# Increasing Arc-Connectivity by Bounded- and Fixed-Size Inversions

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## Abstract

Given an integer $k \geqslant 1$, a digraph $D$ is $k$-arc-strong if the removal of any set of at most $k-1$ arcs of $D$ yields a strongly connected digraph. For a digraph $D$ and some set $X \subseteq V(D)$, the inversion of $X$ is the operation of flipping all arcs both of whose endvertices are in $X$. We initiate the study of establishing arc-connectivity properties by applying inversions of bounded or fixed size.

For fixed-size inversions, we consider the feasibility of the problem by characterizing, for all integers $p \geqslant 2$ and $k \geqslant 1$, the digraphs that can be made $k$-arc-strong by applying inversions of size exactly $p$, provided a minimum size of the digraphs.

For bounded-size inversions, the tractability of the feasibility problem follows easily from a famous theorem of Nash-Williams, so we focus on minimising the number of inversions. We prove that for all integers $p \geqslant 3$ and $k \geqslant 1$ and any $\varepsilon > 0$, there exists a polynomial-time $(4k-2+\varepsilon)$-approximation algorithm for computing the minimum number of inversions of size at most $p$ that make a given digraph $k$-arc-strong. This is in stark contrast to other results on inversion optimization problems. On the other hand, we show that for any $p \geqslant 3$ and $k \geqslant 1$ the problem is NP-hard, and, moreover, APX-hard.

As a result on parameterized complexity, we show that for any $k \geqslant 2$, it is $W[1]$-hard with respect to $p$ to decide whether a given digraph can be made $k$-arc-strong by applying a single inversion of size at most $p$. We also prove that for a given multidigraph, it is $W[1]$-hard with respect to $\ell$ to decide whether it can be made 2-arc-strong by applying $\ell$ inversions of size 2.

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## 1 Introduction

In this paper, a *multidigraph* consists of a vertex set and an arc set, where parallel arcs and digons are allowed (a *digon* being a pair of arcs $\{uv, vu\}$ in opposite directions between the same vertices $u, v$). A multidigraph without parallel arcs is called a *digraph* and a digraph without digons is called an *oriented graph*.

Given a multidigraph $D$ and a set $X \subseteq V(D)$, *inverting $X$ in $D$* means flipping the orientation of all arcs in $D$ that have both endvertices in $X$. We denote by $\mathrm{Inv}(D, X)$ the resulting digraph, and given a collection $\mathcal{X} = (X_i)_{i \in I}$ of subsets of $V(D)$, $\mathrm{Inv}(D; \mathcal{X})$ denotes the digraph obtained after inverting the $X_i$'s one after another. Inversions fall into the category of reconfiguration operations. A general overview of reconfiguration in graph theory can be found in the survey of Nishimura [25]. Despite being a relatively young field in graph theory, the theory of inversions has received significant attention in the last 15 years. Most


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work on inversions has been done with the objective of understanding the minimum number of inversions through which a given oriented graph can be made acyclic. The literature on this subject is already quite rich. In recent years, inversions with different objectives have also been considered. This in particular involves results on the so-called inversion diameter of undirected graphs and inversions with the objective of making an oriented graph highly connected.

It is also natural to consider inversions of bounded and fixed size. In oriented graphs, inversions involving two vertices correspond to arc flips. While general inversions are known to behave rather chaotically, problems involving arc flips tend to be more tractable. It is hence natural to study fixed-size and bounded-size inversions as an intermediate step between these two concepts. Prior to this work, a few articles have considered bounded-size inversions with the objective of making an oriented graph acyclic and in the context of the inversion diameter. So far, to the best of our knowledge, bounded- or fixed-size inversions have not yet been studied in the context of making a given oriented graph highly connected.

The objective of the present article is to fill this gap. As our first main result, for all fixed positive integers $p$ and $k$, we give a characterization of all digraphs that can be made $k$-arc-strong by inversions of size exactly $p$, provided the digraph is sufficiently large compared to $p$ and $k$. Next, we focus on making a digraph $k$-arc-strong by a minimum number of inversions of size at most $p$. We prove that for any fixed $p \geqslant 3$ and $k \geqslant 1$, the problem is APX-hard, but admits a polynomial-time $(4k - 2 + \varepsilon)$-approximation algorithm for arbitrary $\varepsilon > 0$. Finally, we give some results on the parameterized complexity of the problem when considering either $p$ or the solution size as parameters.

In the following, we first give a more extensive overview of previous work and then present our findings in more detail.

## 1.1 Previous work

We first overview the results on making a given oriented graph acyclic by a minimum number of inversions of arbitrary size, the setting that has attracted most attention. Given an oriented graph $D$, we denote by $\mathrm{acinv}(D)$ the minimum number of inversions to make $D$ acyclic. Inversions were first considered by Belkhechine et al. [10]. They focused on tournaments and initiated the study of the asymptotic behaviour of $\mathrm{acinv}(n)$, defined as the maximum of $\mathrm{acinv}(T)$ over all tournaments $T$ on $n$ vertices. Their bounds were later significantly improved by Alon et al. [2] and Aubian et al. [4], who independently showed that $\mathrm{acinv}(n) = (1 - o(1))n$.

