Quantitative Equational Rewriting

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Abstract

Rewriting logic is a logical framework for expressing both concurrent computation and logical deduction using equations and rewrite rules. Quantitative equational reasoning enriches equations with quantitative measures, expressing concepts such as similarity or proximity rather than mere equality of terms. In this article, we bring these two approaches together and propose a quantitative extension of rewriting logic as a flexible formalism for quantitative deduction and computation.

2012 ACM Subject Classification Theory of computation → Equational logic and rewriting

Keywords and phrases Quantitative rewriting, quantitative equational reasoning

Digital Object Identifier 10.4230/LIPIcs.MFCS.2026.76

Related Version *Extended Version*: `https://doi.org/10.35011/risc.26-09`

Funding Partially supported by the Austrian Science Fund (FWF) project P 35530; Rustaveli National Science Foundation of Georgia project FR-25-5957; grant CIPROM/2022/6 funded by Generalitat Valenciana; grant PID2024-162030OB-100 funded by MCIN/AEI/10.13039/501100011033 and ERDF A way of making Europe; The NATO Science for Peace and Security Programme project SymSafe (grant number G6133).

1 Introduction

In many modern applications, approximate and quantitative techniques play an increasingly important role. A quantitative notion of correctness is essential in settings where exact equality is either unattainable or too restrictive to be practically meaningful. In such contexts, approximate forms of equivalence have proved to be more adequate and robust.

Prominent examples arise in the analysis of probabilistic and nondeterministic systems, where classical notions of equivalence have been replaced by metric-based approximations that capture behavioural distance, see, e.g., [14, 31, 23, 16, 47, 25]. These kinds of quantitative notions enable a refined understanding of system behaviour, supporting reasoning about correctness in terms of degrees rather than exact notions as, e.g., in [15, 28, 24]. They also facilitate the integration of quantitative and symbolic techniques, an approach increasingly motivated by developments in neuro-symbolic methods [49, 48].

Recent theoretical advances have led to significant progress in this direction. Methods for quantitative synthesis, modelling, and verification have been developed, and foundational frameworks such as quantitative automata, quantitative algebras, quantitative logics, and quantitative rewriting systems have been systematically investigated, see, e.g., [17, 12, 18, 11, 36, 37, 43, 6, 4, 5, 26, 27, 1, 2].

Modelling and reasoning about dynamic systems requires a framework in which both computation and deduction can be treated in a uniform and mathematically precise way. In the classical setting, rewriting logic [39, 10, 40] is one of such prominent frameworks, where system states are represented as algebraic terms, structural properties of states are described by equations, and state transitions are specified by rewrite rules. A rewriting step could be understood as a concurrent computational transformation or as an inference step in a logical system. Due to its expressive power and logical foundation, rewriting logic (and its executable realisation, the Maude system [13, 21]) has been successfully applied in a wide range of areas, including the formal semantics of programming languages, the specification and verification of concurrent and distributed systems, security protocol analysis and model checking [13, 40, 44].

In this paper, we propose a quantitative extension of rewriting logic aimed at making its expressive and deductive power available for modelling systems that exhibit probabilistic, resource-sensitive, approximate, or otherwise graded behaviour. The resulting framework offers a unified perspective on quantitative deduction and computation. Quantitative information is represented at an abstract level by means of quantales, providing a uniform algebraic structure for combining and comparing degrees. Both equations and rewrite rules are equipped with quantitative information, and deduction as well as computation are carried out relative to this quantitative structure. The main features of the obtained quantitative rewriting logic framework can be summarised as follows:

- *Abstract quantitative foundation.* Following [27], quantitative information is expressed at a high level of abstraction using Lawverean quantales. This accommodates a wide range of special cases, including the standard exact (or *crisp*) setting (the Boolean quantale), fuzzy similarity frameworks (e.g., fuzzy quantales with minimum or product t-norms), structures equipped with abstract notions of distance such as Euclidean, Hamming, or Levenshtein distance (the Lawvere quantale), probabilistic metric spaces, etc.
- *Quantitative rewriting modulo.* We introduce a notion of rewriting modulo quantitative relations, denoted by $\to_{R^\epsilon/E^\epsilon}$, allowing both the rewrite relation R^ϵ and the background equational theory E^ϵ to be quantitative (with exact relations as a special case). Depending on whether R^ϵ and/or E^ϵ is empty, exact, or quantitative, one obtains nine distinct relations as special instances of $\to_{R^\epsilon/E^\epsilon}$. These include some known ones such as syntactic equality, standard equality modulo an equational theory, ordinary rewriting, rewriting modulo, quantitative rewriting, as well as new combinations that, to our knowledge, have not been studied previously.
- *Deduction and operational correspondence.* We investigate deduction in (unsorted, unconditional) quantitative rewriting logic, relating it on the one hand to (the reflexive-transitive closure of) $\to_{R^\epsilon/E^\epsilon}$, and on the other hand to its operational counterpart $\to_{R^\epsilon,E^\epsilon}$ (under certain conditions), which performs rewriting via matching modulo E^ϵ. This extends in a natural way the well-known correspondence between proof-theoretic and operational semantics of rewriting logic from the classical setting to the quantitative one.

Below we provide two examples in which quantitative rewrite rules and quantitative equations arise naturally.

▶ **Example 1.** A distance-bounding protocol allows the verification of both the identity and the physical proximity of a device (the *prover*) to a secure system (the *verifier*). Consider the distance-bounding protocol proposed by Meadows et al. [38], where P is the prover identifier, V is the verifier identifier, N_V is a nonce from the verifier, N_P is a nonce from the prover, $||$ is message concatenation, and $\oplus$ is exclusive-or. By measuring the round-trip time of messages N_V and $N_V \oplus P||N_P$, the verifier establishes an upper bound on distance, effectively preventing attacks. This protocol could be encoded using multiset rewriting and quantitative information using some novel Maude notation, where tVP is the time used to travel the distance between the verifier and the prover and $tXOR$ is the time used for performing exclusive-or.

```
rl mt => n(v,r1) .
rl NV =[tVP]=> NV, (NV xor n(p,r2)) .
rl NV, XOR =[tVP]=> NV, XOR, (XOR xor NV) .
rl NV, XOR, NP => success .
eq X xor X = null [tXOR] .
```

▶ **Example 2.** The *sprinkler* example is a classic Bayesian network involving variables like Cloudy, Rain, Sprinkler, and Wet Grass.

