Segment Watchman Routes

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Abstract

Motivated by applications for robust guarding, we consider a variant of the multiple-watchmen problem that ensures that every point within a polygon P is seen from more than one direction: we search for two routes W_1, W_2, such that every point $p \in P$ is contained in a segment $\overline{w_1 w_2} \subseteq P$ such that $w_1 \in W_1$ and $w_2 \in W_2$. We call such routes *segment watchman routes.*

We show that finding the two routes that are optimal with respect to the min-max criterion is weakly NP-hard even in simple polygons, and that finding the routes that are optimal with respect to the min-sum criterion is NP-hard in polygons with holes. Moreover, we present sufficient conditions for routes to be segment watchman routes, and provide a polynomial-time 2-approximation under both the min-max criterion and the min-sum criterion, both in simple polygons. Finally, we show how to generalize our results for k watchmen.

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1 Introduction

In the classical Watchman Route Problem, introduced by Chin and Ntafos [5, 6], we ask for the shortest route inside a given simple polygon P, such that all points of P are visible from at least one point on the route. Maybe surprisingly, this problem can be solved in polynomial time [25, 26]. In this context, a point $p \in P$ *sees* another point $q \in P$ if the line segment $\overline{pq}$ is fully contained in P.

Carlsson, Nilsson, and Ntafos [4] raised the m-watchmen problem as a natural generalization: we are given m watchmen (with or without given starting points) for which we aim to find routes, such that each point in P is visible from at least one of the m routes. Two common objectives for this problem are to minimize the total length of all m watchman routes (called *min-sum*) and to minimize the length of the longest route assigned to any watchman (called *min-max*). In this paper, we mainly focus on the case $m = 2$.

In the classical setting with m watchmen, the goal is simply to ensure that each point is observed at least once, without any notion of robustness. In practice, however, when robots are used as watchmen, such minimal coverage is often insufficient: robots in multi-robot systems may fail or encounter obstacles, particularly in remote or hazardous regions, and observing a point from multiple perspectives can significantly enhance the reliability and quality of monitoring. To bridge this gap between theory and application, we focus on routes that are robust and effective in real-world scenarios. Specifically, in this paper, we aim to improve coverage quality by ensuring that each point is observed from multiple directions. This is a novel variant of the m-watchmen problem inspired by similar coverage requirements that have been studied for static guards [9, 22] (see related work below). Because each point is seen by each watchman, we will be able to perceive the complete environment even in case of all but one of the robots failing. This redundancy provides robustness (see, e.g., [10]). However, without failures, we cover even another robustness aspect: for certain types of laser scanners, the quality of the attained point clouds depends on the incidence angle (i.e., the angle between incoming laser beam and the local surface normal) [1, 23, 24]. Hence, to be robust against impaired perception with a multi-robot system (our watchmen) equipped with such lasers, we aim to see a point from different directions: for $m = 2$, we require that each point $p \in P$ is seen from two points on the two watchmen's routes that define a segment on which p must lie, that is, p must be seen from opposite sides.

Problem Definition. Let P be a *simple polygonal domain.* A point $p \in P$ is *segment-guarded* by two points $w_1, w_2 \in P$ if each of w_1, w_2 sees p and the segment $\overline{w_1 w_2}$ contains p. We say that two routes W_1, W_2 in P are *segment watchman routes* for P if for every point $p \in P$ there exist two points $w_1 \in W_1$, $w_2 \in W_2$ such that p is segment-guarded by w_1 and w_2.

We always assume that a route is *closed*, that is, it starts and ends at the same point. For a route W in P, let $|W|$ represent its length. We focus on the following two problems:

► **Problem 1** (Min-Max Segment Watchmen Route Problem)**.** *Given a polygonal domain P, find segment watchman routes W_1, W_2 such that* $\max\{|W_1|, |W_2|\}$ *is minimized.*

► **Problem 2** (Min-Sum Segment Watchmen Route Problem)**.** *Given a polygonal domain P, find segment watchman routes W_1, W_2 such that* $|W_1| + |W_2|$ *is minimized.*

In Figure 1, we illustrate a collection of example polygons along with their corresponding segment watchman routes. Note that the routes may overlap. Also note that segment-guarding a point p by two watchmen on routes W_1 and W_2 does not require them to occupy positions $w_1 \in W_1$ and $w_2 \in W_2$ simultaneously.

Our Contribution. We introduce the Segment Watchman Routes Problem, and study it in polygons with and without holes, under both the min-sum and min-max objectives. We provide sufficient conditions for 2-watchmen routes to be segment watchman routes.

On the algorithmic side, we present a 2-approximation algorithm for both the min-max and min-sum objectives. Moreover, we establish NP-hardness for simple polygons under the min-max measure, and for polygons with holes under the min-sum objective.

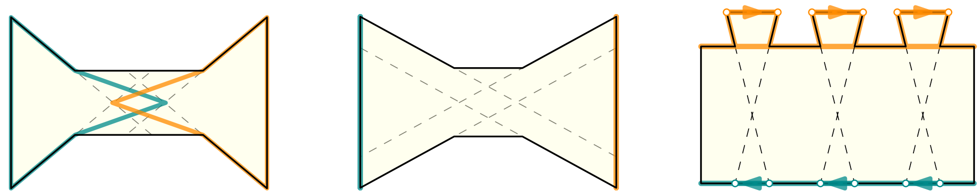


Figure 1 Min-max segment watchman routes (a) may or (b) may not need to overlap. Moreover, (c) segment watchman routes do not "sweep" the polygon, and therefore are not required to see a point at the same time. (Note again that routes are closed, hence, if we depict a segment only, this means that the route goes back and forth along the segment.).

