A Complete Equational Presentation of Qudit Circuits via Polycontrolled PROPs

Colin Blake
Université de Lorraine, CNRS, Inria, LORIA, Nancy, France

Abstract

High-dimensional quantum computation needs a native circuit-level equational theory for qudits. We give the first finite schematic equational theory that is sound and complete for exact unitary qudit circuits in every finite dimension at least two. Circuits are built from local gates, sequential and parallel composition, and value-controls; equality is derivable exactly when the standard unitary denotations agree. For each dimension, a finite list of local bounded-arity axiom schemata presents the theory, and the diagrammatic shapes do not depend on d. Primitive value-control makes control on a chosen basis value part of the language, so local rules generate the internal algebra of controlled operations within the circuit PROP. This gives a finite, dimension-uniform basis for exact equational reasoning about qudit circuits.

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1 Introduction

Qudit circuits make higher-dimensional quantum data visible at circuit level. Basis values of a d-level carrier can be named, controlled, and rewritten directly, which matters both for hardware where extra levels are native and for compilers that exploit those levels as structure. For $d > 2$, high-dimensional carriers are a native resource [32], and multilevel entangling operations have already been demonstrated, for example, in trapped-ion processors [18]. The extra levels also matter when the algorithm is logically qubit-based. Qutrit workspace can implement generalized Toffoli gates without external ancillas and with lower depth, while ququart encodings can shorten three-qubit constructions [15, 23]; the small example in Figure 1 shows the mechanism. Packing two qubits into one four-level system turns two binary controls into one value-control and turns a two-qubit target subspace into an adjacent two-level ququart subspace.

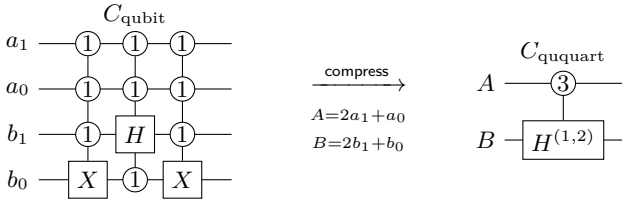


Figure 1 Compressing four qubits into two ququarts via $|a_1a_0\rangle\,|b_1b_0\rangle \mapsto |A\rangle\,|B\rangle$. The branch $a_1 = a_0 = 1$ becomes $A = 3$, while the target subspace $\mathrm{span}\{|01\rangle, |10\rangle\}$ becomes $\mathrm{span}\{|1\rangle, |2\rangle\}$, giving $\mathtt{C}_3(H^{(1,2)})$.

Such compression is useful only if the packed circuits can still be reasoned about locally. A compiler may introduce value-controls, adjacent two-level gates, or swaps between encoded levels; a verifier then has to show that these local changes preserve the intended unitary. This paper studies circuit-level rewriting for qudit circuits.

Quantum circuits provide the standard low-level syntax for finite-dimensional quantum computation [13]: wires carry d-level quantum data, boxes represent unitary gates, and an n-wire circuit denotes a unitary operator on $(\mathbb{C}^d)^{\otimes n}$. A rewrite system for such circuits should therefore be stable under the two ways circuits are built, sequential composition and parallel composition. In PROP language, where objects are wire counts, this asks for local equations between diagrams, closed under those compositions and under the structural symmetries. We ask whether equality of unitary qudit circuits can be axiomatized by a *finite schematic*, syntax-directed equational theory in the circuit PROP.

For qubits ($d = 2$), the answer is known to be yes [6], with later simplifications and minimal presentations [7, 4]. The qubit proofs also identify the pressure point. Control cannot be treated as an afterthought. Without it in the syntax, completeness tends to reintroduce it through rules with arbitrarily many explicit control wires. Controlled PROPs avoid that blow-up by treating control functorially, which restores finite, bounded-arity axiom sets [12] and connects with the categorical account of control constructors in [16].

Graphical calculi provide complementary complete languages for quantum processes, including qudit and finite-dimensional variants of ZX/ZW-style systems [9, 19, 1, 30, 28, 31, 26, 11]. These calculi usually describe a broad class of linear maps. When the endpoint is an ordinary unitary circuit, a proof may finish with a separate reconstruction of a circuit from the whole diagram, or from the matrix it denotes, as in circuit extraction from ZX diagrams [10]. Existing circuit-level qutrit and qudit results are more specialised: phase-gadget methods treat qutrit diagonal gates [29], complete rule sets cover multi-qutrit Clifford fragments [20], and generalized-Clifford formalisms handle multi-qudit computations [21, 22]. The case left open is the full unitary qudit circuit category itself, where wire arities are objects, circuits are morphisms, and, for every $d \geq 2$, one wants a bounded-arity list of local axiom schemata whose shape is uniform in d.

Beyond qubits, the difficulty is to find a qudit-native circuit presentation on which completeness can still be proved by finite local rules. Several binary shortcuts in the qubit completeness proof of [6] become case-sensitive for adjacent two-level gates. A local optical relation may decode either to a single adjacent-level chain or to a mixed row/column configuration guarded by basis controls. The proof therefore needs structured control management while remaining inside a native qudit circuit language.

Our answer is to make each value-control primitive. For every basis value $k \in [d]$, a control functor $\mathtt{C}_k$ sends an endomorphism $f : n \to n$ to its k-controlled extension $\mathtt{C}_k(f) : 1{+}n \to 1{+}n$. The axioms may mention $\mathtt{C}_k(-)$ directly, so the rule set does not need a variable number of control wires.

Main result. For each finite $d \geq 2$, we define a polycontrolled circuit PROP $\mathbf{CQC}_d$ and a finite schematic equational theory QC_d generated by local axiom schemata uniform in d whose instances involve at most three wires; compatibility, commutativity, and exhaustivity are derived as the global control algebra in QC_d. The primitive structure of $\mathbf{CQC}_d$ contains only the same-control functor laws. The main completeness theorem states that QC_d is sound and complete for the standard unitary semantics $[\![-]\!] : \mathbf{CQC}_d \to \mathbf{Qudit}_d$: for any n-qudit circuits $C_1, C_2 : n \to n$, $[\![C_1]\!] = [\![C_2]\!]$ if and only if $\mathrm{QC}_d \vdash C_1 = C_2$.

The completeness proof has three steps. Finite bounded-arity axiom schemata supply the local rules; the control/support schemata derive the control-algebra laws (Theorems 3.9–3.11); Gray-ordered linear-optical transfer proves Theorem 4.17. The transfer encodes n-qudit circuits as d^n-mode single-photon **LOPP** circuits, serialises tensor to match the optical direct-sum tensor, decodes optical derivations back into $\mathbf{CQC}_d$, and proves an encoding/decoding retraction. When $d = 2$, this specializes to the qubit transfer pattern of [6]; for larger d, reflected Gray order keeps optical neighbours adjacent in one qudit coordinate.

2 Qudit circuits as a polycontrolled PROP

Fix a finite dimension $d \geq 2$ and write $[d] = \{0, \ldots, d-1\}$. A PROP is a strict symmetric monoidal category whose objects are natural numbers and whose tensor on objects is addition [24]. PROPs give the ambient circuit calculus used throughout the paper: objects are wire counts and morphisms are diagrams modulo strict symmetric-monoidal coherence. Since all gates considered here are unitary, the only non-empty hom-sets are endomorphism hom-sets.

The extra structure is value control. In ordinary circuit notation one freely draws a control and reads it by its action on a branch; here this operation is part of the categorical syntax. For a detailed analysis of controlled and polycontrolled PROPs and their circuit applications, see [12]; here we use only the control-functor axioms below.

▶ **Definition 2.1.** *For a PROP* $\mathbf{P}$*, let* $\mathbf{P}_{\mathrm{endo}}$ *be the wide subcategory with the same objects and only endomorphism hom-sets. A* control functor *is a functor* $\mathtt{C} : \mathbf{P}_{\mathrm{endo}} \to \mathbf{P}_{\mathrm{endo}}$ *such that* $\mathtt{C}(n) = 1 + n$ *on objects. For every* $f, g : n \to n$*,* $m \geq 0$*, and permutation* $\pi : n \to n$*, it satisfies*

$$\mathtt{C}(\mathrm{id}_n) = \mathrm{id}_{1+n}, \quad \mathtt{C}(g \circ f) = \mathtt{C}(g) \circ \mathtt{C}(f), \quad \mathtt{C}(f \otimes \mathrm{id}_m) = \mathtt{C}(f) \otimes \mathrm{id}_m,$$
$$\mathtt{C}(\pi^{-1} \circ f \circ \pi) = (\mathrm{id}_1 \otimes \pi^{-1}) \circ \mathtt{C}(f) \circ (\mathrm{id}_1 \otimes \pi),$$
$$\mathtt{C}(\mathtt{C}(f)) \circ (\sigma_{1,1} \otimes \mathrm{id}_n) = (\sigma_{1,1} \otimes \mathrm{id}_n) \circ \mathtt{C}(\mathtt{C}(f)).$$

A d*-ary* polycontrolled PROP *is a PROP equipped with control functors* $(\mathtt{C}_k)_{k \in [d]}$.

