Synopsis. Cartesian bicategories are recalled in § 2, followed by the basic concepts of fo bicategories and fo theories in § 3. In § 4 we develop the functorial semantics: we characterise elementary morphisms of models, and prove the Tarski–Vaught test. Diagrammatic Skolemisation is developed in § 5 and the downward Löwenheim–Skolem theorem is proved in § 6. We conclude with § 7. For the reader's convenience, we provide some additional material on fo bicategories in Appendix A.

2 Cartesian Bicategories

All bicategories considered in this paper are *poset-enriched symmetric monoidal categories*: every homset carries a partial order $\leq$, and composition $\fatsemi$ and monoidal product $\otimes$ are monotone. A *poset-enriched symmetric monoidal functor* is a symmetric monoidal functor that preserves the order $\leq$. The notion of *adjoint arrows* amounts to: for $f\colon X \to Y$ and $g\colon Y \to X$, f is *left adjoint* to g, or g is *right adjoint* to f, if $id^{\circ}_X \leq f \fatsemi g$ and $g \fatsemi f \leq id^{\circ}_Y$.

All monoidal categories and functors considered throughout this paper are tacitly assumed to be strict [37]. This is harmless: strictification [37] (see [26, Thm. 2.6.4] for the enriched case) allows to transform any monoidal category into a strict one, enabling the sound use of string diagrams [28, 47]. Particularly relevant for us are the string diagrams for (co)monoids:

$\blacktriangleleft^{\circ}_X =$  $\quad !^{\circ}_X =$  $\quad \blacktriangleright^{\circ}_X =$  $\quad ¡^{\circ}_X =$ 

We recall cartesian bicategories of relations [12]. For brevity, we do not write "of relations".

▶ **Definition 1.** *A* cartesian bicategory *is a poset-enriched symmetric monoidal category* $(\mathbf{C}, \otimes, I)$ *equipped with arrows* $\blacktriangleleft^{\circ}_X\colon X \to X \otimes X$, $!^{\circ}_X\colon X \to I$, $\blacktriangleright^{\circ}_X\colon X \otimes X \to X$ *and* $¡^{\circ}_X\colon I \to X$, *for every object* X, *such that:*

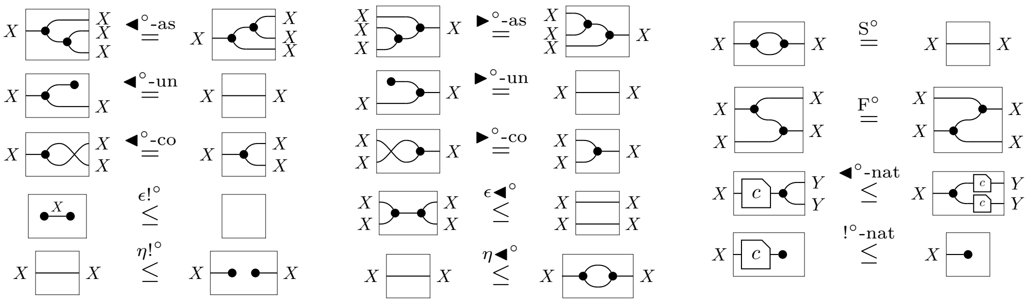
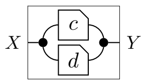
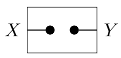
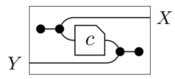
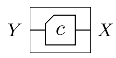


Figure 2 Axioms of Cartesian bicategories.

- *$(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ is a cocommutative comonoid and $(\blacktriangleright^{\circ}_X, \text{¡}^{\circ}_X)$ a commutative monoid, i.e., the equalities ($\blacktriangleleft^{\circ}$-as), ($\blacktriangleleft^{\circ}$-un), ($\blacktriangleleft^{\circ}$-co) and ($\blacktriangleright^{\circ}$-as), ($\blacktriangleright^{\circ}$-un), ($\blacktriangleright^{\circ}$-co) in Figure 2 hold;*
- *every arrow $c\colon X \to Y$ is a lax comonoid homomorphism, i.e., ($\blacktriangleleft^{\circ}$-nat) and ($!^{\circ}$-nat) hold;*
- *comonoids are left adjoints to the monoids, i.e., ($\eta\, \blacktriangleleft^{\circ}$), ($\epsilon\, \blacktriangleleft^{\circ}$), ($\eta !^{\circ}$) and ($\epsilon !^{\circ}$) hold;*
- *monoids and comonoids form special Frobenius bimonoids, i.e., (F°) and (S°) hold;*
- *monoids and comonoids satisfy the expected coherence conditions (see e.g. [8]).*

A morphism of cartesian bicategories *is a poset-enriched strong symmetric monoidal functor preserving monoids and comonoids.*

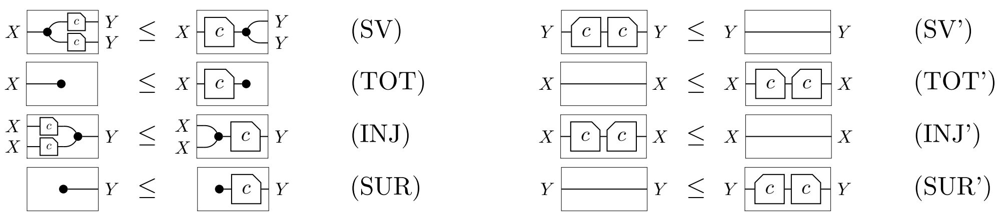
From the (co)monoids one can derive additional structure: each homset $\mathbf{C}[X,Y]$ is a meet-semilattice with $\sqcap$ and $\top$ as shown below. Moreover, there is an iso $(\cdot)^{\dagger}\colon \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$:

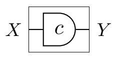
$c \sqcap d \stackrel{\text{def}}{=}$  $\top \stackrel{\text{def}}{=}$  $c^{\dagger} \stackrel{\text{def}}{=}$  (drawn as ).

▶ **Example 2.** **Rel** is a cartesian bicategory with $\otimes$ the cartesian product, I the singleton $1 \stackrel{\text{def}}{=} \{\bullet\}$, $\blacktriangleleft^{\circ}_X$ the diagonal relation on X, $!^{\circ}_X$ the unique function from X to I, and $\blacktriangleright^{\circ}_X$ and ¡°_X the opposites of $\blacktriangleleft^{\circ}_X$ and $!^{\circ}_X$. It follows that $R^{\dagger} = \{(y,x) \mid (x,y) \in R\}$ for any $R \subseteq X \times Y$.

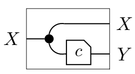
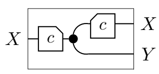
▶ **Example 3.** $\mathsf{Rel}(\mathbf{C})$, the bicategory of relations in $\mathbf{C}$, is a cartesian bicategory if $\mathbf{C}$ is regular. For details on $\mathsf{Rel}(\mathbf{C})$ and its structure, see e.g. [11, Section 2.8].

▶ **Definition 4.** *In a cartesian bicategory, an arrow $c\colon X \to Y$ is* single-valued *if* (SV)*,* total *if* (TOT)*,* injective *if* (INJ) *and* surjective *if* (SUR)*.*



An arrow $c\colon X \to Y$ is a map *if it is single-valued and total, and we draw it as*  *.*

▶ **Lemma 5** ([7])**.** *In a cartesian bicategory,* (SV) *iff* (SV')*,* (TOT) *iff* (TOT')*,* (INJ) *iff* (INJ') *and* (SUR) *iff* (SUR')*. It follows maps $c\colon X \to Y$ have right adjoints, given by $c^{\dagger}\colon Y \to X$.*

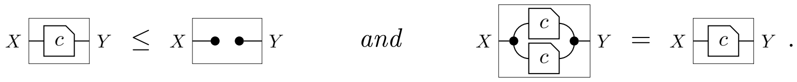
▶ **Lemma 6.** *For any arrow $c\colon X \to Y$, it holds that*  $\leq$  *.*

Moreover, it is an equality when c is injective.

Proof. See Lemma 4.3 in [7]. ◀

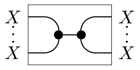
Throughout the paper we uniformly mark with (CB) the usage of the following known results that hold in cartesian bicategories.

▶ **Lemma 7.** *For any arrow* $c\colon X \to Y$*, it holds that*



Proof. See Lemma 4.11 and Lemma 4.12 in [7]. ◀

▶ **Lemma 8.** *Any connected diagram* $c\colon X^n \to X^m$ *made out of* $id^\circ, \sigma^\circ, \blacktriangleleft^\circ, !^\circ, \blacktriangleright^\circ$ *and* $¡^\circ$ *is equivalent to*



.

Proof. This result is known as the *spider lemma* and it holds, more in general, in any symmetric monoidal category whose objects are equipped with special Frobenius algebras, see e.g. [32, 16]. ◀

We conclude this section by recalling a result from [12] that will be useful in Section 6 where, to prove the Löwenheim-Skolem Theorem, we will need to take arbitrary joins.

▶ **Definition 9** ([27], A1.4)**.** *A* geometric *category* $\mathbf{C}$ *is a coherent category s.t. for each* $A \in \mathbf{C}$*, the subobjects of* A *form a complete lattice and pullback functors preserve all joins.*

▶ **Lemma 10** ([12], Remark 3.7)**.** *Let* $\mathbf{C}$ *be a geometric category. Then* $\mathsf{Rel}(\mathbf{C})$ *is monoidally enriched over complete join-semilattices:* $(\bigvee_i f_i) \fatsemi g = \bigvee_i (f_i \fatsemi g)$ *and* $(\bigvee_i f_i) \otimes g = \bigvee_i (f_i \otimes g)$.