Finally, we generalize the concept of being segment-guarded to being *k-hull-guarded*: A point $p \in P$ is *k-hull-guarded* by k points $w_1, w_2, \dots, w_k \in P$ if the convex hull of $w_1, w_2, \dots, w_k$ contains p, and each of $w_1, w_2, \dots, w_k$ sees p. Similarly to segment watchman routes, we say that k routes $W_1, W_2, \dots, W_k$ in P are *k-hull watchman routes* for P if for every point $p \in P$ there exist k points $w_i \in W_i$ for $i \in \{1, \dots, k\}$ such that p is k-hull-guarded by $w_1, w_2, \dots, w_k$. Most of our results for segment watchman routes extend to k-hull watchman routes. In particular, the strategy of our approximation algorithm yields a k-approximation under the min-max criterion, and a 2-approximation under the min-sum criterion. Moreover, the corresponding NP-hardness results carry over as well.

Proofs and details for statements marked with ($\star$) can be found in the full version [2].

Related Work. Carlsson, Nilsson, and Ntafos [4] showed that the m-watchmen problem is NP-hard in simple polygons and provided a polynomial time algorithm for histograms. Polynomial time algorithms for different polygon classes, using either the min-sum or the min-max objective, have also been presented [13, 16, 18, 19]. Nilsson and Packer [17] proposed a 5.969-approximation algorithm to compute min-max 2-watchman routes in simple polygons without given starting points (also called *floating*), and obtained a factor of ≈ 6.922 for the case with given starting points (also called *anchored*). Very recently, Brötzner, Nilsson, and Schmidt [3] improved these approximation factors to $3 + \pi/2 \approx 4.571$ for the floating, and to $2 + \pi/2 \approx 3.571$ for the anchored version.

The robustness requirement we employ for watchman routes in this paper is closely related to the problems of two-sensor visibility and triangle guarding for stationary guards introduced by Efrat, Har-Peled, and Mitchell [9] and Smith and Evans [22], respectively. Both considered two polygons P and Q with $Q \subseteq P$, where the subpolygon Q should be guarded by guards placed in P (assuming that Q's boundary is transparent). In [9], a point $p \in Q$ is *2-guarded at angle α* by two guards g_1, g_2 if $\angle g_1 p g_2 \in [\alpha, \pi - \alpha]$ and both guards see p. Smith and Evans defined a point $p \in Q$ to be *triangle-guarded* by g_1, g_2, g_3 if p is seen by each of the three guards and is contained in the triangle spanned by them. Another variant of robust guarding has recently been established by Das, Filtser, Katz, and Mitchell [7, 8]; and a variant of robustness for a single watchman by Langetepe, Nilsson, and Packer [15].

Huynh, Mitchell, and Polishchuk [14] consider a problem that is similar to our notion of segment watchman routes. They study a version of monitoring a polygon via line segments: sweeping a polygonal domain with variable-length segments whose endpoints are two mobile agents. A point is seen if at some point in time, it lies on a segment between the current positions of the two agents. This differs from our definition of being segment-seen by the temporal constraint, as depicted in Figure 1(c).

2 Preliminaries and Key Lemmas

Let P be a simple polygon with n vertices. Denote by ∂P the boundary of P, and for two points $a, b \in \partial P$ denote by $\partial P[a, b]$ the boundary of P between a and b in counterclockwise order. For two points $a, b \in P$, denote by $\pi(a, b)$ the geodesic path between a and b in P (i.e., the shortest path in P connecting a and b). We assume that P does not contain vertices with an internal angle of exactly 180°, i.e., no three consecutive vertices are on the same line. If P does contain such a vertex, we can simply remove it.

A set $H \subseteq P$ is *geodesically convex* if for any two points $p, q \in H$, the geodesic $\pi(p, q)$ lies in H. The *relative convex hull* $\mathrm{H}(X)$ of a set X of points in a simple polygon P is the smallest geodesically convex set within P that contains X. The relative convex hull $\mathrm{H}(X)$ forms a weakly simple polygon, that is, a polygon whose boundary does not properly self-intersect, although vertices may be repeated. Its boundary is composed of geodesic paths that connect pairs of points in X. When X is a route, we slightly abuse the notation and refer to the boundary of $\mathrm{H}(X)$ as the relative convex hull of X.

A route W in P is called a *watchman route* if every point in P is visible from some point on W. Recall that the length of W is denoted by $|W|$. We say that W is *simple* if it does not cross itself, and that W is *relatively convex* if it coincides with its relative convex hull $\mathrm{H}(W)$.

Let W_1 and W_2 be segment watchman routes for P. By definition, the following holds:

▶ **Observation 3.** *Each of the routes W_1 and W_2 is a watchman route for P.*

Note that Observation 3 also holds for k-hull watchman routes.

The *visibility polygon* of a point $p \in P$ is the set of points in P that are visible from p. The following observation is well known and largely considered folklore; however, we include it for completeness.

▶ **Observation 4** ($\star$). *Let W be a route in P, then W is a watchman route if and only if it visits the visibility polygon of every convex vertex of P.*

We next establish sufficient conditions ensuring that two routes form segment watchman routes. We subsequently extend the argument to the general k-hull setting in Section 5.

▶ **Lemma 5** (The Conditions Lemma for Segment Watchman Routes). *Two routes W_1 and W_2 are segment watchman routes for P if the following conditions hold:*

(i) *Every convex vertex is visited by one of W_1 or W_2.*
(ii) *Both W_1 and W_2 visit the visibility polygon of each convex vertex.*
(iii) *Both W_1 and W_2 are relatively convex.*

Proof. By Observation 4, Condition (ii) implies that W_1 and W_2 are watchman routes. Consider a point $p \in P$. Since both W_1 and W_2 are watchman routes, there exists at least one point on W_1 and at least one point on W_2 that p sees. Consider the two wedges defined by the angles from which p is viewing W_1 and W_2, as visualized in Figure 2: let F_1 be the maximal wedge bounded by two rays starting at p, such that for every ray ρ in F_1 there is a point $w \in W_1$ in this direction that p sees. Note that because P is simple, F_1 is a single wedge. The wedge F_2 is defined analogously for W_2.