The complexity of computing $\mathrm{acinv}(D)$ for a given digraph $D$ was first considered by Bang-Jensen, Costa Ferreira da Silva, and Havet [5]. They proved that it is NP-hard to decide whether $\mathrm{acinv}(D) \leqslant 1$ for a given digraph $D$. However, their proof does not provide a straightforward way to establish hardness for a larger number of inversions. Finally, it was proved in [2] that, indeed, deciding whether $\mathrm{acinv}(D) \leqslant \ell$ is NP-hard for every fixed $\ell \geqslant 1$. Further, in [5], a conjecture on the behavior of acinv under dijoins was made. Given two digraphs $D_1$ and $D_2$, the *dijoin* $D_1 \Rightarrow D_2$ is obtained from vertex-disjoint copies of $D_1$ and $D_2$ by adding one arc from every vertex in $V(D_1)$ to every vertex in $V(D_2)$. The authors conjectured that

$$\mathrm{acinv}(D_1 \Rightarrow D_2) = \mathrm{acinv}(D_1) + \mathrm{acinv}(D_2)$$

holds for all digraphs $D_1$ and $D_2$. While some cases for small inversion numbers were solved in [5], the conjecture was generally refuted independently in [2] and [4]. Some results on special cases of this conjecture have been proved by Behague et al. [9], Wang, Yang, and Lu [27], and Behague and Gaudart-Wifling [8].

Recently, inversions have also been studied beyond the objective of making an oriented graph acyclic. The concept of inversion diameter was introduced by Havet, Hörsch, and Rambaud [17]. Given an undirected simple graph, its *inversion graph* is the graph defined on the set of all its orientations and where two orientations are adjacent if one can be obtained from the other by a single inversion. The *inversion diameter* of a graph is the diameter of its inversion graph. In [17], the authors prove some hardness results on computing the inversion diameter of a graph and show that it is functionally tied to some other graph parameters, namely the star chromatic number, the acyclic chromatic number and the oriented chromatic number. This yields in particular that the inversion diameter is bounded for all proper minor-closed classes of graphs. Further, some more explicit bounds on the inversion diameter of graphs coming from special classes were obtained, for example planar graphs, planar graphs of bounded girth, and graphs of bounded treewidth. The treewidth bound has been shown to be tight by Wang et al. [28]. Recently, several quantitative improvements on bounds in [17] have been obtained by Arana et al. [3].

We now come to another setting that plays a crucial role in this article, namely the idea of applying inversions in order to make the thus obtained oriented graph satisfy certain connectivity properties. To our best knowledge, the only work in which this setting was previously considered is the one of Duron et al. [13]. Let $k \geqslant 1$ be an integer. We say that a graph is *$k$-edge-connected* if the removal of any set of at most $k-1$ edges yields a connected graph. We say that a digraph $D$ is $k$-edge-connected if its underlying graph $\mathrm{UG}(D)$ is $k$-edge-connected. For a multidigraph $D$, we use $\mathrm{sinv}_k(D)$ for the minimum number of inversions which are necessary to make $D$ $k$-arc-strong. In the preceding discussion, we have mainly ignored the question of what happens when instead of considering oriented graphs, we also allow digons or parallel arcs. This is justified by the fact that these objects play no role when considering ways to make a multidigraph acyclic. Indeed, parallel arcs are irrelevant, and the presence of a digon immediately rules out the existence of a feasible solution. Similarly, parallel arcs and digons play no role in the study of inversion diameters. However, when it comes to shooting for a connectivity property, then parallel arcs may make a crucial difference. In particular, it follows directly from a well-known theorem on orientations of Nash-Williams [24] that an oriented graph can be made $k$-arc-strong by inversions if and only if its underlying graph is *$2k$-edge-connected*. As implicitly pointed out in [13], this result can rather easily be generalized to digraphs. On the other hand, it follows directly from a result of Hörsch and Szigeti [20] that it is NP-hard to decide whether a multidigraph can be made 2-arc-strong by applying inversions. In [13], the authors prove that when aiming to minimise the number of necessary inversions, the problem also becomes hard in oriented graphs. More precisely, they prove that for any positive integers $k$ and $\ell$, it is NP-hard to decide whether $\mathrm{sinv}_k(D) \leqslant \ell$ for a given oriented graph $D$. Moreover, they prove the following result, showing that even any acceptable approximation is out of reach.

▶ **Theorem 1** (Duron et al. [13]). *For any two positive integers $k$ and $\ell$, unless $P = NP$ there is no polynomial-time algorithm asserting whether a given oriented graph $D$ can be made $k$-arc-strong by one inversion or cannot be made $k$-arc-strong by at most $\ell$ inversions.*

Next, it was proved in [13] that for every fixed positive integer $k$, the maximum of $\mathrm{sinv}_k(D)$ over all $2k$-edge-connected digraphs $D$ on $n$ vertices is a logarithmic function in $n$. Further, the authors prove a number of upper bounds for the special case of tournaments.

### Inversions of bounded and prescribed size

In all of the above discussion, we have disregarded the size of the sets that have been inverted. It is natural to aim to understand the behavior of inversions when the size of the sets that can be inverted is restricted. In the following, for some positive integer $p$, an $(=p)$-*inversion* (respectively $(\leqslant p)$-inversion) is an inversion of a set of size exactly $p$ (respectively at most $p$). The most restrictive meaningful case to consider is the one of inversions of sets of size 2. In oriented graphs, inversions of size 2 correspond exactly to arc flips, the operation of reversing the orientation of an arc. This operation has been deeply studied from many angles. From an algorithmic perspective, the behavior of inversions has so far been rather chaotic, as exemplified by the fact that deciding whether $\mathrm{acinv}(D) \leqslant 1$ for a given oriented graph $D$ is NP-hard. On the other hand, arc flips are comparatively well understood. In this light, considering inversions of bounded or prescribed size is a natural intermediate step between unbounded inversions and arc flips.