In this syntax, $\mathtt{C}_k(f) : 1 + n \to 1 + n$ is a circuit with one additional distinguished control wire. The equations say that unused wires remain unused, target-wire permutations commute with adding a control, and two nested controls of the same value may be swapped with the two control wires.

For a word $u = u_1 \cdots u_m \in [d]^*$ we write $\mathtt{C}_u(f) := \mathtt{C}_{u_1}(\cdots \mathtt{C}_{u_m}(f) \cdots)$, with $\mathtt{C}_\epsilon(f) = f$. Beyond the single-value control functor axioms, the later proofs use three global principles, compatibility between different values, commutation of different branches, and exhaustivity over all values. These principles are valid for the unitary interpretation of $\mathbf{CQC}_d$ defined below. The structural congruence of circuits contains only the same-control functor laws, and Section 3.3 derives the global principles from the finite axiom system.

▶ **Definition 2.2.** *Let $(\mathtt{C}_k)_{k\in[d]}$ be a family of control functors. For well-typed endomorphisms $f, g : n \to n$, we use the following three control-algebra laws.*

1. *The family is* compatible *when, for any $a, b \in [d]$, the two ways of nesting a- and b-controls around the same circuit agree up to the control-wire symmetry $\sigma_{1,1} : 2 \to 2$, i.e. $\mathtt{C}_a(\mathtt{C}_b(f)) \circ (\sigma_{1,1} \otimes \mathrm{id}_n) = (\sigma_{1,1} \otimes \mathrm{id}_n) \circ \mathtt{C}_b(\mathtt{C}_a(f))$.*
2. *The family is* commutative *when, for distinct $a, b \in [d]$, operations guarded by different values of the same control wire commute, i.e. $\mathtt{C}_a(f) \circ \mathtt{C}_b(g) = \mathtt{C}_b(g) \circ \mathtt{C}_a(f)$.*
3. *The family is* exhaustive *when all value-controlled branches of the same circuit compose to the uncontrolled target operation, i.e. $\mathtt{C}_0(f) \circ \mathtt{C}_1(f) \circ \cdots \circ \mathtt{C}_{d-1}(f) = \mathrm{id}_1 \otimes f$. The product in the exhaustivity law is composed in increasing order of the control value; once commutativity is available, this order is immaterial.*

In the matrix model, these branch laws say that on each computational-basis value of the control wire exactly one guarded operation is active.

▶ **Definition 2.3.** *We define a d-ary polycontrolled PROP to be* support-sensitive *when each endomorphism $f : n \to n$ is assigned a basis-support envelope $\mathrm{supp}(f) \subseteq [d]^n$ such that endomorphisms with disjoint supports commute: if $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing$, then $f \circ g = g \circ f$. Supports must also be stable under the circuit constructors: for $f, g : n \to n$ and $h : m \to m$, $\mathrm{supp}(f \circ g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$, $\mathrm{supp}(f \otimes h) \subseteq (\mathrm{supp}(f) \times [d]^m) \cup ([d]^n \times \mathrm{supp}(h))$, wire symmetries transport supports by the corresponding permutation of basis words, and $\mathrm{supp}(\mathtt{C}_k(f)) = \{k\} \times \mathrm{supp}(f)$.*

▶ Remark 2.4. Support-sensitivity is stronger than commutativity: if $a \neq b$, then $\mathrm{supp}(\mathtt{C}_a(f))$ and $\mathrm{supp}(\mathtt{C}_b(g))$ are disjoint, so the controlled operations commute.

The structural PROP supplies the empty diagram, identity wires, symmetries, sequential composition, and tensor. Set $\mathsf{G}_d := \{\textcircled{\theta} : 0 \to 0 \mid \theta \in \mathbb{R}\} \cup \{-\boxed{H^{(r,r+1)}}- : 1 \to 1 \mid 0 \leq r < d-1\}$. This non-structural gate family consists of scalar phases indexed by θ and adjacent two-level Hadamards indexed by r. A phase on one basis level is expressed as $\mathtt{C}_i(\textcircled{\theta})$, which semantically multiplies $|i\rangle$ by $e^{i\theta}$ and fixes the other basis states.

▶ Remark 2.5. The choice of adjacent two-level generators comes from the transfer. Under reflected Gray order, an adjacent optical beam splitter connects two basis words that differ in one digit by ± 1 (Lemma 4.5). Its qudit decoding acts on two neighbouring levels, possibly under controls, so the transferred equations stay local and bounded in arity. The same choice is close to common qudit gate descriptions, where single-qudit gates are often expressed as rotations on selected pairs of levels [32, 15, 23]. Permutations of basis values can still be built from adjacent swaps.

▶ **Definition 2.6.** *Let $\mathbf{CQC}_d^{\mathrm{raw}}$ be the raw endomorphism syntax generated by the structural PROP operations, the gates in G_d, and the unary constructors $\mathtt{C}_k(-)$ for $k \in [d]$, with no non-endomorphism homs.*

▶ **Definition 2.7.** *Let $\equiv$ be the least congruence, closed under $\circ$, $\otimes$, and every $\mathtt{C}_k(-)$, containing the strict PROP coherence laws and the control-functor equations for each $\mathtt{C}_k$.*

▶ **Definition 2.8.** *Set $\mathbf{CQC}_d(n, n) := \mathbf{CQC}_d^{\mathrm{raw}}(n, n)/\!\equiv$, with $\mathbf{CQC}_d(m, n) := \varnothing$ for $m \neq n$.*

The quotient $\equiv$ contains only structural equations. Compatibility, commutativity, and exhaustivity must still be proved inside QC_d.

Let $\mathbf{Qudit}_d$ be the semantic polycontrolled PROP with $\mathbf{Qudit}_d(n,n) = U(d^n)$ and no non-endomorphism homs. Symmetries are interpreted by wire swaps, and the remaining clauses are

$$\begin{aligned} [\![C_2 \circ C_1]\!] &= [\![C_2]\!]\,[\![C_1]\!]\,, \qquad [\![C_1 \otimes C_2]\!] = [\![C_1]\!] \otimes [\![C_2]\!]\,, \qquad [\![\textcircled{\theta}]\!] = e^{i\theta}, \\ \Big[\!\Big[-\boxed{H^{(r,r+1)}}- \Big]\!\Big] &= \text{Hadamard on span}\{|r\rangle, |r+1\rangle\} \text{ and identity elsewhere}, \\ [\![\mathtt{C}_k(C)]\!] &= |k\rangle\langle k| \otimes [\![C]\!] + \textstyle\sum_{\ell\in[d],\,\ell\neq k} |\ell\rangle\langle\ell| \otimes I. \end{aligned}$$

▶ **Proposition 2.9.** *The interpretation respects the structural congruence and hence factors through a strict symmetric monoidal functor* $[\![-]\!] : \mathbf{CQC}_d \to \mathbf{Qudit}_d$.

Proof. The clauses are the standard matrix interpretation of strict monoidal circuits, so identities, composition, tensor, and symmetries respect the PROP coherence laws. The displayed block formula for $[\![\mathtt{C}_k(C)]\!]$ is functorial in C. It is also stable under padding and target-wire permutations, and it makes two equal-value controls symmetric after swapping the control wires. Thus every generator of the structural congruence has equal denotation, and the interpretation descends to a strict symmetric monoidal functor on $\mathbf{CQC}_d$. ◀

This small signature still reaches all finite-dimensional unitaries.