Sprinkler (S)	Rain (R)	Wet	Dry
T	T	0.99	0.01
T	F	0.90	0.10
F	T	0.40	0.60
F	F	0.00	1.00

This network could be encoded using multiset rewriting and quantitative information using Maude notation.

```
rl rainT, sprinklerT =[0.01]=> dry .
rl rainT, sprinklerT =[0.99]=> wet .
rl rainF, sprinklerT =[0.4]=> wet .
rl rainF, sprinklerT =[0.6]=> dry .
rl rainT, sprinklerF =[0.9]=> wet .
rl rainT, sprinklerF =[0.1]=> dry .
rl rainF, sprinklerF =[0.0]=> wet .
rl rainF, sprinklerF =[1.0]=> dry .
```

But we could consider another variable `cloudy` that is similar to `rain` but up to some level.

```
eq cloudyT = rainT [0.8] .
eq cloudyT = rainF [0.5] .
eq cloudyF = rainT [0.2] .
eq cloudyF = rainF [0.9] .
```

The paper is organised as follows: We introduce the main notions in Sect. 2 and their quantitative counterparts in Sect. 3. In Sect. 4, the quantitative equational rewriting relation $\to_{R^\epsilon/E^\epsilon}$ is defined. In Sect. 5, a quantitative variant of rewriting logic (QRL) is introduced and its provability relation $\longrightarrow\!\!\!\!\blacktriangleright_{R^\epsilon/E^\epsilon}$ is characterised in terms of $\to_{R^\epsilon/E^\epsilon}$. In Sect. 6, we consider computational aspects of QRL, introduce an operational counterpart $\to_{R^\epsilon,E^\epsilon}$ of $\to_{R^\epsilon/E^\epsilon}$, and use its strictly coherent version to characterise provability in QRL. Sect. 7 contains examples that illustrate these relations. Related work and concluding remarks can be found in Sect. 8 and 9, respectively. An extended version of this paper with full proofs is available as a technical report [19].

2 Preliminaries

We assume the reader is familiar with basic notions of term rewriting. For more details we refer to [3].

Given a signature $\mathcal{F}$, i.e., a set of function symbols with fixed arities, and a set $\mathcal{V}$ of variables, we denote by $\mathcal{T}(\mathcal{F},\mathcal{V})$ the set of terms over $\mathcal{F}$ and $\mathcal{V}$. The set of variables occurring in a term t is denoted by $\mathcal{V}(t)$.

A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ which maps all except finitely many elements to themselves. The application of the substitution σ to term t is written $t\sigma$.

The set of positions in a term t is denoted by $Pos(t)$. For a position $p \in Pos(t)$, we write $t|_p$ for the subterm of t at the position p, and $t[s]_p$ for the term obtained from t by replacing the subterm at the position p by the term s.

An $\mathcal{F}$-equation (or just equation) is an unoriented pair $t = t'$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. For a given set E of $\mathcal{F}$-equations, equational logic induces a congruence relation $=_E$ on terms [30]. The E-equivalence class of a term t is denoted by $[t]_E$. An equational theory $(\mathcal{F}, E)$ is a pair with $\mathcal{F}$ a signature and E a set of $\mathcal{F}$-equations.

A *rewrite rule* is an oriented pair $l \to r$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$, where l is not a variable and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. The relation $\to_R$ generated by a set R of rewrite rules is defined as $t \to_R t'$ iff there exists a non-variable position p in t, a rule $l \to r$ in R, and a substitution σ such that $t|_p = l\sigma$ and $t' = t[r\sigma]_p$. The transitive (resp. transitive and reflexive) closure of $\to_R$ is denoted by $\to_R^+$ (resp. $\to_R^*$).

A rewrite theory $\mathcal{R} = (\mathcal{F}, E, R)$ in rewriting logic consists of an equational theory $(\mathcal{F}, E)$ and a set of (possibly conditional) rewrite rules R, which define transition rules over equivalences classes $[t]_E$. The rewriting relation generated by the rewrite rules R *modulo* an equational theory E, called *equational rewriting* relation and written $\to_{R/E}$, is defined as the composition $(=_E; \to_R; =_E)$. Figure 1 provides an alternative definition[1] of the transitive and reflexive closure of $\to_{R/E}$, denoted $\twoheadrightarrow_{R/E}$, which is useful to reason about the equational rewriting relation. Although $\to_{R/E}$ is in general undecidable, *equational rewriting* techniques going back to [46] and including, e.g., [32, 9, 29, 41, 42], provide conditions on R and E that make $\to_{R/E}$ decidable. A key requirement is to have an *E-matching algorithm*: For executability purposes, the relation $\to_{R,E}$ is defined as $t \to_{R,E} t'$ iff there exist a non-variable position p in t, a rule $l \to r$ in R, and a substitution σ such that $t|_p =_E l\sigma$ and $t' = t[r\sigma]_p$. It is a weaker relation, but under certain conditions it can be shown to be equivalent to $\to_{R/E}$. We say that the rewrite relation $\to_{R,E}$ is *strictly* E-coherent iff $\forall u, v, u' : u \to_{R,E} v \wedge u =_E u' \implies \exists v' : u' \to_{R,E} v' \wedge v' =_E v$. If $\to_{R,E}$ is E-coherent, then we have $u \to_{R/E} v \iff \exists w : u \to_{R,E} w \wedge w =_E v$ [41].

3 Quantitative relations

Before providing our quantitative equational rewriting relation $\to_{R^\epsilon/E^\epsilon}$ in Definition 8 below, we first introduce basic notions for quantitative equational theories and quantitative rewriting, following [26, 27].

A (unital) *quantale* $\Omega = (\Omega, \precsim, \otimes, \kappa)$ consists of a monoid $(\Omega, \kappa, \otimes)$ and a complete lattice $(\Omega, \precsim)$ (with join $\vee$ and meet $\wedge$) satisfying the following distributivity laws: $\delta \otimes \left(\bigvee_{i\in I} \varepsilon_i\right) = \bigvee_{i\in I}(\delta \otimes \varepsilon_i)$ and $\left(\bigvee_{i\in I} \varepsilon_i\right) \otimes \delta = \bigvee_{i\in I}(\varepsilon_i \otimes \delta)$. We use Greek letters to denote elements of Ω.

[1] In [40], the replacement inference rule allows rewrites in the matching substitution, which is called "parallelism under one's feet", but it is not considered in this paper.

$$\text{(ReflRL)}\ \frac{}{t \rightarrow_{R/E} t} \qquad \text{(EqRL)}\ \frac{t =_E t' \quad t \rightarrow_{R/E} s \quad s =_E s'}{t' \rightarrow_{R/E} s'} \qquad \text{(ReplRL)}\ \frac{\ell : t \to s \in R}{t\sigma \rightarrow_{R/E} s\sigma}$$

$$\text{(TransRL)}\ \frac{t \rightarrow_{R/E} s \quad s \rightarrow_{R/E} r}{t \rightarrow_{R/E} r} \qquad \text{(CongRL)}\ \frac{t_1 \rightarrow_{R/E} s_1 \quad \cdots \quad t_n \rightarrow_{R/E} s_n}{f(t_1, \ldots, t_n) \rightarrow_{R/E} f(s_1, \ldots, s_n)}$$

Figure 1 Deduction rules for rewriting logic [40].

Table 1 Quantales.

	Generic Ω	Boolean $\mathbb{2}$	Lawvere $\mathbb{L}$	Strong Lawvere $\mathbb{L}^{\max}$	Fuzzy $\mathbb{I}$
Carrier	Ω	$\{0,1\}$	$[0,\infty]$	$[0,\infty]$	$[0,1]$
Order	$\precsim$	$\leqslant$	$\geqslant$	$\geqslant$	$\leqslant$
Join	$\bigvee$	$\exists$	inf	inf	sup
Meet	$\bigwedge$	$\forall$	sup	sup	inf
Tensor	$\otimes$	$\wedge$	$+$	max	left-continuous T-norm
Unit	κ	1	0	0	1

For $n \in \mathbb{N}$ and $\varepsilon \in \Omega$, we denote $\bigotimes_{i=0}^{n} \varepsilon$ by ε^n. When $n = 0$, we have $\varepsilon^n = \kappa$.

