On Positivity of Exponential-Trigonometric Polynomials and Irrationality Exponents

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Abstract

We establish Diophantine hardness results for the decidability of the Positivity Problem for exponential-trigonometric polynomials over computable discrete subfields of the real numbers, and for related questions. We show that any algorithm for deciding either non-negativity, eventual non-negativity, the existence of a zero, or the existence of infinitely many zeros of exponential-trigonometric polynomials over a computable discrete subfield K of the reals containing the number π can be translated into an algorithm for computing the irrationality exponents of all elements of K. As a consequence, we exhibit a computable discrete subfield K of the reals such that all of the aforementioned questions about exponential-trigonometric polynomials over K are undecidable. In particular, we provide the first example of a natural generalisation of the Continuous Skolem Problem that is provably undecidable.

2012 ACM Subject Classification Theory of computation

Keywords and phrases Linear Dynamical Systems, Computability, Computable Numbers, Transcendental Numbers, Irrationality Measure, Irrationality Exponent

Digital Object Identifier 10.4230/LIPIcs.MFCS.2026.65

1 Introduction

An exponential-trigonometric polynomial (ETP) is a real function $f\colon \mathbb{R} \to \mathbb{R}$ of the form

$$f(t) = \sum_{k=0}^{n} P_k(t) e^{t\lambda_k} \tag{1}$$

for distinct complex numbers $\lambda_k \in \mathbb{C}$ and non-zero $P_k \in \mathbb{C}[t]$ for all k. ETPs are also called exponential polynomials or C-finite functions. The numbers λ_k are called the *eigenvalues* or *characteristic roots* of f.

A function f is an ETP if and only if it satisfies a homogeneous linear differential equation

$$f^{(m)}(t) + c_1 f^{(m-1)}(t) + \cdots + c_m f(t) = 0 \tag{2}$$

with constant real coefficients $c_j \in \mathbb{R}$. The eigenvalues of a function f satisfying (2) are the roots of the *characteristic polynomial* $\chi_f(z) = z^m + c_1 z^{m-1} + \cdots + c_m$. The degree of the polynomial P_k in (1) is equal to $m_k - 1$, where m_k is the multiplicity of λ_k as a root of χ_f. ETPs are completely specified by the coefficients c_j in (2) together with initial values $f^{(j)}(0) = u_j$ for $j = 0, \ldots, m-1$.

ETPs abound in applied mathematics, for example in systems control theory as impulse response functions of time-invariant linear systems, or in statistics as probability density functions.

We will study a number of closely related decision problems for ETPs that arise naturally in a wide range of applications. The *(Continuous) Positivity Problem* asks whether a given ETP f is non-negative[1] on the interval $[0, +\infty)$. The *(Continuous) Skolem Problem* asks whether a given ETP f has a zero in the interval $[0, +\infty)$. The *(Continuous) Ultimate Positivity Problem* asks whether a given ETP f is eventually non-negative[1] on the interval $[0, +\infty)$, *i.e.*, if there exists $T > 0$ such that $f(t) \geq 0$ for all $t \geq T$. The *(Continuous) Infinite Zeros Problem* asks whether a given ETP f has infinitely many zeros in the interval $[0, +\infty)$.

The above problems and their discrete-time analogues have received much attention for ETPs over the fields of rational or algebraic numbers, *i.e.* where the coefficients c_j and initial values u_j are rational or algebraic.

Their decidability was first studied by Bell, Delvenne, Jungers, and Blondel [5], who showed decidability of special cases of the problems and proved upper and lower complexity bounds. More recently, Chonev, Ouaknine, and Worrell [11] have shown that the Skolem Problem and the Infinite Zeros Problem for ETPs satisfying a differential equation (2) of order at most eight are decidable, the former subject to Schanuel's Conjecture. They further established a *Diophantine hardness* result for the Infinite Zeros Problem, showing that a decision method for the Infinite Zeros Problem at order nine or higher would yield an algorithm for computing the Lagrange constant of any given real algebraic number – a problem that seems completely out of scope of current techniques. Similar results are known for linear recurrence sequences, the discrete-time analogues of ETPs [27, 26, 20, 19, 21]. It remains a major open question whether any of the above problems are decidable.

To the best of our knowledge, the existing literature on decision problems for ETPs exclusively considers ETPs that satisfy a differential equation of the form (2) with rational or real algebraic coefficients c_j and rational or real algebraic initial values $f^{(j)}(0)$. From a practical perspective, this restriction is not entirely natural: practitioners will likely want to be able to include special transcendental constants such as π in the specification of a function.

The field $\mathbb{Q}(\pi)$ behaves computationally much like the rational or real algebraic numbers: it admits a finite presentation that renders the field operations and (in-)equality comparison computable. Moreover, given the symbolic representation of an element $x \in \mathbb{Q}(\pi)$, we can compute rational approximations of x to any specified accuracy. More generally, it makes sense to study ETPs over any countable subfield K of $\mathbb{R}$ with the above properties. We call any such field K a *computable discrete subfield of* $\mathbb{R}$.

Arguably, much of computational algebra and geometry can be carried out over arbitrary computable discrete subfields K of the reals, for both K and its real closure $\overline{K} \cap \mathbb{R}$ support effective computation of the field operations, equality testing, and inequality testing. Similarly, we can effectively compute the field operations in the algebraic closure $\overline{K}$ and test elements of $\overline{K}$ for equality. In particular, reduction to the *theory of the reals* is a powerful basic tool for establishing the decidability of computational problems in algebra and geometry. The standard decision methods for the theory of the reals, such as cylindrical algebraic decomposition [12], lift to any computable discrete subfield of the reals. Consequently, any decidability result that is provable by reduction to the theory of the reals holds true over any computable discrete subfield of the reals. Further, any algorithm that is formulated in the real RAM model [24] or the BSS model of real computation [7], which are widely used in computational geometry, can be effectively implemented over any computable discrete subfield of the reals.

[1] For the problems we consider, it does not make a difference whether we ask for non-negativity, $f(t) \geq 0$, or (strict) positivity $f(t) > 0$, in the sense that all of our results hold true for either formulation of the problem.

Thus, a large number of problems from algebra and geometry can naturally be formulated over the real numbers and be solved over any computable discrete subfield of the reals by a uniform algorithm that does not depend on the field. Since Positivity and related problems for ETPs are arguably most naturally formulated over the real numbers, it seems very natural to ask whether such a uniform algorithm exists for these problems. Our main results will show that no such algorithm exists. In fact, we can exhibit an explicit computable discrete subfield of the reals over which the above decision problems for ETPs become undecidable – and in many cases neither semi-decidable nor co-semi-decidable.

For a subfield $K \subseteq \mathbb{R}$ we call $f\colon \mathbb{R} \to \mathbb{R}$ an ETP *over* K if the coefficients c_j in (2) all belong to K and if $f^{(j)}(0) \in K$ for $j = 0, \ldots, m-1$. If f is an ETP over K, then its eigenvalues λ_k are (complex) algebraic over K and the polynomials P_k in (1) have complex algebraic coefficients over K.

As our main contribution, we show that the decidability of the above problems for ETPs over a computable discrete subfield K of the reals is linked to the computability of the *irrationality exponents* of the elements of K.

▶ **Definition 1.** *Let α be a real number. The irrationality exponent $\mu(\alpha)$ of α is the supremum of the set of all $\mu \in [0, +\infty)$ such that there exist infinitely many coprime integers p, q with*

$$\left|\alpha - \frac{p}{q}\right| < q^{-\mu}.$$

Intuitively, the irrationality exponent of α measures how well α can be approximated by rational numbers. Rational numbers have irrationality exponent 1, while by Dirichlet's approximation theorem, irrational numbers have irrationality exponent at least 2. The Thue-Siegel-Roth theorem asserts that algebraic irrational numbers have irrationality exponent exactly 2. Almost all numbers, in the sense of Lebesgue measure, have irrationality exponent exactly 2. Numbers with infinite irrationality exponent exist and are called *Liouville numbers*, following Liouville's construction of the first numbers known to be transcendental. In general, the irrationality exponent of a given number can be extremely difficult to find. For example, the irrationality exponent of π is only known to lie in the interval $[2, 7.103205334137\ldots]$ ([28], see also [1]).