▶ **Proposition 2.10.** *For every* $n \geq 0$, $[\![-]\!]$ *is surjective on* n*-wire endomorphisms.*

Proof. For $n = 0$, scalar phases realise $U(1)$. For one qudit, adjacent two-level decompositions reduce an arbitrary element of $U(d)$ to adjacent supports [25, 8, 27, 14], each realised by Euler decompositions into the adjacent Hadamard, level phases, and a scalar phase [25, 2]. For $n > 1$, these one-qudit unitaries together with the controlled scalar $[\![\mathtt{C}_0(\mathtt{C}_0(\textcircled{\pi}))]\!]$ give the exact universal family of [3]: the controlled scalar is entangling for $d \geq 2$, scalar phases supply global phases, and tensors with identities and symmetries place the generators on any chosen wires. ◀

▶ **Definition 2.11.** *An equational theory* Γ *on* $\mathbf{CQC}_d$ *is a set of well-typed equations closed under equivalence closure, sequential composition, tensor, and every control constructor* $\mathtt{C}_k(-)$. *We write* $\Gamma \vdash f = g$ *for the generated congruence.*

▶ **Definition 2.12.** *For each* $d \geq 2$, QC_d *is the equational theory on* $\mathbf{CQC}_d$ *generated by the local axiom schemata displayed in Figures 2 and 3.*

Here QC_d is the congruence generated by the displayed rules. The transfer argument of Section 4 proves that this congruence is exact: Theorem 4.17 identifies it with equality of unitary denotations.

3 A Finite Axiom System for Qudit Circuits

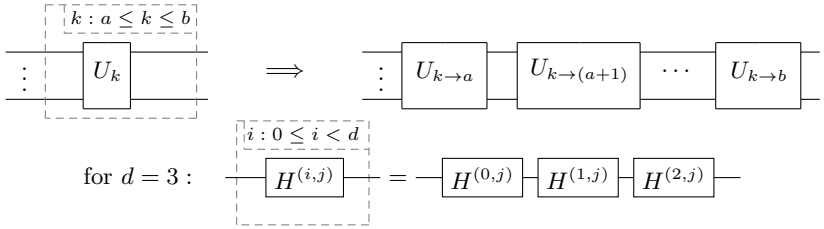
The presentation below consists of the gate algebra of the qudit signature and the finite branch-control rules used by the later normalisation arguments. All equations in Figures 2–3 are read in $\mathbf{CQC}_d$, where the strict PROP laws and same-control coherence equations have already been quotiented out.

3.1 Reading the axiom schemata

The schemata below use a slightly higher-level notation, but no extra generators are being added. The non-structural generators are the gates G_d of Section 2; identities, symmetries, and the control constructors $\mathtt{C}_k(-)$ come from the polycontrolled PROP structure. Here

$X^{(i,j)}$, $H^{(i,j)}$, and $R_x^{(i,j)}(\theta)$ are derived one-qudit circuits: semantically they are, respectively, a transposition of levels i, j, a Hadamard on $\mathrm{span}\{|i\rangle, |j\rangle\}$, and the corresponding two-level beam-splitter rotation. Appendix A.1 gives their recursive definitions. For instance, when $d = 3$, the non-adjacent swap is the adjacent walk $X^{(0,2)} = X^{(1,2)} \circ X^{(0,1)} \circ X^{(1,2)}$; the same pattern gives $H^{(0,2)}$ and $R_x^{(0,2)}(\theta)$.

The figures use *indexed product frames* as meta-notation for finite sequential products. A frame labelled $\vec{k} : \varphi(\vec{k})$ binds the displayed index tuple and expands to the composite of the enclosed diagram over all satisfying tuples, in lexicographic increasing order:



A tuple label such as $(a, b) : 0 \le a < b < d$ is read lexicographically, with a the outer index and b the inner index. Product frames expand on the same wires, so they add repetitions without increasing arity. Read the figures as local support schemata. Increasing d changes the admissible level labels and the finite products expanded by frames, not the arity of the displayed diagrams.

3.2 The axiom schemata

▶ **Theorem 3.1.** *For every $d \ge 2$, QC_d is a finite schematic equational theory on $\mathbf{CQC}_d$. The diagrammatic shape of the rules is independent of d, and every expanded instance acts on at most three wires.*

Proof. The only d-dependent data are level indices and product frames. Product frames expand to sequential products on the same wires, while the displayed diagrams themselves contain at most three wires. Each angle variable ranges over $\mathbb{R}$; finiteness refers to the finite list of rule shapes. ◀

The constructor $\mathtt{C}_k(-)$ is primitive in $\mathbf{CQC}_d$, while the structural congruence contains only the functor laws for a fixed control value. The equations below must therefore generate the algebra relating different control values. Figure 2 gives the gate algebra. Figure 3 lists the bounded control/support schemata used to derive the control laws of Definition 2.2.

Every displayed angle parameter in Figure 3 ranges over $\mathbb{R}$. The inequalities in that figure restrict labels: the S-rules require disjoint active levels, and the B-rules require distinct outer control values $k \neq \ell$.

Before deriving the control algebra, we compare the schemata in Figure 3 with the global laws they generate. If the corresponding control-algebra law, or semantic support-sensitivity, were already present in the syntax, the matching schemata would follow as contextual instances. The next subsection proves the direction needed for completeness: the bounded schemata generate the three control-algebra laws.

▶ **Lemma 3.2.** *If commutativity from Definition 2.2 were admitted as an admissible rewrite in $\mathbf{CQC}_d$, then the schemata* (BPP)*,* (BPπ)*,* (B$\pi\pi$d)*,* (BHπ)*,* (BHH)*,* (BHP)*, and* (B$\pi\pi$s) *would be derivable.*

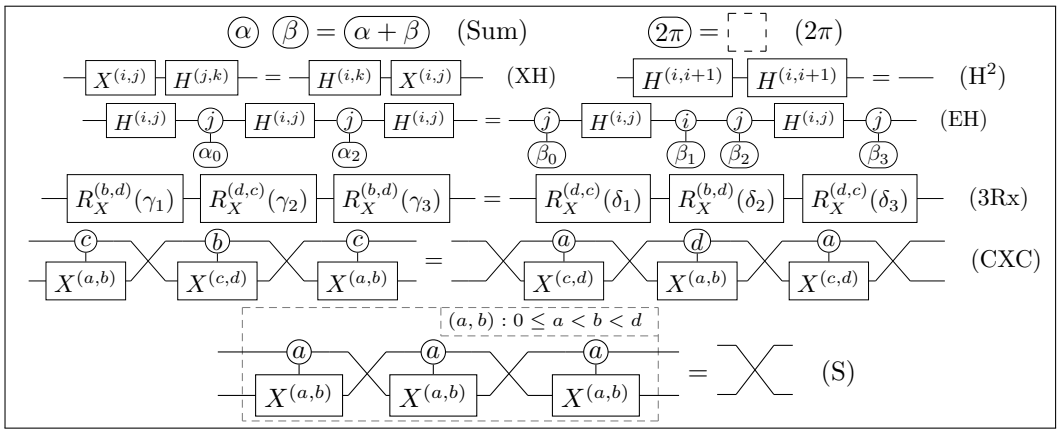


Figure 2 Core local axiom schemata for QC_d. Each displayed equality is a schema: level labels range over $[d]$, and product frames abbreviate finite products over the indices written in their labels. In (EH) and (3Rx) the right-hand-side angles are any admissible choices for the indicated Euler decompositions; Appendix A.1 describes this convention.