The element κ is called the *unit* of the quantale, and $\otimes$ is called its *tensor* (or multiplication). The *top* and *bottom* elements of a quantale are denoted by $\top$ and $\bot$, respectively. Table 1 recalls some well-known quantales.

Quantales in which the unit κ coincides with $\top$ are called *integral quantales*. A quantale is *commutative* if its underlying monoid is. It is *non-trivial* if $\kappa \neq \bot$. It is *cointegral* if $\varepsilon \otimes \delta = \bot$ implies either $\varepsilon = \bot$ or $\delta = \bot$.

We assume our quantales are commutative, integral, cointegral, and nontrivial. Such quantales are called *Lawverean*. Note that the fuzzy quantale $\mathbb{I}$ is Lawverean for the Gödel and product T-norms, but not for the Łukasiewicz T-norm.

▶ **Definition 3** (Quantitative relation [26, 27]). *Let Ω be a Lawverean quantale, and let A be a set. A* quantitative relation[2] *on A over Ω is a ternary relation $R \subseteq A \times \Omega \times A$, which is closed under quantitative weakening, i.e., $(a, \varepsilon, c) \in R$ implies $(a, \zeta, c) \in R$ for every $\zeta \precsim \varepsilon$. The composition $R; S$ of two such quantitative relations R and S is the quantitative relation defined by $(R; S)(a, \varepsilon, b)$ iff there exist $\zeta, \eta \in \Omega$ and $c \in A$ such that $(a, \zeta, c) \in R$, $(c, \eta, b) \in S$, and $\varepsilon \precsim \zeta \otimes \eta$. Powers of R are defined inductively by $R^0 := \{(a, \kappa, a) \mid a \in A\}$ and $R^{n+1} := R^n; R$. The* transitive closure *of R is defined by $R^+ := \bigcup_{n>0} R^n$, and its* reflexive transitive closure *by $R^* := \bigcup_{n\geqslant 0} R^n$. The* transpose *of R is defined by $R^- := \{(t, \varepsilon, s) \mid (s, \varepsilon, t) \in R\}$, and the* symmetric closure *of R by $R \cup R^-$.*

We say that $a \in A$ is in normal form *w.r.t. a quantitative relation R, iff $\bigvee\{\varepsilon \mid (a, \varepsilon, b) \in R$ for some $b \in A\} = \bot$.*

Note that the composition of quantitative relations is associative and monotonous with respect to $\subseteq$ on A. In the rest of the paper, a signature $\mathcal{F}$, a set of variables $\mathcal{V}$, and a Lawverean quantale Ω are assumed to be fixed. For brevity, below Ω is omitted when it is clear from the context.

[2] In [27], the term "Ω-ternary relation" is used.

Let E^ϵ be a set of triples (t, ε, s), where $\varepsilon \in \Omega$ and t and s are terms. We write them as $\varepsilon \Vdash t \approx_{E^\epsilon} s$ and call them Ω-*quantitative equations*, or just quantitative equations for short. Informally, we read $\varepsilon \Vdash t =_{E^\epsilon} s$ as "t and s are at most ε-apart modulo E^ϵ" or "t and s are equal modulo E^ϵ with degree ε". The Ω-*quantitative equational theory* (abbreviated as QET) generated by E^ϵ, denoted by $=_{E^\epsilon}$, is the relation generated from E^ϵ by the rules in Fig. 2, introduced in [26].[3] We call E^ϵ a *presentation* of $=_{E^\epsilon}$. When there is no confusion, we use E^ϵ to refer to $=_{E^\epsilon}$.

$$(\mathsf{Ax}^\epsilon_=)\ \frac{\varepsilon \Vdash t \approx s \in E^\epsilon}{\varepsilon \Vdash t =_{E^\epsilon} s} \qquad (\mathsf{Refl}^\epsilon_=)\ \frac{}{\kappa \Vdash t =_{E^\epsilon} t} \qquad (\mathsf{Symm}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s}{\varepsilon \Vdash s =_{E^\epsilon} t}$$

$$(\mathsf{Trans}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} \qquad \delta \Vdash s =_{E^\epsilon} r}{\varepsilon \otimes \delta \Vdash t =_{E^\epsilon} r} \qquad (\mathsf{Subst}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s}{\varepsilon \Vdash t\sigma =_{E^\epsilon} s\sigma}$$

$$(\mathsf{NExp}^\epsilon_=)\ \frac{\varepsilon_1 \Vdash t_1 =_{E^\epsilon} s_1 \quad \cdots \quad \varepsilon_n \Vdash t_n =_{E^\epsilon} s_n}{\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \Vdash f(t_1, \ldots, t_n) =_{E^\epsilon} f(s_1, \ldots, s_n)}$$

$$(\mathsf{Ord}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s \quad \varepsilon \succsim \delta}{\delta \Vdash t =_{E^\epsilon} s} \qquad (\mathsf{Join}^\epsilon_=)\ \frac{\varepsilon_1 \Vdash t =_{E^\epsilon} s \quad \cdots \quad \varepsilon_n \Vdash t =_{E^\epsilon} s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t =_{E^\epsilon} s}$$

Figure 2 Quantitative equational theory [26].

The $(\mathsf{Join}^\epsilon_=)$ rule also applies to an empty hypothesis, hence $\bot \Vdash t =_{E^\epsilon} s$ holds for any t and s.[4]

An Ω-*quantitative rewrite rule* is an oriented Ω-quantitative equation, written $\varepsilon \Vdash t \mapsto s$. The Ω-*quantitative rewrite relation* $\to_{R^\epsilon}$ generated by a set R^ϵ of Ω-quantitative rewrite rules is defined in Fig. 3, following [27]. We denote the transpose of $\to_{R^\epsilon}$ by $\leftarrow_{R^\epsilon}$, and its symmetric closure by $\leftrightarrow_{R^\epsilon}$.

$$\frac{\varepsilon \Vdash t \mapsto_{R^\epsilon} s}{\varepsilon \Vdash C[t\sigma] \to_{R^\epsilon} C[s\sigma]} \qquad \frac{\varepsilon \Vdash t \to_{R^\epsilon} s \qquad \varepsilon \succsim \delta}{\delta \Vdash t \to_{R^\epsilon} s}$$

Figure 3 Quantitative rewrite relation [27].

Note that the context and substitution used to generate a rewrite step preserve the degree ε associated with the rewrite rule (see Fig. 3). Such relation is called *non-expansive* [27]. A more general relation, called graded rewriting in [27], allows the context and substitution to modify the degree.

▶ **Lemma 4.** *Let R^ϵ be a finite set of triples (l, δ, r), where l and r are terms and δ is an element of the Lawverean quantale Ω, and let $\varepsilon \succ \bot$ be some element of Ω. Then $\varepsilon \Vdash t =_{R^\epsilon} s$ iff there exist $\varepsilon_1, \ldots, \varepsilon_n \in \Omega$ such that $\bigvee_{i=1}^n \varepsilon_i = \varepsilon$ and $\varepsilon_i \Vdash t \leftrightarrow^*_{R^\epsilon} s$ for each i.*

In particular, over a totally ordered quantale, there is a direct correspondence between $=_{R^\epsilon}$ and $\leftrightarrow^*_{R^\epsilon}$.