The statement of our main result requires certain notions of computability for real numbers. A real number x is *computable from below* (or lower semicomputable) if there exists a computable function $\varphi\colon \mathbb{N} \to \mathbb{Q}$ such that $\varphi(n) \leq x$ for all n and $\lim_{n\to\infty} \varphi(n) = x$. It is *computable from above* (or upper semicomputable) if there exists a computable function $\varphi\colon \mathbb{N} \to \mathbb{Q}$ such that $\varphi(n) \geq x$ and $\lim_{n\to\infty} \varphi(n) = x$. It is *computable* if it is both computable from below and from above. Our main result is the following:

▶ **Theorem 2.** *Let K be a computable discrete subfield of the real numbers that contains the number π. Let $\alpha \in K$.*

1. *A semi-decision procedure for either of the following problems over K can be automatically translated into an algorithm that computes the irrationality exponent $\mu(\alpha)$ from above:*
   - **a.** *The Positivity Problem.*
   - **b.** *The Ultimate Positivity Problem.*
   - **c.** *The complement of the Infinite Zeros Problem.*
   - **d.** *The complement of the Continuous Skolem Problem.*
2. *A semi-decision procedure for either of the following problems over K can be automatically translated into an algorithm that computes the irrationality exponent $\mu(\alpha)$ from below:*
   - **a.** *The Infinite Zeros Problem.*
   - **b.** *The complement of the Ultimate Positivity Problem.*

The proof of Theorem 2 is based on ideas from "Diophantine hardness" constructions that have been pioneered by Ouaknine and Worrell [20] in the case of discrete-time linear systems and instantiated by Chonev, Ouaknine, and Worrell [11] in the case of ETPs. Our construction is strongly inspired by one due to Pat McCarthy, given as [14, Example 2]. The basic idea is to consider the sum of two sinusoids with frequencies differing by a ratio α. For irrational α, the functions $\cos\left(\frac{\pi}{\alpha}t\right)$ and $\cos(\pi t)$ do not simultaneously attain the value -1, and the values of the function $2+\cos\left(\frac{\pi}{\alpha}t\right)+\cos(\pi t)$ are strictly positive but have infimum 0. Subtracting a positive function $g(t)$ with $\lim_{t\to\infty} g(t) = 0$ yields a function $h(t) = 2+\cos\left(\frac{\pi}{\alpha}t\right)+\cos(\pi t) - g(t)$ whose eventual positivity depends on the relative rates of decrease of $\inf_{s\in[0,t]}(\cos\left(\frac{\pi}{\alpha}s\right)+\cos(\pi s))$ and $g(t)$. In McCarthy's example, $\alpha = \pi$ and $g(t) = e^{-t}$, and it was shown that only finitely many roots exist. In our work, we consider $g(t) = t^{2-2\mu}$ for some rational $\mu = r/s \geq 2$ and relate the existence of infinitely many sign-changes to the relationship between μ and the irrationality exponent of α. Intuitively, we have

$$2+\cos\left(\tfrac{\pi}{\alpha}t\right)+\cos(\pi t) < t^{2-2\mu} \tag{3}$$

if and only if both $\cos\left(\frac{\pi}{\alpha}t\right)$ and $\cos(\pi t)$ are close to -1, if and only if πt is close to an odd integer multiple $p\pi$ of π and $\frac{\pi}{\alpha}t$ is close to an odd integer multiple $q\pi$ of π. This yields a rational approximation p/q of α. We will show that we have $|\alpha - p/q| < q^{-\mu}$ in this situation, so that the existence of infinitely many zeros yields a lower bound on the irrationality exponent of α.

Conversely, we show that for all $\eta > 0$ and all sufficiently large odd integers p and q with $|\alpha - p/q| < q^{-\mu-\eta}$, the real number $t = p$ satisfies (3). In particular, if α has infinitely many such rational approximations then the function $h(t)$ has infinitely many zeros. Finally, to get rid of the technical requirement that p and q be odd integers, we consider not only the above function, but the function family $2+\cos\left(\frac{\pi}{\alpha}t+j\pi\right)+\cos(\pi t+k\pi) - t^{2-2\mu}$ for $j,k\in\{0,1\}$. We show that μ is a lower bound for the irrationality exponent if and only if all four functions have infinitely many zeros.

The above functions are themselves not ETPs, but for rational $\mu = r/s$ they have the same sign (positive, negative, or zero) as the corresponding ETPs

$$\left(2+\cos\left(\tfrac{\pi}{\alpha}t+j\pi\right)+\cos(\pi t+k\pi)\right)^s t^{2r} - t^{2s}.$$

Formally, the "translations" between algorithms in Theorem 2 are Turing reductions. It can be shown that these reductions are uniform in α subject to the promise that α is irrational, but we will not spell out the proof of this fact in detail.

Theorem 2 yields, for example, a Diophantine hardness result for the very natural field $K = \mathbb{Q}(\pi)$ in the spirit of [11]: an algorithm for either of the problems we are studying over $\mathbb{Q}(\pi)$ would automatically translate to an algorithm for computing arbitrarily good upper bounds for $\mu(\pi)$. Algorithms for Ultimate Positivity or Infinite Zeros would additionally allow the computation of arbitrarily good lower bounds for $\mu(\pi)$. Either result would constitute a major breakthrough in transcendental number theory. While [11] establishes a comparable Diophantine hardness result for the Infinite Zeros Problem over the real algebraic numbers, our results apply also to the Positivity Problem, the Ultimate Positivity Problem, and the Skolem Problem.

Beyond this, we can exhibit a computable discrete subfield of $\mathbb{R}$ for which the above problems become outright undecidable. Becher, Bugeaud, and Slaman [3] have given a complete characterisation of the irrationality exponents of computable numbers: a real number $\mu \geq 2$ is the irrationality exponent of a computable number if and only if it is the

lim sup of a computable sequence. In general, lim sups of computable sequences are neither computable from above nor computable from below. In particular, there exists a computable number whose irrationality exponent is neither computable from above nor from below. This yields:

▶ **Corollary 3.** *There exists a computable discrete subfield K of $\mathbb{R}$ such that:*
1. *Neither the Ultimate Positivity Problem nor its complement are semi-decidable over K.*
2. *Neither the Infinite Zeros Problem nor its complement are semi-decidable over K.*
3. *The Positivity Problem is not semi-decidable over K.*
4. *The complement of the Skolem Problem is not semi-decidable over K.*

Thus, we establish the undecidability of natural generalisations of the Positivity Problem and related problems. Beyond this, our results reveal an obstruction to further progress on these problems over the real algebraic numbers. A solution to, say, the Positivity Problem over the real algebraic numbers, cannot extend to all computable discrete subfields of the reals and thus has to make essential use of the specific Diophantine properties of algebraic numbers. For example, non-negativity of ETPs (over arbitrary real closed fields) cannot be captured by a formula of the theory of the reals. By contrast, Ouaknine and Worrell have shown that the Ultimate Positivity Problem for simple linear recurrences over the integers is complete for the universal fragment of the theory of the reals [21]. Further, by our above remark, non-negativity of ETPs is undecidable in the real RAM model and the BSS model of real computation.

The rest of the paper is structured as follows: In Section 2 we formally define computable discrete subfields of the reals and show that they are closed under adjunction of computable numbers and real algebraic closure. This follows standard ideas and techniques and is included for completeness and to make the paper more self-contained. Theorem 2 is proved in Section 3 and Corollary 3 is proved in Section 4.

2 Computable Discrete Subfields of the Reals

In this section we define computable discrete subfields of the reals, establish some of their basic properties, and show that the class of computable discrete subfields of the reals is closed under adjunction of computable numbers and real closure. The definitions and results in this section are entirely based on standard ideas and techniques from effective algebra (see, *e.g.*, [23, 15, 16, 2]).

Let us introduce some standard notation and terminology. For a ring R, we denote by $R[X]$ and $R(X)$ respectively the ring of polynomials and rational functions in the variable X. For a subfield $K \subseteq \mathbb{R}$ of the reals, we let $\overline{K} \subseteq \mathbb{C}$ denote the (complex) algebraic numbers over K. Accordingly, $\overline{K} \cap \mathbb{R}$ denotes the field of real algebraic numbers over K. The field $\overline{K} \cap \mathbb{R}$ is also called the *real closure* of K, since it is the smallest real closed field that contains K, *cf.* [8].