3 First-Order Bicategories

First-order bicategories (fo bicategories), introduced in [5], are a categorification of Peirce's calculus of relations [40] with a complete equational proof system for fo logic. Here we recall a succinct characterisation [6] and explain how free fo bicategories capture fo theories.

▶ **Definition 11.** *A* fo bicategory *is a cartesian bicategory* $\mathbf{C}$ *such that:*

1. *every homset* $\mathbf{C}[X,Y]$ *is a Boolean algebra* $(\mathbf{C}[X,Y], \wedge, \top, \vee, \bot, \neg)$*;*
2. *for every map* $f\colon X \to Y$ *and arrow* $c\colon Y \to Z$ *it holds that* $f \fatsemi \neg c = \neg(f \fatsemi c)$*.*

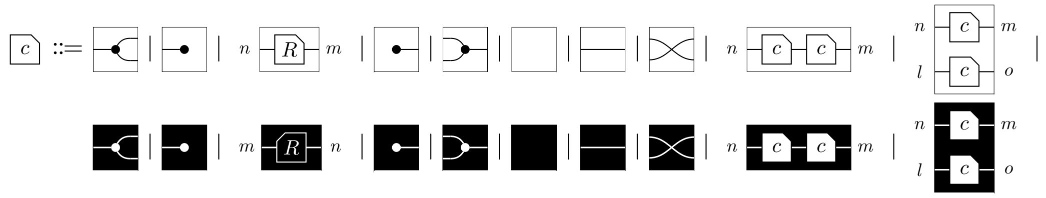
A morphism of fo bicategories *is a cartesian bicategory morphism s.t.* $F(\neg c) = \neg F(c)$*,* $\forall c$*.*

▶ **Example 12.** **Rel** is a fo bicategory with the usual Boolean algebra structure on relations. For a Boolean coherent category $\mathbf{C}$ [27], $\mathsf{Rel}(\mathbf{C})$ is a fo bicategory [6]. This holds, in particular, when $\mathbf{C}$ is additionally a geometric category where regular epis split, as assumed in Theorem 27. Note that this includes any Grothendieck topos satisfying the axiom of choice [20].

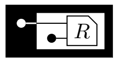
We will now consider an equivalent, diagrammatic presentation of fo bicategories that reveals the underlying algebraic structure, giving a complete equational axiomatisation for fo logic. Given a *monoidal signature* Σ, i.e. a set of symbols $R\colon n \to m$ with arity n and coarity m, $\mathcal{F}_\Sigma$ is the fo bicategory freely generated by Σ. Objects are natural numbers; arrows are terms defined by the following context free grammar

$$\begin{array}{rcccccccccccccccccccc} c & ::= & \blacktriangleleft^{\circ}_{1} & \mid & !^{\circ}_{1} & \mid & R^{\circ} & \mid & ¡^{\circ}_{1} & \mid & \blacktriangleright^{\circ}_{1} & \mid & id^{\circ}_{0} & \mid & id^{\circ}_{1} & \mid & \sigma^{\circ}_{1,1} & \mid & c \fatsemi c & \mid & c \otimes c & \mid \\ & & \blacktriangleleft^{\bullet}_{1} & \mid & !^{\bullet}_{1} & \mid & R^{\bullet} & \mid & ¡^{\bullet}_{1} & \mid & \blacktriangleright^{\bullet}_{1} & \mid & id^{\bullet}_{0} & \mid & id^{\bullet}_{1} & \mid & \sigma^{\bullet}_{1,1} & \mid & c \mathbin{\bullet\!;} c & \mid & c \circledast c & \end{array}$$

where $R \in \Sigma$, quotiented by the equivalence class induced by the precongruence (w.r.t. $\fatsemi, \otimes, \pmb{;}, \circledast$) generated by the axioms in Figure 3. Terms can be represented as string diagrams:

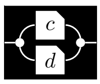
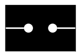


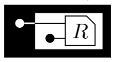
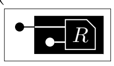
The diagrams in the first line are those of cartesian bicategories, while those on the second line are De Morgan dual to the first. We refer to them as the white and black fragments.

Note that $\mathcal{F}_\Sigma$ features two compositions $\fatsemi$ and $\pmb{;}$, as well as two monoidal products $\otimes$ and $\circledast$, that can be arbitrarily nested. Graphically, this is made clear by the usage of white and black backgrounds. For instance, the term $\mathsf{i}_1^\bullet \pmb{;} ((id_1^\circ \otimes \mathsf{i}_1^\circ) \fatsemi R^\circ)$ is rendered as .

The white fragment corresponds to the $\exists\wedge$-fragment of first-order logic, while the black fragment corresponds to the $\forall\vee$-fragment. The role of the (co)units is that of quantifiers, and hence the diagram above corresponds to the formula $\forall x.\exists y.R(x,y)$. The curious reader may look at Figure 7 in [5] for a full encoding of first-order formulas into $\mathcal{F}_\Sigma$ diagrams of type $n \to 0$. Given this correspondence we will often refer to arrows of $\mathcal{F}_\Sigma$ as *formulas*.

It is not obvious how $\mathcal{F}_\Sigma$ forms a fo bicategory as in Definition 11. By construction, however, $\mathcal{F}_\Sigma$ is a cartesian bicategory. We obtain a join-semilattice on homsets as follows:

$c \sqcup d \stackrel{\text{def}}{=}$  $\qquad \bot \stackrel{\text{def}}{=}$ 

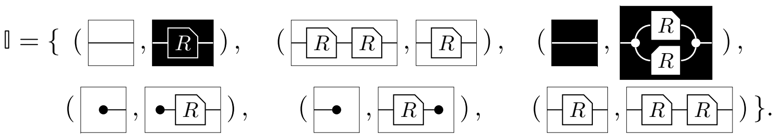
Negation $\neg$ is defined inductively on the syntax (see Definition 32 in Appendix A) and it acts as *photographic negative*, e.g. $\neg($  $) =$ . In general, given a term , we draw its negation as . In [5] it is shown how this structure forms a Boolean algebra on the homsets and, moreover, every map commutes with negation, thus satisfying Definition 11.

In [33] Lawvere introduced functorial semantics. Models of an algebraic theory $\mathbb{T}$ in a cartesian category $\mathbf{C}$ are cartesian functors from the free cartesian category $\mathcal{L}_\mathbb{T}$ on $\mathbb{T}$ to $\mathbf{C}$. In [5] this perspective has been extended to fo theories, replacing cartesian categories with fo bicategories [5].

▶ **Definition 13.** *A* fo theory $\mathbb{T}$ *is a pair* $(\Sigma, \mathbb{I})$ *where* Σ *is a monoidal signature and* $\mathbb{I}$ *is a set of* axioms*: pairs* (c,d) *for* $c, d \colon n \to m$ *in* $\mathcal{F}_\Sigma$*, standing for* $c \leq d$*. We write* $\mathcal{F}_\mathbb{T}$ *for the fo bicategory* $\mathcal{F}_\Sigma$ *quotiented by the axioms in* $\mathbb{I}$ *and* $Q_\mathbb{I} \colon \mathcal{F}_\Sigma \to \mathcal{F}_\mathbb{T}$ *for the quotient.*

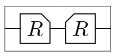
An *interpretation* $\mathcal{I} = (X, \rho)$ of a monoidal signature Σ in $\mathbf{C}$ consists of an object X together with an arrow $\rho(R) \colon X^n \to X^m$ for each generator $R \colon n \to m$ in Σ. This data uniquely determines a morphism of fo bicategories $F_\mathcal{I} \colon \mathcal{F}_\Sigma \to \mathbf{C}$: on objects $F_\mathcal{I}(n) \stackrel{\text{def}}{=} X^n$ and inductively on arrows: see e.g. Definition 33 in Appendix A. Conversely, any morphism $F \colon \mathcal{F}_\Sigma \to \mathbf{C}$ induces an interpretation: $X \stackrel{\text{def}}{=} F(1)$ and $\rho(R) \stackrel{\text{def}}{=} F(R^\circ)$. *Models* of $\mathbb{T} = (\Sigma, \mathbb{I})$ are those interpretations $\mathcal{I}$ that satisfy the axioms in $\mathbb{I}$, i.e. for all $(c,d) \in \mathbb{I}$, $F_\mathcal{I}(c) \leq F_\mathcal{I}(d)$. Hence, models are (in bijective correspondence with) morphisms $F \colon \mathcal{F}_\mathbb{T} \to \mathbf{C}$.

▶ **Example 14** (Dense linear orders without endpoints)**.** Let $\mathbb{DLO}$ be the fo theory $(\Sigma, \mathbb{I})$, where $\Sigma = \{R \colon 1 \to 1\}$ and $\mathbb{I}$ is the following set of pairs:



An interpretation of Σ in $\mathbf{Rel}$ is a set X together with a relation $R \subseteq X \times X$. It is a model of $\mathbb{DLO}$ iff R is irreflexive ($id^{\circ}_X \subseteq \neg R$), transitive ($R \mathbin{;} R \subseteq R$), linear ($id^{\bullet}_X \subseteq R \cup R^{\dagger}$), it has no endpoints ($\top \subseteq \top \mathbin{;} R$ and $\top \subseteq R \mathbin{;} \top$) and is dense ($R \subseteq R \mathbin{;} R$).

