A Forward-Only Construction of Semilinear Inductive Invariants for VAS

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Abstract

The reachability problem for Vector Addition Systems (VAS) is a central decision problem in the theory of infinite-state systems, first solved by Kosaraju and Mayr in the 1980s. An alternative, conceptually simpler approach introduced by Leroux shows that non-reachability is always witnessed by semilinear inductive invariants, yielding a decision procedure by combining an enumeration of runs with a search for such invariants. However, the construction of these invariants relies on a back-and-forth scheme that depends symmetrically on the source and the target. As a result, the invariants are not guaranteed to reflect the structural properties of the VAS, and the construction is difficult to extend to asymmetric models such as Branching VAS.

We introduce a new forward-only construction of semilinear inductive invariants for VAS. Our method builds invariants from the source configuration alone and avoids the need for backward reasoning. This yields invariants that are more canonical and better aligned with the structure of the system. In particular, our method produces periodic inductive invariants for periodic VAS.

Beyond its intrinsic interest, our approach provides a step toward extending invariant-based techniques to Branching VAS.

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1 Introduction

Vector Addition Systems (VAS), equivalently known as Petri nets, are a fundamental model of computation used to represent concurrent processes arising in areas such as distributed systems, chemical reaction networks, and biological systems. A VAS consists of an initial configuration in $\mathbb{N}^d$ together with a finite set of actions in $\mathbb{Z}^d$. The reachability problem asks whether a given target vector can be obtained by iteratively adding actions to the initial configuration while maintaining non-negativity.

The reachability problem for VAS is one of the most fundamental decision problems in the theory of infinite-state systems. It was first posed in the late 1960s [11] and its decidability, established in the early 1980s through the works of Mayr and Kosaraju [12, 21], is widely regarded as a major breakthrough in theoretical computer science. Despite this early result, the computational complexity of the problem has remained open for several decades, until it was finally shown to be Ackermann-complete, more than forty years after its introduction [20, 18, 4].

Two different approaches to the decidability of VAS reachability are known. The first one, often referred to as the KLM decomposition (after Kosaraju, Lambert, Mayr), is based on a structural analysis of runs. It proceeds by decomposing the system into simpler components until a certain sufficient condition for reachability is met, or the decomposition process is exhausted. While powerful, this approach is technically involved, and the underlying condition long appeared somewhat ad hoc, until it was later given a conceptual explanation in terms of ideals of well-quasi-orders [19]. Moreover, refinements of this approach have led to the Ackermann complexity upper bound [20].

A second, more recent approach was introduced by Leroux in the 2010s [15, 16]. It is based on the use of inductive invariants to characterize non-reachability. More precisely, Leroux showed that if a target configuration is not reachable, then there exists a semilinear set containing the initial configuration, closed under the actions of the system, and excluding the target. This yields a decision procedure by combining two semi-algorithms: one enumerating runs, and the other enumerating semilinear sets in search of such inductive invariants.

Numerous extensions of Vector Addition Systems have been introduced over the years, either to model richer computational phenomena or to explore the boundaries of decidability. For several of these extensions, the reachability problem remains decidable – for instance in the presence of nested zero tests [22, 8]. Recently, reachability has also been shown to be decidable for pushdown VAS [9], a long-standing open problem. However, it is still open for well-studied models, including unordered data nets [13] and, in particular, Branching Vector Addition Systems (BVAS) [5, 24]. BVAS extend VAS with branching transitions, allowing one to combine two reachable configurations into a new one. As a result, executions are no longer sequences but trees.

Extending existing techniques for VAS to BVAS is therefore a natural research direction. However, the classical KLM approach appears difficult to generalize to this setting. In contrast, the inductive invariant approach of Leroux seems more promising: recent results show that reachability sets of BVAS enjoy geometric properties similar to those used in Leroux's proof for VAS [2].

A major obstacle, however, lies in the fundamentally asymmetric nature of BVAS. Indeed, Leroux's construction crucially relies on a symmetry property of VAS: by taking the opposite of each action of a VAS, one obtains a reversed VAS in which every run from $\mathbf{s}$ to $\mathbf{t}$ in the original VAS can be reversed into a run from $\mathbf{t}$ to $\mathbf{s}$. This symmetry breaks down in BVAS, where computations are inherently tree-shaped: the initial configuration is placed at the leaves, while the target appears at the root.

Beyond these considerations, Leroux's construction already presents certain limitations even in the setting of plain VAS. His approach follows a back-and-forth scheme: it alternately expands a semilinear over-approximation $\mathbf{S}$ of the forward reachability set from the source and a semilinear over-approximation $\mathbf{T}$ of the backward reachability set from the (unreachable) target, while ensuring that there is no run from $\mathbf{S}$ to $\mathbf{T}$, until the two sets become complementary (which implies that $\mathbf{S}$ is inductive). As a result, the inductive invariant obtained depends equally on the source and the target.

However, in many applications, the source and the target play fundamentally different roles. The source is part of the system specification, fixed once and for all, whereas the target represents a query that may vary. This asymmetry is also reflected in other VAS verification problems, such as coverability or boundedness, which are inherently one-sided.

From this perspective, the back-and-forth nature of Leroux's construction makes the resulting invariants less canonical and harder to relate directly to the structure of the VAS. A good example is the case of periodic VAS, i.e., VAS whose reachability set is periodic, meaning that it contains the zero vector and is closed under addition. Such systems naturally arise as abstractions of population protocols [1] and leaderless communication protocols [6]. The inductive invariants produced by Leroux's method for periodic VAS are not always periodic.

Our contributions.

(i) We propose a new forward-only construction of inductive invariants for VAS. In contrast with Leroux's back-and-forth approach, our construction depends primarily on the reachable configurations from the source rather than on the co-reachable configurations from the target and builds invariants in a purely forward manner.[1] As a result, it yields inductive invariants that more directly reflect the structure of the system.

(ii) We introduce the class of periodic VAS defined as the class of VAS with periodic reachability sets. We provide an effective characterization of those VAS proving that the class of periodic VAS is recursive. We prove that the reachability problem for plain VAS can be reduced to the reachability problem for periodic VAS and we explain how semilinear inductive invariant for plain VAS can be computed from semilinear periodic inductive invariant for periodic VAS.

(iii) We apply our forward-only construction of inductive invariants to the class of periodic VAS and we prove that if the reachability set of a periodic VAS has an empty intersection with a semilinear set of target configurations, then there exists a semilinear periodic inductive invariant that is disjoint from the semilinear target set.

2 Example

These examples illustrate the differences between our forward-only construction and Leroux's back-and-forth construction. A detailed presentation is given in the full version [3].

▶ **Example 2.1.** Consider the VAS with actions $\mathbf{A} \coloneqq \{(0,1,0),(0,-1,1),(1,2,-2)\}$ and initial configuration $\mathbf{C}_0 = \{\mathbf{0}\}$. We take unreachable targets of the form $\mathbf{C}_{\text{bad}} \coloneqq \{(x_{\text{bad}},1,0)\}$ for some $x_{\text{bad}} \geq 1$. The back-and-forth construction yields inductive invariants of the form $\mathbb{N}^3 \setminus ([1,\infty) \times \{(0,0)\} \cup \{(x_{\text{bad}},1,0),(x_{\text{bad}},0,0)\})$, while our forward-only construction returns $\{(0,0,0),(0,1,0)\} \cup ((0,2,0)+\mathbb{N}^3) \cup ((0,0,1)+\mathbb{N}^3)$. In particular, the invariant produced by the back-and-forth construction depends strongly on $\mathbf{C}_{\text{bad}}$, unlike the forward-only one.

▶ **Example 2.2.** Consider the periodic VAS with actions $\mathbf{A} \coloneqq \{5,6\}$ and initial configuration $\mathbf{C}_0 \coloneqq \{0\}$. Let $\mathbf{C}_{\text{bad}} = \{14\}$. The back-and-forth construction yields the invariant $\mathbb{N} \setminus \{1,2,3,4,8,9,14\}$, which is not periodic (it contains 7 but not $7+7=14$). By contrast, the forward-only construction returns the periodic invariant $\mathbb{N} \setminus \{1,2,3,4,7,8,9,13,14\}$.

[1] Our construction still depends on the target but this is unavoidable. Indeed, since the forward reachability set from the source is not semilinear in general, there is no semilinear inductive invariant disjoint from every unreachable target.

3 Preliminaries

This section recalls the definitions of periodic sets and semilinear sets as well as classical results about vector addition systems. In the sequel, the set of natural numbers, integers, non-negative rational numbers, and rational numbers are denoted respectively by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}_{\geq 0}$, and $\mathbb{Q}$. Vectors as well as sets of vectors are denoted in bold face and operations are extended component-wise. We denote by $\mathbf{e}_i$ the *ith unit vector* of $\mathbb{N}^d$ defined by $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for every $j \neq i$. The sum $\mathbf{X}+\mathbf{Y}$ of two sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Q}^d$ is the set $\{\mathbf{x}+\mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}\}$. We also denote by $\mathbf{X}+\mathbf{y}$ and $\mathbf{x}+\mathbf{Y}$ the sets $\mathbf{X}+\{\mathbf{y}\}$ and $\{\mathbf{x}\}+\mathbf{Y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$ and $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Q}^d$. We denote by $\mathbb{Q}\mathbf{X}$ and $\mathbb{Q}_{\geq 0}\mathbf{X}$ the sets $\{\lambda\mathbf{x} \mid (\lambda, \mathbf{x}) \in \mathbb{Q} \times \mathbf{X}\}$ and $\{\lambda\mathbf{x} \mid (\lambda, \mathbf{x}) \in \mathbb{Q}_{\geq 0} \times \mathbf{X}\}$ respectively.