Bounded-size inversions in order to make an oriented graph acyclic were first considered by Yuster [29]. He focused on tournaments and proved a collection of extremal results. For a given oriented graph $D$, we use $\mathrm{acinv}^{=p}(D)$ ($\mathrm{acinv}^{\leqslant p}(D)$) for the minimum number of $(=p)$-inversions ($(\leqslant p)$-inversions) to make $D$ acyclic. The algorithmic complexity of computing the minimum number of bounded-size inversions to make an oriented graph acyclic was studied by Bang-Jensen et al. [7]. While computing a minimum set of arcs whose flipping makes a given oriented graph acyclic is one of Karp's 21 NP-hard problems [21], the corresponding result in tournaments, proved independently by Alon [1] and Charbit, Thomassé, and Yeo [11], answered a long-standing open question. In [7], it was proved that computing $\mathrm{acinv}^{\leqslant p}(D)$ for a given tournament $D$ is NP-hard for any fixed $p \geqslant 2$, using a comparatively easy reduction for $p \geqslant 3$. The existence of a polynomial-time constant factor approximation for the minimum number of arc flips that makes a given digraph acyclic was a long-standing open problem. Finally, it was proved by Guruswami et al. [15] that such an algorithm would contradict the Unique Games Conjecture (UGC). Some simpler hardness proofs were given by Svensson [26] and Guruswami and Lee [16]. It is not difficult to see that for every fixed $p \geqslant 2$, a polynomial-time constant factor approximation for computing $\mathrm{acinv}^{\leqslant p}$ exists if and only if the answer for the arc flip problem is affirmative. Hence, for any $p \geqslant 2$, a constant factor approximation for $\mathrm{acinv}^{\leqslant p}$ can be ruled out under UGC. Next, in [7], parameterized complexity was also considered. Among others, the authors proved the following result.

▶ **Theorem 2** (Bang-Jensen et al. [7]). *Deciding whether a given oriented graph can be made acyclic with a single inversion of size at most p is $W[1]$-hard with respect to p.*

Observe that Theorem 2 is incomparable to the result that it is NP-hard to decide whether a given oriented graph can be made acyclic by a single inversion.

Finally, in [7], fixed-size inversions were also studied. Here, besides the problem of minimising the number of inversions, also the question of the existence of a feasible solution becomes significantly more interesting. Observe that, when considering an oriented graph, a bounded-size inversion can flip a single arc and hence the problem of deciding the existence of a feasible solution reduces to an orientation problem. This is no longer the case for fixed-size inversions. Nevertheless, using methods from linear algebra, the authors of [7] prove a characterization of tournaments that can be made acyclic by fixed-size inversions and use it to give a polynomial-time algorithm for general oriented graphs.

When it comes to bounded-size inversions for other objectives than becoming acyclic, we wish to mention the recent work of Havet, Rambaud, and Silva [18]. They consider a variation of the diameter problem for bounded-size inversions and prove a collection of extremal results.

## 1.2 Our contributions

As far as we are aware, no previous efforts have been made of combining the concept of bounded-size and fixed-size inversions with the objective of making a digraph satisfy connectivity requirements. The objective of the present article is to fill this gap. Our results can be divided into two categories. First, we consider feasibility problems. Here, we deal with the question of which digraphs or multidigraphs can be made $k$-arc-strong by fixed-size inversions, irrespective of the number of required inversions. Afterwards, we focus on bounded-size inversions, and consider the problem of making a directed graph $k$-arc-strong by using a minimum number of such inversions. Here, we give some results on the exact computation and the approximation of the minimum number of inversions. We finally give some results related to parameterized complexity.

### Feasibility

We first consider general multidigraphs and the most basic case, which is $p = 2$. Observe that for $p = 2$, bounded-size and fixed-size inversions give the same means of altering a multidigraph. Moreover, when it comes to feasibility, inversions of size 2 are exactly as powerful as arbitrary inversions. Interestingly, we can obtain the following result as a simple corollary of a result of Hörsch and Szigeti [20]. It rules out the possibility of solving the feasibility problem for general multidigraphs even in the most basic case.

▶ **Theorem 3.** *It is NP-hard to decide whether a given multidigraph can be made* $2$*-arc-strong by applying* $(=2)$*-inversions.*

In the light of Theorem 3, we now focus on digraphs. For any positive integer $k$, a well-known theorem of Nash-Williams [24] states that a graph has a $k$-arc-strong orientation if and only if it is $2k$-edge-connected. In the following, we say that a multidigraph is $2k$-*edge-connected* if its underlying graph is $2k$-edge-connected. Clearly, every digraph that can be made $k$-arc-strong by applying $(=2)$-inversions is in particular $2k$-edge-connected by the theorem of Nash-Williams. Moreover, we immediately obtain that this condition is also sufficient for oriented graphs. Our next result extends this observation in two ways. First, we allow digraphs rather than only oriented graphs. Second, rather than only considering $(=2)$-inversions, we consider orientations of any fixed even size. On the other hand, we need the digraph in consideration to have a certain minimum size. Formally, we prove the following result.