A *computable discrete subfield of the reals* is a countable subfield $K \subseteq \mathbb{R}$ of the real numbers such that the field operations over K are computable, equality over K is decidable, and K embeds computably into $\mathbb{R}$. More formally:

▶ **Definition 4.** *An* effectively presented field *is a countable field K together with a set $D_K \subseteq \mathbb{N}$ and a surjective map $p_k \colon \subseteq D_K \to K$, called the* presentation. *A computable discrete subfield of the reals is an effectively presented field $K \subseteq \mathbb{R}$ such that:*
1. *D_K is a decidable subset of $\mathbb{N}$.*

2. *There exists a computable function* $\texttt{Add}\colon D_K \times D_K \to D_K$ *satisfying*

$$p_K\left(\texttt{Add}(x,y)\right) = p_K(x) + p_K(y).$$

3. *There exists a computable function* $\texttt{Mul}\colon D_K \times D_K \to D_K$ *satisfying*

$$p_K\left(\texttt{Mul}(x,y)\right) = p_K(x) \cdot p_K(y).$$

4. *There exists a partial computable function* $\texttt{Inv}\colon \subseteq D_K \to D_K$ *with* $\mathrm{dom}(\texttt{Inv}) = \{x \in D_K \mid p_K(x) \neq 0\}$ *satisfying* $p_K\left(\texttt{Inv}(x)\right) = 1/p_K(x)$.
5. *The set* $\Delta_K = \{(x,y) \in D_K \times D_K \mid p_K(x) = p_K(y)\}$ *is a decidable subset of* $D_K \times D_K$.
6. *There exists a computable function* $\texttt{Embed}\colon D_K \times \mathbb{N} \to \mathbb{Q}$ *satisfying*

$$|\texttt{Embed}(x,n) - p_K(x)| < 2^{-n}.$$

Observe that in a computable discrete subfield of the reals, *inequality* $x \leq y$ is decidable as well. To decide whether $x \leq y$, first check if $x = y$. If this is the case, then $x \leq y$. If this is not the case then either $x < y$ or $x > y$. We can decide which one is the case by comparing sufficiently good rational approximations of x and y, provided by the function `Embed` above.

Familiar examples of computable discrete subfields of the reals include the rational numbers $\mathbb{Q}$, the real algebraic numbers $\overline{\mathbb{Q}} \cap \mathbb{R}$, or extensions $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ of $\mathbb{Q}$ by finitely many real algebraic numbers $\alpha_1, \ldots, \alpha_m$.

More generally, we can construct new computable discrete subfields of the reals from existing ones by adjunction of computable real numbers and by real closure.

It is easy to see that for every effectively presented field K with decidable equality and computable field operations there exists an effective presentation of the field $K(x)$ of rational functions over K such that equality over $K(X)$ is decidable and the field operations are computable. Let K be a computable discrete subfield of the reals and let α be a computable real number. Fix an effective presentation $p_{K(X)}$ of $K(X)$ as above. We make $K(\alpha)$ into an effectively presented field via the presentation $\mathrm{eval}_\alpha \circ p_{K(X)}$ where $\mathrm{eval}_\alpha\colon \subseteq K(X) \to K(\alpha)$, $\frac{p}{q} \mapsto \frac{p(\alpha)}{q(\alpha)}$, with $\mathrm{dom}(\mathrm{eval}_\alpha) = \left\{\frac{p(X)}{q(X)} \in K(X) \mid q(\alpha) \neq 0\right\}$.

We make $\overline{K} \cap \mathbb{R}$ into an effectively presented field by coding elements $\alpha \in \overline{K} \cap \mathbb{R}$ by a simple polynomial $f \in K[X]$ together with two rational numbers $a < b$ such that α is the unique root of f in the interval $[a,b]$. It is a standard exercise to turn this coding into a partial mapping from $\mathbb{N}$ to $\overline{K} \cap \mathbb{R}$.

We will show that for all computable discrete subfields of $\mathbb{R}$ and all computable real numbers α, the fields $\overline{K} \cap \mathbb{R}$ and $K(\alpha)$ are again computable discrete subfields of $\mathbb{R}$. This follows entirely from standard arguments and is included to make the paper more self-contained. For further background, see any introductory textbook on (effective) algebra such as [9] or [2]. We first observe that polynomial Euclidean division is computable:

▶ **Proposition 5.** *Let* K *be a computable discrete subfield of* $\mathbb{R}$. *Given polynomials* $f, g \in K[X]$ *with* $g \neq 0$ *we can compute the unique polynomials* q, r *such that* $f = qg + r$ *and* $\deg(r) < \deg(g)$.

It follows from the above that we can compute the gcd of two given polynomials, as well as Bézout coefficients satisfying $Uf + Vg = \gcd(f,g)$ via the extended Euclidean algorithm.

▶ **Proposition 6.** *Let* f *and* g *be polynomials. Then* $x \in \mathbb{C}$ *is a root of* $\gcd(f,g)$ *if and only if it is a root of both* f *and* g.

Given any polynomial f, we can compute a simple polynomial g with the same roots as f by taking $g = f/\gcd(f, f')$.

▶ **Proposition 7.** *Let K be a computable discrete subfield of the reals. Let $f, g \in K[X]$ be non-constant polynomials. Let $[a, b]$ be an interval with rational endpoints such that f has a unique and simple root x_0 in $[a, b]$. Then we can decide whether x_0 is a root of g.*

▶ **Proposition 8.** *If α is a computable real number and K is a computable discrete subfield of $\mathbb{R}$, then so is $K(\alpha)$.*

Proof. Recall that an element of $K(\alpha)$ is given as $f(\alpha)$ for a rational function $f \in K(X)$ where $f = p/q$ for polynomials p and q with $q(\alpha) \neq 0$. It is obvious that the field operations are computable with respect to the given presentation. Since α is computable, it is also obvious that $K(\alpha)$ embeds computably into $\mathbb{R}$. It remains to show that equality is decidable. This will in particular allow us to decide for a given $q \in K(X)$ whether $q(\alpha) \neq 0$, which implies that the domain of the presentation of K is a decidable subset of $\mathbb{N}$.

Suppose we are given two rational functions $f_1 = p_1/q_1$ and $f_2 = p_2/q_2$. Then $f_1(\alpha) = f_2(\alpha)$ if and only if

$$p_1(\alpha)q_2(\alpha) - p_2(\alpha)q_1(\alpha) = 0. \tag{4}$$

We now distinguish two cases: either α is transcendental over K or it is algebraic.

In the first case, (4) holds true if and only if the polynomial $p_1q_2 - p_2q_1$ vanishes everywhere, or equivalently, is the zero polynomial in $K[X]$. This can be effectively decided using equality checking over K in order to determine whether the polynomial has a non-zero coefficient.

In the second case, fix a simple polynomial h that vanishes in α together with an isolating interval I. Then $p_1q_2 - p_2q_1$ vanishes in α if and only if shares a root with h in the interval I, which we can effectively decide by Proposition 7. ◀

▶ **Corollary 9.** *Let $\alpha_1, \ldots, \alpha_m$ be computable numbers. Then the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ is a computable discrete subfield of $\mathbb{R}$.*

Observe that the proof of the above fact (Proposition 8) relies on a non-constructive case distinction. Consequently, while it establishes the *existence* of an algorithm for equality testing, it does not necessarily allow us to tell what the algorithm looks like for a given α. For example, it is currently unknown whether the numbers e and π are algebraically independent. By Corollary 9, equality over $\mathbb{Q}(e, \pi)$ is decidable, but we do not know what the algorithm looks like.

Next, we show that the real closure of any computable discrete subfield K of $\mathbb{R}$ is again a computable discrete subfield of $\mathbb{R}$. We start by observing that we can compute the roots of any polynomial with coefficients in K.

▶ **Theorem 10.** *Let K be a computable discrete subfield of $\mathbb{R}$. Given a non-zero polynomial $f \in K[X]$ we can compute a list $\langle(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\rangle$ of pairs of numbers $\alpha_k, \beta_k \in \overline{K} \cap \mathbb{R}$ such that the list $\langle \alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n\rangle$ contains all complex roots of the polynomial f, listed with multiplicity.*

Proof. Let h be a polynomial of degree d. Let $\varepsilon > 0$ be a rational number. A *minimal cover of the zeros of h to accuracy ε* is a list $B_1, \ldots, B_d \subseteq \mathbb{C}$ of d rational square boxes of side-length ε such that each box contains a complex root of h and every complex root of h is contained in one of the boxes.