3.1 Vector Spaces and Space-dimension

A *vector space* is a set $\mathbf{V} \subseteq \mathbb{Q}^d$ that contains the zero vector, such that $\mathbf{V} + \mathbf{V} \subseteq \mathbf{V}$ and such that $\mathbb{Q}\mathbf{V} \subseteq \mathbf{V}$. The vector space *spanned* by a set $\mathbf{X} \subseteq \mathbb{Q}^d$ is the set of finite sums $\lambda_1\mathbf{x}_1 + \cdots + \lambda_k\mathbf{x}_k$ of $k \in \mathbb{N}$ vectors $\mathbf{x}_j \in \mathbf{X}$ scaled by rational numbers $\lambda_j \in \mathbb{Q}$. Recall that every vector space $\mathbf{V} \subseteq \mathbb{Q}^d$ is spanned by a finite set of vectors. The minimal number of vectors spanning $\mathbf{V}$ is called the *dimension* of $\mathbf{V}$ and it is denoted by $\dim(\mathbf{V})$. Let us recall that $\dim(\mathbf{V}) = 0$ if and only if $\mathbf{V} = \{\mathbf{0}\}$. Moreover $\dim(\mathbf{V}) < \dim(\mathbf{W})$ for every vector spaces $\mathbf{V} \subset \mathbf{W}$, and in particular $\dim(\mathbf{V}) \leq d$ for every vector space $\mathbf{V} \subseteq \mathbb{Q}^d$. Finally, if $\mathbf{V}$ is spanned by a set $\mathbf{X} \subseteq \mathbb{Q}^d$ then $\mathbf{V}$ is spanned by a subset of $\dim(\mathbf{V})$ vectors in $\mathbf{X}$.

The *space-dimension* [15, Section 5] of a set $\mathbf{X} \subseteq \mathbb{Q}^d$ is the minimal $r \in \{-1, \ldots, d\}$ such that there exists a sequence $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{Q}^d$ and a sequence $\mathbf{V}_1, \ldots, \mathbf{V}_k$ of vector spaces $\mathbf{V}_j$ such that $\dim(\mathbf{V}_j) \leq r$ and such that $\mathbf{X} \subseteq \mathbf{x}_1 + \mathbf{V}_1 \cup \cdots \cup \mathbf{x}_k + \mathbf{V}_k$. We denote by $\operatorname{sdim}(\mathbf{X})$ the *space-dimension* of $\mathbf{X}$. Observe that $\operatorname{sdim}(\mathbf{X}) = -1$ iff $\mathbf{X}$ is empty and $\operatorname{sdim}(\mathbf{X}) = 0$ iff $\mathbf{X}$ is a non-empty finite set. Note also that $\operatorname{sdim}(\mathbf{X} \cup \mathbf{Y}) = \max\{\operatorname{sdim}(\mathbf{X}), \operatorname{sdim}(\mathbf{Y})\}$, $\operatorname{sdim}(\mathbf{X}) \leq \operatorname{sdim}(\mathbf{Y})$ if $\mathbf{X} \subseteq \mathbf{Y}$, and $\operatorname{sdim}(\mathbf{x}+\mathbf{X}) = \operatorname{sdim}(\mathbf{X})$ for every $\mathbf{x} \in \mathbb{Q}^d$ and $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Q}^d$.

▶ **Example 3.1.** The space dimension of $(\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\})$ is 1. The space dimension of $\{(n, 2^{n+1}) \mid n \in \mathbb{N}\}$ is 2.

The following lemma and corollary shows that the space-dimension of a vector space coincides with its dimension. The proofs are similar to [16, Lemma 7.3] and [16, Lemma 7.4]. They are also given in the full version [3].

▶ **Lemma 3.2.** *Let $\mathbf{P} \subseteq \mathbb{Q}^d$ such that $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. Let $k \in \mathbb{N}_{>0}$, $\mathbf{V}_1, \ldots, \mathbf{V}_k$ be a sequence of vector spaces of $\mathbb{Q}^d$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be a sequence of vectors in $\mathbb{Q}^d$. We have $\mathbf{P} \subseteq \bigcup_{j=1}^{k} \mathbf{x}_j + \mathbf{V}_j$ if, and only if, there exists $j \in \{1, \ldots, k\}$ such that $\mathbf{P} \subseteq \mathbf{x}_j + \mathbf{V}_j$.*

Proof Sketch. We observe that if there exists $\mathbf{p}_0 \in \mathbf{P} \setminus (\mathbf{x}_k + \mathbf{V}_k)$ then $\mathbf{p}_0 + \mathbf{P} \subseteq \bigcup_{j=1}^{k-1} \mathbf{x}_j + \mathbf{V}_j$. This property is obtained by observing that if $\mathbf{p}_0 + \mathbb{N}\mathbf{p}$ where $\mathbf{p} \in \mathbf{P}$ has an infinite intersection with some $\mathbf{x}_j + \mathbf{V}_j$, then $\mathbf{p}_0 + \mathbb{Q}\mathbf{p}$ is included in $\mathbf{x}_j + \mathbf{V}_j$ since $\mathbf{V}_j$ is a vector space. The proof of the lemma is then obtained by induction on k. ◀

▶ **Corollary 3.3.** *We have $\operatorname{sdim}(\mathbf{P}) = \dim(\mathbf{V})$ where $\mathbf{V}$ is the vector space spanned by a non-empty set $\mathbf{P} \subseteq \mathbb{Q}^d$ such that $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$.*

3.2 Periodic Sets and Semilinear Sets

A set $\mathbf{P} \subseteq \mathbb{N}^d$ is said to be *periodic* if it is a submonoid of $(\mathbb{N}^d, +)$, i.e., if it is such that $\mathbf{0} \in \mathbf{P}$ and $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. The periodic set *spanned* by a set $\mathbf{G} \subseteq \mathbb{N}^d$ is the set $\mathrm{Per}(\mathbf{G})$ of finite sums $\mathbf{g}_1 + \cdots + \mathbf{g}_k$ of $k \in \mathbb{N}$ vectors $\mathbf{g}_j \in \mathbf{G}$. Notice that $\mathrm{Per}(\mathbf{G})$ is the minimal periodic set that contains $\mathbf{G}$. A *periodic set* $\mathbf{P} \subseteq \mathbb{N}^d$ is said to be *finitely-generated* if $\mathbf{P} = \mathrm{Per}(\mathbf{G})$ for some finite set $\mathbf{G} \subseteq \mathbb{N}^d$.

A *linear set* is a set $\mathbf{L} \subseteq \mathbb{N}^d$ of the form $\mathbf{b} + \mathrm{Per}(\mathbf{G})$ where $\mathbf{b} \in \mathbb{N}^d$ is called the *base* and $\mathbf{G} \subseteq \mathbb{N}^d$ is a finite set of vectors called the *periods*. A pair $(\mathbf{b}, \mathbf{G})$ is called a *linear-presentation* of $\mathbf{L}$. A *semilinear set* $\mathbf{S} \subseteq \mathbb{N}^d$ is a finite union $\mathbf{L}_1 \cup \ldots \cup \mathbf{L}_k$ of linear sets $\mathbf{L}_j \subseteq \mathbb{N}^d$ where $k \in \mathbb{N}$. A finite set $\{(\mathbf{b}_1, \mathbf{G}_1), \ldots, (\mathbf{b}_k, \mathbf{G}_k)\}$ where $(\mathbf{b}_j, \mathbf{G}_j)$ is a linear-presentations of $\mathbf{L}_j$ is called a *semilinear-presentation* of $\mathbf{S}$. In the sequel semilinear sets (resp. linear sets) are always assumed to be implicitly given by presentations.

Let us recall that the class of semilinear sets is stable by many operations, and in particular by sum, union, intersection, complement, projection, and Cartesian product since a set is semilinear iff it is definable in the Presburger arithmetic [7]. Moreover, the periodic sets spanned by semilinear sets are semilinear since $\mathrm{Per}(\mathbf{X} \cup \mathbf{Y}) = \mathrm{Per}(\mathbf{X}) + \mathrm{Per}(\mathbf{Y})$ for every sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$.

▶ **Example 3.4.** A finitely-generated periodic set is clearly semilinear. The converse is not true since the periodic set $\mathbf{P} = \{(0,0)\} \cup \mathbb{N}^2_{>0}$ is semilinear but not finitely-generated.

In the sequel, we are going to under-approximate linear sets $\mathbf{L}$ by linear sets of the form $\mathbf{L}' \coloneqq \mathbf{L} + \mathbf{q}$ for some $\mathbf{q} \in \mathbb{N}^d$ such that $\mathbf{L}' \subseteq \mathbf{L}$. The following lemma shows that the space-dimension of the difference $\mathbf{L} \setminus \mathbf{L}'$ is strictly smaller than the space-dimension of $\mathbf{L}$ (detailed proofs are given in the full version [3]).