▶ **Theorem 4.** *Let* $k$ *be a positive integer,* $p \geqslant 2$ *be an even integer, and* $D$ *be a* $2k$*-edge-connected digraph of order* $n \geqslant \max\{p+2, 2k+2\}$*. Then* $D$ *can be made* $k$*-arc-strong using* $(=p)$*-inversions.*

The case of fixed odd $p$ is more complicated and obstructions appear. We give a full characterization of the digraphs that cannot be made $k$-arc-strong by inversions of fixed odd size, again assuming a minimum size condition. For every $k \geqslant 1$, we now give a description of $2k$-edge-connected digraphs that cannot be made $k$-arc-strong by applying inversions of fixed odd size. For a graph $G$ and a set $X \subseteq V(G)$, we denote by $d_G(X)$ the number of edges of $G$ with exactly one extremity in $X$. Namely, a $k$-*obstruction* is a digraph $D$, with underlying graph $G$, admitting a partition $(X_1, \ldots, X_r, Y)$ of its vertex set into non-empty parts such that, for $X = \bigcup_{i \in [r]} X_i$, we have:

**(i)** for every $i \in [r]$, $d_G(X_i) = 2k$;
**(ii)** for every $x \in X$ and $y \in Y$, there is exactly one edge between $x$ and $y$ in $G$; and
**(iii)** $d^+_D(X) - \frac{1}{2}|X| \cdot |Y|$ is an odd integer.

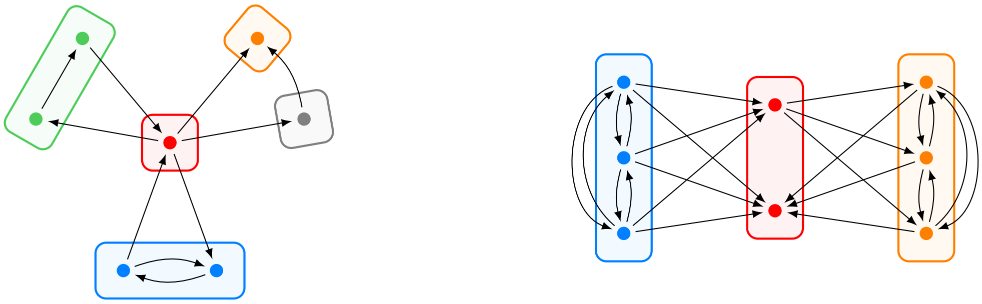

**Figure 1** Two examples of $k$-obstructions, with $k = 1$ (left) and $k = 3$ (right). In both cases, each colour illustrates a part of the partition $(X_1, \ldots, X_r, Y)$, the vertices of $Y$ being in red.

See Figure 1 for an illustration of $k$-obstructions. Note that, for every fixed $k$, there exist arbitrarily large $2k$-edge-connected $k$-obstructions. The following is our main result on feasibility for fixed odd inversions.

▶ **Theorem 5.** *Let $k$ be a positive integer, $p \geqslant 3$ be an odd integer and $D$ be a digraph of order $n \geqslant \max\{p + 2, 4k + 2\}$. Then $D$ can be made $k$-arc-strong using $(=p)$-inversions if and only if $D$ is $2k$-edge-connected and it is not a $k$-obstruction.*

Moreover, we show that obstructions can be recognised in polynomial time. Together with Theorem 4 and Theorem 5, this allows us to conclude that for any fixed $p$ and $k$, we can decide in polynomial time whether a given digraph can be made $k$-arc-strong by inversions of size exactly $p$. Further, this implies the following stronger statement.

▶ **Corollary 6.** *The problem of deciding whether a given directed graph can be made $k$-arc-strong by inversions of size exactly $p$ admits a kernel of size $O(k + p)$.*

Theorems 4 and 5 raise the question whether the minimum size bound is really necessary or just an artefact of a suboptimal proof technique. The following complexity result shows that at least the bound $n \geqslant p + 2$ is crucial in this characterization, even for oriented graphs.

▶ **Theorem 7.** *Given an oriented graph $D$ on $n$ vertices, it is NP-hard to decide whether it can be made strongly connected by inversions of size exactly $n - 1$.*

The statement above follows from a hardness result due to Klostermeyer [22] concerning the closely related problem of *vertex pushing*.

## Exact and Approximate Computation

Here, we focus on the minimum number of inversions of bounded size needed to make a given digraph $k$-arc-strong. First consider the case of inversions of size 2. For oriented graphs, this case corresponds exactly to the operation of arc flipping. This problem can be solved in polynomial time [14, 6] using a powerful orientation theorem of Frank [14]. It is interesting to observe that while the existence of digons does not change the feasibility of an instance, in general digraphs, the number of arc flips necessary to establish $k$-arc-strong connectivity can be different from the number of necessary $(=2)$-inversions. See Figure 2 for an illustration. Nevertheless, we can use the result of Frank in a slightly different way to establish that also the case of digraphs can be solved in polynomial time.

▶ **Theorem 8.** *Given a digraph $D$ and an integer $k$, in polynomial time one can compute the minimum number of $(=2)$-inversions to make $D$ $k$-arc-strong.*

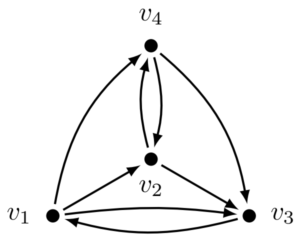


**Figure 2** An example of a digraph where the number of necessary $(=2)$-inversions to make it 2-arc-strong does not correspond to the necessary number of arc reversals. Indeed, the reversal of the arc from $v_1$ to $v_3$ makes the graph 2-arc-strong, but at least two $(=2)$-inversions are required.

Our next result displays a drastic change in the behavior of the problem when considering $p \geqslant 3$. Namely, like for most inversion problems, it turns out that an exact computation of the number of necessary inversions is out of reach. Moreover, the following result also excludes the existence of a PTAS.