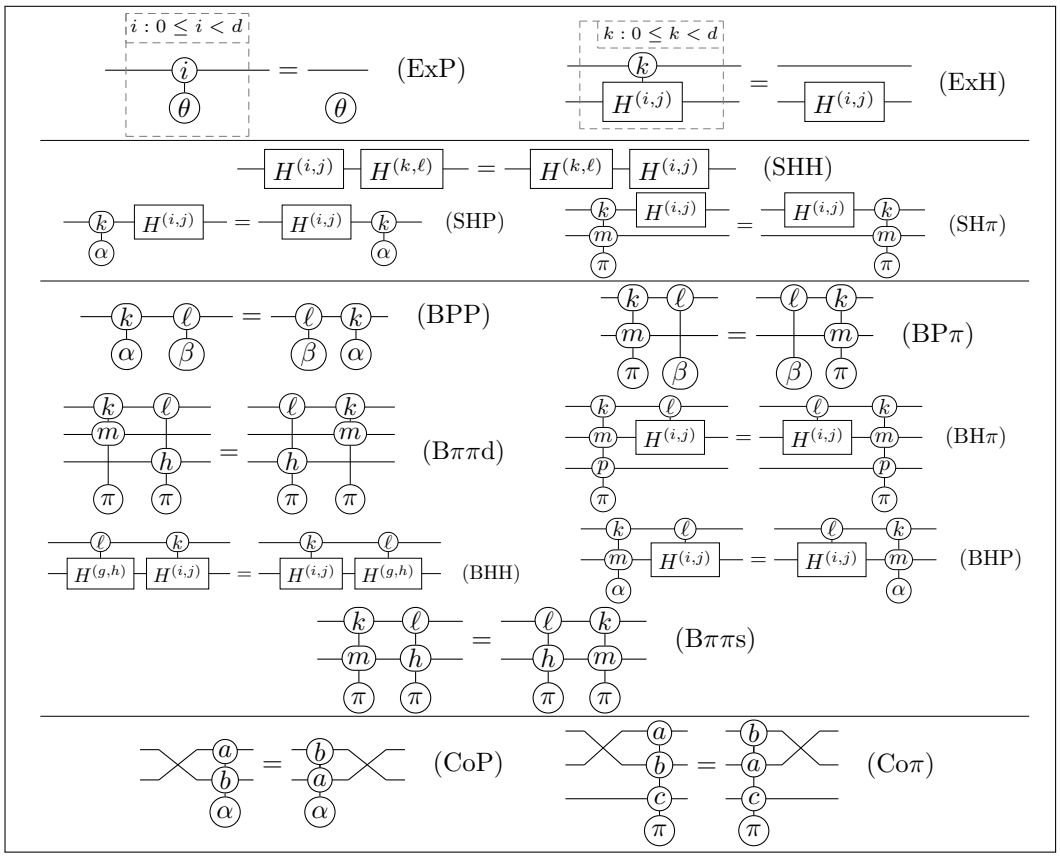


Figure 3 Control-support axiom schemata for QC_d. The prefixes Ex, S, B, and Co mark finite schemata for totalisation, support-locality, branch-commutation, and compatibility. The S-rules use pairwise distinct level labels: i, j, k, and i, j, k, ℓ in (SHH). The B-rules use distinct outer control values $k \neq \ell$.

▶ **Lemma 3.3.** *If* $\mathbf{CQC}_d$ *were assumed to be support-sensitive, with the displayed generators carrying their semantic supports, then the branch-commutation schemata listed in Lemma 3.2 would be derivable. The same assumption would also give* (SHH)*,* (SHP)*, and* (SHπ)*.*

▶ **Lemma 3.4.** *If mixed-control compatibility from Definition 2.2 were admitted as an admissible rewrite in* $\mathbf{CQC}_d$*, then the schemata* (CoP) *and* (Coπ) *would be derivable.*

▶ **Lemma 3.5.** *If exhaustivity from Definition 2.2 were admitted as an admissible rewrite in* $\mathbf{CQC}_d$*, then the totalisation schemata* (ExP) *and* (ExH) *would be derivable.*

Proof of Lemmas 3.2–3.5. The branch-commutation rules are instances of commutativity, up to tensoring with identities and PROP coherence. Support-sensitivity gives the Hadamard commutations because the displayed supports are disjoint: $\{i,j\}$ from $\{k,\ell\}$ or $\{k\}$, and $\{i,j\}\times[d]$ from $\{k\}\times\{m\}$. Compatibility gives (CoP) and (Coπ) by applying it to a scalar phase and to a once-controlled π-phase. Finally, exhaustivity gives (ExP) and (ExH) by taking, respectively, the scalar phase and the derived two-level Hadamard as the controlled circuit. ◀

▶ **Proposition 3.6.** *The semantic polycontrolled PROP* $(\mathbf{Qudit}_d, (\mathsf{C}_k)_{k\in[d]})$ *is support-sensitive.*

Proof. For $U \in \mathbf{Qudit}_d(n,n)$, set $\mathrm{supp}(U) := \{x \in [d]^n \mid U|x\rangle \neq |x\rangle\}$. The support inclusions for composition and tensor, and the formulas for symmetries and controls, follow from their matrix semantics. If U and V have disjoint supports S and T, then they act on different direct summands in the decomposition by S, T, and the remaining basis states, hence commute. ◀

3.3 Derived control-algebra laws

We now derive the three control-algebra laws (Definition 2.2) from the finite schemata in Figures 2–3.

▶ **Definition 3.7.** *Let* $\mathcal{G} = \{\sigma_{1,1},\ \textcircled{\theta},\ -\boxed{H^{(r,r+1)}}-,\ \mathsf{C}_k(\textcircled{\theta}),\ \mathsf{C}_k(\mathsf{C}_\ell(\textcircled{\pi}))\}$*, with the adjacent Hadamard indexed by* $0 \le r < d-1$ *and the controls indexed by* $k,\ell \in [d]$*. A contextual* $\mathcal{G}$*-factor is a morphism* $\mathrm{id}_p \otimes G \otimes \mathrm{id}_q$*, where* $G \in \mathcal{G}$ *and* $p,q \ge 0$*.*

▶ **Lemma 3.8** (Factor normalisation)**.** *Every morphism of* $\mathbf{CQC}_d$ *is provably equal in* QC_d *to a possibly empty finite sequential product of contextual* $\mathcal{G}$*-factors.*

Proof. Apply Separate from Appendix A.2. It expands the derived one-qudit notation, serialises tensor, and pushes controls through composites. When a controlled generator is not already a contextual $\mathcal{G}$-factor, derived equations reduce it to a finite product with fewer outer controls on each factor. The full version gives the termination measure and the omitted diagrammatic reductions. An identity map is sent to the empty product at the same ambient arity, i.e. the PROP unit. ◀

▶ **Theorem 3.9.** *In the equational theory* QC_d *on* $\mathbf{CQC}_d$*, the family* $(\mathsf{C}_k)_{k=0}^{d-1}$ *satisfies compatibility.*

▶ **Theorem 3.10.** *In the equational theory* QC_d *on* $\mathbf{CQC}_d$*, the family* $(\mathsf{C}_k)_{k=0}^{d-1}$ *is commutative.*

▶ **Theorem 3.11.** *In the equational theory* QC_d *on* $\mathbf{CQC}_d$*, the family* $(\mathsf{C}_k)_{k=0}^{d-1}$ *satisfies exhaustivity.*

Proof of Theorems 3.9–3.11. Apply factor normalisation to the controlled composites that occur in the three laws. The resulting products consist of contextual $\mathcal{G}$-factors. Plain symmetry factors are handled by structural naturality; the remaining comparisons are the controlled-factor checks of Lemma A.1.

For compatibility, the two nested-control sides give matching separated products after the outer control wires are swapped. Factors meeting at most one of those wires pass the swap by naturality; factors meeting both wires use the nested-control check of Lemma A.1.

For commutativity, the separated products for $\mathtt{C}_k(f)$ and $\mathtt{C}_\ell(g)$, with $k \neq \ell$, are interchanged one neighbouring pair at a time. Each pair is a branch check from Lemma A.1; induction on the two product lengths moves all factors across.

For exhaustivity, the totalisation check gives the phase and adjacent-Hadamard generators. The identity case follows from the control functor laws, and structural swaps use the fixed presentation (S). The compatibility and commutativity parts give closure under composition, tensor product, and added controls. The omitted factor diagrams and the closure induction are in the full version. ◀

4 Embedding Qudit Circuits into LOPP Circuits

Single-photon linear optics gives a complete equational calculus for finite unitary matrices generated by phases, beam splitters, and swaps [5, 17]. Completeness is proved by encoding qudit circuits into that calculus and decoding the resulting optical derivations. We place the d^n computational basis states of an n-qudit register in reflected Gray order, so adjacent optical generators decode to basis-controlled phases or adjacent two-level qudit gates. The tensor operations do not match. $\mathbf{CQC}_d$ uses Hilbert-space tensor product, while single-photon **LOPP** uses block-diagonal direct sum. The translations are therefore contextual maps on a fixed d^n-mode space. They preserve sequential composition, serialise tensor, and are tied together by an encode-decode retraction in QC_d. We prove the three required statements in that order. Encoding preserves denotations after the Gray-basis change; decoded optical rewrites can be mimicked in QC_d; decoding an encoding retracts to the original qudit circuit.

4.1 Linear optics and Gray-ordered semantics

The decoding map later inspects raw optical subterms, including the consecutive mode interval a subcircuit occupies. The quotient **LOPP** is the language to which the imported completeness theorem applies.

▶ **Definition 4.1.** *Let* $\mathbf{LOPP}^{\mathrm{raw}}$ *be the free endomorphism PROP generated by the empty diagram, identity wires, swap, one-mode phases* $-\boxed{\theta}-$*, and two-mode beam splitters*  *, with* $\mathbf{LOPP}^{\mathrm{raw}}(m,n) = \varnothing$ *for* $m \neq n$*. Let* **LOPP** *be the quotient of* $\mathbf{LOPP}^{\mathrm{raw}}$ *by strict PROP coherence.*

We write **LOPP** for the quotient PROP and LOPP $\vdash$ for derivability in its equational theory.