It follows from a theorem of Specker [25] that we can compute a minimal cover of the zeros of h to accuracy ε for any given polynomial $h \in K[X]$ and any rational number $\varepsilon > 0$.

Consider the simple polynomial $g = f/\gcd(f, f')$. Since g has only simple roots, there exists $\varepsilon > 0$ such that every minimal cover of the zeros of g to accuracy ε consist of pairwise disjoint boxes. The above fact allows us to effectively find such an $\varepsilon > 0$ and a corresponding minimal cover $B_1, \ldots, B_m$. Each of the boxes B_k contains exactly one complex root $\alpha_k + i\beta_k$ of the polynomial f. We now search for a rational $\delta > 0$ and a minimal cover $B'_1, \ldots, B'_d$ of the zeros of g to accuracy δ such that each of the boxes B'_j intersects exactly one of the boxes B_k. This allows us to compute for each box B_k the multiplicity m_k of $\alpha_k + i\beta_k$ as a root of f.

To conclude the proof, it remains to construct algebraic representations of the numbers α_k and β_k. The construction is standard, but we will describe it briefly for the sake of completeness. We can effectively compute polynomials $A, B \in K[X, Y]$ such that $g(x+iy) = A(x, y) + iB(x, y)$. Consider the *resultant* $R(X) = \mathrm{Res}_Y(A, B)(X) \in K[X]$ of A and B with respect to Y. Explicitly, writing

$$A(X, Y) = A_0(X)Y^d + A_1(X)Y^{d-1} + \cdots + A_d(X)$$

and

$$B(X, Y) = B_0(X)Y^e + B_1(X)Y^{e-1} + \cdots + B_e(X)$$

the resultant is the determinant of the *Sylvester matrix*

$$\begin{bmatrix}
A_0(X) & 0 & \ldots & 0 & B_0(X) & 0 & \ldots & 0 \\
A_1(X) & A_0(X) & \ddots & \vdots & B_1(X) & B_0(X) & \ddots & \vdots \\
\vdots & \vdots\ddots & \ddots & 0 & \vdots & \vdots\ddots & \ddots & 0 \\
A_d(X) & A_{d-1}(X) & \ddots & A_0(X) & B_e(X) & B_{e-1}(X) & \ddots & B_0(X) \\
0 & A_d(X) & & A_1(X) & 0 & B_e(X) & & B_1(X) \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & A_d(X) & 0 & \ldots & 0 & B_e(X)
\end{bmatrix} \tag{5}$$

Since K is a computable discrete subfield of $\mathbb{R}$, the resultant is clearly uniformly computable in A and B. The resultant $R(X)$ is a univariate polynomial that vanishes in a given $X_0 \in \mathbb{R}$ if and only if the polynomials $A(X_0, Y) \in K[Y]$ and $B(X_0, Y) \in K[Y]$ have a common root. Indeed, for a fixed $X \in \mathbb{C}$, letting $P_d \subseteq \mathbb{C}[Y]$ denote the space of all polynomials of degree strictly less than d, the matrix (5) is the matrix representation of the linear map

$$\varphi\colon P_e \times P_d \to P_{d+e},\ (U, V) \mapsto A(X, \cdot)U + B(X, \cdot)V$$

with respect to the bases $(Y^e, 0), \ldots, (1, 0), (0, Y^d), \ldots, (0, 1)$ and $Y^{e+d}, \ldots, Y, 1$.

This map is surjective if and only if the ideal generated by $A(X, \cdot)$ and $B(X, \cdot)$ in $\mathbb{C}[Y]$ is equal to all of $\mathbb{C}[Y]$ if and only if $A(X, \cdot)$ and $B(X, \cdot)$ do not have a common root. See also [2, Chapter 4.2] for details.

But, by definition, $A(X_0, Y)$ and $B(X_0, Y)$ have a common root if and only if X_0 is the real part of a complex root of g. It follows that R vanishes precisely in the real parts of the complex roots of g. In general, R is not a simple polynomial, but we can compute the simple polynomial $r = R/\gcd(R, R')$ which has the same roots as R. Isolating intervals for the roots of the polynomial r can again be computed using Specker's result [25]. Algebraic representations of the imaginary parts β_k are obtained analogously, using the resultant $\mathrm{Res}_X(A, B)$ with respect to the variable X. ◀

▶ **Proposition 11.** *Let K be a computable discrete subfield of $\mathbb{R}$. Then so is the field $\overline{K} \cap \mathbb{R}$ of real algebraic numbers over K.*

Proof. Recall that we represent a real algebraic number α over K by a simple polynomial $f \in K[X]$ satisfying $f(\alpha) = 0$ together with rational numbers $a < b$ such that α is the unique root of f in the interval $[a, b]$.

We first show that equality of real algebraic elements is decidable. Suppose we are given algebraic numbers α and β, represented by polynomials $f \in K[X]$ and $g \in K[X]$ and rational intervals I and J. If I and J are disjoint, then α and β are distinct. If the intersection $I \cap J$ is non-empty, then verify if f has a root in $I \cap J$ and if g has a root in $I \cap J$. If either is not the case, α and β must be distinct. If both are the case, apply Proposition 7 to decide whether the root of g in $I \cap J$ is equal to the root of f in $I \cap J$. This is the case if and only if $\alpha = \beta$.

To prove that the domain of the presentation is a decidable subset of of $\mathbb{N}$, we need to show that given a polynomial $f \in K[X]$ and a rational interval $[a, b]$ we can decide whether f is simple and whether f has a unique root in the interval $[a, b]$, counted with multiplicity. The polynomial f is simple if and only if $f = f/\gcd(f, f')$, which is effectively decidable using equality comparison in K. To determine whether the polynomial has a unique root in $[a, b]$ we proceed as follows: By Theorem 10 we can compute a list of all roots of f. Since equality is decidable, we can compute a list of all real roots of f. Since inequality is decidable, we can effectively decide whether the interval $[a, b]$ contains exactly one root. This shows that the domain of the presentation is decidable.

Now, let us show that the field operations are computable.

To compute the sum $\alpha + \beta$ we have to find a simple polynomial h with $h(\alpha + \beta) = 0$ and an interval K such that $\alpha + \beta$ is the unique root of h in the interval K. Let d denote the degree of f and let e denote the degree of g. Then $K[\alpha, \beta]$ is a vector space over K of dimension $d \cdot e$, a basis being given by the set

$$B = \left\{ \alpha^i \beta^j \mid 0 \leq i \leq d-1, 0 \leq j \leq e-1 \right\}.$$

In particular, the set $\left\{ (\alpha + \beta)^k \mid 0 \leq k \leq d \cdot e \right\}$ is linearly dependent over K. Using that K is a computable field with decidable equality we can compute representations of the elements $(\alpha + \beta)^k$ in the basis B. This allows us to find a non-trivial linear relation between these elements using Gaussian elimination. This yields a polynomial $k(x)$ that vanishes in $\alpha + \beta$. Letting $h = k/\gcd(k, k')$, we obtain a simple polynomial that vanishes in $\alpha + \beta$. It remains to compute an interval K such that $\alpha + \beta$ is the unique root of h in the interval K. Using that K computably embeds into $\mathbb{R}$ we can compute arbitrarily good rational approximations to $\alpha + \beta$. Theorem 10 in particular asserts that we can compute the complex roots of h to arbitrary precision. This means that for every rational $\varepsilon > 0$ we can compute a list of disks $D(z_1, \varepsilon)$, $\ldots$, $D(z_{\deg(h)}, \varepsilon)$ in the complex plane, where $z_j \in \mathbb{Q}[i]$, such that every disk contains a complex root of h and every complex root of h is contained in some disk. Since h is simple, we can effectively find an $\varepsilon > 0$ such that the disks $D(z_1, \varepsilon)$, $\ldots$, $D(z_{\deg(h)}, \varepsilon)$ are disjoint. In particular, since h is a real polynomial, any disk intersecting the real line must contain a real root of h. It follows that we can compute a list $I_1, \ldots, I_\ell$ of disjoint rational intervals such that every interval contains a real root of h and such that every real root of h is contained in some interval. Now, $\alpha + \beta$ is contained in exactly one of the intervals I_j, and we can determine j by computing $\alpha + \beta$ to sufficient precision.

The product $\alpha \cdot \beta$ is computed analogously.