▶ **Theorem 9.** *For every fixed $p \geqslant 3$, there exists some $\varepsilon_p > 0$ such that, unless $P = NP$, for every $k \geqslant 1$, in polynomial time one cannot approximate the minimum number of $(\leqslant p)$-inversions to make a digraph $k$-arc-strong within a factor of $1 + \varepsilon_p$.*

Recall that, by Theorem 1, there is no hope to approximate the number of inversions of unbounded size that are necessary to make a given digraph $k$-arc-strong within a constant factor for any $k \geqslant 1$. Moreover, when the objective is to make a given digraph acyclic, then there is also no polynomial-time constant factor approximation for the minimum number of bounded-size inversions due to the result of Guruswami [15]. Surprisingly, the situation is much brighter in our setting. We prove the following result.

▶ **Theorem 10.** *For all integers $k \geqslant 1$ and $p \geqslant 2$ and every $\varepsilon > 0$, there exists a polynomial-time algorithm that computes a $(4k - 2 + \varepsilon)$-approximation for the number of necessary $(\leqslant p)$-inversions to make a given digraph $k$-arc-strong.*

Observe that by Theorem 10, there exists a $(2 + \varepsilon)$-approximation algorithm for any $\varepsilon > 0$ in the case that we aim for a strongly connected digraph. Actually, we will show more technical results that yield slightly better approximation ratios for many special cases. In particular, we obtain the following result.

▶ **Theorem 11.** *For every $\varepsilon > 0$, there exists a polynomial-time algorithm that computes a $(\frac{3}{2} + \varepsilon)$-approximation for the minimum number of $(\leqslant 4)$-inversions to make a given digraph strongly connected.*

### Parameterized Complexity

Besides approximation, parameterized complexity is another common way to make progress on NP-hard problems. In the current setting, the inversion size $p$ and the solution size $\ell$ may serve as parameters. Observe that all the problems in consideration admit trivial XP algorithms in $p$ and $\ell$, which consists of brute force enumerating all appropriate inversions. Hence, it is natural to ask about FPT results. We prove two results in that regard. First, we consider the case that $p$ serves as a parameter and $\ell$ is fixed. We show that for any $k \geqslant 2$, there is no hope for an FPT algorithm even in oriented graphs.

▶ **Theorem 12.** *For every $k \geqslant 2$, it is $W[1]$-hard with respect to $p$ to decide whether a given oriented graph can be made $k$-arc-strong by a single $(\leqslant p)$-inversion.*

Observe that this result is incomparable to the one in [13] stating that it is NP-hard to decide whether a graph can be made $k$-arc-strong by a single inversion.

Next, we consider the case where $p$ is fixed and $\ell$ serves as the parameter. Here, the cases of oriented graphs and digraphs remain open. However, we show that for multidigraphs not even the simplest case $p = 2$ is tractable.

▶ **Theorem 13.** *Given a multidigraph $D$, it is $W[1]$-hard with respect to $\ell$ to decide whether $D$ can be made 2-arc-strong by at most $\ell$ $(=2)$-inversions.*

Observe that this result is incomparable to Theorem 3.

## 2 Technical Overview

The full proofs of our results are postponed to the full version of the article available in [19]. Here, we give sketches of the most interesting proofs in this article.

### 2.1 Fixed-size Inversions

Here, we give an overview of the simple proof of Theorem 4 and the more involved proof of Theorem 5. We also sketch the conclusion of Corollary 6.

A crucial ingredient for the proofs of Theorem 4 and Theorem 5 is the concept of simulating inversions. For a digraph $D$, a set $X \subseteq V(D)$ and a collection $\mathcal{X} \subseteq 2^{V(D)}$, we say that the inversion of $\mathcal{X}$ *simulates* the inversion of $X$ if $\mathrm{Inv}(D; \mathcal{X}) = \mathrm{Inv}(D, X)$. It was proved in [7] that for any even integer $p \geqslant 4$ and any digraph $D$ on at least $p+2$ vertices containing at least one digon or one pair of non-adjacent vertices, the inversion of any $(=2)$-set can be simulated by a sequence of $(=p)$-inversions. Together with the orientation theorem of Nash-Williams, this easily implies Theorem 4.

We now focus on the more interesting case when $p$ is odd. Fix some integer $k \geqslant 1$. We first show that $k$-obstructions indeed cannot be made $k$-arc-strong using $(=p)$-inversions. For this, it suffices to observe that $k$-obstructions are not $k$-arc-strong and, moreover, that any digraph obtained from a $k$-obstruction by applying a $(=p)$-inversion is again a $k$-obstruction.

For the other direction, we first focus on the case $p = 3$. For a contradiction, we suppose that there exists an arc-minimal digraph $D$ that is $2k$-edge-connected, not a $k$-obstruction, and that cannot be made $k$-arc-strong using $(=3)$-inversions. It follows from the minimality of $D$ that for every $a \in A(D)$, we either have that $D - a$ is a $k$-obstruction or the edge corresponding to $a$ is contained in a $2k$-edge-cut of $G$, where $G$ is the underlying graph of $D$. Using these facts, we manage to prove more and more on the structure of $D$ and $G$. In the following, we say that an arc $a \in A(D)$ that is not contained in a digon is *free* is there exists a sequence of $(=3)$-inversions in $D$ simulating the flipping of $a$. Clearly, if all arcs of $D$ are free, we obtain that $D$ can be made $k$-arc-strong by the theorem of Nash-Williams. We can hence choose a non-free arc $uw$ of $D$. We show that there exists a $2k$-edge-cut in $G$ containing the edge corresponding to $uw$. Formally, there exists a partition $(U, W)$ of $V(G)$ such that $u \in U, w \in W$ and $d_G(U) = 2k$. Without loss of generality, we may suppose that $|U \setminus N_G(w)| \geqslant |W \setminus N_G(u)|$. We now define $Y = (W \cap N_G(u)) \setminus N_G(w)$. Through several intermediate claims, we prove results on the structure of $G$ and $D$, finally showing that there exists a partition $(X_1, \dots, X_t)$ of $V(D) \setminus Y$ such that $d_G(X_i) = 2k$ for $i \in [t]$. This easily implies that $D$ is a $k$-obstruction, a contradiction.