▶ **Lemma 3.5.** *For every linear set $\mathbf{L} \subseteq \mathbb{N}^d$ and for every $\mathbf{q} \in \mathbb{N}^d$ such that $\mathbf{L}' \coloneqq (\mathbf{L}+\mathbf{q}) \subseteq \mathbf{L}$, we have:*

$$\mathrm{sdim}(\mathbf{L} \setminus \mathbf{L}') < \mathrm{sdim}(\mathbf{L})$$

▶ Remark 3.6. If $\mathbf{L} = \mathbf{b} + \mathbf{P}$ is a linear set, a vector $\mathbf{q}$ is such that $\mathbf{L} + \mathbf{q} \subseteq \mathbf{L}$ iff $\mathbf{q} \in \mathbf{P}$.

3.3 Vector Addition Systems (VAS)

A *d-dim vector addition system* (*d-VAS* or just *VAS* for short) is a finite set $\mathbf{A}$ of vectors in $\mathbb{Z}^d$ called *actions*. A *configuration* is a vector in $\mathbb{N}^d$. An *initialized VAS*, or just a *VAS* if there is no ambiguity, is a pair $(\mathbf{A}, \mathbf{C}_0)$ where $\mathbf{C}_0 \subseteq \mathbb{N}^d$ is a semilinear set of *initial configurations*. The *reachability set* of a VAS $(\mathbf{A}, \mathbf{C}_0)$ is the set of configurations $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$ defined as the unique minimal (for the inclusion) set of configurations satisfying the following monovariate system of constraints:

$$\mathbf{X} \supseteq \mathbf{C}_0 \tag{1}$$

$$\mathbf{X} \supseteq (\mathbf{X} + \mathbf{a}) \cap \mathbb{N}^d \qquad \forall \mathbf{a} \in \mathbf{A} \tag{2}$$

An *inductive invariant* for an initialized VAS $(\mathbf{A}, \mathbf{C}_0)$, resp. for a VAS $\mathbf{A}$, is a set of configurations $\mathbf{X} \subseteq \mathbb{N}^d$ satisfying (1) and (2), resp. only (2). Clearly, the reachability set of $(\mathbf{A}, \mathbf{C}_0)$ is the minimal for the inclusion inductive invariant for $(\mathbf{A}, \mathbf{C}_0)$. Since semilinear sets are effective for many operations, it follows the class of semilinear inductive invariants (for a initialized VAS or a VAS) is recursive.

The reachability set can be equivalently expressed by introducing the classical notion of runs as follows. We associate with an action $\mathbf{a} \in \mathbf{A}$ the binary relation $\xrightarrow{\mathbf{a}}$ on $\mathbb{N}^d$ defined by $\mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y}$ if $\mathbf{y} = \mathbf{x} + \mathbf{a}$. A *run* ρ of a vector addition system $\mathbf{A}$ is a non-empty word of configurations $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ such that for every $j \in \{1, \ldots, k\}$ there exists an action $\mathbf{a}_j \in \mathbf{A}$ such that $\mathbf{c}_{j-1} \xrightarrow{\mathbf{a}_j} \mathbf{c}_j$. In that case, we say that ρ is a run from $\mathbf{c}_0$ to $\mathbf{c}_k$ labeled by $\sigma = \mathbf{a}_1 \ldots \mathbf{a}_k$ and we write $\mathbf{c}_0 \xrightarrow{\sigma} \mathbf{c}_k$. The configurations $\mathbf{c}_0$ and $\mathbf{c}_k$ are respectively called the *source* and *target* of ρ, and they are denoted by $\mathrm{src}(\rho)$ and $\mathrm{tgt}(\rho)$. The *reachability relation* of a VAS $\mathbf{A}$ is the binary relation $\xrightarrow{\mathbf{A}^*}$ on the configurations defined by $\mathbf{x} \xrightarrow{\mathbf{A}^*} \mathbf{y}$ if there exists a run from $\mathbf{x}$ to $\mathbf{y}$ labeled by a word in $\mathbf{A}^*$. Observe that $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$ is the set of configurations $\mathbf{y} \in \mathbb{N}^d$ such that there exists a run from a configurations $\mathbf{x} \in \mathbf{C}_0$ labeled by a word in $\mathbf{A}^*$ to $\mathbf{y}$.

The reachability problem for VAS takes as input a VAS $(\mathbf{A}, \mathbf{C}_0)$ and a semilinear set $\mathbf{C}_{bad}$ and checks if $\mathbf{C}_{bad}$ has an empty intersection with the reachability set of $(\mathbf{A}, \mathbf{C}_0)$. Based on a simple algorithm that enumerates semilinear inductive invariants and runs, the following theorem shows that the reachability problem for VAS is decidable.

▶ **Theorem 3.7** ([14, 15, 16])**.** *The reachability set of a VAS $(\mathbf{A}, \mathbf{C}_0)$ has an empty intersection with a semilinear set $\mathbf{C}_{bad}$ if and only if there exists a semilinear inductive invariant $\mathbf{I}$ of $(\mathbf{A}, \mathbf{C}_0)$ such that $\mathbf{I} \cap \mathbf{C}_{bad}$ is empty.*

4 Periodic VAS

A VAS $(\mathbf{A}, \mathbf{C}_0)$ is said to be *periodic* if its reachability set is periodic. The following lemma shows that if $\mathbf{C}_0$ is periodic then $(\mathbf{A}, \mathbf{C}_0)$ is periodic. It also shows that if a VAS $(\mathbf{A}, \mathbf{C}_0)$ is periodic, then we can replace $\mathbf{C}_0$ by the semilinear periodic set $\mathrm{Per}(\mathbf{C}_0)$ without modifying the reachability set. Last but not least, since the inclusion of a semilinear set in the reachability set of a VAS is decidable [17], this lemma also proves that the class of periodic VAS is recursive.

▶ **Lemma 4.1.** *Let $(\mathbf{A}, \mathbf{C}_0)$ be a VAS. The following properties are equivalent:*

(i) $(\mathbf{A}, \mathbf{C}_0)$ *is periodic.*

(ii) $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0) = \mathrm{Post}^*_{\mathbf{A}}(\mathrm{Per}(\mathbf{C}_0))$.

(iii) $\mathrm{Per}(\mathbf{C}_0) \subseteq \mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$.

Proof. $(iii) \Rightarrow (ii)$ is trivial since $\mathrm{Per}(\mathbf{C}_0) \subseteq \mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$ implies $\mathrm{Post}^*_{\mathbf{A}}(\mathrm{Per}(\mathbf{C}_0)) \subseteq \mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$. For $(i) \Rightarrow (iii)$, just notice that $\mathbf{C}_0 \subseteq \mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$ implies $\mathrm{Per}(\mathbf{C}_0) \subseteq \mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$ since $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$ is periodic.

Finally let us prove $(ii) \Rightarrow (i)$. Let $\mathbf{P}$ be the reachability set of $(\mathbf{A}, \mathrm{Per}(\mathbf{C}_0))$ and let us prove that $\mathbf{P}$ is periodic. We clearly have $\mathbf{0} \in \mathbf{P}$. Let $\mathbf{p}_1, \mathbf{p}_2 \in \mathbf{P}$. There exist a run ρ_1 from a configuration $\mathbf{x}_1 \in \mathrm{Per}(\mathbf{C}_0)$ to $\mathbf{p}_1$ and a run ρ_2 from a configuration $\mathbf{x}_2 \in \mathrm{Per}(\mathbf{C}_0)$ to $\mathbf{p}_2$. Notice that the word $\rho_1 + \mathbf{x}_2$ obtained from ρ_1 by adding $\mathbf{x}_2$ on each configuration is a run from $\mathbf{x}_1 + \mathbf{x}_2$ to $\mathbf{p}_1 + \mathbf{x}_2$. Symmetrically, $\rho_2 + \mathbf{p}_1$ is a run from $\mathbf{p}_1 + \mathbf{x}_2$ to $\mathbf{p}_1 + \mathbf{p}_2$. We deduce that $\mathbf{p}_1 + \mathbf{p}_2 \in \mathbf{P}$. Therefore $\mathbf{P}$ is periodic. ◀

The following lemma shows that the previous lemma cannot be extended to the projection of VAS reachability sets.

▶ **Lemma 4.2.** *We cannot decide if a VAS $(\mathbf{A}, \mathbf{C}_0)$ with $\mathbf{A} \subseteq \mathbb{N}^d \times \mathbb{N}^e$ for some $d, e \in \mathbb{N}$ is such that the set $\{\mathbf{x} \in \mathbb{N}^d \mid \exists \mathbf{y} \in \mathbb{N}^e, (\mathbf{x}, \mathbf{y}) \in \mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)\}$ is periodic.*

In the rest of this section, we show that periodic VAS naturally occurs when dealing with diagonal semilinear relations (see Section 4.1). We then explain how the reachability problem for plain VAS can be reduced to the reachability problem for periodic VAS (see Section 4.2). Finally, in Section 4.3 we introduce the main result of this paper.