Two models of $\mathbb{DLO}$ are the rationals $(\mathbb{Q}, \sqsubset)$ and the reals $(\mathbb{R}, \prec)$ with their respective standard orders. In our setting, these are morphisms of fo bicategories $F_{\mathbb{Q}}, F_{\mathbb{R}} \colon \mathcal{F}_{\mathbb{DLO}} \to \mathbf{Rel}$ such that $F_{\mathbb{Q}}(1) = \mathbb{Q}, F_{\mathbb{Q}}(R) = \sqsubset$ and $F_{\mathbb{R}}(1) = \mathbb{R}, F_{\mathbb{R}}(R) = \prec$.

As an example, observe that $F_{\mathbb{R}}($$) = \{(x, y) \in \mathbb{R}^2 \mid \exists z \in \mathbb{R}. x \prec z \wedge y \prec z\} = \mathbb{R}^2$.

▶ Remark 15. In $\mathbb{DLO}$, the two no endpoints axioms are, respectively, (SUR) and (TOT). In arbitrary theories, one may also include (SV) and (INJ), thereby avoiding the need for function symbols in standard first-order logic. In this setting, a function symbol f of arity n can instead be represented by a symbol $f \colon n \to 1$ in Σ, together with (SV) and (TOT).

4 Model theory with First-Order Bicategories

Our first novel observation is that monoidal lax natural transformations between morphisms of fo bicategories capture the standard notion of *elementary morphism* of models [51], i.e. functions between models that preserve and reflect the truth of formulas (Theorem 18).

▶ **Definition 16.** *Let $F, G : \mathcal{F}_{\mathbb{T}} \to \mathbf{C}$ be models of $\mathbb{T}$. We say that $\alpha \stackrel{def}{=} \{\alpha_n : F(n) \to G(n)\}_{n \in \mathbb{N}}$ is an* elementary morphism of models *iff it is a* monoidal lax natural transformation *$\alpha \colon F \Rightarrow G$, namely, $\alpha_0 = id^{\circ}_I$, $\alpha_{n+m} = \alpha_n \otimes \alpha_m$ and, for all $\varphi \colon n \to m$ in $\mathcal{F}_{\mathbb{T}}$, $F(\varphi) \mathbin{;} \alpha_m \leq \alpha_n \mathbin{;} G(\varphi)$. We write* $\mathsf{Mod}(\mathbb{T}, \mathbf{C})$ *for the category of models of $\mathbb{T}$ in $\mathbf{C}$ and their morphisms.*

Observe that the first two conditions force α_n to be $\bigotimes_{i=1}^{n} \alpha_1$. Hence, α is uniquely determined by $\alpha_1 \colon F(1) \to G(1)$ which is an arrow of $\mathbf{C}$ from the carrier object of the model F to the one of the model G. When $\mathbf{C} = \mathbf{Rel}$, α_1 is an injective function.

▶ **Lemma 17.** *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a fo theory. Consider models $F, G : \mathcal{F}_{\mathbb{T}} \to \mathbf{C}$ of $\mathbb{T}$ and $\alpha : F \Rightarrow G$ an elementary morphism. Then α_1 is an injective map.*

Proof. By lax naturality, $\blacktriangleleft^{\circ}_{F(1)} \mathbin{;} (\alpha_1 \otimes \alpha_1) \leq \alpha_1 \mathbin{;} \blacktriangleleft^{\circ}_{G(1)}$ and $!^{\circ}_{F(1)} \leq \alpha_1 \mathbin{;} !^{\circ}_{G(1)}$, that is α_1 is single-valued and total, namely a map. For injectivity, i.e., $\alpha_1 \mathbin{;} \alpha_1^{\dagger} \leq id^{\circ}_{F(1)}$, observe that:

$$
\begin{aligned}
\neg(id^{\circ}_{F(1)}) &\stackrel{\text{(TOT')}}{\leq} \neg(id^{\circ}_{F(1)}) \mathbin{;} \alpha_1 \mathbin{;} \alpha_1^{\dagger} \stackrel{(\alpha_1 \text{ lax nat.})}{\leq} \alpha_1 \mathbin{;} \neg(id^{\circ}_{G(1)}) \mathbin{;} \alpha_1^{\dagger} \\
&\stackrel{\text{(Def. 11.2)}}{\leq} \neg(\alpha_1 \mathbin{;} id^{\circ}_{G(1)} \mathbin{;} \alpha_1^{\dagger}) = \neg(\alpha_1 \mathbin{;} \alpha_1^{\dagger})
\end{aligned}
$$
◀

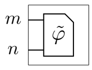
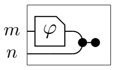
Another elementary, yet illuminating observation is that being a lax natural transformation is equivalent to naturality w.r.t arrows of coarity 0. This reflects the model theoretic definition in [51] of elementary morphisms as those preserving and reflecting the truth of sentences.

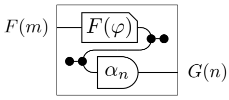
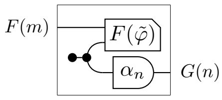
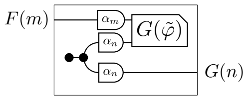
▶ **Theorem 18.** *Let $\mathbb{T}$ be a fo theory, $F, G : \mathcal{F}_{\mathbb{T}} \to \mathbf{C}$ be models and $\alpha = \{\alpha_n : F(n) \to G(n)\}_{n \in \mathbb{N}}$ a family of single-valued arrows. Then $\alpha \colon F \Rightarrow G$ is a lax natural transformation iff $F(\varphi) = \alpha_n \mathbin{;} G(\varphi)$ for all arrows $\varphi : n \to 0$.*

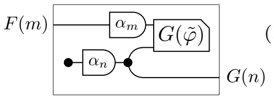
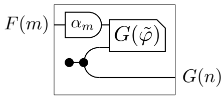
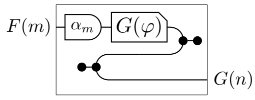
Proof. Assume that α is lax natural. Hence $F(\varphi) \leq \alpha_n \,;\, G(\varphi)$. The following derivation

$$\neg F(\varphi) \overset{(F\text{ morphism})}{=} F(\neg\varphi) \overset{(\alpha\text{ lax nat.})}{\leq} \alpha_n \,;\, G(\neg\varphi) \overset{(G\text{ morphism})}{=} \alpha_n \,;\, \neg G(\varphi) \overset{\substack{\text{(Lem. 17)}\\ \text{(Def. 11.2)}}}{=} \neg(\alpha_n \,;\, G(\varphi))$$

proves the other inequality. Conversely, consider a formula $\varphi : m \to n$ and fix the following:

 $\overset{\text{def}}{=}$  . (5)

Then $F(\varphi) ; \alpha_n \overset{\text{(CB)}}{=}$  $\overset{(5)}{=}$  $\overset{(hp)}{=}$  $\overset{\text{(SV)}}{=}$

 $\overset{(!^\circ\text{-nat})}{\leq}$  $\overset{(5)}{=}$  $\overset{\text{(CB)}}{=} \alpha_m \,;\, G(\varphi)$ ◀

Recall from Example 14, the two models $F_{\mathbb{Q}}, F_{\mathbb{R}} : \mathcal{F}_{\mathbb{DLO}} \to \mathbf{Rel}$ of $\mathbb{DLO}$. Take $\alpha_1 : F_{\mathbb{Q}}(1) \to F_{\mathbb{R}}(1)$ to be the injections of $\mathbb{Q}$ into $\mathbb{R}$ and $\alpha_n = \bigotimes_{i=1}^{n} \alpha_1$. It is well known that $\alpha : F_{\mathbb{Q}} \implies F_{\mathbb{R}}$ is an elementary morphism, but proving this fact is nontrivial. Most of its proofs rely on the Tarski–Vaught test [51], which characterises elementary morphisms as those injective functions closed under existential witnesses for formulas with parameters from the substructure. A simple inductive argument confirms that such a test holds in any fo bicategory, a fact that we shall use in our proof of the downward Löwenheim-Skolem (Theorem 27).

▶ **Lemma 19** (Tarski-Vaught test)**.** *Let $F, G : \mathcal{F}_{\mathbb{T}} \to \mathbf{C}$ be models of $\mathbb{T}$ and $\alpha_1 : F(1) \to G(1)$ an injective map s.t., for each symbol $R : m \to n$ in Σ, $F(R) = \alpha_m \,;\, G(R) \,;\, \alpha_n^\dagger$. TFAE:*

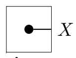
1. *$\alpha : F \Rightarrow G$ is an elementary morphism of models;*
2. *for all $\varphi : m \to n$, $\psi : n \to o$ in $\mathcal{F}_{\mathbb{T}}$, if $F(\varphi) = \alpha_m \,;\, G(\varphi) \,;\, \alpha_n^\dagger$ and $F(\psi) = \alpha_n \,;\, G(\psi) \,;\, \alpha_o^\dagger$ then $F(\varphi \,;\, \psi) \geq \alpha_m \,;\, G(\varphi \,;\, \psi) \,;\, \alpha_o^\dagger$.*

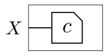
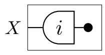
Despite its compositional appearance, this is the diagrammatic counterpart of the classical formulation of the test: closure under composition with $¡^\circ$, i.e. existential quantification, recovers the requirement that existential witnesses with parameters from the substructure remain in the substructure.

The following lemma says that being a model is reflected by elementary morphisms. Given a model $G : \mathcal{F}_{\mathbb{T}} \to \mathbf{C}$, the *induced structure* is the morphism $\overline{G} : \mathcal{F}_{\Sigma} \to \mathbf{C}$ obtained by precomposition with $Q_{\mathbb{I}}$.