Consider the two wedges F_1 and F_2. If p lies on or within one relatively convex route, the corresponding wedge covers 360°. If p does not lie on or within one route, the corresponding wedge covers less than 180° because by Condition (iii) both routes are relatively convex. If at least one of F_1 and F_2 covers 360° around p, then p is segment-guarded: assume that F_2 covers 360° around p, and let w_1 be a point on W_1 that sees p. Then the ray from w_1 in

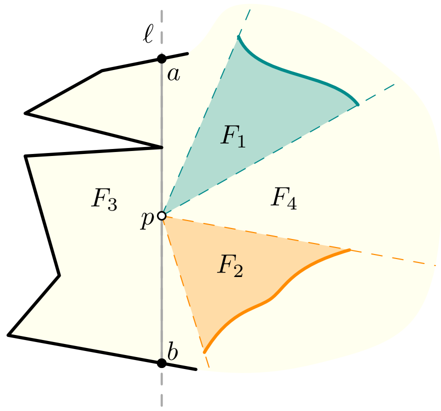


Figure 2 The wedges F_1 and F_2 define the angles from which p is viewing W_1 and W_2, respectively. If F_3 or F_4 is larger than 180°, then there is a convex vertex on the left side of ℓ which is not visited.

the direction of p intersects W_2 at point w_2 that sees p, and thus p is segment-guarded by $\overline{w_1w_2}$. Hence, assume that neither F_1 nor F_2 covers 360° around p. Let F_3 (and, if it exists, F_4) be the maximal wedge(s) bounded by two rays originating at p, such that for every ray ρ within F_3 (and F_4), there is no point $w \in W_1$ or $w \in W_2$ along ρ that is visible from p. The plane around p can then be partitioned into up to four wedges, depending on whether F_1 and F_2 intersect. In the non-intersecting case, the wedges are F_1, F_2, F_3, and F_4. If F_1 and F_2 overlap, the wedges are F_1, F_2, F_3, and a single wedge corresponding to the intersection of F_1 and F_2.

We now argue that neither F_3 nor F_4 can cover more than 180°. This implies that one of the rays through p that define F_1 must meet a point on W_2 in the opposite direction from p, and therefore p is segment-guarded and the claim follows.

Without loss of generality, assume that F_3 covers more than 180° (as shown in Figure 2). Consider a line ℓ through p in F_3 that does not contain an edge of the boundary of P, and assume that ℓ is a vertical line and that both F_1 and F_2 are on the right side of ℓ. Let $\overline{ab}$ be the maximal line segment on ℓ that is contained in P. Then $\overline{ab}$ splits P into at least two subpolygons, and at least one of them, P', is on the left side of $\overline{ab}$. Because P is simple and both W_1 and W_2 do not cross $\overline{ab}$, there are no points of W_1 and W_2 in P'. However, P' must contain a convex vertex v. This yields a contradiction, as by Condition (i), v needs to be visited by at least one of the watchman routes. ◀

In Lemma 5, the conditions imply that W_1 and W_2 are segment watchman routes. However, there exist segment watchman routes that do not fulfill these conditions, see Figure 3. On the other hand, there always exists an optimal solution for the segment watchman route problem in any simple polygon with respect to the min-max and min-sum criterion for which Conditions (i), (ii), and (iii) hold.

▶ **Lemma 6.** *Let P be a simple polygon. If two routes W_1 and W_2 are segment watchman routes for P, then Conditions (i) and (ii) from Lemma 5 hold.*

Proof. First, notice that by Observation 4, Condition (ii) is a necessary condition, that is, if W_1 and W_2 are segment watchman routes then Condition (ii) holds.

Next, we claim that Condition (i) is also a necessary condition, that is, we show that if W_1 and W_2 are segment watchman routes, then every convex vertex of P must be visited by one of W_1 or W_2. Let v be a convex vertex of P. Then v lies on a line segment $\overline{w_1w_2}$ with

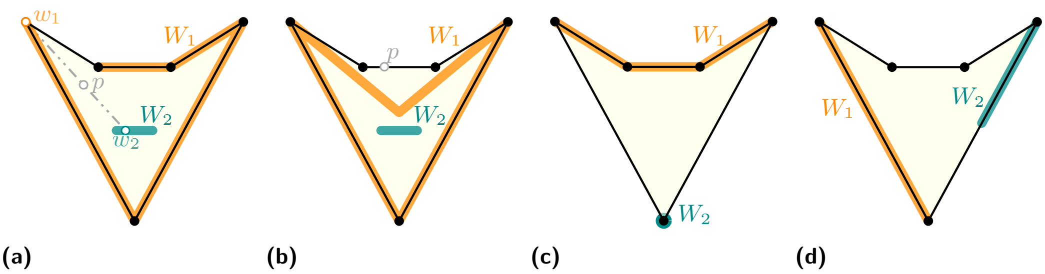


Figure 3 (a) W_1 and W_2 are segment watchman routes (e.g., p lies on $\overline{w_1 w_2}$), but do not fulfill the conditions of Lemma 5. They are not optimal: W_1's relative convex hull $H(W_1)$ (the boundary of the polygon) is shorter than W_1, and $H(W_1)$ together with W_2 are segment watchman routes. (b) W_1 and W_2 do not fulfill Condition (iii) of Lemma 5 and they are not segment watchman routes, since p is not properly guarded. (c) and (d) depict segment watchman routes that fulfill all the conditions of Lemma 5.

$w_1 \in W_1$ and $w_2 \in W_2$, and the segments $\overline{vw_1}$ and $\overline{vw_2}$ are contained in P. As the interior angle at v is strictly smaller than 180°, any line segment in P that contains v has v as one of its endpoints, and therefore v is visited by either W_1 or W_2. ◀

▶ **Lemma 7.** *Let P be a simple polygon. There always exists an optimal solution for the segment watchman route problem in P with respect to the min-max and min-sum criterion for which Conditions (i), (ii), and (iii) from Lemma 5 hold.*

Proof. Let W_1 and W_2 be an optimal solution for the segment watchman route problem in P (with respect to either the min-max or the min-sum criterion). Conditions (i) and (ii) from Lemma 5 hold by Lemma 6.

