The Parameterized Complexity of Maximum Span on Natural Matroid Classes

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Abstract

We study Maximum Span, motivated by the recent *Maximum Span Hypothesis* of Karthik and Khot [SODA 2025], which suggests strong parameterized intractability for finding large structured subsets in vector spaces. Formally, given a matrix M and integers k and t, the task is to decide whether there exists a linearly independent set S of at most k columns such that at least t additional columns of M lie in span(S). Equivalently, the goal is to identify a low-rank witness whose span covers many input columns.

We initiate a systematic study of the parameterized complexity of Maximum Span on natural matroid classes, revealing a diverse complexity landscape. We first show that the problem is polynomial-time solvable on laminar matroids, via a dynamic program over the laminar tree. In sharp contrast, on graphic matroids the problem is W[1]-hard parameterized by $k+t$, and, assuming Gap-ETH, admits no $f(k)\cdot n^{\mathcal{O}(1)}$-time $k^{o(1)}$-approximation. On cographic matroids, we show that the problem is equivalent to deleting at most $k+t$ edges so as to create at least $t+1$ connected components; this yields fixed-parameter tractability parameterized by $k+t$, and W[1]-hardness parameterized by t. On transversal matroids, using a Hall-type interpretation, we prove W[1]-hardness parameterized by $k+t$.

For strict gammoids, we develop a separator-based formulation. We prove W[1]-hardness parameterized by $k+t$, give an XP algorithm parameterized by t, and obtain FPT 2^k-approximation algorithms in both the directed and undirected settings. For general gammoids, we establish W[1]-hardness parameterized by $k+t$, NP-hardness already for $t=1$, and an XP algorithm parameterized by k. Together, these results give a detailed parameterized complexity map for Maximum Span across fundamental matroid classes, ranging from polynomial-time solvability to fixed-parameter algorithms, XP algorithms, approximation algorithms, and strong hardness.

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1 Introduction

Approximation and parameterized complexity are two fundamental approaches for coping with NP-hardness. Their interaction has led to the study of fixed-parameter approximation algorithms, where the goal is to obtain approximation guarantees in running time $f(k) \cdot n^{\mathcal{O}(1)}$ for a parameter k. In parallel, hardness of approximation and parameterized hardness provide complementary lower-bound frameworks. A central question in this area is whether strong inapproximability results for parameterized problems can be based on assumptions intrinsic to parameterized complexity, or at least on gap-free assumptions, rather than on hypotheses that already encode gap structure, as in the framework of [14]. We refer the reader to recent surveys for a broader overview of the field [5, 15, 10].

Given a computational problem φ and a parameter k, we say that φ is *fixed-parameter tractable* (FPT) if it can be solved in time $f(k) \cdot n^c$, where n is the input size, f is a computable function, and c is a constant. Classic examples include Vertex Cover parameterized by solution size [6, 9] and the problem of detecting a simple path of length k in a graph [1]. On the other hand, many fundamental problems, such as k-Clique, are believed not to admit FPT algorithms [6, 9]. This belief is captured by the W-hierarchy; in particular, k-Clique is the canonical W[1]-complete problem. Consequently, a natural direction is to ask whether such hard problems nevertheless admit efficient FPT approximation algorithms.

A prominent assumption in parameterized hardness of approximation is the *Gap Exponential Time Hypothesis* (Gap-ETH), which rules out subexponential-time algorithms for distinguishing fully satisfiable instances of 3-SAT from instances in which every assignment violates a constant fraction of clauses. Gap-ETH has led to many of the first non-trivial parameterized inapproximability results. However, it is also a strong assumption with a gap built into it. This has motivated a major line of work seeking to base parameterized hardness of approximation on weaker, gap-free assumptions such as ETH.

A central conjecture in this direction is the *Parameterized Inapproximability Hypothesis* (PIH), which plays a role analogous to the PCP theorem in parameterized complexity. Informally, PIH asserts that no FPT algorithm can distinguish a satisfiable 2-CSP instance, parameterized by the number of variables, from one in which every assignment violates a constant fraction of constraints; equivalently, it rules out FPT approximation algorithms for the corresponding constraint-satisfaction problem under a parameterized inapproximability hypothesis. For a long time, PIH was known only under Gap-ETH. In a recent breakthrough, Guruswami, Lin, Ren, Sun, and Wu proved PIH under ETH, giving the first derivation of PIH from a gap-free assumption and marking a major advance in parameterized hardness of approximation [11]. See also recent developments in this direction [12, 2].

Motivated by this broader effort to identify intrinsic sources of parameterized intractability for approximation, Karthik and Khot introduced the Maximum Span Hypothesis (MSH) [13]. MSH is formulated through the following linear-algebraic problem.

Maximum Span
Input: A matrix M over a field $\mathbb{F}$ and positive integers k, t.
Parameters: k, t, and $k + t$.
Question: Does there exist a linearly independent set X of at most k columns of M such that at least t further columns of M lie in $\text{span}(X)$?