▶ **Lemma 20.** *Let $G : \mathcal{F}_{\mathbb{T}} \to \mathbf{C}$ be a model of $\mathbb{T}$ and $\overline{G} : \mathcal{F}_{\Sigma} \to \mathbf{C}$ be the induced structure. Consider another $F : \mathcal{F}_{\Sigma} \to \mathbf{C}$. Then, if there is an elementary morphism $\alpha : F \Rightarrow \overline{G}$, then F is also a model of $\mathbb{T}$.*

5 Diagrammatic Skolemisation

Classically, Skolemisation replaces an existentially-quantified formula $\exists x.\, \varphi(x, \bar{y})$ by $\varphi(f(\bar{y}), \bar{y})$ where f is a fresh function symbol, so that existential witnesses, when they exist, are "chosen" by f. In fo bicategories, existential quantification is captured by composition with  , and our aim here is to replace it by a diagrammatic analogue of a partial function: a single-valued arrow. For this, we need the ambient cartesian bicategory to satisfy two properties. *Tabulations* (Definition 21), going back to Carboni and Walters [12], ensure that every arrow $c\colon X \to I$ is witnessed by an injective map to its domain: morally the support of c, represented by a subobject of X. *Choice* (Definition 22), isolated as an axiom in [10], ensures that every total arrow contains a map. Classically (i.e. in **Rel**), the statement "every total relation contains a function" is equivalent to the axiom of choice. Together, the two suffice to construct Skolem witnesses (Lemma 24), which is the technical backbone on which our proof of the downward Löwenheim–Skolem theorem in § 6 relies.

▶ **Definition 21** ([12], Def. 3.1)**.** *A cartesian bicategory has* tabulations *if, for any arrow $c\colon X \to I$, there is an injective map $i\colon Y \to X$ such that*  $=$ .

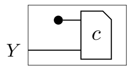
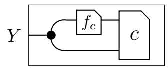
Tabulations give us subobjects, but not selections; for that we also need choice.

▶ **Definition 22** ([10], Def. 10)**.** *A cartesian bicategory has* choice *if, for any total arrow $c\colon X \to Y$ there is a map $f\colon X \to Y$ such that $f \leq c$.*

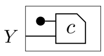
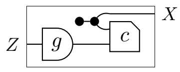
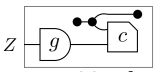
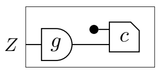
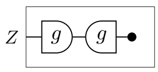
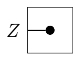
We now isolate sufficient conditions on $\mathbf{C}$ to obtain tabulations and choice in $\mathsf{Rel}(\mathbf{C})$.

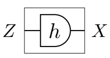
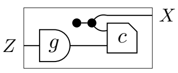
▶ **Lemma 23** ([12], [10])**.** *If $\mathbf{C}$ is a regular category, then $\mathsf{Rel}(\mathbf{C})$ has tabulations. If in $\mathbf{C}$ regular epis split, then $\mathsf{Rel}(\mathbf{C})$ has choice.*

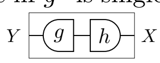
With tabulations and choice in place, we can carry out our diagrammatic Skolemisation. Given a coarity-I arrow $c : X \otimes Y \to I$ – the diagrammatic rendering of a formula $\varphi(\bar{x}, \bar{y})$ with two blocks of variables – we construct a single-valued witness $f_c\colon Y \to X$.

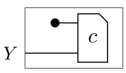
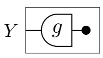
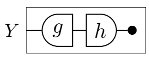
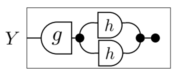
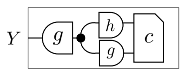
▶ **Lemma 24** (Skolemisation)**.** *In a cartesian bicategory with tabulation and choice, for any $c\colon X \otimes Y \to I$, there is a single-valued $f_c\colon Y \to X$ such that*  $\leq$ .

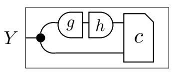
Proof. The proof proceeds in three steps: (i) tabulate c with X discarded to get an injection $g\colon Z \to Y$; (ii) show that a composite of g with c is total, so by choice it admits a map $h\colon Z \to X$ below it; (iii) verify that $g^\dagger \mathbin{;} h\colon Y \to X$ is the desired Skolem witness.

Let $g\colon Z \to Y$ be the tabulation of  and observe that  is total:  $\stackrel{(◀°\text{-un})}{=}$  $\stackrel{(\text{Def. } 21)}{=}$  $\stackrel{(\text{TOT'})}{\geq}$ ; and hence, by Definition 22, there is a map

 $\leq$  . (6)

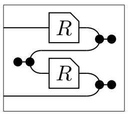
Now, given (INJ) and (SV) in Definition 4, it is immediate that g is injective iff $g^\dagger$ is single-valued. Also the composite of two single-valued arrows is single-valued. Thus,  is single-valued, and we have:

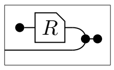
 $\stackrel{(\text{Def. } 21)}{=}$  $\stackrel{(\text{TOT})}{=}$  $\stackrel{(\text{CB})}{=}$  $\stackrel{(6)}{\leq}$ 

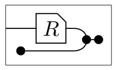
$\stackrel{(\text{CB})}{=}$  ◀

It is instructive to inspect concrete Skolem witnesses in the context of our running example.

▶ **Example 25.** Recall the theory $\mathbb{DLO}$ from Example 14 and its model $F_{\mathbb{R}}\colon \mathcal{F}_{\mathbb{DLO}} \to \mathbf{Rel}$. Consider the following formulas in $\mathcal{F}_{\mathbb{DLO}}$:







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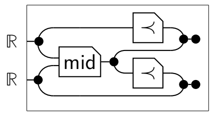
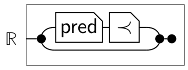
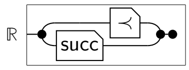
Their interpretations under $F_{\mathbb{R}}$ are, respectively:

$$\{((x,y),\bullet) \mid \exists z \in \mathbb{R}. x \prec z \wedge z \prec y\} \qquad \{(y,\bullet) \mid \exists x \in \mathbb{R}. x \prec y\} \qquad \{(x,\bullet) \mid \exists y \in \mathbb{R}. x \prec y\}.$$

These arrows admit, for instance, the following Skolemisations in **Rel**:







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where $\mathsf{mid} = \{((x,y), \frac{x+y}{2})\} \subseteq \mathbb{R}^2 \times \mathbb{R}$, $\mathsf{pred} = \{(x, x-1)\} \subseteq \mathbb{R} \times \mathbb{R}$, $\mathsf{succ} = \{(x, x+1)\} \subseteq \mathbb{R} \times \mathbb{R}$. Note that while here mid, pred and succ happen to be total, this is not the case in general.

6 Löwenheim-Skolem Theorem

The downward Löwenheim–Skolem theorem [36] is a cornerstone of classical model theory: every infinite first-order structure has a countable elementary substructure. The classical proof proceeds by iterating Skolem closures starting from a base set and taking the union of the resulting chain. In this section we prove a categorical version (Theorem 27) for models valued in $\mathsf{Rel}(\mathbf{C})$ over a Boolean geometric category $\mathbf{C}$ in which regular epimorphisms split. The construction follows the classical recipe: for each formula in the theory, Skolemisation (Lemma 24) supplies a single-valued witness; iteratively closing a base subobject under these witnesses yields a chain of subobjects of the carrier; geometricity of $\mathbf{C}$ supplies the join of this chain, and tabulation extracts it as an injective map. The resulting subobject supports a model structure, and the Tarski–Vaught test (Lemma 19) verifies that the inclusion into the original model is elementary. Although we have most of the theoretical apparatus already in place, the final missing ingredient is the *cardinality* of the substructure produced by the construction. Classically, downward Löwenheim–Skolem yields one with cardinality bounded by the cardinality of the base set plus the cardinality of the signature (with a floor of $\aleph_0$). To state the corresponding bound in a categorical setting we need a suitable notion of "categorical cardinality": λ-presentability from the theory of locally presentable categories [1].

▶ **Definition 26** ([1]). *Let $\mathbf{C}$ be a category. Consider a diagram $D : J \to \mathbf{C}$ where J is a poset. We say that D is* $\lambda-$directed *if for any $S \subseteq J$ of cardinality less than λ, there is an upper bound of S in J. An object A of $\mathbf{C}$ is* $\lambda-$presentable *if $\mathsf{Hom}(A,-)\colon \mathbf{C} \to \mathbf{Set}$ preserves $\lambda-$directed colimits.*

Given a cardinal μ, we denote with μ^+ the successor cardinal of μ, that is the smallest cardinal greater than μ.

We are now ready to state and prove the downward Löwenheim-Skolem theorem for models valued in $\mathsf{Rel}(\mathbf{C})$, where $\mathbf{C}$ is Boolean geometric and it satisfies the axiom of choice.

▶ **Theorem 27** (Downward Löwenheim-Skolem). *Let $\mathbf{C}$ be a Boolean geometric category where regular epis split, $\mathbb{T}$ a first-order theory and $F\colon \mathcal{F}_{\mathbb{T}} \to \mathsf{Rel}(\mathbf{C})$ a model. Let $i\colon A \to F(1)$ be an injective map from a $\lambda-$presentable object A, then there is a model $G_A\colon \mathcal{F}_{\mathbb{T}} \to \mathsf{Rel}(\mathbf{C})$ s.t.*
- *there is an injective map $j\colon A \to G_A(1)$;*
- *there is an elementary morphism of models $\alpha\colon G_A \Rightarrow F$;*
- *and $G_A(1)$ is $(\lambda + (|\Sigma| + \aleph_0)^+)-$presentable in $\mathbf{C}$.*

To prove Theorem 27, we construct the model G_A by mimicking the original proof for **Rel**: first, $\forall n \in \mathbb{N}$, we inductively define injective maps $g_n \colon X_n \to F(1)$ as illustrated below.

