Generalized Snarks, Disjoint Perfect Matchings, and Graph Covers

Filip Filipi
Institute for Applied Mathematics, University of Bonn, Germany

Jan Kratochvíl
Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

Roman Nedela
Department of Mathematics, Faculty of Applied Sciences, West Bohemia University, Pilsen, Czech Republic

Abstract

We explore the interplay among three classical notions in graph theory: edge-colorings, perfect matchings, and graph coverings (locally bijective homomorphisms of graphs). In this paper, we consider undirected graphs in full generality of this notion: in contrast to the standard notion of a simple graph, our graphs may contain loops, semi-edges, and multiple edges. Many well-studied graph concepts, including matchings, edge-colorings, and covering projections, extend naturally to such graphs. Nevertheless, the role of simple graphs for graph covering problems is central, as emphasized in [J. Bok, J. Fiala, N. Jedličková, J. Kratochvíl, and M. Seifrtová. Computational complexity of covering disconnected multigraphs. Discret. Appl. Math., 359:229–243, 2024]. In that work, a relation "being stronger" was defined (a graph A is stronger than a graph B if every simple graph that covers A also covers B), and it was conjectured that if A has no semi-edges, then A is stronger than B if and only if A covers B. In their extended abstract presented at Eurocomb'23, Kratochvíl and Nedela proved this conjecture for 3-regular 1-vertex graphs B (and arbitrary A). They also introduced the notion (A, B)-snark for a simple graph G that demonstrates that A is not stronger than B. We continue this line of research in the current paper.

As the main result, we show that for every graph A, there exists a simple graph D that covers A in such a way that the maximum number of pairwise disjoint perfect matchings equals the maximum number of pairwise disjoint perfect semi-matchings in A, i.e., spanning 1-regular subgraphs. Notably, the proof is constructive. As a corollary, we obtain a necessary condition for A to be stronger than B in general. This condition turns out to be sufficient whenever B is a 1-vertex graph (there are infinitely many of them), which, in particular, proves the aforementioned conjecture of Bok et al. in this case. Finally, we provide a constructive alternative to the existential NP-hardness proof of covering disconnected graphs in Bok et al. for the case when the target graph contains a 1-vertex component which itself determines an NP-hard covering problem.

2012 ACM Subject Classification Mathematics of computing → Graph theory

Keywords and phrases graph, graph cover, perfect matching, NP-completeness

Digital Object Identifier 10.4230/LIPIcs.MFCS.2026.45

Related Version This paper develops the work presented in earlier versions:
Previous Version: `https://arxiv.org/abs/2504.17387v1` [19]
Previous Report: `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-094` [18]

Funding *Filip Filipi*: Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC 2047/2 – 390685813.
Roman Nedela: Supported by the grant VEGA 2/0056/25 of the Ministry of Education, Development, Research and Youth of the Slovak Republic.

Acknowledgements The authors would like to thank the anonymous referees for their helpful comments.

1 Introduction

In this paper, we explore the interplay among three classical notions in graph theory: *edge-colorings*, *perfect matchings* in graphs, and *graph covers*. We study these notions for undirected graphs in a generalized setting: graphs may have multiple edges, loops (edges incident with a single vertex, adding 2 to the degree of the vertex), and semi-edges (edges incident with a single vertex, but contributing only 1 to its degree). Considering this general notion of graphs has now become standard in topological graph theory [20, 23, 24, 26] and mathematical physics [15]. On the other hand, *simple* graphs contain no loops, no semi-edges, and at most one edge incident with the same pair of vertices.

A *(proper) edge-coloring* of a graph is a mapping from its edges to a set (referred to as the set of colors) such that any two edges incident with the same vertex are assigned different colors. A classical result of Vizing [31] says that every simple graph can be edge-colored by as few colors as its maximum degree plus 1. It is NP-complete to decide if the extra color can be avoided [16, 21]. We note that a graph containing a loop does not allow any proper edge-coloring.

A *matching* in a simple graph is a set of edges such that no two of them are incident with the same vertex. A matching is *perfect* if every vertex is incident with exactly one edge of the matching. Clearly, a simple r-regular graph is r-edge colorable if and only if its edge-set can be partitioned into r pairwise disjoint perfect matchings. Edmonds's blossom algorithm [12] witnesses that deciding whether a simple graph contains a perfect matching is polynomial-time solvable. In contrast, for every $k \geq 2$ and $r \geq k + 1$, deciding whether a simple r-regular graph contains k pairwise disjoint perfect matchings is NP-complete [6].

A *graph covering projection* is a surjective incidence-preserving mapping from the vertices and edges of a source graph onto the vertices and edges of a target graph such that for every vertex of the source graph, its edge-neighborhood is mapped bijectively onto the edge-neighborhood of its image (some peculiarities concerning loops are clarified by the formal definition in Section 2). Graph covers are extensively studied in algebraic and topological graph theory. Their application in the theory of local computation was spotted by Angluin [2, 3]. Bodlaender [4] and Abello, Fellows, and Stillwell [1] initiated the study of the computational complexity of deciding the existence of a covering projection between two graphs. However, until recently, the complexity of the covering problems was only considered for graphs without semi-edges. These were later taken into account in [5, 6].

Although the complexity of the H-Cover problem (Input: A graph G. Question: Does G cover H?) for fixed graphs H has been studied in numerous specific and more general settings, the full characterization requested by the authors of [1] remains out of reach. Moreover, taking semi-edges into consideration has led to new challenges. In [7, 8], the authors formulated the *Strong Dichotomy Conjecture* claiming that for every graph H, the H-Cover problem is either polynomial-time solvable for arbitrary input graphs, or NP-complete for simple ones. The conjecture holds true in all cases for which the complexity of covering a graph H is currently known, but the NP-hardness reduction for simple input graphs can sometimes be intricate.

When studying the computational complexity of disconnected graphs in [9, 10], the authors encountered an interesting phenomenon. They proved that H-Cover is NP-complete whenever H'-Cover is NP-complete for at least one component H' of H and, more importantly, that the same implication holds even if only simple graphs are considered on input. In the latter case, however, instead of providing an explicit polynomial time reduction, they only provide an existential proof of its existence. The proof is based on the *being stronger* relation introduced by the authors: a connected graph A is stronger than a connected graph B if every simple graph covering A also covers B. Obviously, due to transitivity of covering projections, if A covers B, then A is stronger than B. In addition to these trivial cases, infinitely many other pairs of A stronger than B are known. However, it is not clear whether the relation "being stronger" is decidable for general pairs A, B. In the journal version [10], the authors state the following conjecture.

▶ **Conjecture 1** (Conj. 9 in [10]). *Let A and B be connected graphs. If A has no semi-edges (and B is arbitrary), then A is stronger than B if and only if A covers B.*

The first steps towards proving this conjecture were made by Kratochvíl and Nedela [18]. They announced a proof of Conjecture 1 for 3-regular 1-vertex graphs B (note that there are just two such graphs: the 1-vertex graph with three semi-edges and the 1-vertex graph with one semi-edge and one loop) and arbitraty (and hence infinitely many) graphs A. In fact, the authors announced a complete description of the graphs A stronger than the considered graphs B. This they achieved by introducing a new notion of *graph semi-covers* (in semi-covering projections, semi-edges may be mapped onto loops as opposed to covering projections, cf. an exact definition in Section 2). The proofs are available on arxiv [19].

Our results and the organization of the paper

In Section 2, we present exact definitions of the notions and add further observations on the relationship of the main characters of the play: generalized snarks, disjoint perfect matchings and semi-matchings, and graph covers and semi-covers. In Section 3, we support Conjecture 1 by showing that it holds for all connected cubic graphs A with at most 4 vertices. Section 4 contains the main result of the paper.

▶ **Theorem 2.** *For every connected graph A, there exists a simple graph D that covers A and the maximum number of pairwise disjoint perfect matchings in D equals the maximum number of pairwise disjoint perfect semi-matchings in A.*

The proof of Theorem 2 is constructive. In Section 5, we use this theorem to extend the results of [18] from 3-regular 1-vertex graphs to all 1-vertex graphs.