4.1 Diagonal Relations

This is a folklore result that the reachability relation of a d-dim VAS $\mathbf{A} \subseteq \mathbb{N}^d$ corresponds to the reachability set of $2d$-dim initialized periodic VAS $(\mathbf{A}', \{(\mathbf{0},\mathbf{0})\})$ defined as follows where $\mathbf{e}_i$ is the ith unit vector (see preliminaries):

$$\mathbf{A}' = (\{\mathbf{0}\} \times \mathbf{A}) \cup \{(\mathbf{e}_i, \mathbf{e}_i) \mid 1 \leq i \leq d\}$$

This observation can be extended to the reflexive and transitive closure of any semilinear *diagonal relation* defined as follows. A binary relation $\mathbf{R}$ over $\mathbb{N}^d$ is said to be *diagonal* if $(\mathbf{x},\mathbf{y}) + (\mathbf{c},\mathbf{c}) \in \mathbf{R}$ for every $(\mathbf{x},\mathbf{y}) \in \mathbf{R}$ and for every $\mathbf{c} \in \mathbb{N}^d$. Diagonal relations are clearly stable by union, intersection, composition, and in particular by transitive and reflexive closure. Let us denote by $\mathbf{R}^*$ the reflexive and transitive closure of a binary relation $\mathbf{R}$.

▶ **Lemma 4.3.** *The reflexive and transitive closure of a diagonal relation is periodic.*

Proof. Let $\mathbf{R}$ be a diagonal reflexive transitive binary relation. Clearly $(\mathbf{0},\mathbf{0}) \in \mathbf{R}$ since $\mathbf{R}$ is reflexive. Let $(\mathbf{x}_1,\mathbf{y}_1)$ and $(\mathbf{x}_2,\mathbf{y}_2)$ be pairs in $\mathbf{R}$. Since $\mathbf{R}$ is diagonal from the membership in $\mathbf{R}$ of these two pairs, it follows that $(\mathbf{x}_1,\mathbf{y}_1) + (\mathbf{x}_2,\mathbf{x}_2)$ and $(\mathbf{x}_2,\mathbf{y}_2) + (\mathbf{y}_1,\mathbf{y}_1)$ are in $\mathbf{R}$. Since $\mathbf{R}$ is transitive, we deduce that $(\mathbf{x}_1+\mathbf{x}_2,\mathbf{y}_1+\mathbf{y}_2)$ is in $\mathbf{R}$. Since this pair is equal to $(\mathbf{x}_1,\mathbf{y}_1) + (\mathbf{x}_2,\mathbf{y}_2)$. We have proved that $\mathbf{R}+\mathbf{R} \subseteq \mathbf{R}$. It follows that $\mathbf{R}$ is periodic. ◀

The following lemma shows that reflexive and transitive closures of semilinear diagonal relations are related to the model of VAS with states, or equivalently to regular expressions of VAS actions, an extension of the model of VAS with the same expressive power [11]. We do not recall the classical definition of regular expressions over a finite alphabet, but we just emphasis that the relation $\xrightarrow{E}$ in the following lemma where E is a regular expression over a $2d$-dim VAS (that plays the role of a finite alphabet) is the binary relation over the pairs in $\mathbb{N}^d \times \mathbb{N}^d$ defined by $(\mathbf{x},\tilde{\mathbf{x}}) \xrightarrow{E} (\mathbf{y},\tilde{\mathbf{y}})$ if there exists a word σ accepted by E such that $(\mathbf{x},\tilde{\mathbf{x}}) \xrightarrow{\sigma} (\mathbf{y},\tilde{\mathbf{y}})$.

▶ **Lemma 4.4.** *For every semilinear diagonal relation R over $\mathbb{N}^d$, we can effectively build a regular expression E over the actions of a $2d$-dim VAS such that for all pairs $(\mathbf{x},\tilde{\mathbf{x}})$ and $(\mathbf{y},\tilde{\mathbf{y}})$ in $\mathbb{N}^d \times \mathbb{N}^d$, we have:*

$$(\mathbf{x}+\tilde{\mathbf{x}})\mathbf{R}^*(\mathbf{y}+\tilde{\mathbf{y}}) \iff (\mathbf{x},\tilde{\mathbf{x}}) \xrightarrow{E} (\mathbf{y},\tilde{\mathbf{y}})$$

Proof. We introduce the finite set A_{swap} of actions of $\mathbb{Z}^d \times \mathbb{Z}^d$ defined as the set of vectors $(\mathbf{e}_i,-\mathbf{e}_i)$ and $(-\mathbf{e}_i,\mathbf{e}_i)$ where i ranges over $\{1,\ldots,d\}$. Notice that for all pairs $(\mathbf{x},\tilde{\mathbf{x}})$ and $(\mathbf{y},\tilde{\mathbf{y}})$ in $\mathbb{N}^d \times \mathbb{N}^d$, we have:

$$(\mathbf{x},\tilde{\mathbf{x}}) \xrightarrow{A_{swap}^*} (\mathbf{y},\tilde{\mathbf{y}}) \iff \mathbf{x}+\tilde{\mathbf{x}} = \mathbf{y}+\tilde{\mathbf{y}}$$

Now, assume that $\mathbf{R}$ is a semilinear diagonal relation given by a semilinear-presentation $\{(b_1,G_1),\ldots,(b_k,G_k)\}$. By replacing G_j by $G_j \cup \bigcup_i \{(\mathbf{e}_i,\mathbf{e}_i)\}$, we do not change the semilinear relation $\mathbf{R}$ denoted by the semilinear-presentation since $\mathbf{R}$ is diagonal. So, w.l.o.g., we can assume that $(\mathbf{e}_i, \mathbf{e}_i) \in G_j$ for every i, j. Let us introduce the function $\delta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d \times \mathbb{Z}^d$ defined by $\delta(\mathbf{x}, \mathbf{y}) = (-\mathbf{x}, \mathbf{y})$ and observe that for all pairs $(\mathbf{x}, \tilde{\mathbf{x}})$ and $(\mathbf{y}, \tilde{\mathbf{y}})$ in $\mathbb{N}^d \times \mathbb{N}^d$ and for every $j \in \{1, \dots, k\}$, we have:

$$(\mathbf{x} + \tilde{\mathbf{x}}, \mathbf{y} + \tilde{\mathbf{y}}) \in b_j + \mathrm{Per}(G_j) \iff (\mathbf{x}, \tilde{\mathbf{x}}) \xrightarrow{A_{swap}^* \delta(b_j) \delta(G_j)^* A_{swap}^*} (\mathbf{y}, \tilde{\mathbf{y}})$$

It follows that the following regular expression E satisfies the lemma.

$$E = A_{swap}^* \cup \Big(\bigcup_{j=1}^{k} A_{swap}^* \delta(b_j) \delta(G_j)^* A_{swap}^*\Big)^* \qquad \blacktriangleleft$$

4.2 From Plain VAS to Periodic VAS

Periodic VAS is a central model for population protocols [1] as well as for communication protocols without leaders [6] since for those models, the set $\mathbf{C}_0$ of initial configurations is a periodic semilinear set of the form $\mathbb{N}(1, 0, \dots, 0)$ where the first counter intuitively counts the number of agents in a given initial state. Even if periodic VAS seems to be a strict subclass of VAS, the reachability problem for plain VAS can be easily reduced to the reachability of a periodic VAS by adding just one extra counter that is unchanged by the VAS actions. Let us explain the construction.

Assume that $\mathcal{V} = (\mathbf{A}, \mathbf{C}_0)$ is a VAS of dimension d, i.e. such that $\mathbf{A} \subseteq \mathbb{N}^d$ is a finite set of actions and $\mathbf{C}_0 \subseteq \mathbb{N}^d$ is a semilinear set of initial configurations given by a semilinear-presentation. We introduce the periodic VAS $\mathcal{V}' = (\mathbf{A}', \mathbf{C}'_0)$ where $\mathbf{A}' = \mathbf{A} \times \{0\}$ and $\mathbf{C}'_0$ is the semilinear periodic set $\mathrm{Per}(\mathbf{C}_0 \times \{1\})$. Now, just observe that the following equality holds.

$$\mathrm{Post}_{\mathbf{A}}^*(\mathbf{C}_0) \times \{1\} = \mathrm{Post}_{\mathbf{A}'}^*(\mathbf{C}'_0) \cap (\mathbb{N}^d \times \{1\}) \tag{3}$$

In particular, the reachability problem for plain VAS that consists in deciding if a semilinear set $\mathbf{C}_{bad} \subseteq \mathbb{N}^d$ has an empty intersection with $\mathrm{Post}_{\mathbf{A}}^*(\mathbf{C}_0)$ reduces to the reachability problem of $(\mathbf{A}', \mathbf{C}'_0)$ for the semilinear set $\mathbf{C}'_{bad}$ defined as $\mathbf{C}_{bad} \times \{1\}$. The computation of inductive invariants for plain VAS also reduces to the computation of inductive invariants for periodic VAS as follows. Assume that $\mathbf{I}'$ is an inductive invariant for the periodic VAS $(\mathbf{A}', \mathbf{C}'_0)$ such that $\mathbf{I}' \cap \mathbf{C}'_{bad}$ is empty. We introduce the semilinear set $\mathbf{I}$ defined as $\{\mathbf{x} \in \mathbb{N}^d \mid (\mathbf{x}, 1) \in \mathbf{I}'\}$. We observe that $\mathbf{I}$ is an inductive invariant for $(\mathbf{A}, \mathbf{C}_0)$ such that $\mathbf{I} \cap \mathbf{C}_{bad}$ is empty.