Finally, suppose that we are given $\alpha \neq 0$ via a simple polynomial f satisfying $f(\alpha) = 0$ and an interval $[a, b]$ such that α is the unique root of f in $[a, b]$.

Write

$$f(x) = x^d + a_1 x^{d-1} + \cdots + a_d.$$

Up to dividing by an appropriate power of x, we may assume that $a_d \neq 0$. Then $1/\alpha$ is a root of the polynomial

$$x^d f(1/x) = 1 + a_1 x + \cdots + a_d x^d.$$

This polynomial is simple, for its roots are of the form $1/z$ where z is a root of f.

Since K computably embeds into $\mathbb{R}$, we can approximate α to any given accuracy. It follows that we can compute a rational interval $[a', b'] \subseteq [a, b]$ with $\alpha \in [a', b']$ and $0 \notin [a', b']$. Then $1/\alpha$ is contained in the interval $[1/b', 1/a']$. Conversely, $1/\alpha$ is the only root of $x^d f(1/x)$ in this interval, for if $1/z$ is contained in the interval $[1/b', 1/a']$, then z is contained in the interval $[a', b']$. ◀

Finally, we remark without proof that standard quantifier elimination techniques such as cylindrical algebraic decomposition [12] can be carried out effectively over (the real closure of) any computable discrete subfield of the reals, so that all sentences of the theory of the reals with constants in K are effectively decidable.

▶ **Theorem 12.** *Let K be a computable discrete subfield of $\mathbb{R}$. Given a formula of the theory of the reals with constants in K, we can compute an equivalent quantifier-free formula with constants in $\overline{K} \cap \mathbb{R}$. In particular, given a sentence Φ of the theory of the reals with constants in K we can decide whether Φ holds true over K.*

3 Exponential-Trigonometric Polynomials and Irrationality Exponents

We will now prove Theorem 2. We recall a few elementary facts:

▶ **Proposition 13.**
1. *We have $1 + \cos(t) = 2\sin^2\left(\frac{t-\pi}{2}\right)$ for all $t \in \mathbb{R}$.*
2. *We have $\sin^2(t) \geq \frac{4t^2}{\pi^2}$ for all $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.*
3. *We have $\sin^2(t) \leq t^2$ for all $t \in \mathbb{R}$.*

The next lemma is our main technical result. As mentioned in the introduction, the construction is strongly inspired by an example due to Pat McCarthy [14, Example 2].

▶ **Lemma 14.** *Let $\alpha > 5/2$ be a real number. Let $\mu \geq 2$ be a real number with $\mu \neq \mu(\alpha)$. Consider the functions*

$$h_{j,k,\mu} \colon (0, +\infty) \to \mathbb{R},\ h_{j,k,\mu}(t) = 2 + \cos\left(\tfrac{\pi}{\alpha} t + j\pi\right) + \cos(\pi t + k\pi) - t^{2-2\mu}$$

where $j, k \in \{0, 1\}$. Then $\mu > \mu(\alpha)$ if and only if all four[2] functions are eventually positive, if and only if all four functions have finitely many zeros.

Proof. For a strictly positive integer m, we have $h_{j,k,\mu}(2m - k) > 0$. Thus, $h_{j,k,\mu}$ has finitely many zeros if and only if it is eventually positive.

[2] In principle, we do not have to consider the function $h_{1,1,\mu}$

Assume that $\mu > \mu(\alpha)$. We show that all four functions have only finitely many zeros. Since the functions are analytic, they have only finitely many zeros in the interval $(0, \alpha]$. Let $t > \alpha$ be a zero of $h_{j,k,\mu}$. Then since $1+\cos(t) \geq 0$ for any t, we have $1+\cos\left(\frac{\pi}{\alpha}t + j\pi\right) \leq t^{2-2\mu}$ and $1 + \cos(\pi t + k\pi) \leq t^{2-2\mu}$.

There exist positive integers p and q with $(2q-j)\alpha \leq t \leq (2q+2-j)$ and $(2p-k) \leq t \leq (2p+2-k)$. Hence $|\frac{t}{\alpha} - (2q+1-j)| \leq 1$ and $|t-(2p+1-k)| \leq 1$.

By Proposition 13(1,2) we have:

$$\begin{aligned}
1 + \cos\left(\tfrac{\pi}{\alpha}t + j\pi\right) &= 2\sin^2\left(\frac{\frac{\pi}{\alpha}t + j\pi - \pi}{2}\right) = 2\sin^2\left(\frac{\pi}{\alpha}\frac{t-(1-j)\alpha}{2}\right) \\
&= 2\sin^2\left(\frac{\pi}{\alpha}\frac{t-(1-j)\alpha}{2} - q\pi\right) = 2\sin^2\left(\frac{\pi}{\alpha}\frac{t-(2q+1-j)\alpha}{2}\right) \\
&\geq 2\frac{(t-(2q+1-j)\alpha)^2}{\alpha^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
1 + \cos(\pi t + k\pi) &= 2\sin^2\left(\frac{\pi t + k\pi - \pi}{2}\right) = 2\sin^2\left(\frac{\pi t + k\pi - \pi}{2} - p\pi\right) \\
&= 2\sin^2\left(\pi\frac{t+k-1-2p}{2}\right) \geq 2\left(t-(2p+1-k)\right)^2.
\end{aligned}$$

We obtain $2\frac{(t-(2q+1-j)\alpha)^2}{\alpha^2} \leq t^{2-2\mu}$ and $2\left(t-(2p+1-k)\right)^2 \leq t^{2-2\mu}$. By rearranging we obtain $|t-(2q+1-j)\alpha| \leq \frac{\alpha}{\sqrt{2}}t^{1-\mu}$ and $|t-(2p+1-k)| \leq \frac{1}{\sqrt{2}}t^{1-\mu}$. The triangle inequality and the fact $t \geq (2q-j)\alpha$ yield:

$$|(2q+1-j)\alpha - (2p+1-k)| < \left(\tfrac{\alpha+1}{\sqrt{2}}\right)t^{1-\mu} \leq \left(\tfrac{\alpha+1}{\sqrt{2}}\right)\left((2q-j)\alpha\right)^{1-\mu}.$$

Since $\alpha > 5/2 > 1+\sqrt{2}$ and $\mu \geq 2$ we have

$$\left(\tfrac{\alpha+1}{\sqrt{2}}\right)\left((2q-j)\alpha\right)^{1-\mu} = \frac{1+1/\alpha}{\sqrt{2}\,\alpha^{\mu-2}}\left(\tfrac{2q+1-j}{2q-j}\right)^{\mu-1}(2q+1-j)^{1-\mu} \leq (2q+1-j)^{1-\mu}$$

for sufficiently large q, so that we obtain $|(2q+1-j)\alpha - (2p+1-k)| < (2q+1-j)^{1-\mu}$ or

$$\left|\alpha - \frac{2p+1-k}{2q+1-j}\right| < (2q+1-j)^{-\mu}.$$

By assumption, this inequality can only hold true for finitely many values of p and q. So, the functions are ultimately positive and have finitely many zeros.

Now assume that $\mu < \mu(\alpha)$. Then there exists $\delta > 0$ such that $\mu+\delta < \mu(\alpha)$. We will show that one of the functions assumes infinitely many negative values. By the infinite pigeonhole principle there must exist $j,k \in \{0,1\}$ (not both equal to 1) and unbounded sequences of positive integers $(p_n)_n$, $(q_n)_n$ such that we have $\left|\alpha - \frac{2p_n+1-k}{2q_n+1-j}\right| < (2q_n+1-j)^{-\mu-\delta}$. By rearranging we obtain $|\alpha(2q_n+1-j) - (2p_n+1-k)| < (2q_n+1-j)^{1-\mu-\delta}$.

Consider the sequence $(t_n)_n$ with $t_n = 2p_n+1-k$. We will show that $h_{j,k,\mu}(t_n)$ is negative for all large n.