▶ **Theorem 3.** *For every 1-vertex graph B and for every connected graph A, A is stronger than B if and only if A semi-covers B.*

This immediately implies that Conjecture 1 holds true for all 1-vertex graphs B.

▶ **Corollary 4.** *For every 1-vertex graph B and for every connected graph A with no semi-edges, A is stronger than B if and only if A covers B.*

Finally, in Section 6, we briefly recall the non-constructive NP-hardness reduction for H-Cover for disconnected graphs H from [10]. We improve upon it by providing an explicit construction for the case when H has a 1-vertex component which itself defines an NP-complete covering problem.

2 Preliminaries

In this section, we present formal definitions of the key notions used in the paper.

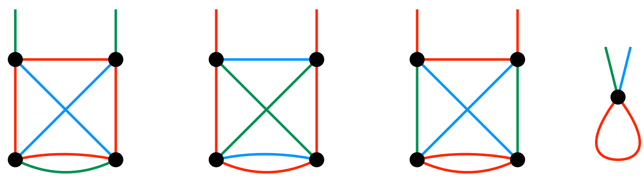

Figure 1 Illustration to the notion of graphs. The 1-vertex graph $F(2,1)$ on the right has two semi-edges and one loop. On the left, three projections from the same graph are depicted by coloring their edges by the colors of their images. The one on the left is a covering projection, the blue edges form a matching and the green edges a semi-matching. The two projections depicted in the middle are semi-covering projections with the preimage of the loop being or containing an open path.

The notion of a graph

As already announced, we consider undirected graphs with multiple edges, loops, and semi-edges. To avoid confusion, we include a formal definition.

▶ **Definition 5.** *A* graph *is a triple $G = (V, \Lambda, \iota)$, where V is a non-empty finite set of* vertices, *$\Lambda = E \cup L \cup S$ is a finite set of edges of G, and $\iota : \Lambda \to \binom{V}{2} \cup V$ is the incidence mapping of edges. There are three distinct types of edges:*

- *the edges of E are called* normal edges *and they satisfy $\iota(e) \in \binom{V}{2}$,*
- *the edges of L are called* loops *and we have $\iota(e) \in V$, and*
- *the edges of S are called* semi-edges *and again $\iota(e) \in V$.*

We say that a vertex u is *incident with* an edge e (and vice versa) if $u \in \iota(e)$ (resp. $u = \iota(e)$) in case of $e \in E$ (resp. $e \in L \cup S$). If G is a graph, its vertex-set will be denoted by $V(G)$, its edge-set by $\Lambda(G)$, and the sets of normal edges, loops, and semi-edges will be denoted by $E(G), L(G)$ and $S(G)$, respectively. The difference between loops and semi-edges is given by their vertex-degree contributions. Loops contribute 2, while semi-edges contribute just 1. Formally, the degree of a vertex $u \in V(G)$ is defined by

$$\deg_G(u) = |\{e \in E(G) : u \in \iota(e)\}| + 2|\{e \in L(G) : u = \iota(e)\}| + |\{e \in S(G) : u = \iota(e)\}|$$

and G is called *d-regular* if $\deg_G(u) = d$ for every vertex $u \in V(G)$. In figures, loops are depicted as closed curves and semi-edges as short segments attached to their vertices (cf. Fig. 1). The 1-vertex graph with s semi-edges and ℓ loops is denoted by $F(s, \ell)$.

A *path* with terminal vertices v_0 and v_k in a graph G is a sequence

$$(e_0,)v_0, e_1, v_1, \ldots, e_k, v_k(, e_{k+1})$$

of distinct vertices and edges such that $\iota(e_i) = \{v_{i-1}, v_i\}$ for every $i = 1, 2, \ldots, k$ and $\iota(e_0) = v_0, \iota(e_{k+1}) = v_k$ for the optional semi-edges e_0 and e_{k+1}. The path is *closed* if it starts with v_0 and ends with v_k, it is *open* if it starts and ends with the semi-edges e_0, e_{k+1}, and it is *half-closed* in the remaining two cases. A closed path with terminal vertices u and v together with an extra normal edge incident with u and v is a *cycle*. A loop is also considered a (degenerate) cycle. A graph is *connected* if any two vertices are connected by a path. Inclusion-wise maximal connected subgraphs are called *components of connectivity* or simply *components* of the graph. For disjoint graphs G, H, we denote their disjoint union by $G + H$. The following graph construction proved useful in further considerations.

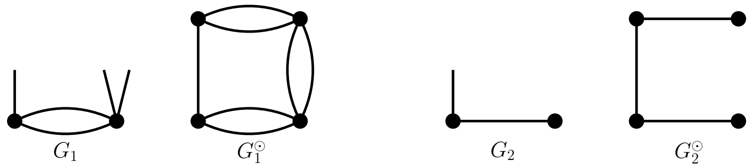

Figure 2 Illustrative examples to Definition 6.

▶ **Definition 6.** *For a graph G, by $G^{\odot}$ we denote the graph with vertex-set $V(G^{\odot}) = \{v_0, v_1 : v \in V(G)\}$, where v_0, v_1 stand for unique copies of v, and edge-set defined as follows*

- *for every normal edge e of G incident with distinct vertices u, v in $V(G)$, we introduce two normal edges: e_0 incident with u_0 and v_0, and e_1 incident with u_1 and v_1;*
- *for every loop e incident with a vertex v of G, we introduce a loop e_0 incident with v_0 and a loop e_1 incident with v_1;*
- *for every semi-edge e incident with a vertex v of G, we introduce a normal edge $\widetilde{e}$ incident with v_0 and v_1.*

Graph covers and semi-covers

▶ **Definition 7.** *A* covering projection *from a graph G to a connected graph H is a pair of mappings $f_V : V(G) \to V(H)$, $f_E : \Lambda(G) \to \Lambda(H)$ such that:*

- *the preimage of a normal edge of H incident with distinct vertices $u, v \in V(H)$ is a matching in G spanning $f_V^{-1}(u) \cup f_V^{-1}(v)$, each edge of the matching being incident with one vertex of $f_V^{-1}(u)$ and with one vertex of $f_V^{-1}(v)$;*
- *the preimage of a loop of H incident with a vertex $u \in V(H)$ is a vertex-disjoint union of cycles in G spanning $f_V^{-1}(u)$; and*
- *the preimage of a semi-edge of H incident with a vertex $u \in V(H)$ is a vertex-disjoint union of semi-edges and normal edges spanning $f_V^{-1}(u)$.*

We write $G \to H$ if G admits a covering projection onto H.

It follows from the definition that the pair of mappings (f_V, f_E) is incidence-preserving because a loop incident with a vertex $u \in V(G)$ can only be mapped onto a loop incident with $f_V(u) \in V(H)$, a semi-edge incident with a vertex $u \in V(G)$ can only be mapped onto a semi-edge incident with $f_V(u) \in V(H)$, and a normal edge incident with vertices $u, v \in V(G)$ can only be mapped onto a loop or a semi-edge incident with $f_V(u) = f_V(v)$ (if $f_V(u)$ and $f_V(v)$ coincide), or to a normal edge incident with $f_V(u)$ and $f_V(v)$ (if these vertices are different). Note that, since we consider only non-empty graphs, both mappings f_V and f_E are surjective. The composition of covering projections is again a covering projection, in particular, if $G_1 \to G_2$ and $G_2 \to G_3$, then $G_1 \to G_3$. It further follows from the definition that $|f_V^{-1}(u)| = |f_V^{-1}(v)|$ whenever u and v are adjacent (connected by a normal edge). Assuming that H is connected, one can use induction on the length of a path with terminal vertices u and v to derive that $|f_V^{-1}(u)| = |f_V^{-1}(v)|$ for any two vertices u, v of H. In such a case, the covering projection f (and G itself) is called a *k-fold cover* of H, for $k = |f_V^{-1}(u)|$. For detailed proofs, and more details in general, the reader is referred to [6,8]. To illustrate the notion of graph covering, consider the case of a 1-vertex target $F(r, 0)$ with no loops and r semi-edges. In a covering projection $f : G \to F(r, 0)$, every vertex $u \in V(G)$ should be incident with exactly one edge (normal edge or semi-edge) from $f_E^{-1}(e)$, for every semi-edge $e \in S(F(r, 0))$. This immediately implies the following observation.