4.3 Periodic Semilinear Inductive Invariants

The main result of the paper is the proof of the following theorem that extends Theorem 3.7 of [14, 15, 16] to periodic VAS. We have seen in Example 2.2 that the back-and-forth construction of [14, 15, 16] may produce non-periodic inductive invariants.

▶ **Theorem 4.5.** *The reachability set of a periodic VAS $(\mathbf{A}, \mathbf{C}_0)$ has an empty intersection with a semilinear set $\mathbf{C}_{bad}$ if and only if there exists a periodic semilinear inductive invariant $\mathbf{I}$ of $(\mathbf{A}, \mathbf{C}_0)$ such that $\mathbf{I} \cap \mathbf{C}_{bad}$ is empty.*

5 Geometry of VAS Reachability Sets

In this section we recall the classical well-quasi-order on the set of runs of a VAS. We also recall the geometrical characterization of the VAS reachability sets by almost semilinear sets. Finally, we introduce the notion of interior vectors of a periodic set, a new notion that provides a way to push vectors in the approximation of almost linear sets $\mathbf{L}$ (called linearization in the sequel) to be in the original set $\mathbf{L}$.

5.1 Well-Quasi-Orders

A binary relation $\sqsubseteq$ on a set S is said to be *almost-full* [23] if for every infinite sequence $(s_n)_{n\in\mathbb{N}}$ of elements in S, there exists $m < n$ such that $s_m \sqsubseteq s_n$. Let us recall that a relation is almost-full if, and only if, for every infinite sequence $(s_n)_{n\in\mathbb{N}}$ of elements in S, there exists an infinite sequence $n_0 < n_1 < \cdots$ such that $s_{n_i} \sqsubseteq s_{n_j}$ for all $i < j$. This characterization is an immediate consequence of the infinite version of Ramsey's Theorem.

A quasi-order that is almost-full is called a *well-quasi-order*. Let us recall that the classical order $\leq$ on $\mathbb{N}$ is a well-quasi-order. Moreover, since the Cartesian product of two well-quasi-orders is a well-quasi-order (*Dickson's lemma*), it follows that the component-wise extension of $\leq$ on $\mathbb{N}^d$ is also a well-quasi-order.

In this subsection, we associate an almost-full relation with each semilinear subset of $\mathbb{N}^d$. These almost-full relations will be used in the sequel (namely, in the proof of Lemma 6.4).

Given a set $\mathbf{S} \subseteq \mathbb{N}^d$, we introduce the binary relation $\leq_{\mathbf{S}}$ on $\mathbf{S}$ defined by $\mathbf{x} \leq_{\mathbf{S}} \mathbf{y}$ if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} + \mathbb{N}(\mathbf{y} - \mathbf{x}) \subseteq \mathbf{S}$. We observe that $\leq_{\mathbf{S}}$ is not necessarily transitive (this is the reason why we need the notion of almost-full relations).

▶ **Lemma 5.1.** *The relation $\leq_{\mathbf{S}}$ is almost-full for every semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$.*

Proof. Let us first prove the lemma for a linear set $\mathbf{L}$ given by a linear presentation $(\mathbf{b}, \mathbf{G})$, i.e such that $\mathbf{L} = \mathbf{b} + \mathrm{Per}(\mathbf{G})$ for $\mathbf{b} \in \mathbb{N}^d$ and $\mathbf{G} = \{\mathbf{g}_1, \ldots, \mathbf{g}_k\}$ is a finite set of vectors in $\mathbb{N}^d$. Let us consider a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of vectors in $\mathbf{L}$. We introduce the mapping $f : \mathbb{N}^k \to \mathbf{L}$ defined by $f(c_1, \ldots, c_k) = \mathbf{b} + \sum_{j=1}^{k} c_j \mathbf{g}_j$. Since $f(\mathbb{N}^k) = \mathbf{L}$, we deduce that for every $n \in \mathbb{N}$, there exists $\mathbf{c}_n \in \mathbb{N}^k$ such that $\mathbf{x}_n = f(\mathbf{c}_n)$. Since the quasi-order $\leq$ on $\mathbb{N}^k$ is almost-full, there exists $m < n$ such that $\mathbf{c}_m \leq \mathbf{c}_n$. It follows that $\mathbf{x}_m \leq_{\mathbf{L}} \mathbf{x}_n$ and we have proved that $\leq_{\mathbf{L}}$ is almost-full.

Now, let us consider a semilinear set $\mathbf{S}$ of the form $\mathbf{L}_1 \cup \ldots \cup \mathbf{L}_k$ where $\mathbf{L}_j$ is a linear set for every j. Let us consider an infinite sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathbf{S}$. There exists j such that $\mathbf{x}_n \in \mathbf{L}_j$ for infinitely many indices $n \in \mathbb{N}$. As $\leq_{\mathbf{L}_j}$ is almost-full, we deduce that there exists $m < n$ such that $\mathbf{x}_m \leq_{\mathbf{L}_j} \mathbf{x}_n$. In particular, as $\mathbf{L}_j \subseteq \mathbf{S}$ we deduce that $\mathbf{x}_m \leq_{\mathbf{S}} \mathbf{x}_n$. We have proved that $\leq_{\mathbf{S}}$ is almost-full. ◀

We introduce the binary relation $\trianglelefteq$ on runs of a VAS $\mathbf{A}$ defined by $\rho \trianglelefteq \rho'$ if $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ for some configurations $\mathbf{c}_0, \ldots, \mathbf{c}_k$ and $\rho' = \rho_0 \ldots \rho_k$ for some runs ρ_j such that $\mathrm{src}(\rho_j), \mathrm{tgt}(\rho_j) \geq \mathbf{c}_j$. Let us recall [10, 19] that $\trianglelefteq$ is a well-quasi-order satisfying the following *amalgamation property*.

▶ **Lemma 5.2** (Amalgamation Property [19])**.** *For all runs $\rho \trianglelefteq \rho_1, \rho_2$ of a VAS $\mathbf{A}$, there exists a run τ of $\mathbf{A}$ such that:*

- $\rho_1, \rho_2 \trianglelefteq \tau$,
- $\mathrm{src}(\rho) + \mathrm{src}(\tau) = \mathrm{src}(\rho_1) + \mathrm{src}(\rho_2)$, *and*
- $\mathrm{tgt}(\rho) + \mathrm{tgt}(\tau) = \mathrm{tgt}(\rho_1) + \mathrm{tgt}(\rho_2)$.

5.2 Almost Semilinear Sets

Thanks to the well-quasi-order $\trianglelefteq$ on VAS runs, the reachability set of a VAS $(\mathbf{A}, \mathbf{C}_0)$ can be decomposed as finite union of sets *called almost linear sets* since they are geometrically close to the linear sets.

An *almost linear set* $\mathbf{L}$ is a set of the form $\mathbf{b}+\mathbf{P}$ where $\mathbf{b}\in\mathbb{N}^d$ and $\mathbf{P}\subseteq\mathbb{N}^d$ is a periodic set such that the *cone* $\mathbb{Q}_{\geq 0}\mathbf{P}$ is definable in $FO(\mathbb{Q}_{\geq 0},+)$.[2]

An *almost semilinear set* is a finite union of almost linear sets.

▶ **Example 5.3.** A semilinear periodic set $\mathbf{P}$ is such that the *cone* $\mathbb{Q}_{\geq 0}\mathbf{P}$ is definable in $FO(\mathbb{Q}_{\geq 0},+)$. The converse is not true since the periodic set $\mathbf{P}=\{(0,0)\}\cup\{(x,y)\in\mathbb{N}^2 \mid x\geq 1\wedge y\leq 2^x\}$ is such that the cone $\mathbb{Q}_{\geq 0}\mathbf{P}$ is definable in $FO(\mathbb{Q}_{\geq 0},+)$ since it is equal to $\{(0,0)\}\cup(\mathbb{Q}_{>0}\times\mathbb{Q}_{\geq 0})$. But $\mathbf{P}$ is not semilinear.

We recall the following theorem.

▶ **Theorem 5.4** ([15]). *The set* $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)\cap\mathbf{S}$ *is almost semilinear for every semilinear set* $\mathbf{S}\subseteq\mathbb{N}^d$ *and for every VAS* $(\mathbf{A},\mathbf{C}_0)$.

We also recall some definitions that provide a way to over-approximate almost linear sets by linear sets (see [15] for more details). The *linearization* of an almost linear set $\mathbf{L}=\mathbf{b}+\mathbf{P}$ is the set $\mathrm{lin}(\mathbf{L})=\mathbf{b}+\mathbf{Q}$ where $\mathbf{Q}\coloneqq(\mathbf{P}-\mathbf{P})\cap\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$ and $\overline{\mathbf{X}}$ where $\mathbf{X}\subseteq\mathbb{Q}^d$ is the classical *topological closure* of $\mathbf{X}$. Let us recall that $\mathrm{lin}(\mathbf{L})$ is a linear set since $\mathbf{Q}$ is a finitely-generated periodic set [15, Lemma 5.1]. A *linearization* of an almost semilinear set $\mathbf{S}$ is a finite set $\{\mathrm{lin}(\mathbf{L}_1),\ldots,\mathrm{lin}(\mathbf{L}_k)\}$ where $\mathbf{L}_1,\ldots,\mathbf{L}_k$ are almost linear sets satisfying $\mathbf{S}=\bigcup_{j=1}^k\mathbf{L}_j$. Since such a decomposition of an almost semilinear set into finite union of almost linear sets is not unique, an almost semilinear set can admit several linearizations.