By Proposition 13(1) we have:

$$\begin{aligned}
&2 + \cos\left(\tfrac{\pi}{\alpha}(2p_n+1-k) + j\pi\right) + \cos(\pi(2p_n+1)) = 1 + \cos\left(\tfrac{\pi}{\alpha}(2p_n+1-k) + j\pi\right) \\
&= 2\sin^2\left(\frac{\frac{\pi}{\alpha}(2p_n+1-k) - (1-j)\pi}{2}\right) = 2\sin^2\left(\frac{\pi}{\alpha}\frac{(2p_n+1-k)-(1-j)\alpha}{2}\right).
\end{aligned}$$

Since $\sin^2$ is π-periodic and by Proposition 13(3), we have:

$$2\sin^2\left(\frac{\pi}{\alpha}\,\frac{(2p_n+1-k)-(1-j)\alpha}{2}\right) = 2\sin^2\left(\frac{\pi}{\alpha}\,\frac{(2p_n+1-k)-(1-j)\alpha}{2} - q_n\pi\right)$$
$$= 2\sin^2\left(\frac{\pi}{\alpha}\,\frac{(2p_n+1-k)-(2q_n+1-j)\alpha}{2}\right)$$
$$\leq \frac{\pi^2}{2\alpha^2}\left((2p_n+1-k)-(2q_n+1-j)\alpha\right)^2 < (2q_n+1-j)^{2-2\mu-2\delta}.$$

Let C be a positive integer with $\alpha < C$. Then we have for all sufficiently large n:

$$\frac{2p_n+1-k}{2q_n+1-j} \leq \alpha + \left|\alpha - \frac{2p_n+1-k}{2q_n+1-j}\right| < C,$$

and hence $2p_n+1-k < C\cdot(2q_n+1-j)$. Since $\mu > 1$, the function $t \mapsto t^{2-2\mu}$ is monotonically decreasing, so that we obtain $(2q_n+1-j)^{2-2\mu} < C^{2\mu+\delta-2}\cdot(2p_n+1-k)^{2-2\mu-\delta}$. In total, letting $t_n = 2p_n+1-k$, we obtain:

$$h_{j,k,\mu}(t_n) < C^{2\mu+\delta-2}\cdot(t_n)^{2-2\mu-\delta} - (t_n)^{2-2\mu} = (t_n)^{2-2\mu}\left(C^{2\mu+\delta-2}\cdot(t_n)^{-\delta} - 1\right). \tag{6}$$

This expression is clearly negative for all large n. So, the function $h_{j,k,\mu}$ is negative at infinitely many values, and hence has infinitely many zeros. ◀

We are now in a position to prove our main theorem.

Proof of Theorem 2. Throughout the proof, fix $\alpha \in K$. We may assume that α is irrational, for otherwise the conclusions of all claims are trivial. Further, $\mu(\alpha) = \mu(\alpha + c)$ for any integer c so that we may assume without loss of generality that $\alpha > 5/2$.

Let $h_{j,k,\mu}$ be the functions from Lemma 14. Let r and s be positive integers. Let

$$\widetilde{h}_{j,k,r,s,b}(t) = \left(2+\cos\left(\tfrac{\pi}{\alpha}t + j\pi\right) + \cos(\pi t + k\pi)\right)^s t^{2r} - t^{2s} + be^{-t}.$$

Then for all $t > 0$ we have $h_{j,k,r/s}(t) \geq 0$ if and only if $\widetilde{h}_{j,k,r,s,0}(t) \geq 0$. In particular, the function $h_{j,k,r/s}$ is eventually positive if and only if the function $\widetilde{h}_{j,k,r,s,0}$ is eventually positive. Observe that the functions $\widetilde{h}_{j,k,r,s,b}(t)$ are ETPs for all j,k,r,s,b and that we can uniformly compute an effective presentation in the form (1) or (equivalently) (2) of $\widetilde{h}_{j,k,r,s,b}$ given j, k, r, s, and b.

Suppose that Ultimate Positivity for ETPs over K is semi-decidable. Then, given $r, s \in \mathbb{Z}$ with $s \neq 0$ and $r/s > 2$, we can semi-decide if $\widetilde{h}_{j,k,r,s,0}$ is eventually positive for all $j,k \in \{0,1\}$. By Lemma 14, if $r/s > \mu(\alpha)$ then $\widetilde{h}_{j,k,r,s,0}$ is eventually positive. Conversely, if $\widetilde{h}_{j,k,r,s,0}$ is eventually positive then by the same lemma we must have $r/s \geq \mu(\alpha)$. This allows us to compute $\mu(\alpha)$ from above.

Now suppose that the complement of the Ultimate Positivity Problem for ETPs over K is semi-decidable. Then, given $r, s \in \mathbb{Z}$ with $s \neq 0$ and $r/s > 2$, we can semi-decide if there exist $j,k \in \{0,1\}$ such that $\widetilde{h}_{j,k,r,s,0}$ is not eventually positive. By the same reasoning as before, if $r/s < \mu(\alpha)$ then this must be the case, and if $\widetilde{h}_{j,k,r,s,0}$ is not eventually positive then we must have $r/s \leq \mu(\alpha)$. Since we assume that α is irrational, we can always safely output the lower bound 2 for $\mu(\alpha)$. It follows that we can compute $\mu(\alpha)$ from below.

Since the functions $\widetilde{h}_{j,k,r,s,0}$ are eventually positive if and only if they have finitely many zeros, the claims for the Infinite Zeros Problem follow immediately from the above.

Now assume that the Positivity Problem for ETPs over K is semi-decidable. Then, given $r, s \in \mathbb{Z}$ with $s \neq 0$ and $r/s > 2$, we can semi-decide if there exists $b > 0$ such that $\widetilde{h}_{j,k,r,s,b}(t)$ is non-negative for all $j, k \in \{0, 1\}$ and all $t \geq 0$. If $r/s > \mu(\alpha)$ then it follows from Lemma 14 that $h_{j,k,r/s}$ is eventually positive and hence bounded from below, so that $\widetilde{h}_{j,k,r,s,b}$ is strictly positive for all sufficiently large b.

Conversely, if $r/s < \mu(\alpha)$ then by the proof of Lemma 14, there exists an unbounded sequence $(t_n)_n$ satisfying (6), *i.e.*, $h_{j,k,r/s}(t_n) < t_n^{2-2r/s}\left(C^{2r/s+\delta-2} \cdot t_n^{-\delta} - 1\right)$, so that

$$\begin{aligned}\widetilde{h}_{j,k,r,s,b}(t_n) &= \left(h_{j,k,r/s}(t_n) + t_n^{2-2r/s}\right)^s t_n^{2r} - t^{2s} + be^{-t_n} \\ &< \left(t_n^{2-2r/s}\left(C^{2r/s+\delta-2} \cdot t_n^{-\delta}\right)\right)^s t_n^{2r} - t^{2s} + be^{-t_n} \\ &= t_n^{2s}\left(C^{2r+\delta s-2s} \cdot t_n^{-\delta s} - 1\right) + be^{-t_n}.\end{aligned}$$

It is easy to see that for all $b \in \mathbb{R}$ there exists N such that the above is negative for all $n \geq N$.

To summarise, if a b exists such that $\widetilde{h}_{j,k,r,s,b}$ is non-negative, then $r/s \geq \mu(\alpha)$, and if $r/s > \mu(\alpha)$ then such a b exists. This allows us to compute $\mu(\alpha)$ from above.

The same argument shows that a semi-decision procedure for the Skolem Problem yields an algorithm for computing $\mu(\alpha)$ from above. ◀

4 Computable Numbers with Uncomputable Irrationality Exponents

A sequence of real numbers $(x_n)_n$ is called *computable* if there exists a computable function $\varphi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{Q}$ such that we have $|\varphi(n, m) - x_n| < 2^{-m}$ for all $n, m \in \mathbb{N}$. Becher, Bugeaud and Slaman have obtained the following complete characterisation of the irrationality exponents of computable numbers:

▶ **Theorem 15** ([3]). *A real number $\mu \geq 2$ is the irrationality exponent of a computable number if and only if there exists a computable sequence of real numbers $(x_n)_n$ with $\mu = \limsup_{n\to\infty} x_n$.*

The following is a folklore observation in computability theory. We include a proof for completeness.

▶ **Proposition 16.** *There exists a computable bounded sequence $(a_n)_n$ of real numbers with $a_n > 2$ such that $\limsup_{n\to\infty} a_n$ is neither lower semicomputable nor upper semicomputable.*

Proof. The Friedberg-Muchnik theorem [13, 17] asserts the existence of computably enumerable subsets $X \subseteq \mathbb{N}$ and $Y \subseteq \mathbb{N}$ of the natural numbers, such that X is not decidable by a Turing machine with oracle Y and such that Y is not decidable by a Turing machine with oracle X. See also, *e.g.*, [18, Theorem 1.6.1] for a proof.