Equivalently, the goal is to find a low-rank witness whose span captures many additional input columns. The two parameters have distinct roles: k measures the size of the witnessing independent set, while t measures the number of additional elements captured by its span.

Even when both parameters are combined, the problem is expected to be highly intractable; indeed, the hypothesis of Karthik and Khot postulates strong parameterized hardness for the associated gap version. Moreover, already the case $t = 1$ contains the problem of detecting a small dependent set: asking whether some independent set of size at most k spans one additional column is equivalent to asking whether the represented matroid contains a circuit of size at most $k + 1$ [3]. This already suggests that parameterizing only by t is unlikely to lead to general tractability; indeed, for gammoids we later prove NP-hardness already for $t = 1$.

This motivates the study of Maximum Span on structured matrices. A first natural direction is to restrict the sparsity pattern of the matrix, for example by bounding the number of nonzero entries in each column. However, sparsity alone does not make the problem easy: matrices with at most two nonzero entries per column already capture graphic matroids via the usual incidence-matrix representation. Thus, rather than treating sparsity as an isolated condition, it is more informative to ask which structural properties of the represented independence system make Maximum Span tractable or hard.

This leads naturally to a matroidal viewpoint. Matroids abstract linear independence while also capturing many combinatorial independence systems arising from graphs, matchings, cuts, and reachability. We therefore study Maximum Span on natural classes of representable matroids. For a matroid $\mathcal{M} = (E, \mathcal{I})$, the problem takes the following form.

Maximum Span on Matroids
Input: A matroid $\mathcal{M} = (E, \mathcal{I})$ and positive integers k, t.
Parameters: k, t, and $k + t$.
Question: Does there exist an independent set $X \in \mathcal{I}$ with $|X| \leq k$ and a set $Y \subseteq E \setminus X$ with $|Y| \geq t$ such that $\mathrm{rank}(X \cup Y) = \mathrm{rank}(X)$?

This formulation captures the original linear-algebraic problem for representable matroids, while allowing us to study how different structural sources of independence affect the complexity of the problem. In this paper, we focus on several fundamental matroid classes: laminar matroids, graphic and cographic matroids, transversal matroids, strict gammoids, and gammoids. These classes cover a range of combinatorial phenomena, including nested capacity constraints, graph connectivity, matchings, cuts, and directed reachability.

Our Contribution

We study Maximum Span across several natural classes of matroids and obtain the following results.

- **Laminar matroids.** We show that Maximum Span is polynomial-time solvable on laminar matroids. The algorithm uses a dynamic program over the laminar tree. The key observation is that an element is spanned by an independent set precisely when some laminar set containing it becomes tight.
- **Graphic matroids.** We prove that Maximum Span on graphic matroids is W[1]-hard when parameterized by $k + t$. The reduction is from Clique and uses the fact that, in a graphic matroid, an edge is spanned by a forest exactly when its endpoints lie in the same connected component. We further show that, assuming Gap-ETH, the optimization version admits no $f(k) \cdot n^{\mathcal{O}(1)}$-time $k^{o(1)}$-approximation.

- **Cographic matroids.** For cographic matroids, we give an equivalent graph-theoretic reformulation: the problem asks whether one can delete at most $k+t$ edges so as to create at least $t+1$ connected components. This yields an FPT algorithm parameterized by $k+t$. On the other hand, we show that the problem is W[1]-hard when parameterized by t alone.
- **Transversal matroids.** For transversal matroids, we use a Hall-type characterization of span to prove W[1]-hardness parameterized by $k+t$. The reduction identifies spanned elements with vertices whose addition does not increase the maximum matching rank.
- **Strict gammoids.** For strict gammoids, we develop a separator-based formulation of the problem. This allows us to prove W[1]-hardness parameterized by $k+t$, while also giving an XP algorithm parameterized by t. We further obtain FPT 2^k-approximation algorithms in both the directed and undirected settings, using important separators and shadow-type arguments.
- **Gammoids.** Finally, for general gammoids, we show that the hardness for strict gammoids carries over, giving W[1]-hardness parameterized by $k+t$. We also prove that the problem is already NP-hard for the fixed value $t=1$, ruling out an XP algorithm parameterized by t unless P = NP. On the positive side, we give a simple XP algorithm parameterized by k by enumerating the witness set and computing ranks via flow.

Overall, our results show that Maximum Span is highly sensitive to the underlying matroid structure: it is polynomial-time solvable on some natural classes, admits parameterized algorithms or approximations on others, and becomes strongly intractable already on several fundamental representable classes.