In order to extend this result for arbitrary odd $p \geqslant 3$, we rely on the fact that $(=3)$-inversions can often be simulated by $(=p)$-inversions for odd $p \geqslant 3$. However, this is slightly more complicated than for the even case. Luckily, we are able to prove the following result showing that the simulations are possible if a small structural extra condition is imposed on $D$. We use $\alpha(D)$ to denote the maximum size of an independent set in $\mathrm{UG}(D)$.

▶ **Lemma 14.** *Let $p \geqslant 3$ be an odd integer and $D$ be a digraph of order $n \geqslant p+2$. If $\alpha(D) \geqslant 3$, then for every triplet $S = \{x, y, z\} \subseteq V(D)$, the inversion of $S$ in $D$ can be simulated by a set of $(=p)$-inversions.*

Together with the result for $p = 3$, Lemma 14 directly implies Theorem 4 for digraphs $D$ with $\alpha(D) \geqslant 3$.

It remains to consider digraphs $D$ with $\alpha(D) \leqslant 2$. We now distinguish the two cases that there exists some $a \in A(D)$ such that $D - a$ is a $k$-obstruction and that $D - a$ is not $2k$-edge-connected for every $a \in A(D)$. Thanks to the additional structural constraint that $\alpha(D) \leqslant 2$, we can now obtain contradictions in either case.

Finally, in order to conclude Corollary 6, we need to prove that we can check efficiently whether a given digraph is a $k$-obstruction. We first show that, given a positive integer $k$, a graph $G$ and a set $X \subseteq V(G)$, we can decide efficiently whether there exists a partition $(X_1, \ldots, X_t)$ of $X$ such that $d_G(X_i) = k$ for all $i \in [t]$. This directly implies that we can efficiently check whether a given set $Y$ can be extended into a certificate for a $k$-obstruction. We conclude by showing that only a small number of candidates for $Y$ need to be considered.

## 2.2 Approximation

Here, we give an overview of the proof of Theorem 10. For the sake of simplicity, we focus on the case $k = 1$, that is, our objective is to make the digraph strongly connected.

The crucial insight of our algorithm is to show that for any digraph $D$, we have that $\mathrm{sinv}_1^{\leqslant 2}(D)$ and $\mathrm{sinv}_1^{\leqslant p}(D)$ are tied in a specific way. More precisely, if $\mathrm{sinv}_1^{\leqslant p}(D)$ is sufficiently large, then $\mathrm{sinv}_1^{\leqslant p}(D)$ is smaller than $\mathrm{sinv}_1^{\leqslant 2}(D)$ by a factor which is linear in $p$.

In order to prove this relationship, we need to show two directions, both of which are nontrivial. We first show the following bound.

▶ **Lemma 15.** *Let $p \geqslant 2$ be an integer and $D$ a 2-edge-connected digraph. Then*

$$\mathrm{sinv}_1^{\leqslant 2}(D) \leqslant (p-1)\,\mathrm{sinv}_1^{\leqslant p}(D).$$

Given integers $k \geqslant 1$ and $p \geqslant 2$, A collection $\mathcal{X} = (X_i)_{i \in I}$ is called a $(k, \leqslant p)$-inverting collection if each $X_i$ has size at most $p$ and the digraph $\mathrm{Inv}(D; \mathcal{X})$ is $k$-arc-strong.

For the proof of Lemma 15, let $\mathcal{X} \subseteq \binom{V(D)}{\leqslant p}$ be a $(1, \leqslant p)$-inverting collection of $D$ and let $D' = \mathrm{Inv}(D; \mathcal{X})$. Observe that the inversion of some $X \in \mathcal{X}$ may lead to the flipping of a number of arcs of $A(D)$ which is quadratic in $p$. Hence, for proving Lemma 15, it is not possible to consider the set of all arcs that are flipped. The crucial insight is that only a small number of these arcs are actually crucial for $D'$ being strongly connected. To this end, let $D'_0$ be an arc-minimal strongly connected spanning subdigraph of $D'$ and let $D_0$ be the subdigraph of $D$ corresponding to $D'_0$. In order to make $D$ strongly connected, it suffices to flip the arcs in $A(D_0)$ that have a different orientation in $A(D'_0)$. Moreover, a well-known result on the density of arc-minimal digraphs [12] implies that every $X \in \mathcal{X}$ can only induce a number of arcs of $A(D_0)$, which is linear in $p$. We can hence obtain a bound for $\mathrm{sinv}_1^{\leqslant 2}(D)$ which is linear in $p$ and $\mathrm{sinv}_1^{\leqslant p}(D)$ by choosing all pairs of vertices corresponding to arcs in $A(D_0)$ that have a different orientation in $A(D'_0)$. We actually give a refined argument to obtain the quantitative bound of Lemma 15.

The proof of the other direction is algorithmic. Given a $(1, \leqslant 2)$-inverting collection, we construct a $(1, \leqslant p)$-inverting collection which is by a factor of roughly $\lfloor \frac{p}{2} \rfloor$ smaller. As a minimum $(1, \leqslant 2)$-inverting collection can be computed in polynomial time by Theorem 8, this yields that our entire proof is algorithmic and hence the desired $(1, \leqslant p)$-inverting collection can be computed in polynomial time.