▶ **Example 5.5.** Let us come back to Example 5.3 and notice that $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}=\mathbb{Q}^2_{\geq 0}$ and $\mathbf{P}-\mathbf{P}=\mathbb{Z}^2$. It follows that $\mathrm{lin}(\mathbf{P})=\mathbb{N}^2$.

▶ **Example 5.6.** The linearization of the linear set $\mathbf{L}=\mathrm{Per}(\{2,3\})$ is the linear set $\mathbb{N}$. Observe that $\mathbb{N}$ is strictly larger than $\mathbf{L}$ since $1\not\in\mathbf{L}$.

Linearizations provide a simple way to over-approximate almost linear sets and almost semilinear sets by linear sets and semilinear sets respectively. The following lemma shows that the space-dimension is unchanged by those approximations.

▶ **Lemma 5.7.** *We have* $\mathrm{sdim}(\mathbf{L})=\mathrm{sdim}(\mathrm{lin}(\mathbf{L}))$ *for every almost linear set* $\mathbf{L}$.

Proof. Assume that $\mathbf{L}=\mathbf{b}+\mathbf{P}$ and observe that $\mathrm{lin}(\mathbf{L})=\mathbf{b}+\mathbf{Q}$ where $\mathbf{Q}$ is the periodic set $(\mathbf{P}-\mathbf{P})\cap\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. Let $\mathbf{V}$ and $\mathbf{W}$ be the vector space spanned by $\mathbf{P}$ and $\mathbf{Q}$. Corollary 3.3 shows that $\mathrm{sdim}(\mathbf{P})=\dim(\mathbf{V})$ and $\mathrm{sdim}(\mathbf{Q})=\dim(\mathbf{W})$. Since $\mathrm{sdim}(\mathbf{L})=\mathrm{sdim}(\mathbf{P})$ and $\mathrm{sdim}(\mathrm{lin}(\mathbf{L}))=\mathrm{sdim}(\mathbf{Q})$, it is sufficient to prove that $\mathbf{V}=\mathbf{W}$. As $\mathbf{P}\subseteq\mathbf{Q}$ we get $\mathbf{P}\subseteq\mathbf{W}$. Hence $\mathbf{V}\subseteq\mathbf{W}$ by minimality of $\mathbf{V}$. For the converse inclusion, the inclusion $\mathbf{P}\subseteq\mathbf{V}$ implies that $\mathbf{P}-\mathbf{P}\subseteq\mathbf{V}$ and $\mathbb{Q}_{\geq 0}\mathbf{P}\subseteq\mathbf{V}$. Since a vector-space is topologically-closed, from $\mathbb{Q}_{\geq 0}\mathbf{P}\subseteq\mathbf{V}$ we derive $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}\subseteq\mathbf{V}$. We have proved $\mathbf{Q}\subseteq\mathbf{V}$. It follows that $\mathbf{W}\subseteq\mathbf{V}$ by minimality of $\mathbf{W}$. We have proved that $\mathbf{V}=\mathbf{W}$. ◀

5.3 Interior Vectors

We now introduce a new technique to refine the approximation thanks to the notion of *interior vectors of a periodic set.* Formally, an *interior vector* of a periodic set $\mathbf{P}\subseteq\mathbb{N}^d$ is a vector $\mathbf{p}\in\mathbb{N}^d$ such that for every $\mathbf{q}\in\mathbf{P}$, there exists $n\in\mathbb{N}_{>0}$ satisfying $n\mathbf{p}\in\mathbf{q}+\mathbf{P}$.

[2] Since $FO(\mathbb{Q}_{\geq 0},+,\leq)$ admits quantifier elimination, a set $\mathbf{X}\subseteq\mathbb{Q}^d_{\geq 0}$ is definable in this logic if, and only if, it is a boolean combination of *half-spaces* $\{(x_1,\ldots,x_d)\in\mathbb{Q}^d_{\geq 0}\mid a_1x_1+\cdots+a_dx_d\sim 0\}$ where $a_1,\ldots,a_d\in\mathbb{Z}$ and $\sim$ is either $\geq$ or $>$.

Interior vectors can be geometrically characterized by the topological interior of the cone $\mathbb{Q}_{\geq 0}\mathbf{P}$ spanned by $\mathbf{P}$. Since this characterization is not used in this paper, we just provide the following results about interior vectors (detailed proofs are given in the full version [3]).

The following lemma shows as direct corollary that any periodic set admits an interior vector.

▶ **Lemma 5.8.** *A vector $\mathbf{p} \in \mathbb{N}^d$ is interior to a periodic set $\mathbf{P} \subseteq \mathbb{N}^d$ if, and only if, there exists a finite sequence $\mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbf{P}$ of vectors spanning the same vector space as $\mathbf{P}$ such that:*

$$\mathbf{p} \in \sum_j \mathbb{Q}_{>0}\mathbf{p}_j$$

The next lemma shows that interior vectors of $\mathbf{P}$ can be obtained just by considering interior vectors of $\mathbf{Q} := (\mathbf{P} - \mathbf{P}) \cap \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. Notice that if $\mathbf{Q}$ is finitely-generated, i.e. a periodic set of the form $\mathbf{Q} = \mathrm{Per}(\mathbf{G})$ for some finite set $\mathbf{G} \subseteq \mathbb{N}^d$, thanks to the previous lemma 5.8, we deduce that the vector $\sum_{\mathbf{g} \in \mathbf{G}} \mathbf{g}$ is an interior vector of $\mathbf{Q}$, and in particular an interior vector of $\mathbf{P}$.

▶ **Lemma 5.9.** *The set of interior vectors of a periodic set $\mathbf{P} \subseteq \mathbb{N}^d$ coincides with the set of interior vectors of the periodic set $\mathbf{Q} := (\mathbf{P} - \mathbf{P}) \cap \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$.*

Interior vectors are used in the sequel to translate a vector $\mathbf{y} \in \mathbf{Q}$ in order to obtain an infinite subset of $\mathbf{P}$ of the form $\mathbf{y} + r\mathbb{N}_{>0}\mathbf{x}$ as shown in the following lemma.

▶ **Lemma 5.10.** *Let $\mathbf{P} \subseteq \mathbb{N}^d$ be a periodic set and let $\mathbf{Q} := (\mathbf{P} - \mathbf{P}) \cap \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. For every $\mathbf{y} \in \mathbf{Q}$, for every interior vector $\mathbf{q}$ of $\mathbf{Q}$, and for every $\mathbf{x} \in \mathbf{q} + \mathbf{Q}$, there exists $r \in \mathbb{N}_{>0}$ such that $\mathbf{y} + r\mathbb{N}_{>0}\mathbf{x} \subseteq \mathbf{P}$.*

6 Forward Semilinear Periodic Inductive Invariants

We are now equipped with the necessary ingredients to prove the main result of the paper, namely Theorem 4.5. Our proof relies on a construction of the desired semilinear periodic inductive invariant $\mathbf{I}$ by means of an algorithm with oracle calls. This algorithm, dubbed `InductiveInvariant`, is defined in Algorithm 1. We don't address whether the oracle calls can be effectively implemented, so the algorithm should be understood only as a convenient presentation of the existence proof. The oracle call at line 4 takes as input a semilinear set $\mathbf{I}$ (stored in the variable `Inv`) and returns a linearization of the almost semilinear set $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I}$. Recall that $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I}$ is almost semilinear by Theorem 5.4. A second oracle call at line 6 selects some vectors $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbb{N}^d$ satisfying a suitable condition. The existence of these vectors is established in Corollary 6.5.

Termination and correctness of this algorithm, under the assumption that we can always find $\mathbf{q}_1, \ldots, \mathbf{q}_k$ at line 6, come from these two lemmas. We assume for the remainder of this section that $(\mathbf{A}, \mathbf{C}_0)$ is a periodic d-VAS and that $\mathbf{C}_{\mathrm{bad}} \subseteq \mathbb{N}^d$ is a semilinear set that is disjoint from $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$. Consider an execution of $\texttt{InductiveInvariant}(\mathbf{A}, \mathbf{C}_0, \mathbf{C}_{\mathrm{bad}})$. As an immediate consequence of the two following claims, we get that the execution terminates and returns a semilinear periodic inductive invariant of $(\mathbf{A}, \mathbf{C}_0)$ that is disjoint from $\mathbf{C}_{bad}$.

▷ Claim 6.1 (Invariant). At the beginning and end of each iteration of the **while**-loop, `Inv` is a semilinear periodic set and $\mathrm{Post}^*_{\mathbf{A}}(\texttt{Inv})$ is disjoint from $\mathbf{C}_{\mathrm{bad}}$.

▷ Claim 6.2 (Termination). Each iteration of the **while**-loop strictly decreases the space-dimension of $\mathrm{Post}^*_{\mathbf{A}}(\texttt{Inv}) \setminus \texttt{Inv}$.

Algorithm 1 $\texttt{InductiveInvariant}(\mathbf{A}, \mathbf{C}_0, \mathbf{C}_{\text{bad}})$.