▶ **Definition 4.2.** *The single-photon interpretation* $[\![-]\!]_{\mathrm{sp}} : \mathbf{LOPP}^{\mathrm{raw}}(m,m) \to U(m)$ *is defined by*

$$[\![C_2 \circ C_1]\!]_{\mathrm{sp}} = [\![C_2]\!]_{\mathrm{sp}}\,[\![C_1]\!]_{\mathrm{sp}}\,, \quad [\![C_1 \otimes C_2]\!]_{\mathrm{sp}} = [\![C_1]\!]_{\mathrm{sp}} \oplus [\![C_2]\!]_{\mathrm{sp}}\,,$$

$$\Big[\!\Big[-\boxed{\theta}-\Big]\!\Big]_{\mathrm{sp}} = (e^{i\theta}), \qquad \Big[\!\Big[\,\raisebox{-0.3em}{}\,\Big]\!\Big]_{\mathrm{sp}} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}.$$

The structural generators are interpreted by $[\![\,]\!]_{\mathrm{sp}} = I_0$, $[\![-]\!]_{\mathrm{sp}} = I_1$, *and* $[\![\times]\!]_{\mathrm{sp}} = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. *More generally, identities and swaps use the corresponding identity and permutation matrices. This interpretation respects strict PROP coherence and therefore descends to* **LOPP**.

The non-structural equations of the calculus are those of the complete single-photon linear-optical theory of [5, 17]; together with strict PROP coherence, they generate LOPP $\vdash$.

▶ **Theorem 4.3** (LOPP completeness [5, 17])**.** *The equational theory* LOPP *is sound and complete for the single-photon semantics: for* $C_1, C_2 : m \to m$, $[\![C_1]\!]_{\mathrm{sp}} = [\![C_2]\!]_{\mathrm{sp}}$ *if and only if* $\mathrm{LOPP} \vdash C_1 = C_2$.

To compare d^n-mode optical circuits with n-qudit circuits, we identify optical modes with computational basis states using a reflected d-ary Gray code.

▶ **Definition 4.4.** *The reflected Gray order* $G_n^d : \{0, \dots, d^n - 1\} \to [d]^n$ *is defined by* $G_0^d(0) = \epsilon$ *and, writing* $k = qd^{n-1} + r$, *by* $G_n^d(k) = q \cdot G_{n-1}^d(r)$ *if* q *is even and* $G_n^d(k) = q \cdot G_{n-1}^d(d^{n-1} - 1 - r)$ *if* q *is odd.*

▶ **Lemma 4.5** (Gray neighbours)**.** *Consecutive Gray words differ in exactly one digit, and that digit changes by* ± 1. *In particular, two adjacent optical modes always correspond to an adjacent two-level qudit transition, possibly under controls on the remaining digits.*

Proof. Induct on n, the case $n = 0$ being empty. Inside a leading block $qd^{n-1}, \dots, (q+1)d^{n-1} - 1$, the suffix is listed either in G_{n-1}^d-order or in reverse order, so the induction hypothesis gives a single adjacent change in the suffix. At the boundary between the q- and $(q+1)$-blocks, reflection makes the terminal suffix of one block equal to the initial suffix of the next; hence only the leading digit changes, from q to $q+1$. Thus every consecutive pair differs in one digit, by ± 1. ◀

For $d = 3, n = 2$, the order is $00, 01, 02, 12, 11, 10, 20, 21, 22$. The edges $02 \to 12$ and $12 \to 11$ show the two cases: a fixed suffix or prefix guards the move, and odd blocks reverse the orientation, while the changed digit stays adjacent. Let $\mathfrak{G}_{n,d} : \mathbb{C}^{d^n} \to (\mathbb{C}^d)^{\otimes n}$ be the permutation unitary determined by $\mathfrak{G}_{n,d}(|t\rangle) = \left|G_n^d(t)\right\rangle$. The native single-photon semantics $[\![-]\!]_{\mathrm{sp}}$ is the one used by LOPP completeness. For comparison with qudit circuits we conjugate it by this fixed Gray-code basis change.

▶ **Definition 4.6.** *For* $C : d^n \to d^n$ *in* **LOPP**, *define its Gray-ordered qudit semantics by* $[\![C]\!]_{\mathrm{LOPP}} := \mathfrak{G}_{n,d} \circ [\![C]\!]_{\mathrm{sp}} \circ \mathfrak{G}_{n,d}^{-1}$.

4.2 Encoding and decoding

Encoding and decoding are defined before quotienting and carry context parameters: encoding records where a qudit circuit sits inside a larger register, and decoding records where a raw optical subcircuit sits in the mode list. The next lemmas check that changing raw representatives does not change the induced encoded or decoded circuits.

Three optical gadget families are used by the encoding. The block permutation $\sigma_{a,n,b}^d \in \mathbf{LOPP}^{\mathrm{raw}}(d^{a+n+b}, d^{a+n+b})$ reorders Gray-coded modes so that, after decoding, it becomes the qudit wire permutation $\mathrm{id}_a \otimes \sigma_{n,b}$. The lift $\mathrm{Lift}_d^p(C)$ copies a d^n-mode circuit C through p additional Gray digits, mirroring the copy on odd reflected blocks. The selected lift $\mathrm{Lift}_{d,j}^p(C)$ copies C only on the branch where the added distinguished digit has value j, with identities on other branches. Appendix A.3 fixes these gadgets; the full version gives the block-swap expansion and the decoding calculations.

▶ **Definition 4.7.** *For a raw qudit circuit $C : n \to n$, the contextual encoding $\mathrm{E}^a_b(C) \in \mathbf{LOPP}^{\mathrm{raw}}(d^{a+n+b}, d^{a+n+b})$ represents C on the middle n qudits of an $(a+n+b)$-qudit register. The recursive clauses are*

$$\begin{aligned}
\mathrm{E}^a_b(C_2 \circ C_1) &:= \mathrm{E}^a_b(C_2) \circ \mathrm{E}^a_b(C_1), \\
\mathrm{E}^a_b(C_1 \otimes C_2) &:= \mathrm{E}^{a+n_1}_b(C_2) \circ \mathrm{E}^a_{b+n_2}(C_1) \quad (C_i : n_i \to n_i), \\
\mathrm{E}^a_b(\mathrm{id}_n) &:= \mathrm{id}_{d^{a+n+b}}, \\
\mathrm{E}^a_b(\textcircled{\theta}) &:= \big(-\boxed{\theta}-\big)^{\otimes d^{a+b}}, \\
\mathrm{E}^a_b(-\boxed{H^{(r,r+1)}}-) &:= \sigma^d_{a,b,1} \circ \mathrm{Lift}^{a+b}_d\big(B^{(i,i+1)}_d\big) \circ \sigma^d_{a,1,b}, \\
\mathrm{E}^a_b(\times) &:= \sigma^d_{a,b,2} \circ \sigma^d_{a+b,1,1} \circ \sigma^d_{a,2,b}, \\
\mathrm{E}^a_b(\mathtt{C}_j(C)) &:= \sigma^d_{a,b,m+1} \circ \mathrm{Lift}^{a+b}_{d,j}(\mathrm{E}^0_0(C)) \circ \sigma^d_{a,m+1,b} \quad (C : m \to m).
\end{aligned}$$

Here $B^{(i,i+1)}_d$ denotes the d-mode optical network implementing the adjacent Hadamard, fixed in Definition A.3 of Appendix A.3. We write $\mathrm{E}(C) := \mathrm{E}^0_0(C)$.

For example, if $g, h : 1 \to 1$, then $\mathrm{E}^0_0(g \otimes h) = \mathrm{E}^1_0(h) \circ \mathrm{E}^0_1(g)$ on the same d^2 modes: tensor is serialised over context branches.