Consider the real numbers $x = \sum_{i\in X} 2^{-i}$ and $y = \sum_{j\in Y} 2^{-j}$. Since X and Y are computably enumerable, the numbers x and y are computable from below. Concretely, there exist computable sequences of finite sets $(X_n)_n$ and $(Y_n)_n$ such that $X = \bigcup_n X_n$ and $Y = \bigcup_n Y_n$. Letting $x_n = \sum_{i\in X_n} 2^{-i}$ and $y_n = \sum_{j\in Y_n} 2^{-j}$, we have $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Since the binary expansion of x encodes membership in X, the number x is not computable from above, not even with a oracle access to Y. Similarly, the number y is not computable from above with oracle access to X.

Now, consider the sequence $(a_n)_n$ where $a_n = x_n - y_n$. By construction, this is a computable sequence of real numbers that converges to the limit $x - y$. In particular, $\limsup_{n\to\infty} a_n = x - y$.

We claim that $x - y$ is neither computable from above nor from below. Indeed, if $x - y$ was computable from above, then given an oracle for Y we could compute the number y and thus compute the number $x = x - y + y$ from above relative to an oracle for Y which contradicts our assumption. Similarly, if $x - y$ was computable from below, then $y - x$ would be computable from above, so that given an oracle for X we could compute the number $y = y - x + x$ from above, contradicting our assumption on X and Y. ◀

Together, Theorem 15 and Proposition 16 yield the existence of a computable real number α whose irrationality exponent is neither computable from above nor from below. By Corollary 9, the field $K = \mathbb{Q}(\pi, \alpha)$ is a computable discrete subfield of the reals. By Theorem 2, the field K witnesses Corollary 3. Bearing in mind the comment after Corollary 9, one might object that while equality in K is decidable, the proof of this fact relies on an a-priori non-constructive case distinction, so that we do not have an *explicit* algorithm for deciding equality of elements in K unless we can determine whether α is algebraic over $\mathbb{Q}(\pi)$, and if so, provide an explicit simple polynomial with coefficients in $\mathbb{Q}(\pi)$ that vanishes in α together with a rational isolating interval. Questions of this form are usually extremely difficult to answer for a given α. Hence, we will address this objection differently: for any given μ as in Theorem 15, we will construct a computable real number α with $\mu(\alpha) = \mu$ such that α is algebraically independent of π by construction. This can be achieved by a straightforward modification of the proof of Theorem 15 that combines the construction in [3] with a diagonalization against all real numbers that are algebraic in π.

We retrace the proof of Theorem 15 given in [3] ("First proof of Theorem 1"), starting with the following fact due to [10]:

▶ **Theorem 17.** *Let $(n_j)_j$ be a sequence of positive integers with $n_{j+1}/n_j \geq 2$. Then the number $\alpha_{(n_j)_j} = \sum_{j=0}^{\infty} 2^{-n_j}$ has irrationality exponent $\limsup_j \frac{n_{j+1}}{n_j}$.*

We estimate the value of α very generously based on its finite approximations:

▶ **Proposition 18.** *Let $(n_j)_j$ be a sequence of positive integers with $n_0 \geq 2$ and $n_{j+1} \geq 2n_j$. Let $\alpha_k = \sum_{j=0}^{k} 2^{-n_j}$. Let $\alpha = \lim_{k\to\infty} \alpha_k$. Then $\alpha \in (\alpha_k + 2^{-n_{k+1}}, \alpha_k + 2^{1-n_{k+1}})$ for all k.*

Proof of Proposition 18. We have

$$\alpha = \sum_{j=0}^{\infty} 2^{-n_j} = \alpha_k + \sum_{j=k+1}^{\infty} 2^{-n_j} = \alpha_k + \sum_{j=0}^{\infty} 2^{-n_{k+1+j}}.$$

We immediately obtain $\alpha > \alpha_k + 2^{-n_{k+1}}$. By induction we obtain for $j \geq 0$

$$n_{k+1+j} \geq 2^j n_{k+1} \geq (j+1)n_{k+1}.$$

This yields

$$\alpha \leq \alpha_k + \sum_{j=0}^{\infty} 2^{-(j+1)n_{k+1}} = \alpha_k + \frac{2^{-n_{k+1}}}{1 - 2^{-n_{k+1}}} < \alpha_k + 2^{1-n_{k+1}}.$$ ◀

We obtain the following modified version of the relevant direction of Theorem 15:

▶ **Theorem 19.** *Let $\mu > 2$ be the* $\limsup$ *of a computable sequence of real numbers. Then there exists a computable number α such that α is algebraically independent of π and $\mu(\alpha) = \mu$.*

Proof. Let $(b_n)_n$ be a computable sequence of real numbers with $\mu = \limsup_{n\to\infty} b_n$. By (computably) passing to a subsequence if need be, we may assume that $b_n > 2$ for all n. Consider the field $K = \mathbb{Q}(\pi)$. By Proposition 11 the field $\overline{K} \cap \mathbb{R}$ is a computable discrete subfield of the reals. In particular, the presentation $p\colon D \to \overline{K} \cap \mathbb{R}$ has a decidable domain $D \subseteq \mathbb{N}$. It follows that we can compute a sequence $(x_n)_n$ containing all (codes[3] of) elements of $\overline{K} \cap \mathbb{R}$.

We will construct a computable sequence $(n_j)_j$ of positive integers with $n_{j+1}/n_j \geq b_{j+1}$ and $(n_{j+1} - 3)/n_j < b_{j+1}$. Let $\alpha = \sum_{j=0}^{\infty} 2^{-n_j}$. Then by Theorem 17 we have $\mu(\alpha) = \limsup_{j\to\infty} \frac{n_{j+1}}{n_j}$. On the one hand, we have $\limsup_{j\to\infty} \frac{n_{j+1}}{n_j} \geq \limsup_{j\to\infty} b_{j+1} = \mu$. On the other hand, we have $\limsup_{j\to\infty} \frac{n_{j+1}}{n_j} = \limsup_{j\to\infty} \frac{n_{j+1}-3}{n_j}$ and $\limsup_{j\to\infty} \frac{n_{j+1}-3}{n_j} \leq \limsup_{j\to\infty} b_{j+1} = \mu$. Thus, $\mu(\alpha) = \mu$.

We further ensure that we have $x_k \notin (\alpha_k + 2^{-n_{k+1}}, \alpha_k + 2^{1-n_{k+1}})$ for all $k \geq 0$. This ensures that α is algebraically independent of π: By construction, $\mu(\alpha) > 2$, so that α must be irrational. If α is algebraic over $\mathbb{Q}(\pi)$, then $\alpha = x_{k-1}$ for some k, contradicting Proposition 18.

We let $n_0 = 2$. Suppose that $n_0, \ldots, n_k$ have been defined. Let n be the smallest integer satisfying $n/n_k \geq b_k$. Since inequality in $\overline{K} \cap \mathbb{R}$ is decidable, we can decide whether x_k is contained in the interval $(\alpha_k + 2^{-n}, \alpha_k + 2^{1-n})$.

If $x_k \notin (\alpha_k + 2^{-n}, \alpha_k + 2^{1-n})$, we let $n_{k+1} = n$. Then all required properties hold true by construction.

If $x_k \in (\alpha_k + 2^{-n}, \alpha_k + 2^{1-n})$, we let $n_{k+1} = n + 2$. Then $n_{k+1}/n_k \geq b_k$ and $(n_{k+1} - 3)/n_k < b_k$. Further, $x_k > \alpha_k + 2^{-n} = \alpha_k + 2^{2-n_{k+1}} > \alpha_k + 2^{1-n_{k+1}}$ so that $x_k \notin (\alpha_k - 2^{-n_{k+1}}, \alpha_k + 2^{1-n_{k+1}})$. ◀

Theorem 19 yields the existence of a computable number α, algebraically independent of π, such that $\mu(\alpha)$ is neither computable from above nor from below. Thus, the field $K = \mathbb{Q}(\pi, \alpha)$ witnesses Corollary 3. Unlike in our previous example, we have an explicit equality-comparison algorithm available for K: As described in Section 2, elements of K are coded by rational functions in $\mathbb{Q}(X, Y)$ via the evaluation map $\mathbb{Q}(X, Y) \to \mathbb{Q}(\pi, \alpha)$. Transcendence of π and algebraic independence of π and α guarantee that the kernel of this map is trivial, so that equality comparison in K reduces to equality comparison in $\mathbb{Q}(X, Y)$.