[3] Note that $=_{E^\epsilon}$ is indeed a quantitative relation: rule $(\mathsf{Ord}^\epsilon_=)$ ensures closure under quantitative weakening.
[4] In [26] an infinitary Archimedean rule is included, which has no effect on $=_{E^\epsilon}$ whenever the presentation E^ϵ is finite. Therefore, we do not consider it here.

$$\frac{\varepsilon_1 \Vdash t =_{E^\epsilon} t' \qquad \delta \Vdash t' \to_{R^\epsilon} s' \qquad \varepsilon_2 \Vdash s' =_{E^\epsilon} s}{\varepsilon_1 \otimes \delta \otimes \varepsilon_2 \Vdash t \to_{R^\epsilon/E^\epsilon} s}$$

$$\frac{\varepsilon \Vdash t \to_{R^\epsilon/E^\epsilon} s \qquad \varepsilon \succsim \delta}{\delta \Vdash t \to_{R^\epsilon/E^\epsilon} s} \qquad \frac{\varepsilon_1 \Vdash t \to_{R^\epsilon/E^\epsilon} s \quad \cdots \quad \varepsilon_n \Vdash t \to_{R^\epsilon/E^\epsilon} s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t \to_{R^\epsilon/E^\epsilon} s}$$

Figure 4 Quantitative equational rewrite relation $\to_{R^\epsilon/E^\epsilon}$.

▶ **Corollary 5.** *Let R^ϵ be a finite set of triples (l, δ, r), where l and r are terms and δ is an element of the Lawverean quantale Ω whose order $\precsim$ is total, and let $\varepsilon \succ \perp$ be some element of Ω. Then $\varepsilon \Vdash t =_{R^\epsilon} s$ if and only if $\varepsilon \Vdash t \leftrightarrow^*_{R^\epsilon} s$.*

The notion of distance between terms can be seen as a particular case of quantitative equational theories.

▶ **Definition 6** (Ω-metric space [33]). *Let Ω be a quantale, X a set. An* Ω-metric *over X is a map $d\colon X \times X \to \Omega$ satisfying* (i) $d(x, x) = \kappa$ *(reflexivity) and* (ii) $d(x, y) \succsim d(x, z) \otimes d(z, y)$ *(triangle inequality) for all elements $x, y, z \in X$.*

The pair (X, d) is called an Ω-metric space. *An Ω-metric d is* symmetric *if it also satisfies $d(x, y) = d(y, x)$ for all $x, y \in X$.*

In particular, a symmetric $\mathbb{L}$-metric d is just an extended pseudo-metric – that is, d is a metric except that it can take the value ∞ and $d(x, y) = 0$ does not necessarily imply $x = y$. Thus, Ω-metric spaces can be viewed as generalised metric spaces. Lemma 7 shows that a distance can be defined as a QET.

▶ **Lemma 7.** *Let Ω be a Lawverean quantale, and let d be a symmetric Ω-metric over $T(\mathcal{F}, \mathcal{V})$. Assume that d is closed under contexts and substitutions, i.e. $d(C[t\sigma], C[s\sigma]) \succsim d(t, s)$ holds for all terms t, s, contexts C and substitutions σ. Then d can be viewed as a QET, i.e., there exists some QET E_d such that $d(t, s) = \bigvee\{\varepsilon \mid \varepsilon \Vdash t =_{E_d} s\}$ for all terms t and s.*

Similarly, any quantitative equational theory E induces an Ω-metric over the term algebra: $d_E(t, s) = \bigvee\{\varepsilon \mid \varepsilon \Vdash t =_E s\}$ for all terms t and s.

4 Quantitative rewriting modulo quantitative equations

In this section we define a notion of quantitative equational rewriting that generalises the relation $\to_{R/E}$ to take into account that both R and E could be quantitative. This relation is denoted by $\to_{R^\epsilon/E^\epsilon}$.

▶ **Definition 8** (Quantitative equational rewriting $\to_{R^\epsilon/E^\epsilon}$). *Given a set R^ϵ of Ω-quantitative rewrite rules and a (possibly empty) set E^ϵ of Ω-quantitative equations, the relation $\to_{R^\epsilon/E^\epsilon}$ is defined by rules in Fig. 4.*

The general definition above can be restricted along different dimensions to obtain some existing and some new notions of quantitative rewriting and quantitative equational reasoning. Below we provide precise definitions of the notions of trivial/crisp R^ϵ and E^ϵ using quantitative ternary relations, which will be needed to formalise the correspondence between $\to_{R^\epsilon/E^\epsilon}$ and previous notions of rewriting and equational reasoning. Crisp relations are sometimes called *exact*.

▶ **Definition 9** (Trivial/crisp relation). *A quantitative equational theory E^ϵ is called* trivial *if $=_{E^\epsilon}$ coincides with $=_\emptyset$ (the theory induced by an empty set of quantitative equations), and* crisp *if $=_{E^\epsilon}$ coincides with a theory induced by a set of quantitative equations with degree κ. Similarly, a quantitative term rewrite relation R^ϵ is trivial if $\to_{R^\epsilon} = \to_\emptyset$, and crisp if it coincides with a rewrite relation induced by a set of quantitative rewrite rules with degree κ.*

▶ Remark 10. Let E^ϵ be a crisp equational theory. If $\varepsilon \Vdash t =_{E^\epsilon} s$ for some $\varepsilon \succ \bot$ then $\kappa \Vdash t =_{E^\epsilon} s$. The other direction holds trivially for all ε since quantitative relations are closed under weakening. Let E be the non-quantitative theory obtained as the image of E^ϵ under the projection $\pi\colon (t, \kappa, s) \mapsto (t, s)$. Then $\varepsilon \Vdash t =_{E^\epsilon} s$ for $\varepsilon \succ \bot$ iff $t =_E s$.

Similarly, a crisp rewrite relation R^ϵ satisfies $\varepsilon \Vdash t \to_{R^\epsilon} s$ for $\varepsilon \succ \bot$ iff $\kappa \Vdash t \to_{R^\epsilon} s$ iff $t \to_{\pi(R^\epsilon)} s$.

Thus, crisp quantitative rewrite relations (resp. equational theories) and non-quantitative (i.e., standard) rewrite relations (resp. equational theories) can be considered the same. When we want to emphasise that we are talking about crisp R^ϵ and E^ϵ, we will use the notation R and E, respectively.

Table 2 describes the various relations obtained as particular cases of $\to_{R^\epsilon/E^\epsilon}$. References to well-known standard notions are omitted.

Table 2 Quantitative Equational Rewrite Relations.

Relation	**Empty R**	**Crisp R**	**Arbitrary R^ϵ**
Empty E	Syntactic equality $=$	Standard rewriting $\to_R$	Quantitative rewriting $\to_{R^\epsilon}$ [27, 35]
Crisp E	Equality modulo standard theory $=_E$	Standard rewriting modulo standard theory $\to_{R/E}$	Quantitative rewriting modulo standard theory $\to_{R^\epsilon/E}$
Arbitrary E^ϵ	Equality modulo quantitative theory $=_{E^\epsilon}$	Standard rewriting modulo quantitative theory $\to_{R/E^\epsilon}$ [20]	Quantitative rewriting modulo quantitative theory $\to_{R^\epsilon/E^\epsilon}$

5 Quantitative Rewriting Logic (QRL)

Below we introduce a quantitative counterpart of the version of rewriting logic in Fig. 1.