▶ **Lemma 4.8.** *For all $a, b \geq 0$, if raw qudit circuits $C, C' : n \to n$ satisfy $C \equiv C'$ in the structural congruence defining $\mathbf{CQC}_d$, then $\mathrm{LOPP} \vdash \mathrm{E}^a_b(C) = \mathrm{E}^a_b(C')$.*

Proof. The structural congruence is generated by strict PROP coherence and by the value-control functor laws. We check the generators. The recursive clauses for E^a_b send the unit, associativity, interchange, and symmetry equations of strict PROP coherence to the corresponding raw **LOPP** coherence equations, with the qudit symmetry implemented by the block permutations $\sigma^d_{a,n,b}$. For the control-functor equations, selected lifts implement the same projector algebra on the distinguished Gray digit: identity, composition, strength, naturality with respect to target permutations, and the same-control nested-swap law all give equal single-photon matrices after encoding. Completeness of the single-photon **LOPP** theory gives the displayed derivability; the full version gives the clause-by-clause raw calculation. ◀

▶ **Lemma 4.9.** *The contextual encoding preserves semantics: for $C : n \to n$ in $\mathbf{CQC}_d$, $[\![\mathrm{E}^a_b(C)]\!]_{\mathrm{LOPP}} = I_{d^a} \otimes [\![C]\!] \otimes I_{d^b}$. In particular, $[\![\mathrm{E}(C)]\!]_{\mathrm{LOPP}} = [\![C]\!]$.*

Proof. By Lemma 4.8, choose any raw representative and argue by induction on it. The identity, composition, and phase clauses are immediate from the definition of E^a_b. The tensor clause serialises $C_1 \otimes C_2$ as two actions on disjoint context branches, so the Gray-ordered matrix is $I_{d^a} \otimes [\![C_1]\!] \otimes [\![C_2]\!] \otimes I_{d^b}$. Ordinary lifts duplicate a local gate on every Gray branch; selected lifts keep only the branch with the required control value; and block permutations move the relevant Gray blocks to the qudit wire on which the generator acts. The lift and block-permutation constructions are in the full version. ◀

Decoding goes in the opposite direction, reading a raw **LOPP** subcircuit in a consecutive mode interval $t, \ldots, t+\ell-1$ inside a d^n-mode system. When $G^d_n(t) = u\,v\,w$ and $G^d_n(t+1) = u\,(v+\varepsilon)\,w$, with $\varepsilon \in \{-1, +1\}$, write $\Lambda^u_w(g) := \sigma_{|u|,|w|,1} \circ \mathtt{C}_{uw}(g) \circ \sigma_{|u|,1,|w|}$ for the one-qudit gate g placed on the changed digit and controlled by the surrounding basis word.

▶ **Definition 4.10.** *For $0 \le t$ and $t+\ell \le d^n$, the contextual decoding $\mathrm{D}_n^t : \mathbf{LOPP}^{\mathrm{raw}}(\ell,\ell) \to \mathbf{CQC}_d^{\mathrm{raw}}(n,n)$ reads a raw optical subcircuit as occupying global modes $t,\dots,t+\ell-1$. Its recursive clauses are*

$$
\begin{aligned}
\mathrm{D}_n^t(C_2 \circ C_1) &:= \mathrm{D}_n^t(C_2) \circ \mathrm{D}_n^t(C_1),\\
\mathrm{D}_n^t(C_1 \otimes C_2) &:= \mathrm{D}_n^{t+\ell_1}(C_2) \circ \mathrm{D}_n^t(C_1) \quad (C_1 : \ell_1 \to \ell_1),\\
\mathrm{D}_n^t(\,\cdot\,) &:= \mathrm{id}_n, \qquad \mathrm{D}_n^t(-) := \mathrm{id}_n,\\
\mathrm{D}_n^t(-\boxed{\theta}-) &:= \mathtt{C}_{G_n^d(t)}(\textcircled{\theta}),\\
\mathrm{D}_n^t(\times) &:= \Lambda_w^u(-\boxed{X^{(v,v+\varepsilon)}}-),\\
\mathrm{D}_n^t(\overset{\theta}{\times}) &:= \Lambda_w^u(-\boxed{R_X^{(v,v+\varepsilon)}(\theta)}-).
\end{aligned}
$$

In the last two clauses, u,v,w,ε are determined by the Gray-neighbour lemma for the consecutive modes $t,t+1$. The figures $X^{(v,v+\varepsilon)}$ and $R_x^{(v,v+\varepsilon)}(\theta)$ use $\varepsilon=+1$ for the adjacent transition $v \leftrightarrow v+1$ and $\varepsilon=-1$ for $v \leftrightarrow v-1$; the displayed order records the direction of the Gray edge. We write $\mathrm{D}(C) := \mathrm{D}_n^0(C)$ for a circuit on all d^n modes.

For $d=3, n=2$, mode 4 has Gray word 11, so its phase decodes to a phase controlled on both qutrits being 1; beam splitters on modes $2,3$ and $3,4$ decode to adjacent transitions along $02 \leftrightarrow 12$ and $12 \leftrightarrow 11$, with the unchanged digit as control.

The lemmas below keep track of the interval $t,\dots,t+\ell-1$ and its Gray orientation. Tensor is read on consecutive intervals as $\mathrm{D}_n^{t+\ell_1}(C_2) \circ \mathrm{D}_n^t(C_1)$; odd Gray blocks reverse the local order. The remaining checks are local phases, swaps, and beam splitters.

▶ **Lemma 4.11.** *For every n,t, the decoding of a one-mode phase is supported on the single basis word $G_n^d(t)$, and the decoding of a swap or beam splitter on consecutive modes $t,t+1$ is supported on the two Gray-neighbouring basis words $G_n^d(t)$ and $G_n^d(t+1)$.*

Proof. The one-mode phase clause above is $\mathtt{C}_{G_n^d(t)}(\textcircled{\theta})$, so its support is exactly the branch word $G_n^d(t)$. For swaps and beam splitters, the Gray-neighbour lemma writes the adjacent Gray words as $u\,v\,w$ and $u\,(v+\varepsilon)\,w$. The decoding clause is then $\Lambda_w^u(g) = \sigma_{|u|,|w|,1} \circ \mathtt{C}_{uw}(g) \circ \sigma_{|u|,1,|w|}$, where g is the adjacent X- or R_x-gate on the changed digit. Hence the only non-identity action is on those two basis words. ◀

▶ **Lemma 4.12.** *Let $P[-]$ be a raw* **LOPP** *context whose hole occupies a consecutive mode interval $t,\dots,t+\ell-1$ inside a d^n-mode circuit. If $C,C' : \ell \to \ell$ are equal by strict PROP coherence in $\mathbf{LOPP}^{\mathrm{raw}}$, then $\mathrm{QC}_d \vdash \mathrm{D}_n^0(P[C]) = \mathrm{D}_n^0(P[C'])$. In particular, $\mathrm{QC}_d \vdash \mathrm{D}(C) = \mathrm{D}(C')$ for coherent d^n-mode circuits C,C'.*

Proof. The proof is an induction on the strict-PROP coherence step inside the fixed raw context. Units and associativity follow directly from the recursive clauses for D. For interchange, the two optical factors occupy disjoint consecutive intervals; by Lemma 4.11, their decodings act on disjoint Gray supports, so Theorem 3.10 commutes the two decoded factors. The symmetry equations reduce to the decoding of adjacent optical swaps, which gives the controlled adjacent transposition of the two corresponding Gray labels. The same disjoint-support and transposition equations handle the surrounding raw context $P[-]$. The full version gives the induction over raw contexts. ◀

▶ **Lemma 4.13.** *The optical gadgets used in the encoding decode as follows. For every $n,p \ge 0$, every raw $C : d^n \to d^n$ in* **LOPP**, *and every $j \in [d]$, $\mathrm{QC}_d \vdash \mathrm{D}_{n+p}^0(\mathrm{Lift}_d^p(C)) = \mathrm{id}_p \otimes \mathrm{D}_n^0(C)$ and $\mathrm{QC}_d \vdash \mathrm{D}_{n+1+p}^0(\mathrm{Lift}_{d,j}^p(C)) = \mathrm{id}_p \otimes \mathtt{C}_j(\mathrm{D}_n^0(C))$. For all $a,n,b \ge 0$, $\mathrm{QC}_d \vdash \mathrm{D}_{a+n+b}^0(\sigma_{a,n,b}^d) = \mathrm{id}_a \otimes \sigma_{n,b}$.*

Proof. For $\mathrm{Lift}_d^p(C)$, the d^p Gray blocks contain copies of C, with the local order reflected on odd blocks. Decoding one full block gives $\mathrm{D}_n^0(C)$ guarded by the corresponding p-digit Gray word; the reflected-block calculation gives the same decoded circuit. Distinct-branch commutation reorders these guarded copies, and exhaustivity removes the full family of guards, giving $\mathrm{id}_p \otimes \mathrm{D}_n^0(C)$. The selected lift keeps only the blocks with outer digit j, so the same calculation leaves $\mathrm{id}_p \otimes \mathtt{C}_j(\mathrm{D}_n^0(C))$. Finally, $\sigma_{a,n,b}^d$ permutes whole Gray blocks. Decoding its adjacent optical swaps gives the corresponding controlled adjacent transpositions, whose product is $\mathrm{id}_a \otimes \sigma_{n,b}$. The full version gives the even/odd block and adjacent-swap calculations. ◀

▶ **Theorem 4.14** (Mimicking). *If* $\mathrm{LOPP} \vdash C_1 = C_2$ *for* $C_1, C_2 : d^n \to d^n$, *then* $\mathrm{QC}_d \vdash \mathrm{D}(C_1) = \mathrm{D}(C_2)$.