5 Conclusion and Future Work

We have studied the Positivity Problem, the Ultimate Positivity Problem, the Skolem Problem, and the Infinite Zeros Problem for exponential-trigonometric polynomials over computable discrete subfields of the reals, establishing that all of these problems are undecidable in general. Moreover, we have shown that an algorithm for either of these problems for ETPs over the very natural field $\mathbb{Q}(\pi)$ would entail a major breakthrough in transcendental number theory. For ETPs over the algebraic numbers, a comparable result was known for the Infinite Zeros Problem, but not for the other problems we have considered.

More sharply, we have shown that in the case of the Ultimate Positivity Problem and the Infinite Zeros Problem, neither the problem nor its complement are semi-decidable. It is clear that the complement of the Positivity Problem is semi-decidable, for we can detect if an ETP attains a negative value by computing a sufficiently good numerical approximation of the function with guaranteed error bounds. This leaves open the question whether the

[3] Recall that real algebraic elements over K are coded by a polynomial and an isolating interval.

Continuous Skolem Problem is semi-decidable, *i.e*, if we can semi-decide if a given ETP has a zero. This reduces to the problem of deciding whether a given ETP has a zero on a given *bounded* interval $[0, T]$. Chonev, Ouaknine, and Worrell [11] have shown that the latter is decidable for ETPs over the algebraic numbers subject to Schanuel's Conjecture. This suggests the following question:

Open Problem 1. Let K be a computable discrete subfield of the reals. Is it decidable whether a given ETP over K has a zero on a given bounded interval $[0, T]$ with rational endpoints?

Similarly, one may wonder whether the potential uncomputability of irrationality exponents is the only obstruction to computability, *e.g.*, whether Positivity becomes computable relative to an oracle for irrationality exponents. This is however a very difficult question since it subsumes for example the Positivity Problem over $\mathbb{Q}$ where all irrationality exponents are known to be equal to 1.

The existing literature on continuous and discrete time ETPs contains decidability results for low-order ETPs [27, 26, 11, 20] and "mathematical hardness" results for higher-order ones [11, 20]. By contrast, our hardness and undecidability results do not put a bound on the order of the ETPs. Thus, it is consistent with our Corollary 3 that all of the problems we have considered are decidable for ETPs of any fixed order, but that there is no uniform algorithm for deciding any of the problems for all orders. The result would be much more satisfactory if it applied to ETPs with an explicit, ideally sharp, bound on the order. This brings us to our next open problem:

Open Problem 2. Is there a computable discrete subfield of the reals K such that there exists an integer $n \in \mathbb{N}$ with the property that the Positivity Problem for ETPs over K is decidable for ETPs of order n and undecidable for ETPs of order $n + 1$?

References

1. Max A. Alekseyev. On convergence of the Flint Hills series, 2011. arXiv:1104.5100.
2. Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in Real Algebraic Geometry*. Springer-Verlag, Berlin, Heidelberg, 2006.
3. Verónica Becher, Yann Bugeaud, and Theodore A. Slaman. The irrationality exponents of computable numbers. *Proceedings of the American Mathematical Society*, 144(4):1509–1521, 2016. doi:10.1090/proc/12841.
4. Jason P. Bell and Stefan Gerhold. On the positivity set of a linear recurrence. *Israel Journal of Mathematics*, 157:333–345, 2007. doi:10.1007/s11856-006-0015-1.
5. Paul C. Bell, Jean-Charles Delvenne, Raphaël M. Jungers, and Vincent D. Blondel. The continuous Skolem-Pisot problem. *Theoretical Computer Science*, 411(40):3625–3634, 2010. doi:10.1016/j.tcs.2010.06.005.
6. Jean Berstel and Maurice Mignotte. Deux propriétés décidables des suites récurrentes linéaires. *Bulletin de la Société Mathématique de France*, 104:175–184, 1976. doi:10.24033/bsmf.1823.
7. Lenore Blum, Mike Shub, and Steve Smale. On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines. *Bulletin of the American Mathematical Society*, 21(1), 1989. doi:10.1090/S0273-0979-1989-15750-9.
8. Jacek Bochnak, Michel Coste, and Roy Marie-Françoise. *Real Algebraic Geometry*. Springer Berlin, Heidelberg, 1998. doi:10.1007/978-3-662-03718-8.
9. Siegfried Bosch. *Algebra*. Springer-Verlag Berlin Heidelberg, 2023. doi:10.1007/978-3-662-67464-2.
10. Yann Bugeaud. Diophantine approximation and Cantor sets. *Mathematische Annalen*, 341:677–684, 2008. doi:10.1007/s00208-008-0209-4.

11 Ventsislav Chonev, Joël Ouaknine, and James Worrell. On the Zeros of Exponential Polynomials. *J. ACM*, 70(4):26:1–26:26, 2023. doi:10.1145/3603543.

12 George E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decompostion. In H. Brakhage, editor, *Automata Theory and Formal Languages*, pages 134–183, Berlin, Heidelberg, 1975. Springer Berlin Heidelberg. doi:10.1007/3-540-07407-4_17.

13 Richard M. Friedberg. Two recursively enumerable sets of incomparable degrees of unsolvability (solution of Post's problem, 1944). *Proc. Nat. Acad. Sci. U.S.A.*, 43(2):236–238, 1957. doi:10.1073/pnas.43.2.236.

14 Bernard Hanzon and Finbarr Holland. Non-negativity Analysis for Exponential-Polynomial-Trigonometric Functions on $[0, \infty)$. In Wolfgang Arendt, Joseph A. Ball, Jussi Behrndt, Karl-Heinz Förster, Volker Mehrmann, and Carsten Trunk, editors, *Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations*, pages 399–412, Basel, 2012. Springer Basel. doi:10.1007/978-3-0348-0297-0_21.

15 A. H. Lachlan and E. W. Madison. Computable Fields and Arithmetically Definable Ordered Fields. *Proceedings of the American Mathematical Society*, 24(4):803–807, 1970. doi:10.2307/2037328.

16 E. W. Madison. A Note on Computable Real Fields. *The Journal of Symbolic Logic*, 35(2):239–241, 1970. doi:10.2307/2270515.

17 Albert Muchnik. On the unsolvability of the problem of reducibility in the theory of algorithms. *Dokl. Akad. Nauk SSSR (N.S.)*, 108:194–197, 1956.

18 André Nies. *Computability and Randomness*. Oxford University Press, 2009.

19 Joël Ouaknine and James Worrell. On the positivity problem for simple linear recurrence sequences,. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 318–329. Springer, 2014. doi:10.1007/978-3-662-43951-7_27.

20 Joël Ouaknine and James Worrell. Positivity problems for low-order linear recurrence sequences. In Chandra Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 366–379. SIAM, 2014. doi:10.1137/1.9781611973402.27.

21 Joël Ouaknine and James Worrell. Ultimate positivity is decidable for simple linear recurrence sequences. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 330–341. Springer, 2014. doi:10.1007/978-3-662-43951-7_28.

22 Joël Ouaknine and James Worrell. On linear recurrence sequences and loop termination. *ACM SIGLOG News*, 2(2):4–13, 2015. doi:10.1145/2766189.2766191.

23 Michael O. Rabin. Computable Algebra, General Theory and Theory of Computable Fields. *Transactions of the American Mathematical Society*, 95(2):341–360, 1960. doi:10.2307/1993295.

24 Michael Ian Shamos. *Computational Geometry*. PhD thesis, Yale University, 1978.

25 Ernst Specker. The fundamental theorem of algebra in recursive analysis. In B. Dejon and P. Henrici, editors, *Constructive Aspects of the Fundamental Theorem of Algebra*, pages 321–329. Wiley-Interscience, London, 1969.

26 R. Tijdeman, M. Mignotte, and T.N. Shorey. The distance between terms of an algebraic recurrence sequence. *Journal für die reine und angewandte Mathematik*, 349:63–76, 1984. URL: http://eudml.org/doc/152622.

27 N.K. Vereshchagin. Occurrence of zero in a linear recursive sequence. *Mat. Zametki*, 38(2):177–189, 1985.

28 Doron Zeilberger and Wadim Zudilin. The irrationality measure of π is at most 7.103205334137.... *Moscow Journal of Combinatorics and Number Theory*, 9(4), 2020. doi:10.2140/moscow.2020.9.407.