▶ **Example 8.** For every integer r, a graph covers the 1-vertex graph $F(r, 0)$ if and only if it is r-regular and r-edge-colorable.

▶ **Example 9.** A simple graph G covers the lollipop graph $F(1,1)$ if and only if it is 3-regular and has a perfect matching. (In every covering projection, the preimage of the semi-edge of $F(1,1)$ is a perfect matching. On the other hand, if G has a perfect matching $M \subset E(G)$, then $G - M = (V(G), E(G) \setminus M)$ is a 2-regular graph, i.e., a vertex-disjoint union of cycles. Thus, the edges of $G - M$ can be mapped onto the loop of $F(1,1)$ and the edges of M onto the semi-edge.)

As we will be primarily interested in graph covers which are simple, the following construction will be useful.

▶ **Theorem 10.**

1. *Let H be a graph with at most d parallel normal edges incident with the same pair of vertices such that for every vertex, the number of semi-edges incident with this vertex plus twice the number of loops incident with it is at most q. Then for every even integer $p \geq \max\{d, q+1\}$, there exists a simple p-fold cover of H.*
2. *If, moreover, H has no semi-edges, then a simple p-fold cover of H exists for every integer $p \geq \max\{d, q+1\}$.*

Proof. Let p be large enough, as specified by the assumptions. For every vertex $u \in V(H)$, take p copies of u and denote them by $u_1, u_2, \ldots, u_p$. In a covering projection, these vertices will be mapped onto u.

If vertices $u \neq v \in V(H)$ are connected by $d \leq p$ parallel edges, add edges of a d-regular bipartite graph with classes of bipartition $\{u_i : i = 1, 2, \ldots, p\}$ and $\{v_i : i = 1, 2, \ldots, p\}$. Such a bipartite graph always exists, since the complete bipartite graph $K_{p,p}$ can be properly edge-colored by p colors, and we include edges of d of these colors. In a covering projection, edges of one color will be mapped onto one of the parallel edges incident with u and v in H.

If H has no semi-edges at a vertex, say u, let ℓ be the number of loops at u in H. Add edges $u_i u_{i+j}$, $i = 1, 2, \ldots, p, j = 1, 2, \ldots, \ell$, addition in the subscripts being modulo p. This adds 2ℓ "vertical" edges incident with every vertex u_i, and in a covering projection, edges $u_i u_{i+j}$ will be mapped onto the j-th loop at $u \in V(H)$.

If a vertex u is incident with ℓ loops and $s > 0$ semi-edges, which means $s + 2\ell \leq q$, we add edges on $u_1, \ldots, u_p$ as follows. It is well known that the edges of the complete graph on p vertices (note that p is even in this case) can be properly colored by $p - 1 \geq s + 2\ell$ colors. Consider such a coloring by colors $c_1, \ldots, c_{p-1}$ for the complete graph on vertices $u_1, \ldots, u_p$ and discard the edges of colors c_i for $i > s + 2\ell$. In a covering projection, edges of colors c_{2i-1} and c_{2i} will be mapped onto the i-th loops incident with vertex u, for $i = 1, 2, \ldots, \ell$, and edges of color $c_{2\ell+j}$ onto the j-th semi-edge incident with u, for $j = 1, 2, \ldots, s$. ◀

Kratochvíl and Nedela introduced the following generalization of the notion of graph covering projection in [18].

▶ **Definition 11.** *A* semi-covering projection *from a graph G to a connected graph H is a pair of mappings $f_V : V(G) \to V(H)$, $f_E : \Lambda(G) \to \Lambda(H)$ such that:*

- *the preimage of a normal edge of H incident with distinct vertices $u, v \in V(H)$ is a matching in G spanning $f_V^{-1}(u) \cup f_V^{-1}(v)$, each edge of the matching being incident with one vertex of $f_V^{-1}(u)$ and with one vertex of $f_V^{-1}(v)$;*
- *the preimage of a loop of H incident with a vertex $u \in V(H)$ induces a 2-regular subgraph of G spanning $f_V^{-1}(u)$; and*
- *the preimage of a semi-edge of H incident with a vertex $u \in V(H)$ induces a 1-regular subgraph of G spanning $f_V^{-1}(u)$.*

We write $G \rightsquigarrow H$ if G admits a semi-covering projection onto H.

Note that the only distinction from Definition 7 is that semi-edges are now allowed to map onto loops. Every covering projection is also a semi-covering projection and if the domain graph G has no semi-edges, then the notions equal. Similarly as for covering projections, any semi-covering projection is incidence-preserving and surjective. Also, the composition of two semi-covering projections results in another semi-covering projection. Examples of covering and semi-covering projections are depicted in Fig. 1. Note here that $F(3,0)$ semi-covers but does not cover $F(1,1)$.

If H is a connected graph and $f : A \to H$, $g : B \to H$ are (semi-)covering projections, then by $(f+g) : (A+B) \to H$ we denote their disjoint union which is itself a (semi-)covering projection.

So far, we have only considered connected target graphs. The situation gets more complicated for graphs that are not connected. The authors of [10] explore three possible ways of generalizing Definition 7 to disconnected target graphs and argue that imposing the condition on sizes of preimages is the most relevant one. We adopt this notion as described by the following definition.

▶ **Definition 12.** *Let G and H be two not necessarily connected graphs. A mapping $f = (f_V, f_E) : G \to H$ is a covering projection if there exists an integer k such that*

- *for every component G_i of G, the restriction of f to G_i is a covering projection of G_i to some component of H; and*
- *for every vertex $u \in V(H)$, $|f_V^{-1}(u)| = k$.*

In [9, 10], the authors show that the H-Cover problem is solvable in polynomial time if so is the covering problem determined by every component of H. It is relevant to our further considerations that they also show that H-Cover is NP-complete for simple input graphs if so is the covering problem determined by at least one component of H. The somewhat surprising fact is that they were only able to prove the existence of an NP-hardness reduction, but were not able to give an explicit one. The reduction itself is based on the notion described in the next subsection. The aim of our paper is to provide an explicit NP-hardness reduction, at least in the case when H has a 1-vertex component that determines an NP-hard covering problem.

The “being stronger” relation

▶ **Definition 13** (Def. 4 in [9]). *A connected graph A is said to be* stronger *than a connected graph B, denoted by $A \triangleright B$, if it holds that every simple graph covers A only if it also covers B. Formally,*

$$A \triangleright B \quad \Leftrightarrow \quad \forall G \text{ simple graph} : ((G \to A) \Rightarrow (G \to B)).$$

If A is simple, then $A \triangleright B$ if and only if $A \to B$ (put $G := A$ in the definition). This relation becomes interesting when A does not cover B. Let us look at an example.

▶ **Example 14.** Every simple 3-regular 3-edge-colorable graph has a perfect matching. Therefore, Examples 8 and 9 show that $F(3,0) \triangleright F(1,1)$ (note that this fact follows straightforwardly from the forthcoming Proposition 15). On a different note, every simple 3-regular graph that contains a perfect matching covers $F(1,1)$. In fact, simple 3-regular non-3-edge-colorable graphs that contain a perfect matching are exactly the simple graphs witnessing $F(1,1) \not\triangleright F(3,0)$.

The introduction of the notion of semi-covering projections is a promising step towards understanding the “being stronger” relation.

▶ **Proposition 15.** *For any two graphs A and B, if $A \rightsquigarrow B$, then $A \triangleright B$.*

Proof. For any graph G, if $G \to A$ and $A \rightsquigarrow B$, the composition of these projections implies that $G \rightsquigarrow B$. If G is simple, this further implies that $G \to B$. ◀

Our Theorem 3, which will be proved in Section 5, shows that whenever B is a 1-vertex graph, the converse of the implication in Proposition 15 is also true.