Proof. Induct on the LOPP-derivation. Strict PROP steps use Lemma 4.12. The non-structural axioms decode to the matching controlled phase, swap, and Euler rules of QC_d. For the three-beam-splitter axiom, same-digit Gray moves give (3Rx), while two-coordinate corners use its mixed controlled form. The axiom-by-axiom derivations are in the full version. ◀

4.3 Completeness via transfer

The transfer uses the raw context kept by E_b^a and D_n^t. The encoding parameters a, b place the qudit circuit inside a larger Gray register; the decoding parameter t records the optical interval being read inside d^n modes. These data are needed before quotienting, since optical tensor is read as consecutive mode intervals.

▶ **Theorem 4.15.** *For all* $n, a, b \geq 0$ *and all* $C : n \to n$ *in* $\mathbf{CQC}_d$, $\mathrm{QC}_d \vdash \mathrm{D}_{a+n+b}^0(\mathrm{E}_b^a(C)) = \mathrm{id}_a \otimes C \otimes \mathrm{id}_b$.

The internal retraction compensates for the serialisation of tensor in the encoding. In a fixed surrounding register, encoding followed by decoding returns the original circuit with only identity padding.

Proof. Choose a raw representative by Lemma 4.8 and argue by structural induction; the case analysis is in the full version. For empty diagrams and identities, the clauses for E and D both return identity maps. Scalar phases collapse by exhaustivity, adjacent Hadamards use Lemma 4.13, and wire symmetries use the block-permutation equation. Composition and tensor follow from the recursive clauses. For $\mathtt{C}_j(C')$, the selected-lift equation gives $\mathrm{id}_a \otimes \mathtt{C}_j(\mathrm{D}(\mathrm{E}(C'))) \otimes \mathrm{id}_b$, which the induction hypothesis reduces to $\mathrm{id}_a \otimes \mathtt{C}_j(C') \otimes \mathrm{id}_b$. ◀

▶ **Corollary 4.16.** *For every* $C : n \to n$ *in* $\mathbf{CQC}_d$, $\mathrm{QC}_d \vdash \mathrm{D}(\mathrm{E}(C)) = C$.

Proof. Take $a = b = 0$ in the previous theorem; the identity padding is then the monoidal unit on both sides. ◀

The control/support schemata give the global control laws used by the transfer. The Gray encoding sends qudit circuits to **LOPP** circuits with the same denotation, **LOPP** completeness gives an optical derivation, and Theorem 4.14 decodes that derivation back into QC_d. The encode-decode retraction then identifies $\mathrm{D}(\mathrm{E}(C))$ with C.

▶ **Theorem 4.17** (Completeness). *For every* $d \geq 2$, QC_d *is sound and complete for exact unitary qudit semantics: for all* $C_1, C_2 : n \to n$, $[\![C_1]\!] = [\![C_2]\!]$ *if and only if* $\mathrm{QC}_d \vdash C_1 = C_2$.

Proof. Soundness is checked axiom by axiom against the displayed unitary semantics. The case-by-case soundness checks are in the full version. phase arithmetic, 2π-periodicity, and the Euler/rotation rules are the corresponding one- or two-level matrix identities. For the support and branch-commutation cells, the controlled summands are selected by disjoint basis projectors; compatibility permutes the projectors on the two control wires; totalisation sums the d branch projectors to the identity. If $[\![C_1]\!] = [\![C_2]\!]$, then Lemma 4.9 gives equal single-photon semantics for the encodings after Gray conjugation, so $\mathbf{LOPP} \vdash \mathrm{E}(C_1) = \mathrm{E}(C_2)$. Theorem 4.14 and Corollary 4.16 yield $\mathbf{QC}_d \vdash C_1 = \mathrm{D}(\mathrm{E}(C_1)) = \mathrm{D}(\mathrm{E}(C_2)) = C_2$. ◀

5 Conclusion

We have given a finite schematic, bounded-arity equational theory for exact unitary qudit circuits, uniform in the dimension and including global phases. Primitive value-control keeps the schemata within three wires, while compatibility, commutativity, and exhaustivity are derived internally. The Gray-ordered **LOPP** transfer preserves circuits through mimicking and encode-decode retraction. Reflected Gray order, selected lifts, and derived control algebra keep optical locality compatible with native tensor structure.

The displayed diagrams have dimension-independent arity. The dependence on d is carried by level labels, product-frame ranges, and the reflected Gray order used by the transfer. For higher-dimensional compilation, the calculus gives direct equations for value controls and adjacent two-level operations. Natural next steps include orienting rule fragments as rewrite systems and specialising the presentation to fault-tolerant or hardware-native qudit gate sets.

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A Proof Reductions

This appendix contains the finite reductions used in Sections 3 and 4. The full version gives the omitted diagrams, Gray-code gadgets, and soundness checks.

A.1 Derived Notation and Angles

The non-adjacent two-level gates in the axiom schemata are abbreviations in $\mathbf{CQC}_d$. Write $H^{(r,r+1)}$ for the adjacent Hadamard generator $-\boxed{H^{(r,r+1)}}-$, and set $X^{(r,r+1)} := H^{(r,r+1)} \circ \mathtt{C}_{r+1}(\textcircled{\pi}) \circ H^{(r,r+1)}$. Let $X^{(i,i)} := \mathrm{id}_1$. For $i < j$, define $S_{i,i} := \mathrm{id}_1$ and $S_{i,j} := \prod_{k=0}^{j-i-1} X^{(i+k,i+k+1)}$, where $\prod_{k=0}^{m} f_k = f_m \circ \cdots \circ f_0$, then set $X^{(i,j)} := S_{i+1,j} \circ X^{(i,i+1)} \circ S_{i+1,j}^{-1}$. For $i > j$, put $X^{(i,j)} := X^{(j,i)}$. For Hadamards, set $H^{(i,i)} := \mathrm{id}_1$; if $i < j$, set $H^{(i,j)} := X^{(j,i+1)} \circ H^{(i,i+1)} \circ X^{(j,i+1)}$; and if $i > j$, set $H^{(i,j)} := X^{(j,i)} \circ H^{(j,i)} \circ X^{(j,i)}$. For $i \neq j$, define $R_x^{(i,j)}(\theta) := H^{(i,j)} \circ \mathtt{C}_i(\textcircled{\theta}) \circ \mathtt{C}_j(\textcircled{-\theta}) \circ H^{(i,j)}$. Product frames are expanded in the order of their displayed index tuples.

For each instance of (EH), choose an admissible tuple of right-hand-side angles making the displayed two-level Euler identity hold on $\mathrm{span}\{|i\rangle, |j\rangle\}$, acting as the identity outside that subspace. For each instance of (3Rx), choose an admissible tuple of XZX Euler angles for the same real 3×3 rotation as the displayed ZXZ product. At degenerate Euler points this tuple need not be unique. The full version fixes canonical extraction formulas and proves their soundness, including the mixed controlled instance used in the transfer proof.