Given an arrow $\varphi \colon 1 + m_\varphi \to 0$ in $\mathcal{F}_\mathbb{T}$, take $f^F_\varphi \colon F(1)^{m_\varphi} \to F(1)$ to be the single-valued arrow obtained from the Skolemisation of $F(\varphi)$ (Lemma 24) and consider the arrow $\boxed{\bullet\!-\!\boxed{i^{m_\varphi}}\!-\!\boxed{f^F_\varphi}-}\ F(1)$ where $i^{m_\varphi} \colon A^{m_\varphi} \to F(1)^{m_\varphi}$ is defined as $i^{m_\varphi} \stackrel{\text{def}}{=} \bigotimes_{i=1}^{m_\varphi} i$. Since $\mathbf{C}$ is a geometric category, by Lemma 10, we can take the join $\bigvee_\varphi \boxed{\bullet\!-\!\boxed{i^{m_\varphi}}\!-\!\boxed{f^F_\varphi}-}\ F(1)$ in $\mathsf{Rel}(\mathbf{C})$ and, since the latter has tabulations, there is an injective map $g_0 \colon X_0 \to F(1)$ such that

$$\boxed{\bullet\!-\!\boxed{g_0}-}\ F(1) \ = \bigvee_{\varphi} \boxed{\bullet\!-\!\boxed{i^{m_\varphi}}\!-\!\boxed{f^F_\varphi}-}\ F(1) \quad \text{and thus,} \quad \boxed{\bullet\!-\!\boxed{i^{m_\varphi}}\!-\!\boxed{f^F_\varphi}-}\ F(1) \ \leq \ \boxed{\bullet\!-\!\boxed{g_0}-}\ F(1)\ .$$

For the induction step, given an injective map $g_n \colon X_n \to F(1)$, we define $g_{n+1} \colon X_{n+1} \to F(1)$ to be the tabulation such that

$$\boxed{\bullet\!-\!\boxed{g_{n+1}}-}\ F(1) \ = \bigvee_{\varphi} \boxed{\bullet\!-\!\boxed{g_n^{m_\varphi}}\!-\!\boxed{f^F_\varphi}-}\ F(1) \quad \text{and thus,} \quad \boxed{\bullet\!-\!\boxed{g_n^{m_\varphi}}\!-\!\boxed{f^F_\varphi}-}\ F(1) \ \leq \ \boxed{\bullet\!-\!\boxed{g_{n+1}}-}\ F(1)\ . \tag{7}$$

Now, from all these $g_n \colon X_n \to F(1)$, again by the fact that $\mathsf{Rel}(\mathbf{C})$ has tabulations and arbitrary joins, we construct the injective map $\alpha_1 \colon G_A(1) \to F(1)$ such that

$$\boxed{\bullet\!-\!\boxed{\alpha_1}-}\ F(1) \ = \bigvee_{n \in \mathbb{N}} \boxed{\bullet\!-\!\boxed{g_n}-}\ F(1) \quad \text{and thus,} \quad \boxed{\bullet\!-\!\boxed{g_n}-}\ F(1) \ \leq \ \boxed{\bullet\!-\!\boxed{\alpha_1}-}\ F(1)\ . \tag{8}$$

▶ Remark 28. In **Rel**, $G_A(1)$ is the subset of $F(1)$ defined as $\bigcup_{n\in\mathbb{N}} X_n$ where $X_0 \stackrel{\text{def}}{=} \bigcup_\varphi \{f^F_\varphi(a_1, \dots, a_{m_\varphi}) \mid a_i \in A\}$ and $X_{n+1} \stackrel{\text{def}}{=} \bigcup_\varphi \{f^F_\varphi(a_1, \dots, a_{m_\varphi}) \mid a_i \in X_n\}$. It is thus immediate that its cardinality is bounded by $\lambda + |\Sigma| + \aleph_0$. More in general, the following holds in any Boolean geometric category. The proof, that we omit for reasons of space, follows from several results from the theory of locally presentable categories.

▶ **Proposition 29.** *$G_A(1)$ is $(\lambda + (|\Sigma| + \aleph_0)^+)$–presentable in $\mathbf{C}$.*

Now we define an interpretation $\mathcal{I} = (X, \rho)$ of Σ in $\mathsf{Rel}(\mathbf{C})$: the object X is $G_A(1)$ and, for all $R \colon m \to n$ in Σ, $\rho(R) \stackrel{\text{def}}{=} \alpha_m \,;\, \overline{F}(R) \,;\, \alpha_n^\dagger$, where $\alpha_m \stackrel{\text{def}}{=} \bigotimes_{i=1}^m \alpha_1$ and $\overline{F} \colon \mathcal{F}_\Sigma \to \mathsf{Rel}(\mathbf{C})$ is $Q_\mathbb{I} \,;\, F$. The interpretation $\mathcal{I}$ gives rise to a morphism of fo bicategories $G_A \colon \mathcal{F}_\Sigma \to \mathsf{Rel}(\mathbf{C})$ such that $G_A(n) = G_A(1)^n$ and $G_A(R) = \alpha_m \,;\, \overline{F}(R) \,;\, \alpha_n^\dagger$. The latter allows us to exploit the Tarski-Vaught test (Lemma 19) to prove that $\alpha \colon G_A \Rightarrow \overline{F}$ is an elementary morphism. First we fix $G_A(m)\ \boxed{-\boxed{f^{G_A}_\varphi}-}\ G_A(1) \stackrel{\text{def}}{=} G_A(m)\ \boxed{-\boxed{\alpha_m}\!-\!\boxed{f^F_\varphi}\!-\!\boxed{\alpha_1}-}\ G_A(1)$ and we observe the following.

▶ **Lemma 30.** *For every $\varphi \colon 1 + m \to 0$,* $G_A(m)\ \boxed{-\boxed{\alpha_m}\!-\!\boxed{f^F_\varphi}-}\ F(1) \ \leq \ G_A(m)\ \boxed{-\boxed{f^{G_A}_\varphi}\!-\!\boxed{\alpha_1}-}\ F(1)\ .$

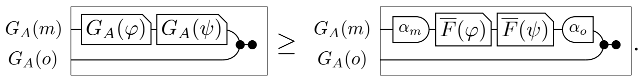
▶ **Example 31.** To illustrate the construction so far, consider the model $F_\mathbb{R} \colon \mathcal{F}_{\mathbb{DLO}} \to \mathbf{Rel}$ from Example 14. Let $A = \{0\}$ and let $i \colon \{0\} \to F_\mathbb{R}(1)$ be the canonical inclusion.

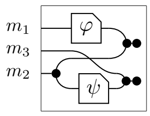
The theory $\mathbb{DLO}$ enjoys a syntactic property – irrelevant to detail here – which allows us to restrict the $\varphi \colon 1 + m_\varphi \to 0$ in the above construction to only the three formulas appearing in Example 25. Assuming that their Skolemisation yields precisely the three functions mid, pred, and succ from that example, we obtain that $G_A(1)$ is the set of dyadic rationals $\mathbb{Z}[\frac{1}{2}] = \{\frac{n}{2^m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\}$. The unique symbol $R \colon 1 \to 1$ in Σ is then interpreted as the order $\prec$ on $\mathbb{R}$ restricted to $\mathbb{Z}[\frac{1}{2}]$. To prove that this defines an elementary submodel of $F_\mathbb{R}$, we use Lemma 30, which ensures that $\mathbb{Z}[\frac{1}{2}]$ is closed under the operations mid, pred, and succ.

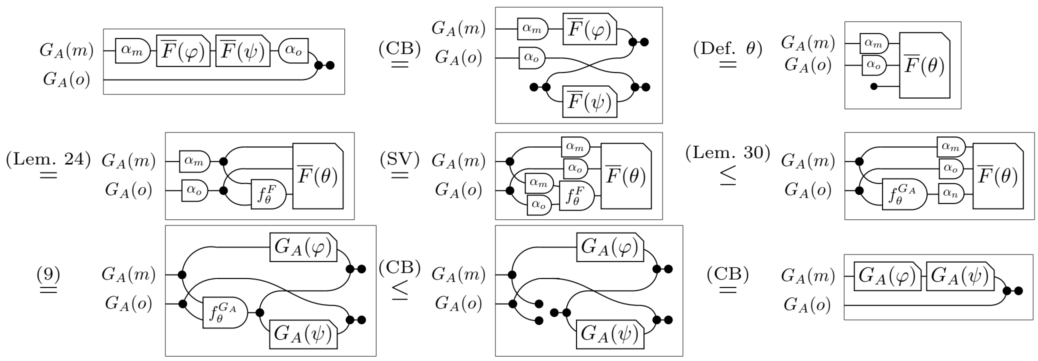
To conclude that α is an elementary morphism from G_A to $\overline{F}$, by the Tarski-Vaught test, it is enough to prove that for all $\varphi : m \to n$ and $\psi : n \to o$ in $\mathcal{F}_\Sigma$, if

$$G_A(\varphi) = \alpha_m \mathbin{;} \overline{F}(\varphi) \mathbin{;} \alpha_n^\dagger \text{ and } G_A(\psi) = \alpha_n \mathbin{;} \overline{F}(\psi) \mathbin{;} \alpha_o^\dagger \tag{9}$$

then $G_A(\varphi) \mathbin{;} G_A(\psi) \geq \alpha_m \mathbin{;} \overline{F}(\varphi) \mathbin{;} \overline{F}(\psi) \mathbin{;} \alpha_o^\dagger$. The latter is equivalent to

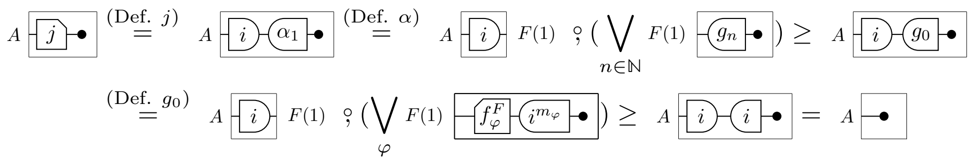


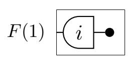
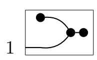
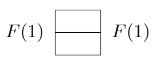
This is proved by the following derivation, where $\theta \overset{\text{def}}{=}$ .