Disjoint perfect matchings

The problem of determining the maximum number of pairwise disjoint perfect matchings in simple regular graphs that additionally satisfy specified connectivity constraints was extensively studied due to its connection to Berge-Fulkerson conjecture and Seymour's subsequent generalization; see [22] for a recent overview. We remark that the notion of matchings naturally extends to *semi-matchings* in general graphs.

▶ **Definition 16.** *A* semi-matching *in a graph is a vertex-disjoint union of normal edges and semi-edges. A spanning semi-matching is called a* perfect semi-matching. *The maximum number of pairwise edge-disjoint perfect semi-matchings in a graph G will be denoted by* $\mathrm{dps}(G)$.

We reserve the notion of a *matching* for a vertex-disjoint union of normal edges, and analogously for *perfect matchings.* The maximum number of pairwise edge-disjoint perfect matchings in a graph G will be denoted by $\mathrm{dpm}(G)$. By definition, if G has no semi-edges (and in particular, if G is simple), then $\mathrm{dps}(G) = \mathrm{dpm}(G)$. It is also worth noting that an r-regular graph G is r-edge-colorable if and only if $\mathrm{dps}(G) = r$. The following observations are crucial for the constructions further on.

▶ **Observation 17.** *For any two connected graphs A, B, if $A \rightsquigarrow B$, then* $\mathrm{dps}(A) \geq \mathrm{dps}(B)$.

Proof. Let f be a semi-covering projection from A onto B. The preimage of a perfect semi-matching of B is a perfect semi-matching in A. ◀

▶ **Proposition 18.** *For any two integers s, ℓ, a graph G semi-covers $F(s, \ell)$ if and only if it is $(s + 2\ell)$-regular and* $\mathrm{dps}(G) \geq s$.

Proof. The "only if" implication follows from Observation 17. For the "if" implication, suppose that G is $(s + 2\ell)$-regular and $\mathrm{dps}(G) \geq s$. Let $M_1, \ldots, M_s$ be s pairwise edge-disjoint perfect semi-matchings in G. Construct a semi-covering f as follows. Since $F(s, \ell)$ has only one vertex, the f_V part of f is uniquely determined: every vertex of G is mapped onto the only vertex of $F(s, \ell)$. First, for every $i = 1, 2, \ldots, s$, map the edges of M_i onto the i-th semi-edge of $F(s, \ell)$. The graph $G' = (V(G), \Lambda(G) \setminus \bigcup_{i=1}^{s} M_i)$ is 2ℓ-regular, and so is $G'^{\odot}$ (see Definition 6). Since $G'^{\odot}$ has no semi-edges, its edge-set can be partitioned into pairwise edge-disjoint spanning 2-regular subgraphs (2-factors) [27]. Map the edges of the i-th 2-factor that are fully contained in one copy of G' and the semi-edges of this copy that gave rise to normal edges of this 2-factor onto the i-th loop of $F(s, \ell)$. ◀

3 Small cubic graphs

In this section, we support Conjecture 1 by showing that it holds for all connected cubic graphs A with at most 4 vertices. Fig. 3 shows all connected cubic graphs with at most 4 vertices without semi-edges (there are 5 of them on 4 vertices and 2 on 2 vertices) and all connected cubic graphs with at most 2 vertices (that are the candidates for the B graph in the $A \triangleright B$ relation). Furthermore, the following theorem shows that for all pairs of graphs A, B from Fig. 3, $A \triangleright B$ if and only if A semi-covers B.

▶ **Theorem 19.** *For any two distinct graphs A, B from Fig. 3, the following claims hold:*

1. *If there is a green arc from A to B, then $A \to B$, and hence also $A \rightsquigarrow B$ and $A \triangleright B$.*
2. *If there is a purple arc from A to B, then $A \rightsquigarrow B$ and hence also $A \triangleright B$.*
3. *No other two graphs are in any of these relations, except those that follow from transitivity and the fact that a green arc implies a purple arc between the same pair of graphs (cf. Proposition 15).*

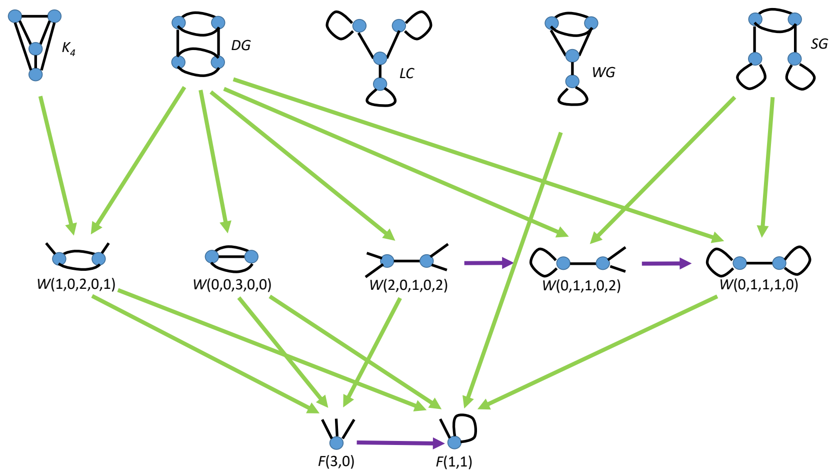

Figure 3 Cubic graphs with at most 4 vertices.

Proof. The proof is somewhat technical and involves constructions of generalized snarks as witnesses of $A \not\triangleright B$ for several pairs of graphs A, B. It will appear in the journal version of the paper. In the meantime, an interested reader may find it in [19]. ◀

4 Simple covers with a prescribed number of disjoint perfect matchings

In this section, we prove Theorem 2. To do this, we provide three different graph augmentations, each of which produces a graph cover that preserves the maximum number of pairwise disjoint perfect semi-matchings while, at the same time, makes the graph "simpler": the first one removes semi-edges, the second one loops, and the last one reduces edge multiplicities.

▶ **Proposition 20.** *For every connected graph A, the graph $A^{\odot}$ covers A, and* $\mathrm{dpm}(A^{\odot}) = \mathrm{dps}(A^{\odot}) = \mathrm{dps}(A)$.

Proof. We define a graph mapping $h : A^{\odot} \to A$ by $h_V = (\mathrm{id} + \mathrm{id})_V$ and h_E given as follows. For every $e \in E(A) \cup L(A)$, we put $h_E(e_0) = h_E(e_1) = (\mathrm{id} + \mathrm{id})_E(e) = e$, and for every $e \in S(A)$, we put $h_E(\tilde{e}) = e$. This way, each $e \in E(A) \cup L(A)$ has its preimage under h_E equal to the preimage of the edge-part of the covering projection $(\mathrm{id} + \mathrm{id}) : (A + A) \to A$, i.e., the respective condition of Definition 7 is satisfied. It is thus sufficient to check that

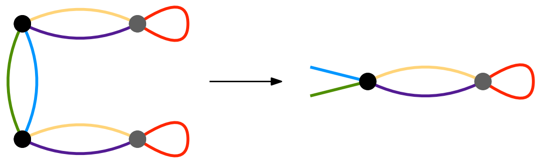

Figure 4 Illustration to the proof of Proposition 20.

for a semi-edge $e \in S(A)$ incident with a vertex $v \in V(A)$, the preimage $h_E^{-1}(e) = \{\tilde{e}\}$ is a vertex-disjoint union of semi-edges and normal edges spanning $h_V^{-1}(v) = \{v_0, v_1\}$. This is true, and so $A^{\odot} \to A$.