Now let a digraph $D$ and a minimum $(1, \leqslant 2)$-inverting collection $\mathcal{X}$ of $D$ be given. Further, let $D' = \mathrm{Inv}(D; \mathcal{X})$. Our objective is, roughly speaking, to invert a collection of $\lfloor \frac{p}{2} \rfloor$ elements of $\mathcal{X}$ by a single $(\leqslant p)$-inversion without executing unwanted arc flips that would destroy the connectivity of $D'$. To this end, let $D'_0$ be an arc-minimally strongly connected spanning subdigraph of $D'$ and let $D_0$ be the subdigraph of $D$ corresponding to $D'_0$. We will choose our $(\leqslant p)$-inversions so that, among the arcs in $A(D_0)$, we invert exactly those that have a different orientation in $D'_0$. Clearly, applying these inversions to $D$ yields a strongly connected digraph.

In order to choose a first $(\leqslant p)$-inversion, we define an auxiliary graph $H$ on $\mathcal{X}$, where two sets $X_1, X_2 \in \mathcal{X}$ are linked by an edge if the subdigraph of $D_0$ induced by $X_1 \cup X_2$ does not contain an arc that has a different orientation in $D'_0$. Now, using results on the structure of arc-minimally strongly connected digraphs, we can conclude that $H$ does not contain a large independent set. Hence, applying a famous result of Ramsey, assuming that $\mathcal{X}$ is sufficiently large, we obtain that $H$ contains a clique of size $\lfloor \frac{p}{2} \rfloor$. The union of the sets in this clique serves as the first $(\leqslant p)$-inversion set in our collection. We now recursively keep applying this argument and deleting the corresponding sets from $\mathcal{X}$ until the size of $\mathcal{X}$ is too small to make the Ramsey argument feasible. We then invert the remaining sets in $\mathcal{X}$ one by one. The number of inversions in this last part is bounded by a function depending only on $p$, which results in the additional $\varepsilon$ in the approximation factor.

## 2.3 Inapproximability

Here, we give an overview of the proof of Theorem 9, focusing on the case $p = 3$ and $k = 1$. Our reduction is based on a path packing problem in bipartite graphs. For a graph $G$, a *$P_3$-packing* in $G$ is a collection of pairwise vertex-disjoint subgraphs of $G$ each of which is isomorphic to $P_3$. The problem $P_3$ PACKING (P3P) takes as input a graph $G$ and the objective is to find a $P_3$-packing of maximum size in $G$. It has been proved in [23] that P3P is APX-hard even when the input graph is bipartite and we are guaranteed a $P_3$-packing in $G$ that is linear in the number of vertices of $G$. The following result states that, given a bipartite graph $G$, we can compute a corresponding instance of our inversion problem.

▶ **Lemma 16.** *In polynomial time, given a bipartite graph $G$, one can compute an oriented graph $D$ whose size is polynomial in the size of $G$ such that for $n = |V(G)|$, $x = \mathrm{sinv}_1^{\leqslant 3}(D)$ and $y$ being the maximum size of a $P_3$-packing in $G$, we have $x = \lceil \frac{n-y}{2} \rceil$.*

Observe that our reduction goes from a maximization to a minimization problem. For this reason, concluding the considered case of Theorem 9 from Lemma 16 is slightly technical, though not conceptually difficult. The existence of a packing of linear size is crucial to avoid the objective function for the inversion problem being dominated by an additional constant.

We now give an overview of the construction proving Lemma 16. Let a bipartite graph $G$ with a bipartition $(A, B)$ be given. We now construct an oriented graph $D$ as follows. First, we let $V(D)$ contain $V(G)$. Next, for every $v \in V(G)$, we let $V(D)$ contain a vertex $w_v$. Further, for every $Q \in \binom{V(G)}{2}$, we let $V(D)$ contain a vertex $w_Q$. We let $W$ be the set that consists of $w_v$ for all $v \in V(G)$ and $w_Q$ for all $Q \in \binom{V(G)}{2}$.

We then add arbitrary arcs to $D$ such that $D[W]$ is a strongly connected tournament. Further, for every $a \in A$, we add an arc from $a$ to $w_a$. For every $b \in B$, we add an arc from $w_b$ to $b$. Next, for every $Q \in \binom{V(D)}{2}$ and every $a \in Q \cap A$, we add an arc from $a$ to $w_Q$ and for every $Q \in \binom{V(D)}{2}$ and every $b \in Q \cap B$, we add an arc from $w_Q$ to $b$. Finally, for every $ab \in E(G)$, we let $A(D)$ contain an arc from $a$ to $b$.

The crucial insight for the correctness of the reduction is that the problem of making $D$ strongly connected by (=3)-inversions is equivalent to finding a collection of sets in $\binom{V(D)}{3}$ that induce connected subgraphs of $\mathrm{UG}(D)$ and cover $V(G)$. The most efficient way to do this is finding vertex-disjoint 3-paths in $G$. However, we also need to control how efficiently the remaining vertices can be dealt with once a $P_3$-packing has been used. This is the reason for introducing the new vertices $w_Q$ for all $Q \in \binom{V(G)}{2}$. This ensures that every set of two vertices in $V(G)$ can be dealt with by a single inversion. Hence the total number of inversions needed only depends on the size of the initial $P_3$-packing used and not on the structure obtained after deleting the packing. This finishes the overview of Lemma 16 and Theorem 9.

While the construction can easily be extended for higher connectivity, we are not aware of an easy way to adapt this result for larger inversions. Therefore, for $p \geqslant 4$, we give a proof using similar ideas from a restricted version of the hypergraph matching problem.