Since $F\colon \mathcal{F}_{\mathbb{T}} \to \mathsf{Rel}(\mathbf{C})$ is a model of $\mathbb{T}$ and $\alpha\colon G_A \Rightarrow \overline{F}$ is an elementary morphism then, by Lemma 20, also G_A is a model of $\mathbb{T}$.

We are only left to prove that there is an injective map $j\colon A \to G_A(1)$. By taking $j \overset{\text{def}}{=} i \mathbin{;} \alpha_1^\dagger$ this amounts to the following. Observe that, since α is injective, $\alpha^\dagger$ is single-valued; and since α is single-valued, $\alpha^\dagger$ is injective. Thus j is both single-valued and injective, being the composition of two arrows with each property. It remains to show that j is total. It holds that:



where the last inequality comes from the fact that the join contains  when $\varphi =$  and $f_\varphi^F =$  .

7 Conclusions and Future work

Following Lawvere's tradition of functorial semantics, we have shown that lax natural transformations between functors of first-order bicategories recover precisely the classical model-theoretic notion of elementary morphism (Theorem 18). Building on this correspondence, we extended several cornerstone results of model theory, including the Tarski-Vaught test (Lemma 19), Skolemisation (Lemma 24), and the Löwenheim-Skolem theorem (Theorem 27).

Beyond the generalisations, the value of our work lies in the use of first-order bicategories as a semantic framework. This approach supports reasoning that is both equational and point-free, aligning with Tarski's vision of relational algebra [49, 50]. These features make the framework particularly well suited for formalisation [31, 42, 43, 44] and, in particular, the use of diagrams is crucial in several recently developed proof assistants [30, 2, 45, 13, 21, 48, 35]. Taken together, these results suggest that first-order bicategories provide not only a conceptual unification of model-theoretic ideas, but also a promising foundation for their formalisation.

At this point, it is worth highlighting a more concrete connection with existing approaches to mechanised reasoning. In particular, double pushout graph rewriting [17] provides an effective framework for implementing equational reasoning on string diagrams, as developed in [4] and realised in [29]. As shown in [9], revisiting a classical result from database theory [14], the diagrams of the white fragment of $\mathcal{F}_\Sigma$ can be identified with hypergraphs – indeed, with models – and logical entailment corresponds precisely to the existence of suitable morphisms. This correspondence is what allows rewriting techniques to be effective in this fragment. Extending this perspective to the whole of $\mathcal{F}_\Sigma$ would require a combinatorial characterisation of its diagrams, which remains an open problem, although recent progress in this direction has been reported in [46]. Our expectation is that, as in [9], the appropriate notion of model morphism might be the key in uncovering the relevant combinatorial structures.

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A Appendix to Section 3

In this appendix we report the full set of axioms for $\mathcal{F}_{\Sigma}$, both in term and diagrammatic form, and we recall some additional definition.

▶ **Definition 32.** *The operation $\neg(\cdot)\colon \mathcal{F}_{\Sigma}[n,m] \to \mathcal{F}_{\Sigma}[n,m]$ is inductively defined on the syntax as follows:*

$$\begin{array}{cccc}
\neg id^{\circ}_0 \stackrel{def}{=} id^{\bullet}_0 & \neg id^{\circ}_1 \stackrel{def}{=} id^{\bullet}_1 & \neg id^{\bullet}_0 \stackrel{def}{=} id^{\circ}_0 & \neg id^{\bullet}_1 \stackrel{def}{=} id^{\circ}_1 \\
\neg \sigma^{\circ}_{1,1} \stackrel{def}{=} \sigma^{\bullet}_{1,1} & \neg R^{\circ} \stackrel{def}{=} (R^{\bullet})^{\dagger} & \neg \sigma^{\bullet}_{1,1} \stackrel{def}{=} \sigma^{\circ}_{1,1} & \neg R^{\bullet} \stackrel{def}{=} (R^{\circ})^{\dagger} \\
\neg \blacktriangleleft^{\circ}_1 \stackrel{def}{=} \blacktriangleleft^{\bullet}_1 & \neg \blacktriangleright^{\circ}_1 \stackrel{def}{=} \blacktriangleright^{\bullet}_1 & \neg \blacktriangleleft^{\bullet}_1 \stackrel{def}{=} \blacktriangleleft^{\circ}_1 & \neg \blacktriangleright^{\bullet}_1 \stackrel{def}{=} \blacktriangleright^{\circ}_1 \\
\neg !^{\circ}_1 \stackrel{def}{=} !^{\bullet}_1 & \neg \text{¡}^{\circ}_1 \stackrel{def}{=} \text{¡}^{\bullet}_1 & \neg !^{\bullet}_1 \stackrel{def}{=} !^{\circ}_1 & \neg \text{¡}^{\bullet}_1 \stackrel{def}{=} \text{¡}^{\circ}_1 \\
\neg(c \mathbin{;^{\circ}} d) \stackrel{def}{=} \neg c \mathbin{;^{\bullet}} \neg d & \neg(c \otimes d) \stackrel{def}{=} \neg c \mathbin{\otimes^{\bullet}} \neg d & \neg(c \mathbin{;^{\bullet}} d) \stackrel{def}{=} \neg c \mathbin{;^{\circ}} \neg d & \neg(c \mathbin{\otimes^{\bullet}} d) \stackrel{def}{=} \neg c \otimes \neg d
\end{array}$$

▶ **Definition 33.** *Given an interpretation $\mathcal{I} = (X, \rho)$ of a monoidal signature Σ in $\mathbf{C}$, the unique morphism $F_{\mathcal{I}}\colon \mathcal{F}_{\Sigma} \to \mathbf{C}$ is defined on objects as $F_{\mathcal{I}}(1) \stackrel{def}{=} X$ and inductively on arrows as follows:*

$$\begin{array}{cccc}
F_{\mathcal{I}}(\blacktriangleleft^{\circ}_1) \stackrel{def}{=} \blacktriangleleft^{\circ}_X & F_{\mathcal{I}}(!^{\circ}_1) \stackrel{def}{=} !^{\circ}_X & F_{\mathcal{I}}(\blacktriangleright^{\circ}_1) \stackrel{def}{=} \blacktriangleright^{\circ}_X & F_{\mathcal{I}}(\text{¡}^{\circ}_1) \stackrel{def}{=} \text{¡}^{\circ}_X \\
F_{\mathcal{I}}(id^{\circ}_0) \stackrel{def}{=} id^{\circ}_1 & F_{\mathcal{I}}(id^{\circ}_1) \stackrel{def}{=} id^{\circ}_X & F_{\mathcal{I}}(\sigma^{\circ}_{1,1}) \stackrel{def}{=} \sigma^{\circ}_{X,X} & F_{\mathcal{I}}(R^{\circ}) \stackrel{def}{=} \rho(R) \\
\multicolumn{2}{c}{F_{\mathcal{I}}(c \mathbin{;^{\circ}} d) \stackrel{def}{=} F_{\mathcal{I}}(c) \mathbin{;^{\circ}} F_{\mathcal{I}}(d)} & \multicolumn{2}{c}{F_{\mathcal{I}}(c \otimes d) \stackrel{def}{=} F_{\mathcal{I}}(c) \otimes F_{\mathcal{I}}(d)} \\
F_{\mathcal{I}}(\blacktriangleleft^{\bullet}_1) \stackrel{def}{=} \neg \blacktriangleleft^{\circ}_X & F_{\mathcal{I}}(!^{\bullet}_1) \stackrel{def}{=} \neg !^{\circ}_X & F_{\mathcal{I}}(\blacktriangleright^{\bullet}_1) \stackrel{def}{=} \neg \blacktriangleright^{\circ}_X & F_{\mathcal{I}}(\text{¡}^{\bullet}_1) \stackrel{def}{=} \neg \text{¡}^{\circ}_X \\
F_{\mathcal{I}}(id^{\bullet}_0) \stackrel{def}{=} \neg id^{\circ}_1 & F_{\mathcal{I}}(id^{\bullet}_1) \stackrel{def}{=} \neg id^{\circ}_X & F_{\mathcal{I}}(\sigma^{\bullet}_{1,1}) \stackrel{def}{=} \neg \sigma^{\circ}_{X,X} & F_{\mathcal{I}}(R^{\bullet}) \stackrel{def}{=} \neg \rho(R)^{\dagger} \\
\multicolumn{2}{c}{F_{\mathcal{I}}(c \mathbin{;^{\bullet}} d) \stackrel{def}{=} \neg(\neg F_{\mathcal{I}}(c) \mathbin{;^{\circ}} \neg F_{\mathcal{I}}(d))} & \multicolumn{2}{c}{F_{\mathcal{I}}(c \mathbin{\otimes^{\bullet}} d) \stackrel{def}{=} \neg(\neg F_{\mathcal{I}}(c) \otimes \neg F_{\mathcal{I}}(d))}
\end{array}$$