Of course, $\mathrm{dps}(A^{\odot}) = \mathrm{dpm}(A^{\odot})$ as $A^{\odot}$ has no semi-edges. If M is a perfect matching in $A^{\odot}$, then $M' = \{e : \tilde{e} \in M\} \cup \{e : e_0 \in M\}$ is a perfect semi-matching in A. If two matchings of $A^{\odot}$ are disjoint, then so are the induced matchings in A. Consequently, $\mathrm{dps(A)} \geq \mathrm{dpm}(A^{\odot})$. The other inequality follows from Observation 17. ◀

For a graph G, we denote by $\mathrm{m}(G)$ the number of unordered pairs of vertices having at least one normal edge in G incident with both of them.

▶ **Proposition 21.** *If B is a graph cover of a connected graph A such that $\mathrm{dps}(B) = \mathrm{dps}(A)$, then there exists a graph cover C of A with $\mathrm{dps}(C) = \mathrm{dps}(A)$ which, furthermore, has no loops and satisfies $|S(C)| = |S(B)|$, and $|E(C)| - \mathrm{m}(C) = |E(B)| - \mathrm{m}(B)$.*

Proof. To construct C, we find a sequence $B = B_{|L(B)|}, B_{|L(B)|-1}, \ldots, B_0 = C$ of covers of A such that, for each i, $|E(B_i)| - \mathrm{m}(B_i) = |E(B)| - \mathrm{m}(B), |L(B_i)| = i, |S(B_i)| = |S(B)|$, and $\mathrm{dps}(B_i) = \mathrm{dps}(A)$. This way, C will have no loops.

By the assumptions, the graph $B_{|L(B)|} = B$ has the required properties. To proceed using induction, assume that $|L(B)| \geq i > 0$ is given and we know that B_i satisfies the statement of the proposition. We construct B_{i-1}.

Theorem 10 provides us with a simple p-fold cover P of A for p being large enough and even. If necessary, we relabel the vertices and edges of P to be disjoint from the ones of B_i. Let there be covering projections $f : B_i \to A$ and $g : P \to A$. Since $|L(B_i)| = i > 0$, there is a loop $\bar{e} \in L(B_i)$ with some vertex $\bar{u} \in V(B_i)$ incident with it. In P, we pick an edge $\hat{e} \in g_E^{-1}(f_E(\bar{e}))$. Since P is simple, the edge $\hat{e}$ has to be a normal edge incident with some distinct vertices $\hat{u}, \hat{u}' \in g_V^{-1}(f_V(\bar{u}))$.

Let B_{i-1} be the graph (cf. Fig. 5) defined by $V(B_{i-1}) = V(B_i + P)$ and

$$E(B_{i-1}) = E(B_i + P) \cup \{\tilde{e}, \tilde{e}'\} \setminus \{\hat{e}\};\ L(B_{i-1}) = L(B_i + P) \setminus \{\bar{e}\};\ S(B_{i-1}) = S(B_i + P)$$

where $\tilde{e}, \tilde{e}'$ are new normal edges incident with vertices $\bar{u}, \hat{u}$ and $\bar{u}, \hat{u}'$, respectively. Due to P being simple, $|E(B_{i-1})| - \mathrm{m}(B_{i-1}) = |E(B_i)| - \mathrm{m}(B_i) = |E(B)| - \mathrm{m}(B), |L(B_{i-1})| = |L(B_i)| - 1 = i - 1$, and $|S(B_{i-1})| = |S(B_i)| = |S(B)|$.

Let $h : B_{i-1} \to A$ be the map obtained by taking a covering projection $(f+g) : (B_i+P) \to A$ and then modifying it as follows: instead of the edges $\bar{e}, \hat{e}$ (for which $(f+g)_E(\bar{e}) = f_E(\bar{e})$, $(f+g)_E(\hat{e}) = g_E(\hat{e}) = f_E(\bar{e})$), we define $h_E(\tilde{e}) = h_E(\tilde{e}') = f_E(\bar{e})$. This way, for every $e \in \Lambda(A) \setminus \{f_E(\bar{e})\}$, the preimage $h_E^{-1}(e) = (f+g)_E^{-1}(e)$ satisfies the corresponding condition from Definition 7. It now suffices to check such a condition for the edge $f_E(\bar{e})$. Covering projections force loops to map to loops, thus $f_E(\bar{e})$ is a loop and we need to show that $h_E^{-1}(f_E(\bar{e})) = (f+g)_E^{-1}(f_E(\bar{e})) \cup \{\tilde{e}, \tilde{e}'\} \setminus \{\bar{e}, \hat{e}\}$ is a vertex-disjoint union of cycles spanning $h_V^{-1}(f_V(\bar{u})) = (f+g)_V^{-1}(f_V(\bar{u}))$. This is true for $(f+g)_E^{-1}(f_E(\bar{e}))$ and, as can be checked on the vertices $\bar{u}, \hat{u}, \hat{u}'$, is preserved upon the substitution of $\{\bar{e}, \hat{e}\}$ with $\{\tilde{e}, \tilde{e}'\}$. Consequently, $B_{i-1} \to A$.

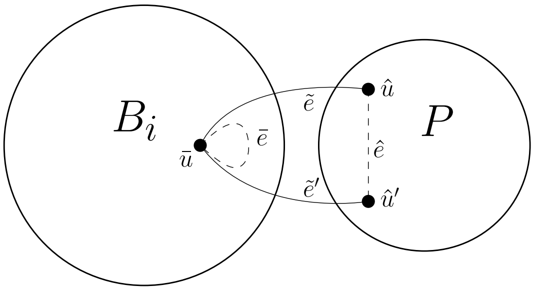

Figure 5 An illustration for the definition of a graph B_{i-1}.

To examine $\mathrm{dps}(B_{i-1})$, first observe that whenever M is a perfect semi-matching in B_{i-1}, then $\tilde{e}, \tilde{e}' \notin M$: They share an incident vertex $\bar{u}$, and so M, being a semi-matching, cannot contain both of them. Since M should be perfect and the edges $\tilde{e}, \tilde{e}'$ separate $V(P)$ from the remaining vertices of B_{i-1}, if exactly one of these edges was contained, then $|V(P)|$ would be odd as there are no semi-edges in P. But P is a p-fold cover for p even, thus $|V(P)|$ is even.

Let now $\mathcal{M}$ be a system of pairwise disjoint perfect semi-matchings in B_{i-1} such that $\mathrm{dps}(B_{i-1}) = |\mathcal{M}|$. Since none of the edges $\tilde{e}, \tilde{e}'$ is contained in a semi-matching from $\mathcal{M}$, we can restrict them to the edge-set of B_i. This yields $\mathrm{dps}(B_{i-1}) = |\mathcal{M}| = |\{M \cap \Lambda(B_i) \mid M \in \mathcal{M}\}| \le \mathrm{dps}(B_i) = \mathrm{dps}(A)$. The other inequality follows from Observation 17. ◀

▶ **Proposition 22.** *If C is a graph cover of a connected graph A such that $\mathrm{dps}(C) = \mathrm{dps}(A)$, then there exists a graph cover D of A with $\mathrm{dps}(D) = \mathrm{dps}(A)$ which, furthermore, has the same number of loops and semi-edges as the graph C and for any two vertices there is at most one normal edge incident with both of them.*

Proof. To construct D, we find a sequence $C = C_{|E(C)|-\mathrm{m}(C)}, C_{|E(C)|-\mathrm{m}(C)-1}, \dots, C_0 = D$ of covers of A such that, for each i, $|E(C_i)| - \mathrm{m}(C_i) = i$, $|L(C_i)| = |L(C)|$, $|S(C_i)| = |S(C)|$, and $\mathrm{dps}(C_i) = \mathrm{dps}(A)$. This way, in D there will be no two vertices which would have more than one normal edge incident with both of them.

By the assumptions, the graph $C_{|E(C)|-\mathrm{m}(C)} = C$ has the required properties. To proceed using induction, assume that $|E(C)| - \mathrm{m}(C) \ge i > 0$ is given and we know that C_i satisfies the statement of the proposition. We construct C_{i-1}.