Input: A periodic d-VAS $(\mathbf{A}, \mathbf{C}_0)$ and a semilinear set $\mathbf{C}_{\text{bad}} \subseteq \mathbb{N}^d \setminus \text{Post}^*_{\mathbf{A}}(\mathbf{C}_0)$.
Output: A semilinear periodic inductive invariant of $(\mathbf{A}, \mathbf{C}_0)$ that is disjoint from $\mathbf{C}_{bad}$.

1: $\texttt{Inv} \leftarrow \text{Per}(\mathbf{C}_0)$
2: **while** $\text{Post}^*_{\mathbf{A}}(\texttt{Inv}) \not\subseteq \texttt{Inv}$ **do**
3: ▷ *The existence of* $\{\mathbf{L}_1, \ldots, \mathbf{L}_k\}$ *at line 4 is ensured by Theorem 5.4* ◁
4: Let $\{\mathbf{L}_1, \ldots, \mathbf{L}_k\}$ be a linearization of $\text{Post}^*_{\mathbf{A}}(\texttt{Inv}) \setminus \texttt{Inv}$
5: ▷ *The existence of* $\mathbf{q}_1, \ldots, \mathbf{q}_k$ *at line 6 is ensured by Corollary 6.5* ◁
6: Let $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbb{N}^d$ such that $\mathbf{L}'_i \coloneqq (\mathbf{L}_i + \mathbf{q}_i) \subseteq \mathbf{L}_i$ for each $i \in \{1, \ldots, k\}$ and $\text{Post}^*_{\mathbf{A}}(\text{Per}(\texttt{Inv} \cup \mathbf{L}'_1 \cup \cdots \cup \mathbf{L}'_k)) \subseteq (\texttt{Inv} \cup \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_k) \setminus \mathbf{C}_{\text{bad}}$
7: $\texttt{Inv} \leftarrow \text{Per}(\texttt{Inv} \cup \mathbf{L}'_1 \cup \cdots \cup \mathbf{L}'_k)$
8: **return** $\texttt{Inv}$

Proof. Consider an iteration of the **while**-loop. Let $\mathbf{I}$ be the value of $\texttt{Inv}$ at the beginning (line 2) and let $\mathbf{J}$ be its value at the end (line 7). We need to prove that $\text{sdim}(\text{Post}^*_{\mathbf{A}}(\mathbf{J}) \setminus \mathbf{J}) < \text{sdim}(\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I})$.

According to the body of the **while**-loop, there exist a linearization $\{\mathbf{L}_1, \ldots, \mathbf{L}_k\}$ of $\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I}$ and $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbb{N}^d$ with $(\mathbf{L}_i + \mathbf{q}_i) \subseteq \mathbf{L}_i$ for each $i \in \{1, \ldots, k\}$ such that, firstly, $\mathbf{J} = \text{Per}(\mathbf{I} \cup \mathbf{L}'_1 \cup \cdots \cup \mathbf{L}'_k)$ where $\mathbf{L}'_i \coloneqq (\mathbf{L}_i + \mathbf{q}_i)$ for each i, and secondly, $\text{Post}^*_{\mathbf{A}}(\mathbf{J}) \subseteq (\mathbf{I} \cup \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_k)$. By definition of linearizations of almost semilinear sets, there exist almost linear sets $\mathbf{H}_1, \ldots, \mathbf{H}_k$ such that $\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I} = \bigcup_{i=1}^k \mathbf{H}_i$ and $\mathbf{L}_i = \text{lin}(\mathbf{H}_i)$ for each $i \in \{1, \ldots, k\}$. We derive from Lemma 5.7 that $\text{sdim}(\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I}) = \max_i \text{sdim}(\mathbf{H}_i) = \max_i \text{sdim}(\mathbf{L}_i)$. Let us now upper-bound $\text{sdim}(\text{Post}^*_{\mathbf{A}}(\mathbf{J}) \setminus \mathbf{J})$. Observe that $\mathbf{J} \supseteq (\mathbf{I} \cup \mathbf{L}'_1 \cup \cdots \cup \mathbf{L}'_k)$ and recall that $\text{Post}^*_{\mathbf{A}}(\mathbf{J}) \subseteq (\mathbf{I} \cup \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_k)$. It follows that $\text{Post}^*_{\mathbf{A}}(\mathbf{J}) \setminus \mathbf{J}$ is contained in $(\mathbf{L}_1 \setminus \mathbf{L}'_1) \cup \cdots \cup (\mathbf{L}_k \setminus \mathbf{L}'_k)$, hence, $\text{sdim}(\text{Post}^*_{\mathbf{A}}(\mathbf{J}) \setminus \mathbf{J}) \leq \max_i \text{sdim}(\mathbf{L}_i \setminus \mathbf{L}'_i)$. By Lemma 3.5, $\text{sdim}(\mathbf{L}_i \setminus \mathbf{L}'_i) < \text{sdim}(\mathbf{L}_i)$ for each $i \in \{1, \ldots, k\}$. We derive that $\text{sdim}(\text{Post}^*_{\mathbf{A}}(\mathbf{J}) \setminus \mathbf{J}) < \max_i \text{sdim}(\mathbf{L}_i) = \text{sdim}(\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I})$. ◁

The remainder of this section is devoted to the proof that some $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbb{N}^d$ satisfying the condition at line 6 always exist (see Corollary 6.5). We will need the following easy consequence of Lemma 5.2.

▶ **Lemma 6.3.** *For every runs $\rho \trianglelefteq \rho'$ of a VAS $\mathbf{A}$ and for every $r \in \mathbb{N}$, there exists a run τ of $\mathbf{A}$ such that $\text{src}(\tau) = \text{src}(\rho) + r(\text{src}(\rho') - \text{src}(\rho))$, and $\text{tgt}(\tau) = \text{tgt}(\rho) + r(\text{tgt}(\rho') - \text{tgt}(\rho))$.*

▶ **Lemma 6.4.** *Let $\mathbf{I} \subseteq \mathbb{N}^d$ be a semilinear periodic set and consider a linearization $\{\mathbf{L}_1, \ldots, \mathbf{L}_k\}$ of the almost semilinear set $\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I}$. Let $(\mathbf{b}_i, \mathbf{G}_i)$ be a linear-presentation of $\mathbf{L}_i$ for each $i \in \{1, \ldots, k\}$. For every semilinear set $\mathbf{T} \subseteq \mathbb{N}^d$ such that $\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \subseteq \mathbf{T}$, there exists $n \in \mathbb{N}$ such that $\text{Post}^*_{\mathbf{A}}(\text{Per}(\mathbf{I} \cup \mathbf{L}'_1 \cup \cdots \cup \mathbf{L}'_k)) \subseteq \mathbf{T}$ where $\mathbf{L}'_i \coloneqq (\mathbf{L}_i + \mathbf{q}_i)$ and $\mathbf{q}_i \coloneqq n \sum_{\mathbf{g} \in \mathbf{G}_i} \mathbf{g}$ for each $i \in \{1, \ldots, k\}$.*

Proof. Consider a semilinear set $\mathbf{T} \subseteq \mathbb{N}^d$ such that $\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \subseteq \mathbf{T}$. For each $n \in \mathbb{N}$, define $\mathbf{S}_n \coloneqq \text{Per}(\mathbf{I} \cup \mathbf{L}'_{1,n} \cup \cdots \cup \mathbf{L}'_{k,n})$ where $\mathbf{L}'_{i,n} \coloneqq (\mathbf{L}_i + \mathbf{q}_{i,n})$ and $\mathbf{q}_{i,n} \coloneqq n \sum_{\mathbf{g} \in \mathbf{G}_i} \mathbf{g}$ for each $i \in \{1, \ldots, k\}$. Suppose by contradiction that $\text{Post}^*_{\mathbf{A}}(\mathbf{S}_n) \not\subseteq \mathbf{T}$ for every $n \in \mathbb{N}$.

Define $\mathbf{Q}_i \coloneqq \text{Per}(\mathbf{G}_i)$ and note that $\mathbf{L}_i = \mathbf{b}_i + \mathbf{Q}_i$ for each $i \in \{1, \ldots, k\}$. Since $\{\mathbf{L}_1, \ldots, \mathbf{L}_k\}$ is a linearization of $\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I}$, there exist periodic sets $\mathbf{P}_1, \ldots, \mathbf{P}_k \subseteq \mathbb{N}^d$ such that $\text{Post}^*_{\mathbf{A}}(\mathbf{I}) \setminus \mathbf{I} = \bigcup_{i=1}^k \mathbf{b}_i + \mathbf{P}_i$ and $\mathbf{Q}_i = (\mathbf{P}_i - \mathbf{P}_i) \cap \overline{\mathbb{Q}_{\geq 0} \mathbf{P}_i}$ for each $i \in \{1, \ldots, k\}$.

By hypothesis, for every $n \in \mathbb{N}$, there is a run ρ_n from some $\mathbf{s}_n \in \mathbf{S}_n$ to some $\mathbf{t}_n \in (\mathbb{N}^d \setminus \mathbf{T})$. Moreover, according to the definition of $\mathbf{S}_n$, we can write each $\mathbf{s}_n$ under the form $\mathbf{s}_n = \mathbf{i}_n + \sum_{j \in J_n} (\mu_{j,n}(\mathbf{b}_j + \mathbf{q}_{j,n}) + \mathbf{u}_{j,n})$ for some $\mathbf{i}_n \in \text{Per}(\mathbf{I}) = \mathbf{I}$, some subset $J_n \subseteq \{1, \ldots, k\}$

and, for each $j \in J_n$, some $\mu_{j,n} \in \mathbb{N}$ with $\mu_{j,n} > 0$ and some $\mathbf{u}_{j,n} \in \mathrm{Per}(\mathbf{G}_j)$. Extracting a subsequence if necessary, we may assume that $J_m = J_n$ for all $m, n \in \mathbb{N}$, and we let J denote their common value.