$$\begin{array}{rcl@{\qquad}rcl}
\blacktriangleleft^{\circ}_1 \fatsemi (id^{\circ}_1 \otimes \blacktriangleleft^{\circ}_1) & \overset{(\blacktriangleleft^{\circ}\text{-as})}{=} & \blacktriangleleft^{\circ}_1 \fatsemi (\blacktriangleleft^{\circ}_1 \otimes id^{\circ}_1) & (id^{\circ}_1 \otimes \blacktriangleright^{\circ}_1) \fatsemi \blacktriangleright^{\circ}_1 & \overset{(\blacktriangleright^{\circ}\text{-as})}{=} & (\blacktriangleright^{\circ}_1 \otimes id^{\circ}_1) \fatsemi \blacktriangleright^{\circ}_1 \\
\blacktriangleleft^{\circ}_1 \fatsemi (id^{\circ}_1 \otimes !^{\circ}_1) & \overset{(\blacktriangleleft^{\circ}\text{-un})}{=} & id^{\circ}_1 & (id^{\circ}_1 \otimes \text{¡}^{\circ}_1) \fatsemi \blacktriangleright^{\circ}_1 & \overset{(\blacktriangleright^{\circ}\text{-un})}{=} & id^{\circ}_1 \\
\blacktriangleleft^{\circ}_1 \fatsemi \sigma^{\circ}_{1,1} & \overset{(\blacktriangleleft^{\circ}\text{-co})}{=} & \blacktriangleleft^{\circ}_1 & \sigma^{\circ}_{1,1} \fatsemi \blacktriangleright^{\circ}_1 & \overset{(\blacktriangleright^{\circ}\text{-co})}{=} & \blacktriangleright^{\circ}_1
\end{array}$$

$$(\blacktriangleleft^{\circ}_1 \otimes id^{\circ}_1) \fatsemi (id^{\circ}_1 \otimes \blacktriangleright^{\circ}_1) \overset{(\mathrm{F}^{\circ})}{=} (id^{\circ}_1 \otimes \blacktriangleleft^{\circ}_1) \fatsemi (\blacktriangleright^{\circ}_1 \otimes id^{\circ}_1) \qquad \blacktriangleleft^{\circ}_1 \fatsemi \blacktriangleright^{\circ}_1 \overset{(\mathrm{S}^{\circ})}{=} id^{\circ}_1$$

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\text{¡}^{\circ}_1 \fatsemi !^{\circ}_1 & \overset{(\epsilon !^{\circ})}{\leq} & id^{\circ}_0 & \blacktriangleright^{\circ}_1 \fatsemi \blacktriangleleft^{\circ}_1 & \overset{(\epsilon \blacktriangleleft^{\circ})}{\leq} & (id^{\circ}_1 \otimes id^{\circ}_1) & c \fatsemi \blacktriangleleft^{\circ}_m & \overset{(\blacktriangleleft^{\circ}\text{-nat})}{\leq} & \blacktriangleleft^{\circ}_n \fatsemi (c \otimes c) \\
id^{\circ}_1 & \overset{(\eta !^{\circ})}{\leq} & !^{\circ}_1 \fatsemi \text{¡}^{\circ}_1 & id^{\circ}_1 & \overset{(\eta \blacktriangleleft^{\circ})}{\leq} & \blacktriangleleft^{\circ}_1 \fatsemi \blacktriangleright^{\circ}_1 & c \fatsemi !^{\circ}_m & \overset{(!^{\circ}\text{-nat})}{\leq} & !^{\circ}_n
\end{array}$$

$$\begin{array}{rcl@{\qquad}rcl}
\blacktriangleleft^{\bullet}_1 ; (id^{\bullet}_1 \circledast \blacktriangleleft^{\bullet}_1) & \overset{(\blacktriangleleft^{\bullet}\text{-as})}{=} & \blacktriangleleft^{\bullet}_1 ; (\blacktriangleleft^{\bullet}_1 \circledast id^{\bullet}_1) & (id^{\bullet}_1 \circledast \blacktriangleright^{\bullet}_1) ; \blacktriangleright^{\bullet}_1 & \overset{(\blacktriangleright^{\bullet}\text{-as})}{=} & (\blacktriangleright^{\bullet}_1 \circledast id^{\bullet}_1) ; \blacktriangleright^{\bullet}_1 \\
\blacktriangleleft^{\bullet}_1 ; (id^{\bullet}_1 \circledast !^{\bullet}_1) & \overset{(\blacktriangleleft^{\bullet}\text{-un})}{=} & id^{\bullet}_1 & (id^{\bullet}_1 \circledast \text{¡}^{\bullet}_1) ; \blacktriangleright^{\bullet}_1 & \overset{(\blacktriangleright^{\bullet}\text{-un})}{=} & id^{\bullet}_1 \\
\blacktriangleleft^{\bullet}_1 ; \sigma^{\bullet}_{1,1} & \overset{(\blacktriangleleft^{\bullet}\text{-co})}{=} & \blacktriangleleft^{\bullet}_1 & \sigma^{\bullet}_{1,1} ; \blacktriangleright^{\bullet}_1 & \overset{(\blacktriangleright^{\bullet}\text{-co})}{=} & \blacktriangleright^{\bullet}_1
\end{array}$$

$$(\blacktriangleleft^{\bullet}_1 \circledast id^{\bullet}_1) ; (id^{\bullet}_1 \circledast \blacktriangleright^{\bullet}_1) \overset{(\mathrm{F}^{\bullet})}{=} (id^{\bullet}_1 \circledast \blacktriangleleft^{\bullet}_1) ; (\blacktriangleright^{\bullet}_1 \circledast id^{\bullet}_1) \qquad \blacktriangleleft^{\bullet}_1 ; \blacktriangleright^{\bullet}_1 \overset{(\mathrm{S}^{\bullet})}{=} id^{\bullet}_1$$

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
!^{\bullet}_1 ; \text{¡}^{\bullet}_1 & \overset{(\epsilon \text{¡}^{\bullet})}{\leq} & id^{\bullet}_1 & \blacktriangleleft^{\bullet}_1 ; \blacktriangleright^{\bullet}_1 & \overset{(\epsilon \blacktriangleright^{\bullet})}{\leq} & id^{\bullet}_1 & \blacktriangleleft^{\bullet}_n ; (c \circledast c) & \overset{(\blacktriangleleft^{\bullet}\text{-nat})}{\leq} & c ; \blacktriangleleft^{\bullet}_m \\
id^{\bullet}_0 & \overset{(\eta \text{¡}^{\bullet})}{\leq} & \text{¡}^{\bullet}_1 ; !^{\bullet}_1 & \blacktriangleright^{\bullet}_1 ; \blacktriangleleft^{\bullet}_1 & \overset{(\eta \blacktriangleright^{\bullet})}{\leq} & (id^{\bullet}_1 \circledast id^{\bullet}_1) & !^{\bullet}_n & \overset{(!^{\bullet}\text{-nat})}{\leq} & c ; !^{\bullet}_m
\end{array}$$

$$a \fatsemi (b ; c) \overset{(\delta_l)}{\leq} (a \fatsemi b) ; c \qquad (a ; b) \fatsemi c \overset{(\delta_r)}{\leq} a ; (b \fatsemi c)$$

$$\begin{array}{rcl@{\quad}rcl@{\qquad}rcl@{\quad}rcl}
id^{\circ}_{n+m} & \overset{(\tau\sigma^{\circ})}{\leq} & \sigma^{\circ}_{n,m} ; \sigma^{\bullet}_{m,n} & \sigma^{\bullet}_{n,m} \fatsemi \sigma^{\circ}_{m,n} & \overset{(\gamma\sigma^{\circ})}{\leq} & id^{\bullet}_{n+m} & id^{\circ}_n & \overset{(\tau R^{\circ})}{\leq} & R^{\circ} ; R^{\bullet} & R^{\bullet} \fatsemi R^{\circ} & \overset{(\gamma R^{\circ})}{\leq} & id^{\bullet}_m \\
id^{\circ}_{n+m} & \overset{(\tau\sigma^{\bullet})}{\leq} & \sigma^{\bullet}_{n,m} ; \sigma^{\circ}_{m,n} & \sigma^{\circ}_{n,m} \fatsemi \sigma^{\bullet}_{m,n} & \overset{(\gamma\sigma^{\bullet})}{\leq} & id^{\bullet}_{n+m} & id^{\circ}_m & \overset{(\tau R^{\bullet})}{\leq} & R^{\bullet} ; R^{\circ} & R^{\circ} \fatsemi R^{\bullet} & \overset{(\gamma R^{\bullet})}{\leq} & id^{\bullet}_n
\end{array}$$