Theorem 10 provides us with a simple p-fold cover P of A for p being large enough and even. If necessary, we relabel the vertices and edges of P to be disjoint from the ones of C_i. Let there be covering projections $f : C_i \to A$ and $g : P \to A$. Since $|E(C_i)| > \mathrm{m}(C_i)$, we may find vertices $\bar{u}, \bar{v} \in V(C_i)$ that are joined by more than one normal edge. Let $\bar{e}$ be one of these normal edges. In P, we pick an edge $\hat{e} \in g_E^{-1}(f_E(\bar{e}))$. Due to P being simple, such edge has to be a normal edge incident with some vertices $\hat{u} \in g_V^{-1}(f_V(\bar{u}))$ and $\hat{v} \in g_V^{-1}(f_V(\bar{v}))$.

Let C_{i-1} be the graph (cf. Fig. 6) defined by $V(C_{i-1}) = V(C_i + P)$ and

$$E(C_{i-1}) = E(C_i + P) \cup \{\tilde{e}, \tilde{e}'\} \setminus \{\bar{e}, \hat{e}\};\ L(C_{i-1}) = L(C_i + P);\ S(C_{i-1}) = S(C_i + P)$$

where $\tilde{e}, \tilde{e}'$ are new normal edges incident with vertices $\bar{u}, \hat{v}$ and $\bar{v}, \hat{u}$, respectively. Due to P being simple, this manipulation leaves us with $|E(C_{i-1})| - \mathrm{m}(C_{i-1}) = |E(C_i)| - \mathrm{m}(C_i) - 1 = i - 1$, $|L(C_{i-1})| = |L(C_i)| = |L(C)|$, and $|S(C_{i-1})| = |S(C_i)| = |S(C)|$.

Let $h : C_{i-1} \to A$ be obtained by taking a covering projection $(f + g) : (C_i + P) \to A$ and then modifying it as follows: instead of the edges $\bar{e}, \hat{e}$ (for which $(f + g)_E(\bar{e}) = f_E(\bar{e})$ and $(f + g)_E(\hat{e}) = g_E(\hat{e}) = f_E(\bar{e})$), we define $h_E(\tilde{e}) = h_E(\tilde{e}') = f_E(\bar{e})$. This way, for every $e \in \Lambda(A) \setminus \{f_E(\bar{e})\}$, the preimage $h_E^{-1}(e) = (f + g)_E^{-1}(e)$ satisfies its corresponding condition

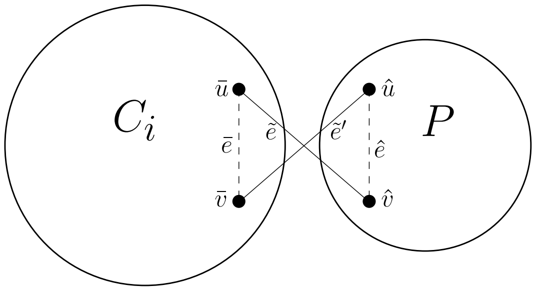

Figure 6 An illustration for the definition of a graph C_{i-1}.

of Definition 7. It now suffices to check that the preimage $h_E^{-1}(f_E(\bar{e})) = (f+g)_E^{-1}(f_E(\bar{e})) \cup \{\tilde{e}, \tilde{e}'\} \setminus \{\bar{e}, \hat{e}\}$ also satisfies its corresponding spanning condition on $h_V^{-1}(f_V(\bar{u})) \cup h_V^{-1}(f_V(\bar{v}))$ given by this definition. This holds for $(f+g)_E^{-1}(f_E(\bar{e}))$ and by checking the individual cases of $f_E(\bar{e})$ being a normal edge, a loop, or a semi-edge, one can examine the vertices $\bar{u}, \hat{u}, \bar{v}, \hat{v}$ to witness that this remains to hold even after substituting $\{\bar{e}, \hat{e}\}$ with $\{\tilde{e}, \tilde{e}'\}$. As a consequence, $C_{i-1} \to A$.

The graph P is a p-fold cover for p even, and so $|V(P)|$ is even. As argued in Proposition 21, the edges $\tilde{e}, \tilde{e}'$ separate the vertices of the simple graph P from the other vertices of C_{i-1}, and so it impossible for a perfect semi-matching to contain exactly one of these edges.

Let $\mathcal{M}$ be a system of pairwise disjoint perfect semi-matchings in C_{i-1} such that $\mathrm{dps}(C_{i-1}) = |\mathcal{M}|$. If $\tilde{e}$ nor $\tilde{e}'$ is contained in a matching from $\mathcal{M}$, then the system of restrictions $\mathcal{M}' = \{M \cap \Lambda(C_i) \mid M \in \mathcal{M}\}$ consists of $|\mathcal{M}|$ pairwise disjoint perfect semi-matchings in C_i. Otherwise, according to the previous paragraph, there is $\tilde{M} \in \mathcal{M}$ which contains both $\tilde{e}$ and $\tilde{e}'$. It is now possible to correct the restriction of $\tilde{M}$ by letting $\mathcal{M}' := \left\{M \cap \Lambda(C_i) \mid M \in \mathcal{M}, M \neq \tilde{M}\right\} \cup \{(\tilde{M} \cap \Lambda(C_i)) \cup \{\bar{e}\}\}$. Either way, $\mathrm{dps}(C_{i-1}) = |\mathcal{M}| = |\mathcal{M}'| \leq \mathrm{dps}(C_i) = \mathrm{dps}(A)$. The other inequality follows from Observation 17. ◀

Proof of Theorem 2. Observe that if a graph G covers a connected graph H, then there also is a component G' of G which covers H and $\mathrm{dps}(G') = \mathrm{dps}(G)$. Using this observation, we now finish the main proof. The graph $B := A^{\odot}$ has no semi-edges. Furthermore, by Proposition 20, it covers A and $\mathrm{dps}(B) = \mathrm{dps}(A)$. By applying Proposition 21 to a properly chosen component of the graph B, we obtain a graph C that covers A, has no semi-edges, no loops, and $\mathrm{dps}(C) = \mathrm{dps}(A)$. Finally, Proposition 22 applied to a properly chosen component of C provides a simple cover D of A such that $\mathrm{dpm}(D) = \mathrm{dps}(D) = \mathrm{dps}(A)$. ◀

5 Generalized snarks

A *snark* is a "non-trivial" simple 3-regular graph that is not 3-edge-colorable. Even though this notion got its name from Martin Gardner in 1976 [14], these graphs became famous already after it was shown by Peter G. Tait in 1880 that the 4-color theorem holds if and only if all snarks[1] are non-planar [28,30]. Besides the 4-color theorem, it often happens that snarks are of interest as a possible source of counter examples to famous conjectures and are studied extensively, cf. [11,13,17,25,29,32]. The "non-triviality" conditions are heavily context-dependent and are set to serve the corresponding goals, possibly restricting the class of considered snarks as much as possible. Nevertheless, in general, snarks are considered

[1] In here, snarks are assumed to be 2-connected.

to be at least 2-connected. By Petersen's theorem [27], every simple 3-regular 2-connected graph contains a perfect matching. Therefore, setting these "non-triviality" conditions to "contains a perfect matching" gives a weakening of snarks. By Example 14, these weakened snarks become exactly the simple graphs that cover $F(1,1)$ but do not cover $F(3,0)$. Using this perspective, to describe graphs that "allow some structural property and disallow other", the following notion of *generalized snarks* was introduced by Kratochvíl and Nedela in [18]. For connected graphs A, B, a simple graph G is called an (A,B)-*snark* if $G \to A$ but $G \not\to B$. The connection to (non) k-edge-colorability of simple k-regular graphs is clear in the case when $B = F(k,0)$ and $A = F(r,\ell)$, where $k = r + 2\ell$. In [18], the existence of (A,B)-snarks was resolved whenever $B \in \{F(1,1), F(3,0)\}$. Our Theorem 3 extends this result to all 1-vertex graphs B.