▶ **Definition 11** (Quantitative rewrite theory). *A* quantitative rewrite theory *$\mathcal{R}$ over $\mathcal{F}$, $\mathcal{V}$, and Ω, Ω-QRT for short, is a pair (R^ϵ, E^ϵ), where*

- *R^ϵ is a finite set of labelled Ω-quantitative rewrite rules $\ell : t \to_\rho s$, where ℓ is a label, $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\mathcal{V}(s) \subseteq \mathcal{V}(t)$, and $\rho \in \Omega$ (called the* weight *of the rule),*
- *E^ϵ is a finite set of Ω-quantitative equations over terms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$.*

Given $\mathcal{R} = (R^\epsilon, E^\epsilon)$, the sentences that $\mathcal{R}$ proves are triples of the form $(t, \varepsilon, s) \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \Omega \times \mathcal{T}(\mathcal{F}, \mathcal{V})$, written as $\varepsilon \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$, which are obtained by finite application of the QRL deduction rules in Fig. 5.

These rules extend those of rewriting logic in Fig. 1 with quantitative information and include new rules $(\mathsf{Ord}^\epsilon_{\mathsf{RL}})$ and $(\mathsf{Join}^\epsilon_{\mathsf{RL}})$, which are the counterpart of the rules $(\mathsf{Ord}^\epsilon_{=})$ and $(\mathsf{Join}^\epsilon_{=})$ in Figure 2. When $\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$, we say that δ is the *degree* of reduction.

$$(\mathsf{Refl}^{\epsilon}_{\mathsf{RL}})\ \frac{}{\kappa \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} t} \qquad (\mathsf{Eq}^{\epsilon}_{\mathsf{RL}})\ \frac{\varepsilon_1 \Vdash t =_{E^\epsilon} t' \quad \delta \Vdash t' \twoheadrightarrow_{R^\epsilon/E^\epsilon} s' \quad \varepsilon_2 \Vdash s' =_{E^\epsilon} s}{\varepsilon_1 \otimes \delta \otimes \varepsilon_2 \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} s}$$

$$(\mathsf{NExp}^{\epsilon}_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t_1 \twoheadrightarrow_{R^\epsilon/E^\epsilon} s_1 \quad \cdots \quad \delta_n \Vdash t_n \twoheadrightarrow_{R^\epsilon/E^\epsilon} s_n}{\delta_1 \otimes \cdots \otimes \delta_n \Vdash f(t_1,\ldots,t_n) \twoheadrightarrow_{R^\epsilon/E^\epsilon} f(s_1,\ldots,s_n)}$$

$$(\mathsf{Repl}^{\epsilon}_{\mathsf{RL}})\ \frac{\ell : t \to_\rho s \in R}{\rho \Vdash t\sigma \twoheadrightarrow_{R^\epsilon/E^\epsilon} s\sigma} \qquad (\mathsf{Trans}^{\epsilon}_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} s \quad \delta_2 \Vdash s \twoheadrightarrow_{R^\epsilon/E^\epsilon} r}{\delta_1 \otimes \delta_2 \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} r}$$

$$(\mathsf{Ord}^{\epsilon}_{\mathsf{RL}})\ \frac{\varepsilon \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} s \quad \varepsilon \succsim \delta}{\delta \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} s} \qquad (\mathsf{Join}^{\epsilon}_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} s \ \cdots\ \delta_n \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} s}{\delta_1 \vee \cdots \vee \delta_n \Vdash t \twoheadrightarrow_{R^\epsilon/E^\epsilon} s}$$

Figure 5 Deduction rules for Quantitative Rewriting Logic.

The resulting framework is general and flexible. The standard rewriting logic is recovered by taking crisp rules and equations, which would correspond to $\twoheadrightarrow_{R/E}$ (see Remark 10). A generic weighted variant of rewriting logic is obtained by taking equations crisp but keeping nontrivial rule weights (i.e, $\twoheadrightarrow_{R^\epsilon/E}$). Varying the underlying quantale structure introduces an additional dimension of generality, enabling the framework to capture a wide range of rewriting-based systems.

For a quantitative relation R, we denote by $R^{\mathbf{J}}$ the closure of R by *finite* joins. The theorem below characterises $\twoheadrightarrow_{R^\epsilon/E^\epsilon}$ in terms of $\to_{R^\epsilon/E^\epsilon}$.

▶ **Theorem 12.** *The following relations hold:* $\to^*_{R^\epsilon/E^\epsilon} \subseteq \twoheadrightarrow_{R^\epsilon/E^\epsilon} \subseteq \left(\to^*_{R^\epsilon/E^\epsilon} \cup =_{E^\epsilon}\right)^{\mathbf{J}}$.

To avoid confusion, below we use $\equiv$ for the meta-equality between relations.

▶ **Theorem 13.** $\twoheadrightarrow_{R^\epsilon/E^\epsilon} \equiv \left(\to^*_{R^\epsilon/E^\epsilon} \cup =_{E^\epsilon}\right)^{\mathbf{J}} \equiv \left(\to^*_{R^\epsilon/E^\epsilon} ; =_{E^\epsilon}\right)^{\mathbf{J}}$.

Note that if $\precsim$ is total (like e.g., in Boolean, Fuzzy, Lawvere, Strong Lawvere quantales), then $\left(\to^*_{R^\epsilon/E^\epsilon} \cup =_{E^\epsilon}\right)^{\mathbf{J}} \equiv \left(\to^*_{R^\epsilon/E^\epsilon} \cup =_{E^\epsilon}\right)$ and $\left(\to^*_{R^\epsilon/E^\epsilon} ; =_{E^\epsilon}\right)^{\mathbf{J}} \equiv \left(\to^*_{R^\epsilon/E^\epsilon} ; =_{E^\epsilon}\right)$.

6 Computing with Quantitative Rewriting Logic

The relation $\twoheadrightarrow_{R^\epsilon/E^\epsilon}$ captures the logical, deductive aspect of QRL. From a computational perspective, however, it is impractical, just as its standard counterpart $\twoheadrightarrow_{R/E}$ is. For this reason, in the classical setting, the operationally better-behaved relation $\to_{R,E}$ was introduced as an executable counterpart of $\to_{R/E}$. Under some conditions, the reflexive-transitive closure of $\to_{R,E}$ can be related to $\twoheadrightarrow_{R/E}$. (See the notion of strict E-coherence in Section 2.) Following the same idea, we introduce a quantitative relation $\to_{R^\epsilon,E^\epsilon}$ and investigate its relationship with $\twoheadrightarrow_{R^\epsilon/E^\epsilon}$.

▶ **Definition 14** (Quantitative equational rewriting $\to_{R^\epsilon,E^\epsilon}$). *Given an Ω-QRT (R^ϵ, E^ϵ), the relation $\to_{R^\epsilon,E^\epsilon}$ is the quantitative relation defined by the inference rules in Fig. 6, where C is an arbitrary context. We call the leftmost rule* main rule*, and the rightmost one* weakening *rule.*

$$\frac{\ell : t \to_{\rho} s \in R^{\epsilon} \qquad \varepsilon \Vdash t\sigma =_{E^{\epsilon}} r}{\rho \otimes \varepsilon \Vdash C[r] \to_{R^{\epsilon},E^{\epsilon}} C[s\sigma]} \qquad\qquad \frac{\varepsilon \Vdash t \to_{R^{\epsilon},E^{\epsilon}} s \quad \varepsilon \succsim \delta}{\delta \Vdash t \to_{R^{\epsilon},E^{\epsilon}} s}$$

Figure 6 Quantitative equational rewrite relation $\to_{R^{\epsilon},E^{\epsilon}}$.