$$\begin{array}{rcl@{\qquad}rcl}
id^{\circ}_{n+m} & \overset{(\circledast^{\circ})}{\leq} & id^{\circ}_n \circledast id^{\circ}_m & id^{\bullet}_n \otimes id^{\bullet}_m & \overset{(\otimes^{\bullet})}{\leq} & id^{\bullet}_{n+m} \\
(a ; b) \otimes (c ; d) & \overset{(\nu^{\circ}_l)}{\leq} & (a \otimes c) ; (b \circledast d) & (a \circledast c) \fatsemi (b \otimes d) & \overset{(\nu^{\bullet}_l)}{\leq} & (a \fatsemi b) \circledast (c \fatsemi d) \\
(a ; b) \otimes (c ; d) & \overset{(\nu^{\circ}_r)}{\leq} & (a \circledast c) ; (b \otimes d) & (a \otimes c) \fatsemi (b \circledast d) & \overset{(\nu^{\bullet}_r)}{\leq} & (a \fatsemi b) \circledast (c \fatsemi d)
\end{array}$$

$$\begin{array}{rcl@{\quad}rcl@{\qquad}rcl@{\quad}rcl}
id^{\circ}_n & \overset{(\tau \blacktriangleleft^{\circ})}{\leq} & \blacktriangleleft^{\circ}_n ; \blacktriangleright^{\bullet}_n & \blacktriangleright^{\bullet}_n \fatsemi \blacktriangleleft^{\circ}_n & \overset{(\gamma \blacktriangleleft^{\circ})}{\leq} & id^{\bullet}_{n+n} & id^{\circ}_{n+n} & \overset{(\tau \blacktriangleright^{\circ})}{\leq} & \blacktriangleright^{\circ}_n ; \blacktriangleleft^{\bullet}_n & \blacktriangleleft^{\bullet}_n \fatsemi \blacktriangleright^{\circ}_n & \overset{(\gamma \blacktriangleright^{\circ})}{\leq} & id^{\bullet}_n \\
id^{\circ}_n & \overset{(\tau !^{\circ})}{\leq} & !^{\circ}_n ; \text{¡}^{\bullet}_n & \text{¡}^{\bullet}_n \fatsemi !^{\circ}_n & \overset{(\gamma !^{\circ})}{\leq} & id^{\bullet}_0 & id^{\circ}_0 & \overset{(\tau \text{¡}^{\circ})}{\leq} & \text{¡}^{\circ}_n ; !^{\bullet}_n & !^{\bullet}_n \fatsemi \text{¡}^{\circ}_n & \overset{(\gamma \text{¡}^{\circ})}{\leq} & id^{\bullet}_n \\
id^{\circ}_n & \overset{(\tau \blacktriangleleft^{\bullet})}{\leq} & \blacktriangleleft^{\bullet}_n ; \blacktriangleright^{\circ}_n & \blacktriangleright^{\circ}_n \fatsemi \blacktriangleleft^{\bullet}_n & \overset{(\gamma \blacktriangleleft^{\bullet})}{\leq} & id^{\bullet}_{n+n} & id^{\circ}_{n+n} & \overset{(\tau \blacktriangleright^{\bullet})}{\leq} & \blacktriangleright^{\bullet}_n ; \blacktriangleleft^{\circ}_n & \blacktriangleleft^{\circ}_n \fatsemi \blacktriangleright^{\bullet}_n & \overset{(\gamma \blacktriangleright^{\bullet})}{\leq} & id^{\bullet}_n \\
id^{\circ}_n & \overset{(\tau !^{\bullet})}{\leq} & !^{\bullet}_n ; \text{¡}^{\circ}_n & \text{¡}^{\circ}_n \fatsemi !^{\bullet}_n & \overset{(\gamma !^{\bullet})}{\leq} & id^{\bullet}_0 & id^{\circ}_0 & \overset{(\tau \text{¡}^{\bullet})}{\leq} & \text{¡}^{\bullet}_n ; !^{\circ}_n & !^{\circ}_n \fatsemi \text{¡}^{\bullet}_n & \overset{(\gamma \text{¡}^{\bullet})}{\leq} & id^{\bullet}_0
\end{array}$$

$$\begin{array}{rcl@{\quad}rcl}
(\blacktriangleleft^{\bullet}_n \otimes id^{\circ}_n) \fatsemi (id^{\circ}_n \otimes \blacktriangleright^{\circ}_n) & \overset{(\mathrm{F}^{\bullet}{}_{\circ})}{=} & (id^{\circ}_n \otimes \blacktriangleleft^{\circ}_n) \fatsemi (\blacktriangleright^{\bullet}_n \otimes id^{\circ}_n) & (\blacktriangleleft^{\circ}_n \otimes id^{\circ}_n) \fatsemi (id^{\circ}_n \otimes \blacktriangleright^{\bullet}_n) & \overset{(\mathrm{F}^{\circ}{}_{\bullet})}{=} & (id^{\circ}_n \otimes \blacktriangleleft^{\bullet}_n) \fatsemi (\blacktriangleright^{\circ}_n \otimes id^{\circ}_n) \\
(\blacktriangleleft^{\circ}_n \circledast id^{\bullet}_n) ; (id^{\bullet}_n \circledast \blacktriangleright^{\bullet}_n) & \overset{(\mathrm{F}_{\bullet}{}^{\circ})}{=} & (id^{\bullet}_n \circledast \blacktriangleleft^{\bullet}_n) ; (\blacktriangleright^{\circ}_n \circledast id^{\bullet}_n) & (\blacktriangleleft^{\bullet}_n \circledast id^{\bullet}_n) ; (id^{\bullet}_n \circledast \blacktriangleright^{\circ}_n) & \overset{(\mathrm{F}_{\circ}{}^{\bullet})}{=} & (id^{\bullet}_n \circledast \blacktriangleleft^{\circ}_n) ; (\blacktriangleright^{\bullet}_n \circledast id^{\bullet}_n)
\end{array}$$

Figure 3 Axioms for $\mathcal{F}_{\Sigma}$. Here a, b, c, d are properly typed terms.

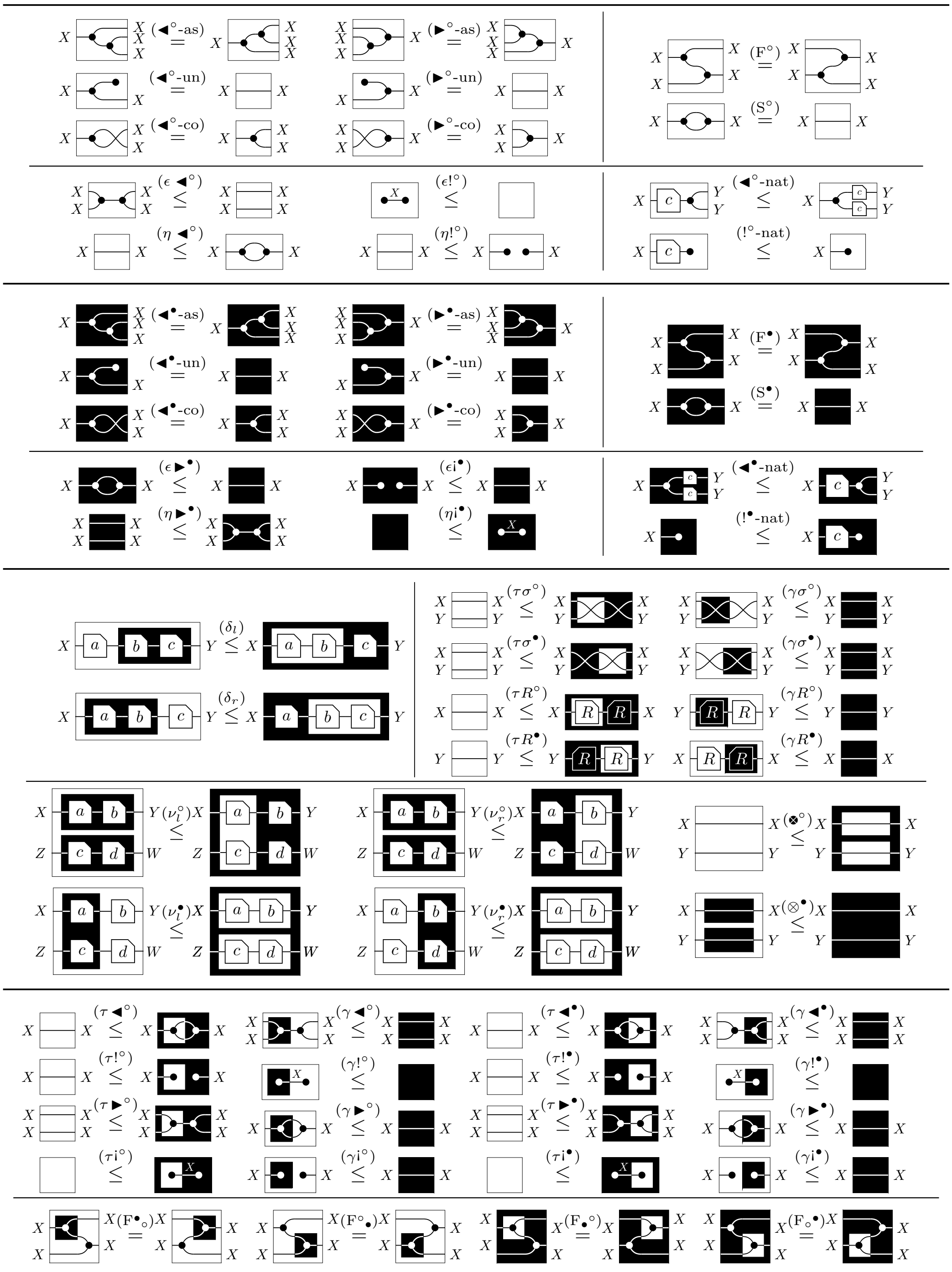

Figure 4 Diagrammatic axioms for $\mathcal{F}_\Sigma$. Here a, b, c, d are properly typed terms.