A.2 Control Algebra

Compatibility, commutativity, and exhaustivity all use the normalisation Separate. It returns, at the arity of the input term, a possibly empty sequential product of contextual $\mathcal{G}$-factors as in Definition 3.7. Expand $X^{(i,j)}$, $H^{(i,j)}$, and $R_x^{(i,j)}(\theta)$ into adjacent gates. Let ε_n denote the empty product at arity n. The clauses for identity, composition, and tensor are

$$
\begin{aligned}
\textsc{Separate}(\mathrm{id}_n) &= \varepsilon_n,\\
\textsc{Separate}(T_1 \circ T_2) &= \textsc{Separate}(T_1) \circ \textsc{Separate}(T_2),\\
\textsc{Separate}(T_1 \otimes T_2) &= (\textsc{Separate}(T_1) \otimes \mathrm{id}_{|T_2|}) \circ (\mathrm{id}_{|T_1|} \otimes \textsc{Separate}(T_2)).
\end{aligned}
$$

When a control reaches a composite, tensor, adjacent target swap, or generator, use

$$
\begin{aligned}
\textsc{Separate}(\mathtt{C}_k(T_1 \circ T_2)) &= \textsc{Separate}(\mathtt{C}_k(T_1)) \circ \textsc{Separate}(\mathtt{C}_k(T_2)),\\
\textsc{Separate}(\mathtt{C}_k(T_1 \otimes T_2)) &= (\textsc{Separate}(\mathtt{C}_k(T_1)) \otimes \mathrm{id}_{|T_2|}) \circ (\mathrm{id}_1 \otimes \sigma_{|T_1|,|T_2|}) \circ\\
&\quad (\textsc{Separate}(\mathtt{C}_k(T_2)) \otimes \mathrm{id}_{|T_1|}) \circ (\mathrm{id}_1 \otimes \sigma_{|T_2|,|T_1|}),\\
\textsc{Separate}(\mathtt{C}_k(\sigma_{1,1})) &= \textsc{Separate}\Big(\textstyle\prod_{0\le a<b<d} \big(\mathtt{C}_k(\mathtt{C}_a(X^{(a,b)})) \circ (\mathrm{id}_1 \otimes \sigma_{1,1}) \circ\\
&\qquad \mathtt{C}_k(\mathtt{C}_a(X^{(a,b)})) \circ (\mathrm{id}_1 \otimes \sigma_{1,1}) \circ \mathtt{C}_k(\mathtt{C}_a(X^{(a,b)}))\big)\Big),\\
\textsc{Separate}(\mathtt{C}_k(\mathrm{id}_n)) &= \varepsilon_{n+1},\\
\textsc{Separate}(\mathtt{C}_k(\textcircled{\alpha})) &= \mathtt{C}_k(\textcircled{\alpha}),\\
\textsc{Separate}(\mathtt{C}_k(H^{(r,r+1)})) &= \textsc{Separate}(\mathrm{CHD}_{k,r}),\\
\textsc{Separate}(\mathtt{C}_{k\ell}(\textcircled{\pi})) &= \mathtt{C}_{k\ell}(\textcircled{\pi}),\\
\textsc{Separate}(\mathtt{C}_{k\ell}(\textcircled{\beta})) &= \textsc{Separate}(\mathrm{P2D}_{k,\ell,\beta}),\\
\textsc{Separate}(\mathtt{C}_{k\ell m}(\textcircled{\alpha})) &= \textsc{Separate}(\mathrm{P3D}_{k,\ell,m,\alpha}),\\
\textsc{Separate}(\mathtt{C}_k(T)) &= \textsc{Separate}(\mathtt{C}_k(\textsc{Separate}(T))) \quad \text{otherwise.}
\end{aligned}
$$

For a controlled block permutation, first write the block permutation as a product of adjacent target swaps. The adjacent-swap clause is (S) with the outer k-control placed on the three non-structural factors in each (a,b)-term; the two occurrences of $\mathrm{id}_1 \otimes \sigma_{1,1}$ are the target crossings displayed in the left-hand side of (S). If F is already a contextual $\mathcal{G}$-factor and none of the controlled clauses applies, set $\textsc{Separate}(F) = F$. Subscripts such as $k\ell m$ abbreviate nested controls in the input of a recursive clause. The terms $\mathrm{CHD}_{k,r}$, $\mathrm{P2D}_{k,\ell,\beta}$, and $\mathrm{P3D}_{k,\ell,m,\alpha}$ denote the right-hand sides of the derived rules CHD, P2D, and P3D proved in the full version. The phase and π-phase clauses already produce factors in $\mathcal{G}$. The other derived clauses replace controlled-generator cases by finite products of the same kind. The local comparisons below therefore involve the non-structural cores $\textcircled{\theta}$, $H^{(r,r+1)}$, $\mathtt{C}_a(\textcircled{\theta})$, and $\mathtt{C}_a(\mathtt{C}_b(\textcircled{\pi}))$, tensored with identities. Plain adjacent swaps are moved by symmetric-monoidal naturality.

▶ **Lemma A.1** (Controlled factor checks). *Let $\mathcal{G}^\circ$ be $\mathcal{G}$ without the adjacent-swap core, and let $L, M : n \to n$ be contextual $\mathcal{G}^\circ$-factors. In QC_d, $\mathtt{C}_s(L) \circ \mathtt{C}_t(M) = \mathtt{C}_t(M) \circ \mathtt{C}_s(L)$ for $s \neq t$, and $\mathtt{C}_s(\mathtt{C}_t(L)) \circ (\sigma_{1,1} \otimes \mathrm{id}_n) = (\sigma_{1,1} \otimes \mathrm{id}_n) \circ \mathtt{C}_t(\mathtt{C}_s(L))$. If $P : n \to n$ is a contextual factor whose core is either a phase or an adjacent Hadamard, then $\mathtt{C}_0(P) \circ \cdots \circ \mathtt{C}_{d-1}(P) = \mathrm{id}_1 \otimes P$.*

Proof. For branch commutation, target-wire symmetries reduce the placement of L and M to finitely many adjacent-window offsets. Figure 3 contains the primitive B-rules used in this table, including the Hadamard cases (BHπ), (BHP), and (BHH). The full version derives the remaining pairwise commutations, both for the other coincident core pairs and for the shifted offsets. The S-rules give the disjoint-active-level cases.

For nested controls, factors meeting at most one of the two distinguished control wires pass the outer swap by structural naturality. The primitive two-wire comparisons are (CoP) and (Coπ); the full version derives the corresponding checks for Hadamard factors and higher controlled phases from the separated expansions. Totalisation is (ExP) for phases and (ExH) for adjacent Hadamards. Tensoring with identities and applying PROP coherence gives the stated contextual instances. ◀

The post-control-algebra derivations also use the mixed controlled form of the three-rotation identity (3Rx); its derivation is in the full version.

A.3 Gray-Code Encoding Gadgets

Section 4 uses the following raw **LOPP** gadgets.

▶ **Definition A.2.** *For $r \geq 0$, let $\rho_r : d^r \to d^r$ be the mode permutation reversing the order of the d^r modes. If $C : d^m \to d^m$, set $\mathtt{Rev}(C) := \rho_m \circ C \circ \rho_m$, and put $\mathtt{Rev}^q(C) = C$ for even q, $\mathtt{Rev}^q(C) = \mathtt{Rev}(C)$ for odd q. The ordinary lift is given by $\mathrm{Lift}_d^0(C) := C$ and $\mathrm{Lift}_d^{p+1}(C) := \bigotimes_{q=0}^{d-1} \mathtt{Rev}^q(\mathrm{Lift}_d^p(C))$. For $j \in [d]$, let $C_j^{(q)} := C$ if $q = j$, and $C_j^{(q)} := \mathrm{id}_{d^m}$ otherwise. The selected lift is given by $\mathrm{Lift}_{d,j}^0(C) := \bigotimes_{q=0}^{d-1} \mathtt{Rev}^q(C_j^{(q)})$ and $\mathrm{Lift}_{d,j}^{p+1}(C) := \bigotimes_{q=0}^{d-1} \mathtt{Rev}^q(\mathrm{Lift}_{d,j}^p(C))$. Finally, $\sigma_{a,n,b}^d : d^{a+n+b} \to d^{a+n+b}$ is the raw permutation that sends the Gray word $u\,v\,w$, with $|u| = a$, $|v| = n$, and $|w| = b$, to $u\,w\,v$.*

The full version expands $\sigma_{a,n,b}^d$ into adjacent optical swaps and proves the decoding equations of Lemma 4.13.

▶ **Definition A.3.** *Write $P(\phi) : 1 \to 1$ for the* **LOPP** *phase shifter of angle ϕ, and $B(\theta) : 2 \to 2$ for the* **LOPP** *beam splitter of angle θ. Set $H_{\mathrm{BS}} := (\mathrm{id}_1 \otimes P(-\pi/2)) \circ B(\pi/4) \circ (\mathrm{id}_1 \otimes P(-\pi/2))$. For $0 \leq i < d-1$, define $B_d^{(i,i+1)} := \mathrm{id}_i \otimes H_{\mathrm{BS}} \otimes \mathrm{id}_{d-i-2} : d \to d$. Its single-photon denotation, in the Gray order used by the encoding, is the adjacent two-level Hadamard $H^{(i,i+1)}$; the matrix check is in the full version.*