We show that $\mathrm{H}(W_1)$ and $\mathrm{H}(W_2)$ are segment watchman routes. Since $|\mathrm{H}(W_i)| \le W_i$, this would prove our claim. Clearly, since Condition (i) holds for W_1 and W_2, it also holds for $\mathrm{H}(W_1)$ and $\mathrm{H}(W_2)$. Condition (iii) holds by definition. We therefore only need to prove that Condition (ii) still holds. Let v be a convex vertex of P, and assume that W_i visits the visibility polygon of v. Let w be a point on W_i that sees v. The segment $\overline{vw}$ is contained in P, and because $\mathrm{H}(W_i)$ contains W_i, its boundary must cross $\overline{vw}$ at a point w'. We conclude that $\mathrm{H}(W_i)$ also visits the visibility polygon of v, and thus all conditions from Lemma 5 hold, and both $\mathrm{H}(W_1)$ and $\mathrm{H}(W_2)$ are segment watchman routes. ◀

3 Approximation Algorithms for Segment Watchman Routes

We begin, as in the previous section, by considering segment watchman routes. We generalize the ideas developed in this section in Section 5.

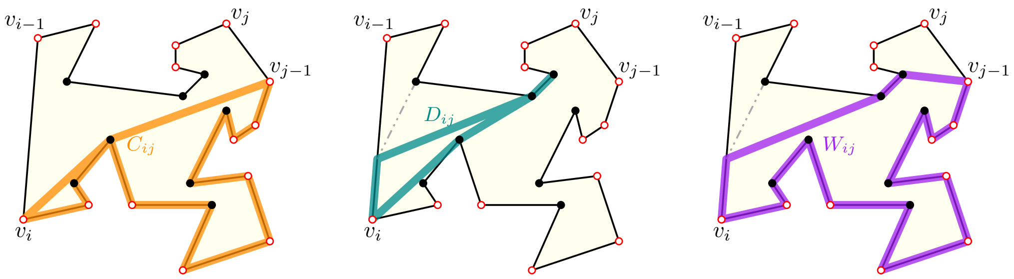
Let P be a simple polygon with t convex vertices. We enumerate the convex vertices in counterclockwise order $V_C = \{v_0, \dots, v_{t-1}\}$, with v_0 chosen arbitrarily. In the following, we assume without loss of generality that indices are counted modulo t.

▶ **Lemma 8** ($\star$). *Let $X \subseteq V_C$ be a subset of the convex vertices of P. Then $\mathrm{H}(X)$ is a shortest route in P that visits X.*

▶ **Lemma 9** ($\star$). *Let C be a route that visits a subset $X \subset V_C$ of the convex vertices of P, and D be a route that sees another subset $Y \subset V_C$. Then the route $W = \mathrm{H}(C \cup D)$ visits the convex vertices in X and sees the convex vertices in Y.*

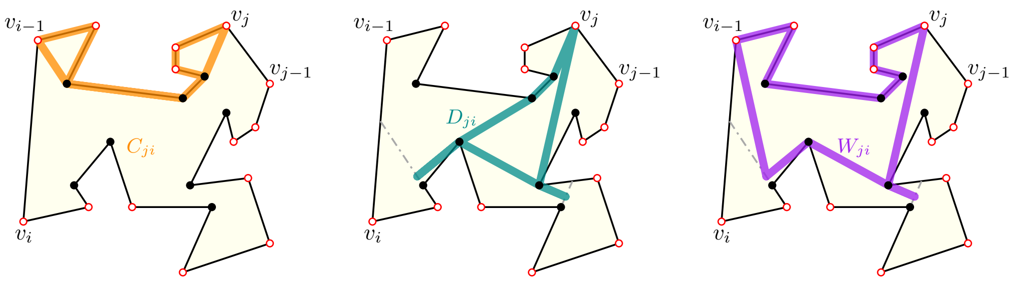
Let $v_i, v_j \in V_C$ be two different convex vertices and let $C_{ij} = \mathrm{H}(\{v_i, v_{i+1}, \ldots, v_{j-1}\})$ be a shortest route that visits the convex vertices $v_i, v_{i+1}, \ldots, v_{j-1}$ as guaranteed by Lemma 8. The route $C_{ji} = \mathrm{H}(\{v_j, v_{j+1} \ldots, v_{i-1}\})$ is then a shortest route that visits the convex vertices $v_j, v_{j+1} \ldots, v_{i-1}$. Figures 4 and 5 exemplify the routes defined in this subsection. Denote $C_P = \partial P$, and note that it is the shortest route that visits all convex vertices of P.

Let D_{ij} be the shortest route that starts and ends at v_i, and that sees all the convex vertices $v_j, \ldots, v_{i-1}$. The route D_{ji} is then the shortest route that starts and ends at v_j, and that sees all the convex vertices $v_i, \ldots, v_{j-1}$. Let D_P be the shortest floating watchman route in P (that is, the shortest watchman route without a given starting point).



(a) **(b)** **(c)**

Figure 4 Examples of the routes used in Section 3: (a) $C_{ij} = \mathrm{H}(\{v_i, v_{i+1}, \ldots, v_{j-1}\})$ denotes a shortest route that visits the convex vertices $v_i, v_{i+1}, \ldots, v_{j-1}$. (b) D_{ij} is a shortest route that starts and ends at v_i and that sees all the convex vertices $v_j, \ldots, v_{i-1}$. (c) depicts $W_{ij} = \mathrm{H}(C_{ij} \cup D_{ij})$.