## 2.4 Parameterized Complexity

Here, we give an overview of the proof of Theorem 12, focusing on the case $k = 2$. Our reduction is based on the problem MultiColored Clique (MCC). An instance of MCC consists of a graph $G$ and a partition $(V_1, \dots, V_t)$ of $V(G)$. The goal is to decide whether there is a set $\{v_1, \dots, v_t\} \subseteq V(G)$ such that $v_i \in V_i$ for $i \in [t]$ and $G[\{v_1, \dots, v_t\}]$ is a clique. It is well-known that MCC is W[1]-hard with respect to $t$. We now show how to compute an instance $(D, p)$ of our inversion problem from an instance $(G, (V_1, \dots, V_t))$ of MCC.

Let $T$ be an arbitrary 2-arc-strong tournament of order 6. We start from the vertex set $V(G)$, and we add a disjoint copy $\tilde{T}$ of $T$. Next, for every integer $i \in [r]$, we add a new copy $T_i$ of $T$, and we put 2 arcs in both directions between $V(T_i)$ and $V(\tilde{T})$. We further add a new vertex $x_i$ with 2 arcs from $x_i$ to $V(T_i)$, and 1 arc from $V(T_i)$ to $x_i$. Next, for every vertex $v \in V_i$, we add 2 arcs in both directions between $v$ and $V(T_i)$, and an arc from $x_i$ to $v$.

For every pair of integers $1 \leqslant i < j \leqslant r$, we add a new copy $T_{i,j}$ of $T$, together with 2 arcs from $V(T_{i,j})$ to $V(\tilde{T})$. Furthermore, for every pair of vertices $(u, v) \in V_i \times V_j$ which are adjacent in $G$, we add a vertex $z_{u,v}$ with the arcs $z_{u,v}u, z_{u,v}v$, and 2 arcs in both directions between $z_{u,v}$ and $V(T_{i,j})$. We denote by $Z_{i,j}$ the set of such vertices. Finally, we set $p = 2t + \binom{t}{2}$.

In order to see that $(G, (V_1, \dots, V_t))$ and $(D, p)$ are indeed equivalent, observe that every set $X \subseteq V(D)$ such that $\mathrm{Inv}(D, X)$ is 2-arc-connected needs to contain $x_i$ for $i \in [t]$ and a vertex in $Z_{i,j} \cup V(T_{i,j})$ for all distinct $i, j \in [t]$ and at least one neighbor of all these vertices. By the choice of our budget, we obtain that $X$ consists of these vertices and a vertex $v_i \in V_i$ for all $i \in [t]$. It follows that $\mathrm{Inv}(D, X)$ is 2-arc-strong if and only if we can choose the vertex $z_{v_i,v_j}$ for the unique vertex in $Z_{i,j} \cup V(T_{i,j})$ for all distinct $i, j \in [t]$. By construction, this is equivalent to $(G, (V_1, \dots, V_t))$ being a yes-instance of MCC.

In the full version, we prove a quantatively stronger statement using ETH by relying on a more general problem called Partitioned Subgraph Isomorphism. The reduction proving Theorem 13 is based on the same problem.

# 3 Conclusion

We have given a characterization of digraphs for which a certain arc-connectivity can be obtained by applying inversions of fixed size, assuming a moderate minimum size of the graph. Further, we have proved a collection of algorithmic results for the case when the same objective is supposed to be obtained using a minimum number of bounded-size inversions. Our work leaves many questions open.

For Theorem 4 and Theorem 5, it follows from Theorem 7 that the condition $n \geqslant p+2$ cannot be omitted. However, it would be interesting to see whether the condition $n \geqslant 4k+2$ is really crucial.

► **Question 17.** *Are there integers $p \geqslant 3$ and $k \geqslant 1$ and a $2k$-edge-connected digraph $D$ on $n$ vertices with $n \geqslant p+2$ such that $D$ cannot be made $k$-arc-strong with $(=p)$-inversions and $p$ is even or $D$ is not a $k$-obstruction?*

For the approximation questions, it would be interesting to better understand the influence of the inversion size and the connectivity requirement on the best approximation factor. An important question is whether the connectivity requirement can be eliminated from the approximation factor.

► **Question 18.** *Is there a constant $\alpha > 1$ such that, for any integers $p \geqslant 2$ and $k \geqslant 1$, one can approximate $\mathrm{sinv}_k^{\leqslant p}$ in digraphs within a factor of $\alpha$ in polynomial time?*

Our results indicate that increasing the value of $p$ does not make the approximation significantly worse. We conjecture the following hardness result that, roughly speaking, states that the approximation does at least not get easier when increasing $p$.

► **Conjecture 19.** *There exists $\varepsilon > 0$ such that, unless $P = NP$, for any $p \geqslant 3$ and any $k \geqslant 1$, there does not exist a polynomial-time $(1+\varepsilon)$-approximation algorithm for $\mathrm{sinv}_k^{\leqslant p}$ in digraphs.*

Also the parameterized complexity section leaves open questions. In Theorem 12, the case of strong connectivity remains open.

► **Question 20.** *Is there an algorithm that, given a digraph $D$, decides whether $\mathrm{sinv}_1^{\leqslant p}(D) \leqslant 1$ holds in time $f(p) \cdot n^{O(1)}$ for some computable function $f$?*

The parameterization by $\ell$ is even less understood. In particular, Theorem 13 only handles multidigraphs. We pose the following question:

► **Question 21.** *Is there an algorithm that, given a digraph $D$, decides whether $\mathrm{sinv}_k^{\leqslant p}(D) \leqslant \ell$ holds in time $f(\ell) \cdot n^{g(k,p)}$ for some computable functions $f, g$?*

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