As $\mathbf{I}$ and $\mathbb{N}^d \setminus \mathbf{T}$ are semilinear, the relations $\leq_{\mathbf{I}}$ and $\leq_{\mathbb{N}^d \setminus \mathbf{T}}$ are almost-full (see Lemma 5.1). The usual order $\leq$ on $\mathbb{N}$ is a well-quasi-order, and so is the relation $\trianglelefteq$ on runs of $\mathbf{A}$ (see Section 5.1). It is also well-known that for every finite subset $\mathbf{G} \subseteq \mathbb{N}^d$, the binary relation $\preceq_{\mathbf{G}}$ on $\mathrm{Per}(\mathbf{G})$ defined by $\mathbf{u} \preceq_{\mathbf{G}} \mathbf{v}$ if $(\mathbf{v} - \mathbf{u}) \in \mathrm{Per}(\mathbf{G})$, is a well-quasi-order. So we can find an increasing pair $m, n \in \mathbb{N}$ with $m < n$ such that $\mathbf{i}_m \leq_{\mathbf{I}} \mathbf{i}_n$, $\mathbf{t}_m \leq_{\mathbb{N}^d \setminus \mathbf{T}} \mathbf{t}_n$, $\mu_{j,m} \leq \mu_{j,n}$, $\rho_m \trianglelefteq \rho_n$ and $\mathbf{u}_{j,m} \preceq_{\mathbf{G}_j} \mathbf{u}_{j,n}$ for each $j \in J$. Let us define, $\mathbf{y}_j \coloneqq \mu_{j,m}\mathbf{q}_{j,m} + \mathbf{u}_{j,m}$ and $\mathbf{x}_j \coloneqq \mu_{j,n}\mathbf{q}_{j,n} - \mu_{j,m}\mathbf{q}_{j,m} + \mathbf{u}_{j,n} - \mathbf{u}_{j,m}$, for each $j \in J$. Observe that $\mathbf{y}_j \in \mathbf{Q}_j$ and $\mathbf{x}_j \in (\nu \sum_{\mathbf{g} \in \mathbf{G}_j} \mathbf{g}) + \mathbf{Q}_j$, where $\nu \coloneqq n\mu_{j,n} - m\mu_{j,m} > 0$. We derive from Lemmas 5.8 and 5.10 that there exists $r_j \in \mathbb{N}_{>0}$ such that $\mathbf{y}_j + r_j\mathbb{N}_{>0}\mathbf{x}_j \subseteq \mathbf{P}_j$. By defining $r \coloneqq \prod_{j \in J} r_j$, we get that $(\mathbf{y}_j + r\mathbf{x}_j) \in \mathbf{P}_j$ for each $j \in J$.

Recall that $\rho_m \trianglelefteq \rho_n$. By Lemma 6.3, there exists a run τ of $\mathbf{A}$ such that $\mathrm{src}(\tau) = \mathbf{s}_m + r(\mathbf{s}_n - \mathbf{s}_m)$, and $\mathrm{tgt}(\tau) = \mathbf{t}_m + r(\mathbf{t}_n - \mathbf{t}_m)$. Observe that $\mathrm{tgt}(\tau) \in (\mathbb{N}^d \setminus \mathbf{T})$ since $\mathbf{t}_m, \mathbf{t}_n \in (\mathbb{N}^d \setminus \mathbf{T})$ and $\mathbf{t}_m \leq_{\mathbb{N}^d \setminus \mathbf{T}} \mathbf{t}_n$. So to get a contradiction, there only remains to show that $\mathrm{src}(\tau)$ is in $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$. Indeed, if this is the case, by prepending to τ a run from $\mathbf{I}$ to $\mathrm{src}(\tau)$, we get a run from $\mathbf{I}$ to $\mathbb{N}^d \setminus \mathbf{T}$, contradicting the assumption that $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I}) \subseteq \mathbf{T}$.

We now show that $\mathrm{src}(\tau)$ is in $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$. Notice that $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$ is periodic since $\mathbf{I}$ is periodic (see Lemma 4.1). So it is enough to express $\mathrm{src}(\tau)$ as a sum of elements of $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$. We have $\mathbf{s}_m = \mathbf{i}_m + \sum_{j \in J} (\mu_{j,m}\mathbf{b}_j + \mathbf{y}_j)$ and $\mathbf{s}_n - \mathbf{s}_m = \mathbf{i}_n - \mathbf{i}_m + \sum_{j \in J} (\mu_{j,n} - \mu_{j,m})\mathbf{b}_j + \sum_{j \in J} \mathbf{x}_j$. It follows that

$$\mathrm{src}(\tau) = \mathbf{i}_m + r(\mathbf{i}_n - \mathbf{i}_m) + \sum_{j \in J} (\mu_{j,m} - 1 + r(\mu_{j,n} - \mu_{j,m}))\, \mathbf{b}_j + \sum_{j \in J} (\mathbf{b}_j + \mathbf{y}_j + r\mathbf{x}_j) \ .$$

The term $\mathbf{i}_m + r(\mathbf{i}_n - \mathbf{i}_m)$ belongs to $\mathbf{I} \subseteq \mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$ since $\mathbf{i}_m, \mathbf{i}_n \in \mathbf{I}$ and $\mathbf{i}_m \leq_{\mathbf{I}} \mathbf{i}_n$. For each $j \in J$, we have $\mu_{j,m} \geq 1$, $\mu_{j,n} \geq \mu_{j,m}$ and $\mathbf{b}_j \in \mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$, hence, by periodicity, $(\mu_{j,m} - 1 + r(\mu_{j,n} - \mu_{j,m}))\, \mathbf{b}_j$ is in $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$. Finally, the integer r was chosen so that $(\mathbf{y}_j + r\mathbf{x}_j) \in \mathbf{P}_j$ for each $j \in J$, hence, $\mathbf{b}_j + \mathbf{y}_j + r\mathbf{x}_j$ is in $\mathbf{b}_j + \mathbf{P}_j \subseteq \mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$. We have written $\mathrm{src}(\tau)$ as a sum of elements of $\mathrm{Post}^*_{\mathbf{A}}(\mathbf{I})$, which completes the proof. ◀

▶ **Corollary 6.5.** *In Algorithm 1, one can always find* $\mathbf{q}_1, \ldots, \mathbf{q}_k$ *at line 6.*

Proof. By Claim 6.1, the value of `Inv` at line 6 is a semilinear periodic set and $\mathrm{Post}^*_{\mathbf{A}}(\mathtt{Inv})$ is disjoint from $\mathbf{C}_{\mathrm{bad}}$. The corollary follows from Lemma 6.4 instantiated with the semilinear set $\mathbf{T} \coloneqq (\mathtt{Inv} \cup \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_k) \setminus \mathbf{C}_{\mathrm{bad}}$. ◀

▶ Remark 6.6. As already observed in Section 4.2, semilinear inductive invariants for a plain VAS $(\mathbf{A}, \mathbf{C}_0)$ and a set $\mathbf{C}_{\mathrm{bad}}$ can be derived from semilinear inductive invariants for a periodic VAS $(\mathbf{A}', \mathbf{C}'_0)$ and a set $\mathbf{C}'_{\mathrm{bad}}$ obtained by introducing an extra counter that is untouched by $\mathbf{A}'$. Instead of applying Algorithm 1 to $(\mathbf{A}', \mathbf{C}'_0)$ and $\mathbf{C}'_{\mathrm{bad}}$, one may apply a variant of that algorithm obtained by just removing the operator Per each time it is used. In fact, notice that if $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are subsets of $\mathbb{N}^d$, and if $\mathbf{X}' \coloneqq \mathrm{Per}(\mathbf{X} \times \{1\})$, $\mathbf{Y}' \coloneqq \mathrm{Per}(\mathbf{Y} \times \{1\})$, and $\mathbf{Z}' \coloneqq \mathrm{Per}(\mathbf{Z} \times \{1\})$, then $\mathbf{Z} = \mathbf{X} \cup \mathbf{Y}$ if, and only if, $\mathbf{Z}' = \mathbf{X}' + \mathbf{Y}'$.

7 Conclusion

The approach of Leroux [15, 16] to solve the VAS reachability problem is based on the fundamental property (recalled in Theorem 3.7) that every semilinear set $\mathbf{S}$ containing the reachability set of a VAS $(\mathbf{A}, \mathbf{C}_0)$ also contains a semilinear inductive invariant $\mathbf{I}$ for $(\mathbf{A}, \mathbf{C}_0)$. Let us call this property the "semilinear inductive invariant" property.

In this paper, we have shown that $\mathbf{I}$ can in addition be required to be periodic when the VAS is periodic. Moreover, our method builds invariants solely from the set $\mathbf{C}_0$ of initial configurations and avoids the need for backward reasoning. We believe that such a forward approach is more suitable than the back-and-forth approach for attacking the branching VAS reachability problem, which is still open. We discuss potential extensions of the "semilinear inductive invariant" property in the full version [3].

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