▶ **Lemma 15.** *For all terms u, v and all $\alpha \in \Omega$, if $\alpha \Vdash u \to_{R^{\epsilon},E^{\epsilon}} v$, then $\alpha \Vdash u \to_{R^{\epsilon}/E^{\epsilon}} v$.*

The following theorem states that $\to^*_{R^{\epsilon},E^{\epsilon}}$ is included into $\relbar\!\!\!\blacktriangleright_{R^{\epsilon}/E^{\epsilon}}$:

▶ **Theorem 16** (Operational rewriting implies provability)**.** *Let $(R^{\epsilon}, E^{\epsilon})$ be an Ω-QRT. For all terms t, s and all $\delta \in \Omega$, if $\delta \Vdash t \to^*_{R^{\epsilon},E^{\epsilon}} s$, then $\delta \Vdash t \relbar\!\!\!\blacktriangleright_{R^{\epsilon}/E^{\epsilon}} s$.*

The converse of Lemma 15 does not hold, as shown by the following example (for simplicity, we omit labels in rules).

▶ **Example 17.** Consider the Lawvere quantale and a signature with a binary function symbol f and two constants a, b. Let $R^{\epsilon} = \{1 \Vdash f(a, b) \mapsto a\}$ and $E^{\epsilon} = \{2 \Vdash f(b, f(a, a)) \approx f(a, f(a, b))\}$. By Definition 8, we have $3 \Vdash f(b, f(a, a)) \to_{R^{\epsilon}/E^{\epsilon}} f(a, a)$. However, $f(b, f(a, a))$ is in normal form with respect to $\to_{R^{\epsilon},E^{\epsilon}}$ and, hence, $\varepsilon \Vdash f(b, f(a, a)) \to_{R^{\epsilon},E^{\epsilon}} f(a, a)$ does not hold for any $\varepsilon \in [0, \infty)$.

Extending R^{ϵ} with the rule $\varepsilon \Vdash f(b, f(a, a)) \mapsto f(a, a)$ for $\varepsilon \leqslant 3$ (i.e., $\varepsilon \succsim 3$) makes the step $3 \Vdash f(b, f(a, a)) \to_{R^{\epsilon},E^{\epsilon}} f(a, a)$ possible. Using $\varepsilon < 3$, however, would change the $\to_{R^{\epsilon}/E^{\epsilon}}$ relation; $\varepsilon = 3$ is the only value that retains $\to_{R^{\epsilon}/E^{\epsilon}}$ while also enabling the desired $\to_{R^{\epsilon},E^{\epsilon}}$ step.

To further relate the $\to_{R^{\epsilon}/E^{\epsilon}}$ and $\to_{R^{\epsilon},E^{\epsilon}}$ relations, we introduce a notion of quantitative strict coherence, which is a counterpart of strict coherence from [41]:

▶ **Definition 18** (Quantitative strict coherence)**.** *We say that the rewrite relation $\to_{R^{\epsilon},E^{\epsilon}}$ is* strictly E^{ϵ}-coherent *iff for all terms t, s, t' and for all $\varepsilon_1, \varepsilon_2 \in \Omega$, if $\varepsilon_1 \Vdash t \to_{R^{\epsilon},E^{\epsilon}} s$ and $\varepsilon_2 \Vdash t =_{E^{\epsilon}} t'$, then there exist a term s' and $\delta_1, \delta_2 \in \Omega$ such that $\delta_1 \Vdash t' \to_{R^{\epsilon},E^{\epsilon}} s'$, $\delta_2 \Vdash s' =_{E^{\epsilon}} s$, and $\varepsilon_1 \otimes \varepsilon_2 = \delta_1 \otimes \delta_2$. Graphically:*

$$\begin{tikzcd}
t \arrow[r, "\varepsilon_1", "\to_{R^{\epsilon},E^{\epsilon}}"'] \arrow[d, equal, "\varepsilon_2"', "E^{\epsilon}"] & s \arrow[d, dashed, equal, "\delta_2"', "E^{\epsilon}"] \\
t' \arrow[r, dashed, "\delta_1", "\to_{R^{\epsilon},E^{\epsilon}}"'] & s'
\end{tikzcd} \qquad \varepsilon_1 \otimes \varepsilon_2 = \delta_1 \otimes \delta_2$$

▶ **Theorem 19.** *If $\to_{R^{\epsilon},E^{\epsilon}}$ is strictly E^{ϵ}-coherent, then for all terms t and s and $\varepsilon \in \Omega$, if $\varepsilon \Vdash t \to_{R^{\epsilon}/E^{\epsilon}} s$, then there exists a term r and $\delta_1, \delta_2 \in \Omega$ such that $\delta_1 \Vdash t \to_{R^{\epsilon},E^{\epsilon}} r$, $\delta_2 \Vdash r =_{E^{\epsilon}} s$, and $\varepsilon = \delta_1 \otimes \delta_2$. Graphically:*

$$\begin{tikzcd}
t \arrow[r, "\varepsilon", "\to_{R^{\epsilon}/E^{\epsilon}}"'] \arrow[dr, dashed, "\delta_1", "\to_{R^{\epsilon},E^{\epsilon}}"'] & s \arrow[d, dashed, equal, "\delta_2"', "E^{\epsilon}"] \\
& r
\end{tikzcd} \qquad \varepsilon = \delta_1 \otimes \delta_2$$

▶ **Theorem 20.** *If $\to_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then $\to_{R^\epsilon/E^\epsilon} \equiv (\to_{R^\epsilon,E^\epsilon};=_{E^\epsilon})$.*

▶ **Example 21.** Let Ω be the Lawvere quantale, $E^\epsilon = \{1 \Vdash x+y \approx y+x,\ 2 \Vdash (x+y)+z \approx x+(y+z)\}$, and $R^\epsilon = \{3 \Vdash a+b \mapsto a\}$. Then we have

$$\begin{aligned} &4 \Vdash a+(a+b) \to_{R^\epsilon/E^\epsilon} a+a \quad \text{and} \quad 4 \Vdash a+(a+b) \to_{R^\epsilon,E^\epsilon} a+a,\\ &6 \Vdash b+(a+a) \to_{R^\epsilon/E^\epsilon} a+a \quad \text{but } b+(a+a) \text{ is in } \to_{R^\epsilon,E^\epsilon}\text{-normal form.}\end{aligned}$$

However, if we extend R^ϵ with the rule $\delta \Vdash (a+b)+X \mapsto a+X$ for an arbitrary $\delta \le 3$, then $\to_{R^\epsilon,E^\epsilon}$ will become E^ϵ-coherent and we get $3+\delta \Vdash b+(a+a) \to_{R^\epsilon,E^\epsilon} a+a$, which, by weakening, gives $6 \Vdash b+(a+a) \to_{R^\epsilon,E^\epsilon} a+a$.