▶ **Proposition 23.** *For all pairs of integers s, ℓ, and for an arbitrary connected $(s+2\ell)$-regular graph A, the following conditions are equivalent:*

1. $A \triangleright F(s,\ell)$.
2. $\mathrm{dps}(A) \geq s$.
3. $A \rightsquigarrow F(s,\ell)$.

Proof.

1. $\Rightarrow$ 2. We will prove that $\mathrm{dps}(A) < s$ implies that $A \not\triangleright F(s,\ell)$ (for arbitrary ℓ). Suppose $\mathrm{dps}(A) < s$. Theorem 2 states that there exists a simple graph D such that $D \to A$ and $\mathrm{dpm}(D) = \mathrm{dps}(A) < s$. Since D is a simple graph, we have $\mathrm{dps}(D) = \mathrm{dpm}(D)$, and thus, $\mathrm{dps}(D) < s = \mathrm{dps}(F(s,\ell))$. It follows from Observation 17 that D does not semi-cover $F(s,\ell)$, and consequently, $D \not\to F(s,\ell)$. The graph D is thus a witness for $A \not\triangleright F(s,\ell)$.
2. $\Rightarrow$ 3. This implication follows from Proposition 18.
3. $\Rightarrow$ 1. This implication follows from Proposition 15. ◀

Proof of Theorem 3. This is expressed by the equivalence of 1. and 3. in Proposition 23 ◀

It should be mentioned that for general A and B, it is unknown whether the relation $A \triangleright B$ (or, equivalently, the existence of a generalized snark for A and B) is decidable, because no bound on the size of a smallest (A,B)-snark is known. Proposition 23 implies that this question is decidable for 1-vertex graphs B. When considering A or B (or both) as part of the input, we can determine the complexity of deciding $A \triangleright B$ for 1-vertex graphs B precisely.

▶ **Proposition 24.** *The problem of deciding $A \triangleright F(s,\ell)$ is*

1. *in the class NP when all A, s and ℓ are part of the input,*
2. *decidable in polynomial time if $s \leq 1$ or $s = 2$ and $\ell = 0$ (A is part of the input),*
3. *NP-complete for every fixed s, ℓ such that $s \geq 2$ and $s + \ell \geq 3$ (A is part of the input).*

Proof. For a graph A, to be stronger than $F(s,\ell)$, it must be, in particular, $(s+2\ell)$-regular. This necessary condition can be verified in polynomial time. For the following, we assume that A is indeed $(s+2\ell)$-regular.

1. By Proposition 23, $A \triangleright F(s,\ell)$ is the same question as $\mathrm{dps}(A) \geq s$, for which a polynomially verifiable certificate is a collection of s pairwise edge-disjoint perfect semi-matchings in A.
2. If $s = 0$, every 2ℓ-regular graph A satisfies $\mathrm{dps}(A) \geq s$, and hence the answer is always "yes". If $s = 1$, checking $\mathrm{dps}(A) \geq 1$ means deciding whether A has a perfect semi-matching. This can be clearly performed in polynomial time using any algorithm for deciding the existence of a perfect matching in simple graphs. The exceptional case $s = 2, \ell = 0$ also reduces to deciding the existence of a perfect matching, since a 2-regular graph contains 2 disjoint perfect semi-matchings if and only if it contains at least one.
3. For any fixed $s \geq 2$ and $\ell \geq 3 - s$, deciding $\mathrm{dpm}(G) \geq s$ is NP-complete even for simple graphs G on input [6], and the claims follows. ◀

We will conclude this section by observing that Theorem 2 provides a necessary condition for $A \triangleright B$, which, however, may not be sufficient for larger graphs B.

▶ **Corollary 25.** *If A and B are connected graphs such that $\mathrm{dps}(A) < \mathrm{dps}(B)$, then $A \not\triangleright B$.*

Proof. According to Observation 17, the graph D from Theorem 2 is an (A, B)-snark. ◀

6 Explicit NP-hardness reduction for covering disconnected graphs

In [9, 10], the authors show that for a disconnected graph H, the H-Cover problem is NP-complete for simple input graphs whenever the H_i-Cover problem is NP-complete for simple input graphs for at least one component H_i of H. However, they only show that a polynomial reduction exists; the proof is non-constructive. We present how to use our results to design an explicit reduction when H has a 1-vertex component which itself determines an NP-complete covering problem.

First, we briefly recall the idea of the reduction from [10]. Let the components of H be $H_1, H_2, \ldots, H_q$ and without loss of generality assume that H_1-Cover is NP-complete for simple input graphs. Given a simple graph G_1 as input of H_1-Cover, we construct a simple graph G to be an input of H-Cover as a disjoint union of (several copies of) G_1 and simple covers of the other components of H. For every $j = 2, 3, \ldots, q$, we let G_j be a simple cover of H_j that does not cover H_1 if $H_j \not\triangleright H_1$, and an arbitrary simple cover of H_j otherwise. Once we have decided what the graphs $G_2, G_3, \ldots, G_q$ are, we take adequate numbers of disjoint copies of $G_1, G_2, \ldots, G_q$ so that the constructed covering projection $G \to H$ can be k-fold for a suitable k. If G_1 covers H_1, this covering projection is constructed by mapping all copies of G_j onto H_j, for $j = 1, 2, \ldots, q$. The argument for the reverse implication exploits the "being stronger" relation. Suppose that G covers H via a covering projection f. Then there exists a sequence of distinct indices $j_1, j_2, \ldots, j_p$ such that $f(G_1) = H_{j_1}$, $f(G_{j_i}) = H_{j_{i+1}}$ for $i = 1, 2, \ldots, p-1$, and $f(G_{j_p}) = H_1$. Now we see that G_{j_p} covers H_1, which, due to the choice of G_{j_p}, means that $H_{j_p} \triangleright H_1$. But then the simple graph $G_{j_{p-1}}$ that covers H_{j_p}, must also cover H_1. We continue by backward induction to finally deduce that G_1 covers H_1.

The problem (resp. feature) of this reduction is that we do not know what G_j to take, since we do not know how to test $H_j \triangleright H_1$ in general. Despite that, this proves that a polynomial reduction exists (the sizes of the graphs H_j, $j = 1, 2, \ldots, q$, and consequently also of G_j, $j = 2, 3, \ldots, q$ are constant with respect to the size of the input, which is G_1). However, for a 1-vertex graph H_1, we can answer the $H_j \triangleright H_1$ question using Proposition 23, and in the negative case, construct an (H_j, H_1)-snark G_j via Theorem 2.

7 Conclusion

We have fully characterized all pairs of connected graphs A, B for which $A \triangleright B$, provided that B is a 1-vertex graph, and we have shown that this characterization implies that Conjecture 1 is true for such graphs B. In those cases when $A \not\triangleright B$, we exploited the nature of pairwise disjoint perfect semi-matchings to construct an explicit construction of (A, B)-snarks and we argued that this construction also produces generalized snarks for every pair of connected graphs satisfying $\mathrm{dps}(A) < \mathrm{dps}(B)$. Of course, the full characterization of all pairs of connected graphs A, B such that $A \triangleright B$ remains the main open problem. We would like to explicitly mention some special cases and related questions.

▶ **Problem 26.** *Characterize all graphs A such that $A \triangleright K_4$.*

We have decided to single out the complete graph on 4 vertices, not only because it is the smallest simple cubic graph but because it also has an interesting relation to another well-studied notion in graph theory, the so-called *1-perfect codes* (also known as independent efficient dominating sets). It follows from the definition that a graph covers K_4 if and only if its vertex-set can be partitioned into 4 disjoint 1-perfect codes.

▶ **Problem 27.** *Is the relation $A \triangleright B$ decidable? (Here connected graphs A and B are both part of the input.)*

Note that an affirmative answer to this problem would provide an explicit NP-hardness reduction for the covering of disconnected graphs. The approach of [10], as described in Section 6, would enable an explicit construction of a requested generalized snark by an exhaustive search algorithm, once the existence of such a snark is guaranteed. However, this search would not have guaranteed running time. On the other hand, an affirmative answer to the following question would not only provide a running time estimate, but also imply an affirmative answer to the previous problem.