(a) **(b)** **(c)**

Figure 5 Complementary routes to those shown in Figure 4: (a) C_{ji}, (b) D_{ji}, and (c) W_{ji}.

We define $W_{ij} \stackrel{\text{def}}{=} \mathrm{H}(C_{ij} \cup D_{ij})$, i.e., W_{ij} is the relative convex hull of the route obtained by connecting the two routes C_{ij} and D_{ij} at v_i. Then, we construct our approximate solution by choosing the pair

$$(W_1, W_2) = \operatorname*{arg\,min}_{i \neq j} \big\{ \max\{|W_{ij}|, |W_{ji}|\}, \max\{|C_P|, |D_P|\} \big\}.$$

By Lemmas 5 and 9, the pair (W_1, W_2) constitutes a feasible solution to the segment watchman routes problem.

For two routes W_1 and W_2, the subsequent Lemmas 10 and 11 establish upper bounds relative to the length of an optimal solution for P on both the longer of W_1 and W_2, and on their sum. We start with the min-max objective. For this, let $\mathrm{OPT}(P)$ denote the length of an optimal solution for a polygon P under the min-max criterion.

▶ **Lemma 10.** $\max\{|W_1|, |W_2|\} \leq 2 \cdot \mathrm{OPT}(P)$.

Proof. Let W_1^* and W_2^* be two segment watchman routes with $\max\{|W_1^*|, |W_2^*|\} = \mathrm{OPT}(P)$. Without loss of generality, we may assume that W_1^* and W_2^* are as short as possible.

If W_1^* or W_2^* visits all convex vertices of P, then (C_P, D_P) is an optimal solution to the problem and the theorem therefore holds. Hence, for the remainder of this proof, we assume that W_1^* visits some fixed convex vertex v_i and W_2^* visits a different fixed convex vertex v_j.

Since W_1^* visits v_i and it either sees or visits the convex vertices $v_j, \ldots, v_{i-1}$ by construction, we have that $|D_{ij}| \leq |W_1^*|$. Similarly, W_2^* visits v_j and it either sees or visits the convex vertices $v_i, \ldots, v_{j-1}$, yielding $|D_{ji}| \leq |W_2^*|$. We distinguish the following cases.

W_1^* and W_2^* do not intersect. Because W_1^* and W_2^* do not intersect, the two convex vertices v_i and v_j can be chosen so that W_1^* visits $v_i, \ldots, v_{j-1}$ by increasing index (modulo t) and sees the remaining ones, whereas W_2^* visits $v_j, \ldots, v_{i-1}$ and sees the remaining ones. From this, it follows that $|C_{ij}| \leq |W_1^*|$ and $|C_{ji}| \leq |W_2^*|$. We obtain that

$$\begin{aligned}\max\{|W_1|, |W_2|\} &\leq \max\{|\mathrm{H}(C_{ij} \cup D_{ij})|, |\mathrm{H}(C_{ji} \cup D_{ji})|\} \\ &\leq \max\{2|W_1^*|, 2|W_2^*|\} = 2 \cdot \max\{|W_1^*|, |W_2^*|\} = 2 \cdot \mathrm{OPT}(P). \quad (1)\end{aligned}$$

W_1^* and W_2^* intersect. Because W_1^* and W_2^* intersect and together visit all the vertices, we have $|C_P| \leq |W_1^* \cup W_2^*| = |W_1^*| + |W_2^*|$ and $|D_P| \leq \min\{|W_1^*|, |W_2^*|\}$, as both W_1^* and W_2^* are watchman routes. We obtain that

$$\begin{aligned}\max\{|W_1|, |W_2|\} &\leq \max\{|C_P|, |D_P|\} \leq \max\{|W_1^*| + |W_2^*|, \min\{|W_1^*|, |W_2^*|\}\} \\ &\leq 2 \cdot \max\{|W_1^*|, |W_2^*|\} = 2 \cdot \mathrm{OPT}(P). \quad (2)\end{aligned}$$

Since the case distinction is exhaustive, this concludes the proof. ◀

Note that in fact, for the min-max criterion we may also let $W_2 = C_P$ to avoid computing a floating shortest watchman route. We proceed with the min-sum objective. For this, let $\mathrm{OPT}^\Sigma(P)$ denote the size of an optimal solution for P under the min-sum criterion.

▶ **Lemma 11.** $|W_1| + |W_2| \leq 2 \cdot \mathrm{OPT}^\Sigma(P)$.

Proof. The proof proceeds along the same lines as that of Lemma 10. In particular, it suffices to replace Equation (1) and Equation (2) with the following two equations, respectively.

W_1^* and W_2^* do not intersect.

$$|W_1| + |W_2| \leq |\mathrm{H}(C_{ij} \cup D_{ij})| + |\mathrm{H}(C_{ji} \cup D_{ji})| \leq 2 \cdot (|W_1^*| + |W_2^*|) = 2 \cdot \mathrm{OPT}^\Sigma(P).$$

W_1^* and W_2^* intersect.

$$|W_1| + |W_2| \leq |C_P| + |D_P| \leq |W_1^*| + |W_2^*| + \min\{|W_1^*|, |W_2^*|\} \leq 3/2 \cdot \mathrm{OPT}^\Sigma(P).$$ ◀

Using Lemmas 10 and 11, we obtain the following theorem.

▶ **Theorem 12.** *Given a simple polygon P with n vertices, out of which t are convex, one can compute in $\mathcal{O}(n^4 + t^2n^3)$ time a 2-approximation for the segment watchman routes problem under both criteria.*

Proof. Feasibility follows directly from Lemma 5, while the approximation guarantees are established by Lemmas 10 and 11.