Under the top-position restriction, the relation $\to_{R^\epsilon,E^\epsilon}$ becomes strictly E^ϵ-coherent. Therefore, from Theorem 20, we get the following result:

▶ **Theorem 22.** *If rewrite rules are permitted to apply only on the top position (i.e., the context C in Fig. 3 and Def. 14 is empty), then $\to_{R^\epsilon/E^\epsilon} \equiv (\to_{R^\epsilon,E^\epsilon};=_{E^\epsilon})$.*

This theorem is also interesting from the practical point of view because many concurrent systems are specified as topmost rewrite theories.

The next theorem is the main result in this section: it indicates that deduction in QRL can be modelled by quantitative rewriting via the $\to_{R^\epsilon,E^\epsilon}$ relation:

▶ **Theorem 23** (Modelling deduction via operational rewriting)**.** *If the relation $\to_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then $\twoheadrightarrow_{R^\epsilon/E^\epsilon} \equiv \left(\to^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}\right)^{\mathbf{J}}$.*

Also here, for total quantales we have $\left(\to^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}\right)^{\mathbf{J}} \equiv \left(\to^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}\right)$. The computational mechanism of $\to_{R^\epsilon,E^\epsilon}$ is matching modulo E^ϵ. A modification of the quantitative unification algorithm from [22] to matching problems can provide a complete solving method for a specific E^ϵ. Alternatively, matching modulo E^ϵ can be seen as a matching problem in the combination of the exact and inexact quantitative parts of E^ϵ. It has been investigated in [19].

7 Examples

Consider the Lawvere quantale $\mathbb{L}$ and quantitative rewrite theory (R^ϵ, E^ϵ), where

$$\begin{aligned} R^\epsilon &= \{\ell_1 : a \to_2 c,\ \ell_2 : f(h(x),h(b)) \to_2 g(x,x)\},\\ E^\epsilon &= \{0 \Vdash f(x,y) \approx f(y,x),\ 1 \Vdash a \approx b,\ 2 \Vdash h(x) \approx p(x)\}.\end{aligned}$$

The quantale is total. Hence, we have $\left(\to^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}\right)^{\mathbf{J}} \equiv \left(\to^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}\right)$. The relation $\to_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent. Below we show two examples that illustrate the relation stated in Theorem 23:

Example 24:

- a proof of $5 \Vdash f(p(b),h(a)) \twoheadrightarrow_{R^\epsilon/E^\epsilon} g(a,b)$ (i.e., $(f(p(b),h(a)),5,g(a,b)) \in \twoheadrightarrow_{R^\epsilon/E^\epsilon}$), and
- a proof of $(f(p(b),h(a)),5,g(a,b)) \in \to^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}$.

Example 25:

- a proof of $6 \Vdash f(p(b),h(a)) \twoheadrightarrow_{R^\epsilon/E^\epsilon} g(a,c)$ (i.e., $(f(p(b),h(a)),6,g(a,c)) \in \twoheadrightarrow_{R^\epsilon/E^\epsilon}$), and
- a proof of $(f(p(b),h(a)),6,g(a,c)) \in \to^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}$.

▶ **Example 24.**

Proving $5 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, b)$

$$\dfrac{eq_1 \qquad \dfrac{\ell_2 : f(h(x), h(b)) \to_2 g(x, x) \in R^\epsilon \quad \sigma = \{x \mapsto a\}}{2 \Vdash f(h(a), h(b)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)}\,(\mathsf{Repl}^\epsilon_{\mathsf{RL}}) \qquad eq_2}{5 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, b)}\,(\mathsf{Eq}^\epsilon_{\mathsf{RL}})$$

where $eq_1 = 2 \Vdash f(h(a), h(b)) =_{E^\epsilon} f(p(b), h(a))$ and $eq_2 = 1 \Vdash g(a, a) =_{E^\epsilon} g(a, b)$. They can be proved using rules of QET in Fig. 2 as follows:

Proof of eq_1**.**

$$\dfrac{\dfrac{\dfrac{0 \Vdash f(x, y) \approx f(y, x) \in E^\epsilon}{0 \Vdash f(x, y) =_{E^\epsilon} f(y, x)}\,(\mathsf{Ax}^\epsilon_{=})}{0 \Vdash f(h(a), h(b)) =_{E^\epsilon} f(h(b), h(a))}\,(\mathsf{Subst}^\epsilon_{=}) \qquad eq_{11}}{2 \Vdash f(h(a), h(b)) =_{E^\epsilon} f(p(b), h(a))}\,(\mathsf{Trans}^\epsilon_{=})$$

Proof of eq_{11}**.**

$$\dfrac{\dfrac{\dfrac{2 \Vdash h(x) \approx p(x) \in E^\epsilon}{2 \Vdash h(x) =_{E^\epsilon} p(x)}\,(\mathsf{Ax}^\epsilon_{=})}{2 \Vdash h(b) =_{E^\epsilon} p(b)}\,(\mathsf{Subst}^\epsilon_{=}) \qquad \dfrac{}{0 \Vdash h(a) =_{E^\epsilon} h(a)}\,(\mathsf{Refl}^\epsilon_{=})}{2 \Vdash f(h(b), h(a)) =_{E^\epsilon} f(p(b), h(a))}\,(\mathsf{NExp}^\epsilon_{=})$$

Proof of eq_2**.**

$$\dfrac{\dfrac{}{0 \Vdash a =_{E^\epsilon} a}\,(\mathsf{Refl}^\epsilon_{=}) \qquad \dfrac{1 \Vdash a \approx b \in E^\epsilon}{1 \Vdash a =_{E^\epsilon} b}\,(\mathsf{Ax}^\epsilon_{=})}{1 \Vdash g(a, a) =_{E^\epsilon} g(a, b)}\,(\mathsf{NExp}^\epsilon_{=})$$

Proving $(f(p(b), h(a)), 5, g(a, b)) \in \to^*_{R^\epsilon, E^\epsilon} ; =_{E^\epsilon}$

We use rules in Fig. 6. Applying

$\ell_2 : f(h(x), h(b)) \to_2 g(x, x)$ with $\sigma = \{x \mapsto a\}$

and

$2 \Vdash f(h(x), h(b))\sigma =_{E^\epsilon} f(p(b), h(a))$ (which has been proved above as eq_1)

gives

$4 \Vdash f(p(b), h(a)) \to_{R^\epsilon, E^\epsilon} g(a, a)$.

For $g(a, a)$, we have

$1 \Vdash g(a, a) =_{E^\epsilon} g(a, b)$,

which is also proved above (as eq_2). Then we get

$4 \Vdash f(p(b), h(a)) \to_{R^\epsilon, E^\epsilon} g(a, a)$ and $1 \Vdash g(a, a) =_{E^\epsilon} g(a, b)$,

which imply $(f(p(b), h(a)), 5, g(a, b)) \in \to^*_{R^\epsilon, E^\epsilon} ; =_{E^\epsilon}$.