▶ **Problem 28.** *Determine a computable function $s(-,-)$ (if it exists) such that for any two connected graphs A, B such that $A \not\triangleright B$, there exists an (A, B)-snark of size at most $s(||A||, ||B||)$.*

The generalized snarks for the case of 1-vertex graphs B that we have constructed in Section 4 are of order which is polynomial in the sizes of A and B.

References

1. James Abello, Michael R. Fellows, and John C. Stillwell. On the complexity and combinatorics of covering finite complexes. *Australian Journal of Combinatorics*, 4:103–112, 1991. URL: `http://ajc.maths.uq.edu.au/pdf/4/ocr-ajc-v4-p103.pdf`.
2. Dana Angluin. Local and global properties in networks of processors (extended abstract). In Raymond E. Miller, Seymour Ginsburg, Walter A. Burkhard, and Richard J. Lipton, editors, *Proceedings of the 12th Annual ACM Symposium on Theory of Computing, April 28-30, 1980, Los Angeles, California, USA*, pages 82–93. ACM, 1980. `doi:10.1145/800141.804655`.
3. Dana Angluin and A. Gardiner. Finite common coverings of pairs of regular graphs. *Journal of Combinatorial Theory B*, 30(2):184–187, 1981. `doi:10.1016/0095-8956(81)90062-9`.
4. Hans L. Bodlaender. The classification of coverings of processor networks. *Journal of Parallel Distributed Computing*, 6(1):166–182, 1989. `doi:10.1016/0743-7315(89)90048-8`.
5. Jan Bok, Jiří Fiala, Petr Hliněný, Nikola Jedličková, and Jan Kratochvíl. Computational complexity of covering multigraphs with semi-edges: Small cases. In Filippo Bonchi and Simon J. Puglisi, editors, *46th International Symposium on Mathematical Foundations of Computer Science, MFCS 2021, Tallinn, Estonia, August 23-27, 2021*, volume 202 of *LIPIcs*, pages 21:1–21:15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. `doi:10.4230/LIPIcs.MFCS.2021.21`.
6. Jan Bok, Jiří Fiala, Petr Hliněný, Nikola Jedličková, and Jan Kratochvíl. Computational complexity of covering multigraphs with semi-edges: Small cases. *J. Comput. Syst. Sci.*, 156:103714, 2026. `doi:10.1016/J.JCSS.2025.103714`.
7. Jan Bok, Jiří Fiala, Nikola Jedličková, Jan Kratochvíl, and Pawel Rzążewski. List covering of regular multigraphs. In Cristina Bazgan and Henning Fernau, editors, *Combinatorial Algorithms - 33rd International Workshop, IWOCA 2022, Trier, Germany, June 7-9, 2022, Proceedings*, volume 13270 of *Lecture Notes in Computer Science*, pages 228–242. Springer, 2022. `doi:10.1007/978-3-031-06678-8_17`.

8 Jan Bok, Jiří Fiala, Nikola Jedličková, Jan Kratochvíl, and Pawel Rzążewski. List covering of regular multigraphs with semi-edges. *Algorithmica*, 86(3):782–807, 2024. doi:10.1007/S00453-023-01163-7.

9 Jan Bok, Jiří Fiala, Nikola Jedličková, Jan Kratochvíl, and Michaela Seifrtová. Computational complexity of covering disconnected multigraphs. In Evripidis Bampis and Aris Pagourtzis, editors, *Fundamentals of Computation Theory - 23rd International Symposium, FCT 2021, Athens, Greece, September 12-15, 2021, Proceedings*, volume 12867 of *Lecture Notes in Computer Science*, pages 85–99. Springer, 2021. doi:10.1007/978-3-030-86593-1_6.

10 Jan Bok, Jiří Fiala, Nikola Jedličková, Jan Kratochvíl, and Michaela Seifrtová. Computational complexity of covering disconnected multigraphs. *Discret. Appl. Math.*, 359:229–243, 2024. doi:10.1016/J.DAM.2024.07.035.

11 Gunnar Brinkmann, Jan Goedgebeur, Jonas Hägglund, and Klas Markström. Generation and properties of snarks. *J. Comb. Theory B*, 103(4):468–488, 2013. doi:10.1016/J.JCTB.2013.05.001.

12 Jack Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965. doi:10.4153/CJM-1965-045-4.

13 Miquel A. Fiol, Giuseppe Mazzuoccolo, and Eckhard Steffen. Measures of edge-uncolorability of cubic graphs. *Electron. J. Comb.*, 25(4):4, 2018. doi:10.37236/6848.

14 Martin Gardner. Mathematical games. *Scientific American*, 234(4):126–131, 1976. URL: http://www.jstor.org/stable/24950334.

15 Ezra Getzler and Mikhail M. Kapranov. Modular operads. *Compositio Mathematica*, 110(1):65–125, 1998. doi:10.1023/A:1000245600345.

16 Ian Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981. doi:10.1137/0210055.

17 Martin Kochol. Snarks without small cycles. *J. Comb. Theory B*, 67(1):34–47, 1996. doi:10.1006/JCTB.1996.0032.

18 Jan Kratochvíl and Roman Nedela. Graph covers and generalized snarks (extended abstract). In Daniel Král and Jaroslav Nešetřil, editors, *Proceedings of EUROCOMB'23*, pages 681–687. MUNI Press, 2023. doi:10.5817/CZ.MUNI.EUROCOMB23-094.

19 Jan Kratochvíl and Roman Nedela. Graph covers and semi-covers: Who is stronger?, 2025. doi:10.48550/arXiv.2504.17387v1.

20 Jin Ho Kwak and Roman Nedela. Graphs and their coverings. *Lecture Notes Series*, 17, 2007.

21 Daniel Leven and Zvi Galil. NP-completeness of finding the chromatic index of regular graphs. *Journal of Algorithms*, 4(1):35–44, 1983. doi:10.1016/0196-6774(83)90032-9.

22 Yulai Ma, Davide Mattiolo, Eckhard Steffen, and Isaak H. Wolf. Edge-connectivity and pairwise disjoint perfect matchings in regular graphs. *Comb.*, 44(2):429–440, 2024. doi:10.1007/S00493-023-00078-9.

23 Aleksander Malnič, Dragan Marušič, and Primož Potočnik. Elementary abelian covers of graphs. *Journal of Algebraic Combinatorics*, 20(1):71–97, 2004. doi:10.1023/B:JACO.0000047294.42633.25.

24 Aleksander Malnič, Roman Nedela, and Martin Škoviera. Lifting graph automorphisms by voltage assignments. *European Journal of Combinatorics*, 21(7):927–947, 2000. doi:10.1006/eujc.2000.0390.

25 Edita Máčajová and Martin Škoviera. Superposition of snarks revisited. *Eur. J. Comb.*, 91:103220, 2021. doi:10.1016/j.ejc.2020.103220.

26 Roman Nedela and Martin Škoviera. Regular embeddings of canonical double coverings of graphs. *Journal of Combinatorial Theory, Series B*, 67(2):249–277, 1996. doi:10.1006/jctb.1996.0044.

27 Julius Petersen. Die theorie der regulären graphs. *Acta Mathematica*, 15:193–220, 1891.

28 Julius Petersen. Sur le théorème de Tait. *L'Intermédiaire des Mathématiciens*, 5:225–227, 1898.

29 Eckhard Steffen. 1-factor and cycle covers of cubic graphs. *J. Graph Theory*, 78(3):195–206, 2015. doi:10.1002/jgt.21798.

30 P. G. Tait. 10. remarks on the previous communication. *Proceedings of the Royal Society of Edinburgh*, 10:729–729, 1880. doi:10.1017/S0370164600044643.

31 V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz.*, 3:25–30, 1964.

32 Martin Škoviera and Peter Varša. Deciding whether four perfect matchings can cover the edges of a snark is NP-complete. *Theor. Comput. Sci.*, 988:114374, 2024. doi:10.1016/j.tcs.2023.114374.