Moreover, we can compute D_P in $\mathcal{O}(n^4)$ time [25, 26]. The route C_{ij} is a concatenation of $\partial P[v_i, v_{j-1}]$ and the geodesic $\pi(v_i, v_{j-1})$, and can thus be computed in linear time for any $i, j \in 0, \ldots, t-1$. The routes D_{ij} can be computed in $\mathcal{O}(n^3)$ time by modifying the algorithm of Jiang and Tan [26]. Since there are $\mathcal{O}(t^2)$ such routes (all possible pairs of convex vertices), the total running time for computing all those routes W_{ij} is $\mathcal{O}(t^2n^3)$. ◀

4 Computational Complexity

We now study the computational complexity of computing shortest segment watchman routes. We show that the problem is weakly NP-hard under the min-max objective, even when restricted to simple polygons. For the min-sum objective, we demonstrate that the problem is NP-hard in polygons with holes.

4.1 Min-Max Segment Watchman Routes in Simple Polygons

We reduce from Multiway Number Partitioning [12]. In particular, we consider the problem of partitioning a set of numbers into two subsets with equal sum. This problem, commonly referred to as Partition, is known to be weakly NP-hard.

▶ **Theorem 13** ($\star$). *The Min-Max Segment Watchman Routes Problem is weakly NP-hard, even in simple polygons.*

Proof Sketch. We construct a star-shaped polygon, as visualized in Figure 6. The length of a spike's boundary (i.e., the path $v_{2i-1}, v_{2i}, v_{2i+1}$) represents the value α_i from the Partition instance φ, and let T denote the sum of all values. Both watchmen start in the bottommost vertex v_0, and thus need to return to it. A min-max segment watchman route of length $T/2 + \varepsilon$ exists if and only if there exists a partition of φ into two sets of equal sum. ◀

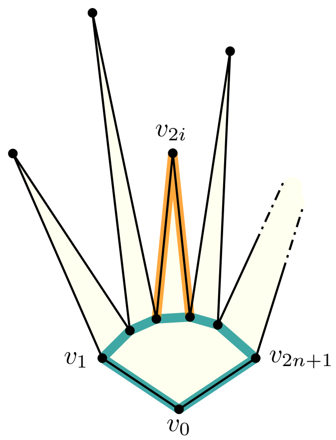


Figure 6 High-level idea of the type of polygon utilized in the NP-hardness reduction.

4.2 Min-Sum Segment Watchman Routes in Polygons with Holes

For polygons with holes, we lack conditions as those in Lemma 5 for simple polygons. We overcome this in our NP-hardness proof, which is based on a reduction from the Euclidean Traveling Salesman Problem, by constructing feasible solutions whose lengths sharply distinguish optimal from non-optimal TSP tours. As a result, computing an optimal segment watchman route necessitates solving the TSP instance.

▶ **Theorem 14.** *The Min-Sum Segment Watchman Routes Problem is NP-hard in polygons with holes.*

Proof. We reduce from the Euclidean Traveling Salesman Problem [11, 21]. For a given set $\mathcal{P}$ of points in the plane, we construct a polygonal domain with a niche for each point on the boundary of the convex hull of $\mathcal{P}$ and a hole for each point in the interior of $\mathcal{P}$.

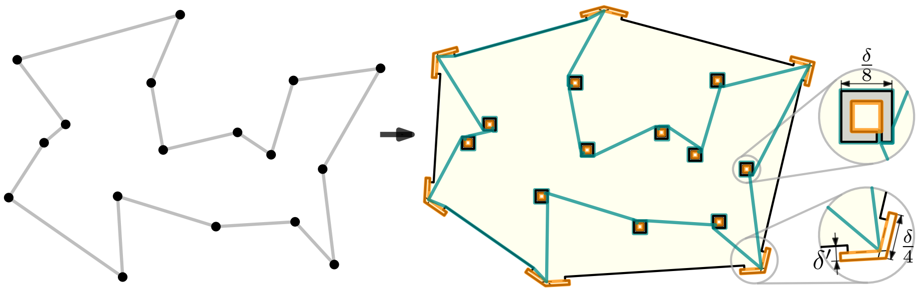


Figure 7 Schematic illustration of the reduction from a TSP instance to a polygonal domain. The left figure shows a TSP tour on the input point set, while the right figure shows the corresponding routes in the constructed polygon. Feasible watchman routes, including the detours around the convex hull of each hole (taken by both watchmen), are shown in green. For each niche and hole, W_1 additionally visits all convex vertices; these maximal detours are shown in orange.

We may assume that for any pair of points, the distance between the two points is different from the distance between any other pair of points. Now let $\Delta = \min_{p,q,p',q' \in \mathcal{P}} d(p,q) - d(p',q')$ be the smallest difference among all pairwise distances, and set $\delta \ll \Delta/|\mathcal{P}|$. With this, we can construct our polygonal domain, as depicted in Figure 7.

The outer polygon P is a modification of the convex hull of $\mathcal{P}$, where each corner is replaced by a niche. Each such niche adds two essential cuts to P that every route needs to visit, and the essential cuts lie outside the convex hull (of $\mathcal{P}$). Similarly, each polygonal hole is shaped such that it "hides" some part of the interior of P, hence, adding an additional cut that every route needs to visit. As every route of a segment watchman route is also a watchman route for the domain considered, every route needs to visit the essential cuts of the four niches and the "door" of every hole.

Each hole H is a polygon with 10 vertices. The convex hull of H is a square with side length $\delta/8$. In the inner part of the hole lies an inner square, which is connected to the outside of H by a narrow tunnel, the *door*. See the top lens in Figure 7 for an illustration. To see the inner square of H, a watchman needs to walk through the corresponding door.

To place a niche at a corner of the outer polygon, eight additional vertices are necessary. For this, on each edge incident to a convex hull corner, a segment of length $\delta/4$ is cut off and parallel shifted outwards by $\delta' < \delta/8$. Then, to create the essential cut, the shifted corner is connected with the convex hull edges by three edgers of length $\delta'/2$ on each side, creating a chain of two convex vertices, the first of them is incident to the shifted corner edge, and two reflex vertices. See the bottom lens in Figure 7. All edges are parallel or orthogonal to the convex hull edges. The total length that is added to the outer boundary of the polygon is $2\delta/4 + 6\delta'/2 < \delta$. With this, each niche adds two essential cuts to the outer polygon which intersect in the interior of P, but outside the convex hull.