▶ **Example 25.**

Proving $6 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)$

$$\frac{T_1 \qquad T_2}{6 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)}\ (\mathsf{Trans}^\epsilon_{\mathsf{RL}})$$

where T_1 denotes the proof tree for $4 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)$:

$$\frac{eq_1 \qquad \dfrac{\begin{array}{c}\ell_2 : f(h(x), h(b)) \to_2 g(x, x) \in R^\epsilon \\ \sigma = \{x \mapsto a\}\end{array}}{2 \Vdash f(h(a), h(b)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)}\ (\mathsf{Repl}^\epsilon_{\mathsf{RL}}) \qquad eq_2}{4 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)}\ (\mathsf{Eq}^\epsilon_{\mathsf{RL}})$$

with $eq_1 = 2 \Vdash f(h(a), h(b)) =_{E^\epsilon} f(p(b), h(a))$ and $eq_2 = 0 \Vdash g(a, a) =_{E^\epsilon} g(a, a)$. They both can be proved using rules in Fig. 2. In fact, eq_1 is the same as eq_1 from Example 24 and has been proved there, while eq_2 has a one-step proof using the $(\mathsf{Refl}^\epsilon_=)$ rule.

The tree T_2 is the proof tree for $2 \Vdash g(a, a) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)$:

$$\frac{\dfrac{}{0 \Vdash a \longrightarrow_{R^\epsilon/E^\epsilon} a}\ (\mathsf{Refl}^\epsilon_{\mathsf{RL}}) \qquad \dfrac{\ell_1 : a \to_2 c \in R}{2 \Vdash a \longrightarrow_{R^\epsilon/E^\epsilon} c}\ (\mathsf{Repl}^\epsilon_{\mathsf{RL}})}{2 \Vdash g(a, a) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)}\ (\mathsf{NExp}^\epsilon_{\mathsf{RL}})$$

Proving $(f(p(b), h(a)), 6, g(a, c)) \in \to^*_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}$

As in Example 24, applying $\ell_2 : f(h(x), h(b)) \to_2 g(x, x))$ with $\sigma = \{x \mapsto a\}$ and $2 \Vdash f(h(x), h(b))\sigma =_{E^\epsilon} f(p(b), h(a))$ gives

$$4 \Vdash f(p(b), h(a)) \to_{R^\epsilon,E^\epsilon} g(a, a). \tag{1}$$

Next step: application of $\ell_1 : a \to_2 c$ to $g(a, a)$ gives

$$2 \Vdash g(a, a) \to_{R^\epsilon,E^\epsilon} g(a, c). \tag{2}$$

From (1) and (2) we get

$$6 \Vdash f(p(b), h(a)) \to^*_{R^\epsilon,E^\epsilon} g(a, c),$$

from which and $0 \Vdash g(a, c) =_{E^\epsilon} g(a, c)$ we get $(f(p(b), h(a)), 6, g(a, c)) \in \to^*_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}$.

8 Related work

Since the notion of quantitative equational reasoning (resp. quantitative rewriting) includes as a particular case the standard, non-quantitative version (see Remark 10), it is clear that some of the relations shown in Table 2 correspond to well-known relations. For example, syntactic equality, standard term rewriting and standard equational reasoning can be obtained as particular cases of $\to_{R^\epsilon/E^\epsilon}$ by considering resp. both R^ϵ and E^ϵ empty, E^ϵ empty and R^ϵ crisp, and R^ϵ empty and E^ϵ crisp. If R^ϵ is a set of quantitative rewrite rules and E^ϵ is empty we obtain the notion of quantitative term rewriting from [27]; fuzzy term rewriting [35] can be seen as a particular case.

The notion of weighted term rewriting [2] is also a ternary relation but the degrees are not taken from a quantale, instead they are elements of a set with a monoidal structure and a partial order. The weighted rewriting relation is non-expansive by definition. A notion of weighted rewriting can be obtained as a particular case of quantitative rewriting by relaxing the conditions on the quantale. To our knowledge, no notion of *equational* weighted rewriting has been defined in prior work.

Starting with Peterson & Stickel's seminal paper [46] there has been extensive work on rewriting modulo equational theories (also called equational rewriting, or rewriting modulo), see for example [32, 9, 29, 41, 42]. If both R and E are crisp, $\rightarrow_{R^\epsilon/E^\epsilon}$ corresponds to the standard notion of equational rewriting, which is the foundation of programming languages such as Maude [42]. More recently, Real-Time Maude [44] and model checkers such as PRISM [34] have been developed, which are based on notions of quantitative rewriting modulo crisp theories, i.e., the relation denoted $\rightarrow_{R^\epsilon/E}$ in Table 2.

Fuzzy rewriting [20], which has application in systems working on partial information, is a notion of rewriting where the matching algorithm uses a fuzzy (or approximate) notion of equality. In other words, fuzzy rewriting is standard rewriting using fuzzy matching. Since fuzzy equality can be seen as a particular case of quantitative equality, fuzzy rewriting [20] (restricted to similarity relations) is a particular case of $\rightarrow_{R/E^\epsilon}$. More generally, the notion of rewriting modulo quantitative equational theories $\rightarrow_{R/E^\epsilon}$ obtained as a particular case of $\rightarrow_{R^\epsilon/E^\epsilon}$ could be used to provide a semantics in settings where the notion of approximation involves equational reasoning. To the best of our knowledge, no definitions of quantitative rewriting with fuzzy matching exist in prior work. Such a notion is a particular case of $\rightarrow_{R^\epsilon/E^\epsilon}$ that could be used as a basis for the definition of probabilistic or real time versions of fuzzy programming languages.

The notion of quantitative rewriting modulo quantitative equational theories $\rightarrow_{R^\epsilon/E^\epsilon}$, which is the subject of this paper, subsumes all the other relations mentioned in Table 2. It could be used to provide a semantics to computations where both the notion of equality and the notion of rewriting require quantification, e.g., to specify computations in neural networks or to specify notions of protocol analysis that take resources into account.

9 Concluding remarks

In this work, we introduced a quantitative extension of rewriting logic, where both computation and deduction explicitly account for quantitative information. We defined a system of inference rules for quantitative rewrite sentences and proved its relation to the reflexive-transitive closure of quantitative rewriting modulo a quantitative equational theory (the $\rightarrow_{R^\epsilon/E^\epsilon}$ relation). We also presented an operational counterpart (the $\rightarrow_{R^\epsilon,E^\epsilon}$ relation), based on quantitative matching, and showed that it adequately models provability under the property of strict coherence. This establishes quantitative analogues of classical results while providing a framework for reasoning about quantitative system behaviour.

Quantitative matching modulo E^ϵ, as used in $\rightarrow_{R^\epsilon,E^\epsilon}$, can be solved by a modification of the algorithm from [22] for a particular class of equational theories. In [19], the problem was also studied from the perspective of combining exact and inexact quantitative matching, where the component theories admit complete matching procedures and satisfy conditions typically required for combining matching algorithms in the classical setting.

There are several promising directions for future work. A natural next step is to consider fully general quantitative rewrite theories as in the classical setting [40], allowing sorts, conditional rules, and parallelism. It would be interesting to relate such theories with conditional rules to the recently developed formalisms of Lawvere logics [7, 8], where quantitative consequence relations can be expressed via classical consequence relations over linear, polynomial, or rational inequalities. Integrating our framework with real-time rewrite frameworks such as Real-Time Maude [45] could enable modelling of probabilistic timed systems, cost-aware scheduling, and other time-sensitive quantitative features. Finally, relaxing the non-expansiveness requirement and adopting a graded quantitative rewriting approach [27], in which the effect of a rewrite may depend on its context, would further enhance expressivity, making the framework suitable for systems whose behaviour is both quantitative and context-dependent.

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