We obtain feasible watchman routes by adapting the TSP tour on $\mathcal{P}$ to visit the extensions in the holes and walking once around the convex hull of every hole. The latter adds a detour of $\leq \delta/2$. We denote these two routes as W_1 and W_2. Now, extend W_1 so that

- For every niche, W_1 visits all convex vertices of the niche, adding a detour of $\leq \delta$
- For every hole H, W_1 visits all vertices in the inner part of H, adding a detour of $\leq \delta/2$

We can prove that W_1 and W_2 are segment watchman routes in the polygonal domain. First, consider a point p that lies in the interior of one of the routes, say W_1. By construction, it is seen from W_2, i.e., there exist $w_2 \in W_2$ that sees p. Then the ray from w_2 in the direction of p intersects W_1 at a point w_1 that sees p, and thus p is segment-guarded by $\overline{w_1 w_2}$.

Now consider a point p that lies outside of each of the routes W_1 and W_2. Then p lies inside a simple polygon P' bounded by the two routes and the line ℓ through one of the edges of the outer polygon. Let ℓ_p be the line parallel to ℓ that contains p. The line ℓ either intersects two of the essential cuts of the outer polygon, or it intersects one of the essential cuts on one side of the edge extension and creates a pocket that contains a convex vertex on the other side of the edge extension. Therefore, ℓ_p intersects both routes on one side of p before it hits an edge of the polygonal domain, and at least one of the routes in the other direction. Hence, p is segment-guarded.

Next, let T_{OPT} be an optimal solution to the TSP in $\mathcal{P}$, and let T_2 be the shortest solution to TSP in $\mathcal{P}$ that visits the points in a different order than T_{OPT}. We argue that the length of each of the routes W_1 and W_2 is shorter than T_2. Each route W_i takes a detour of at most δ per corner niche and at most δ per hole. With this, the following inequality holds:

$$|W_2| < |W_1| \leq (T_{OPT} + |\mathcal{P}| \cdot \delta) < (T_{OPT} + \Delta) \leq T_2.$$

Thus, contracting the loops around the holes and the detour at the niches yields a shortest route that visits all points in $\mathcal{P}$.

The routes W_1 and W_2 are feasible segment watchman routes, each of length at least T_{OPT}, but strictly shorter than T_2. The optimal segment watchman routes also need to visit all essential cuts of the holes and niches, and have, hence, also length at least T_{OPT}. Moreover, they are at most as long as W_1 and W_2, hence, we need to solve the TSP on $\mathcal{P}$ to find the optimal segment watchman routes. ◀

5 Generalization to k-Hull Watchman Routes

We now generalize our results on segment watchman to k-hull watchman routes. To this end, we prove sufficient conditions equivalent to those from Lemma 5 for $k = 2$ to all $k \geq 2$:

▶ **Lemma 15** ($\star$) (The Conditions Lemma for k-Hull Watchman Routes)**.** *The routes $W_1, \ldots, W_k$, $k \geq 2$, are k-hull watchman routes for P if the following conditions hold:*

(i$_k$) *Every convex vertex is visited by one of $W_1, \ldots, W_k$.*

(ii$_k$) *The visibility polygon of each convex vertex is visited by all of the routes $W_1, \ldots, W_k$.*

(iii$_k$) *The routes $W_1, \ldots, W_k$ are relatively convex.*

5.1 Approximation Algorithms for k-Hull Watchman Routes

In order to generalize the algorithms from Section 3, we establish the following crucial lemma.

▶ **Lemma 16** ($\star$)**.** *Let X_1 and X_2 be subsets of V_C. Then $\mathrm{H}(X_1)$ and $\mathrm{H}(X_2)$ cross each other iff there exist vertices v_a, v_b, v_c, v_d such that $v_a, v_c \in X_1$, $v_b, v_d \in X_2$, and $a < b < c < d$.*

Consider a subset $X \subseteq V_C$ of the convex vertices of P, and by Lemma 8, let $C_X = \mathrm{H}(X)$ be a shortest route that visits the vertices in X. Let v_X be the vertex in X with the smallest index, and denote by D_X the shortest route that starts and ends at v_X, and that sees all the convex vertices in $V_C \setminus X$. Recall that $C_P = \partial P$ is a watchman route, D_P is the shortest floating watchman route in P, and that $|D_P| \leq |C_P|$. We define $W_X \stackrel{\text{def}}{=} \mathrm{H}(C_X \cup D_X)$ as the relative convex hull of the routes C_X and D_X.

We construct our approximate solutions by iterating over all possible ways to partition the set V_C of convex vertices into subsets $X_1, \ldots, X_m$ for $m \leq k$ such that the routes $C_{X_1}, \ldots, C_{X_m}$ are disjoint. By Lemma 16, two routes C_{X_i} and C_{X_j} are disjoint if and only if X_i and X_j are "non-crossing" with respect to the order along ∂P. We say that two sets

$X, Y \subseteq V_C$ are *non-crossing* if there exist two indices $0 \leq a, b \leq t-1$ such that (i) the indices of vertices in X are in $[a, b-1]$ and (ii) the indices of vertices in Y are in $[b, a-1]$. We say that k subsets $X_1, \ldots, X_k \subseteq V_C$ are a *non-crossing partition* of V_C if for a convex $|V_C|$-gon the relative convex hulls of the vertices corresponding to the subsets X_i do not intersect. It is well-known that the number of non-crossing partitions of size k for a set of t points is equal to the Narayana number $N(k,t) = \frac{1}{t}\binom{t}{k}\binom{t}{k-1}$ [20] with $\frac{1}{t}\binom{t}{k}\binom{t}{k-1} = \mathcal{O}(t^{2k-2})$.

Min-Max Variant. Let $\mathcal{S}^m$ be the set of all possible non-crossing partitions of size m for the set V_C. For the approximated solution, we iterate over all the non-crossing partitions and pick the best option, as follows. For every length $2 \leq m \leq k$, set:

$$(W_1, W_2, \ldots, W_m) = \operatorname*{arg\,min}_{\{X_1,\ldots,X_m\} \in \mathcal{S}^m} \max\{|W_{X_1}|, |W_{X_2}|, \ldots, |W_{X_m}|\}, \tag{3}$$

and set $W_{m+1} = \cdots = W_k = D_P$. Then, we take the optimal set over all choices of m. Finally, if $\max_i |W_i| > |C_P|$, we pick $W_1 = C_P$ and $W_2 = W_3 = \cdots = W_k = D_P$.

By Lemmas 9 and 15, $(W_1, W_2, \ldots, W_k)$ is a feasible solution for the k-hull watchman routes problem. If $\mathrm{OPT}_k(P)$ denotes the size of an optimal solution for the min-max k-hull watchman routes problem in P, we obtain the following lemma:

▶ **Lemma 17** ($\star$). $\max\left\{|W_1|, |W_2|, \ldots, |W_k|\right\} \leq k \cdot \mathrm{OPT}_k(P)$.

We thus derive the following theorem. Its proof mirrors that of Theorem 12: feasibility follows from Lemma 15, and the approximation guarantee from Lemma 17. Moreover, the number of partitions is $\mathcal{O}(t^{2k-2})$.

▶ **Theorem 18.** *Given a simple polygon P with n vertices, out of which t are convex, one can compute a k-approximation for the k-hull watchman route problem under the min-max criterion in $\mathcal{O}(n^4 + t^{2k-2} \cdot n^3)$ time.*

Min-Sum Variant. We now generalize our approximation algorithm for the min-sum objective. Here, instead of setting the routes as in Equation (3), we set

$$(W_1, W_2, \ldots, W_m) = \operatorname*{arg\,min}_{\{X_1,\ldots,X_m\} \in \mathcal{S}^m} \sum_{i=1}^{m} \left|W_{X_i}\right|. \tag{4}$$

In addition, we define D_{X_i} slightly differently here: it is the shortest floating watchman route that sees all convex vertices in $V_C \setminus X_i$, without being required to visit any convex vertex. We denote by $\mathrm{OPT}_k^{\Sigma}(P)$ the size of an optimal solution for the Min-Sum k-hull-Watchman Routes Problem in P.

▶ **Lemma 19** ($\star$). $\sum_{i=1}^{k} |W_i| \leq 2 \cdot \mathrm{OPT}_k^{\Sigma}(P)$.

We obtain the following theorem. Again, our proof is similar to our proof of Theorem 12: using Lemma 15 for feasibility and Lemma 19 for the approximation factor, and where the number of partitions is $\mathcal{O}(t^{2k-2})$. However, the running time increases slightly, as D_X is now defined to be a floating watchmen route, whose computation requires $\mathcal{O}(n^4)$ time.

▶ **Theorem 20.** *Given a simple polygon P with n vertices, out of which t are convex, one can compute a 2-approximation for the k-hull watchmen route problem under the min-sum criterion in $\mathcal{O}(t^{2k-2} \cdot n^4)$ time.*

5.2 NP-Hardness Results for k-Hull Watchman Routes

By leveraging similar arguments as in the proof of Theorem 13, we straightforwardly obtain NP-hardness for the min-max variant of our problem with k watchmen by reducing from Multiway Number Partitioning instead of Partition.

▶ **Corollary 21.** *The* Min-Max *k*-hull Watchman Routes Problem *is* NP*-hard, even in simple polygons.*

Moreover, we extend the proof of Theorem 14 by constructing k routes, each of which visits every hole and every corner niche, and traverses the convex hull of each hole. Additionally, for every corner niche and every hole, at least one of these k routes visits all convex vertices. This straightforwardly establishes:

▶ **Corollary 22.** *The* Min-Sum *k*-hull Watchman Routes Problem *is* NP*-hard in polygons with holes for $k \geq 2$.*

6 Conclusions and Future Work

In this paper, we studied segment watchman routes in simple polygons. First, we identified sufficient conditions for two routes to constitute valid segment watchman routes. Leveraging these conditions, we then designed a 2-approximation algorithm for segment watchman routes for both the min-max and min-sum optimization objectives.

Moreover, we establish both the NP-hardness of the Min-Max Segment Watchman Routes Problem even in simple polygons, as well as the NP-hardness of the Min-Sum k-hull Watchman Routes Problem in polygons with holes, highlighting the inherent computational difficulty of the problem.

Finally, we extended all our results to k-hull watchman routes, obtaining a k- and a 2-approximation algorithm for k-hull watchman routes for the min-max and min-sum optimization objectives, respectively.

Interior k-Hulls. In the problems considered in this paper, the k-hull (for all $k > 2$) defined by the points on the watchman routes that see a point $p \in P$ is not required to be contained in P. One can naturally define variants of the two problems in which this additional containment condition is imposed. However, the conditions of Lemma 15 are not sufficient in this setting, see Figure 8: the three routes satisfy all conditions of the lemma. The point p is seen from exactly one point on each of the orange and violet routes. As p is a

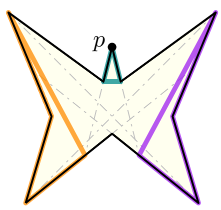


Figure 8 An example for the conditions from Lemma 15 not being sufficient for interior-k-hull watchman routes.

convex vertex, the corresponding point on the green route must be p itself. Since the triangle determined by these three points intersects the boundary of P, it is not fully contained in the polygon. Finding sufficient conditions for this variant, as well as developing approximation algorithms, remains an open problem.

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