2 Preliminaries

In this section, we present notations and foundational results from matroid theory used throughout the paper. As several matroids considered here arise from graphs, we begin by recalling the necessary graph-theoretic notions. All graphs are assumed to be simple and undirected unless stated otherwise. For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. For a set $S \subseteq V(G)$, the subgraph of G *induced* by S, denoted by $G[S]$, is defined as the subgraph of G with vertex set S and edge set $\{uv \in E(G) : u, v \in S\}$ (both for directed and undirected graphs). For any graph $G=(V,E)$ and $u \in V$, a vertex $v \in V$ is said to be a *neighbor* of u if $uv \in E(G)$. The *open neighborhood* of u is denoted by $N(u)$ and it is defined as $N(u) = \{v \in V : uv \in E(G)\}$. For a vertex $v \in V(G)$ and an edge $e \in E(G)$, $G-v$ denotes the graph obtained by removing v along with all edges incident to it, while $G-e$ denotes the graph obtained by removing only the edge e from G. Similarly, for a vertex subset $S \subseteq V(G)$ and an edge subset $F \subseteq E(G)$, $G-S$ denotes the graph obtained by removing all vertices in S along with their incident edges, and $G-F$ denotes the graph obtained by removing all edges in F from G. A graph is said to be *connected* if there exists u–v path for every $u, v \in V(G)$. In an undirected graph G, a *connected component* C of G is a maximal connected subgraph of G. In a directed graph D, a maximal subgraph C is said to be *strongly connected component* if there exists a directed path from u to v and vice versa for every vertex $u, v \in V(C)$. For a connected graph G, an edge $e \in E(G)$ is called a *bridge* if $G-e$ is disconnected. A set $M \subseteq E(G)$ is called a *matching* in G if no two edges in M are incident to a common vertex. We say that M *saturates* a vertex set $S \subseteq V(G)$ if every vertex in S is incident to some edge in M. A graph G is called *bipartite* if its vertex set can be partitioned into two sets A and B such that every edge in $E(G)$ has one endpoint in A and the other in B. We now begin to give preliminaries on matroid theory.

Preliminaries on Matroid Theory

We use standard terminology on matroids; see Oxley [16] and the textbook of Cygan et al. [6, Chapter 12]. We begin with the definitions of matroids, bases, representable matroids, rank, span, and circuits.

▶ **Definition 1** (Matroids). *A pair $\mathcal{M} = (E, \mathcal{I})$ is a* matroid *if E is a finite ground set and $\mathcal{I} \subseteq 2^E$ satisfies the following axioms:*
(1) $\emptyset \in \mathcal{I}$.
(2) *If $A' \subseteq A$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$.*
(3) *If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.*
The sets in $\mathcal{I}$ are the independent sets *of $\mathcal{M}$.*

An inclusion-wise maximal independent set is a *basis* of the matroid $\mathcal{M}$. All bases have the same cardinality, called the *rank* of $\mathcal{M}$. For a matroid $\mathcal{M} = (E, \mathcal{I})$, the *rank function* is $\mathrm{rank} : 2^E \to \mathbb{Z}_{\geq 0}$, where for any subset $X \subseteq E$, $\mathrm{rank}(X)$ denotes the maximum size of an independent subset of X. In particular, if X is independent, then $\mathrm{rank}(X) = |X|$.

A set is called *dependent* if it is not independent. The *span*, or closure, of a set $X \subseteq E$ is $\mathrm{span}_{\mathcal{M}}(X) = \{e \in E : \mathrm{rank}(X \cup \{e\}) = \mathrm{rank}(X)\}$. When the matroid is clear from context, we write simply $\mathrm{span}(X)$. Equivalently, if X is independent, then $\mathrm{span}(X)$ consists of the elements of X together with all elements $y \in E \setminus X$ such that $X \cup \{y\}$ is dependent.

A *circuit* of $\mathcal{M}$ is an inclusion-wise minimal dependent set. We will use the following standard fact. If X is independent and $e \in \mathrm{span}(X) \setminus X$, then $X \cup \{e\}$ contains a unique circuit C_e, called the fundamental circuit of e with respect to X, and $e \in C_e$. Moreover, for every $x \in C_e \setminus \{e\}$, the set $(X \setminus \{x\}) \cup \{e\}$ is independent and has the same span as X.

The proofs of the results marked (★) are either straightforward or a proof sketch is provided just above the corresponding result.

3 Laminar Matroids

In this section, we show that Maximum Span is polynomial-time solvable on laminar matroids. A family $\mathcal{F} \subseteq 2^E$ is called *laminar* if for every two sets $A, B \in \mathcal{F}$, either $A \subseteq B$, or $B \subseteq A$, or $A \cap B = \emptyset$. A laminar matroid on ground set E is given by a laminar family $\mathcal{F} \subseteq 2^E$ and a capacity function $b : \mathcal{F} \to \mathbb{N}$. A set $X \subseteq E$ is *independent* if and only if $|X \cap F| \leq b(F)$ for every $F \in \mathcal{F}$. We first state the problem Maximum Span on Laminar Matroids formally.

Maximum Span on Laminar Matroids
Input: a ground set E, a laminar family $\mathcal{F}$, a capacity function $b : \mathcal{F} \to \mathbb{N}$, and positive integers k, t.
Parameters: k, t, $k + t$.
Question: Does there exist an independent set $X \subseteq E$ with $|X| \leq k$ and a set $Y \subseteq E \setminus X$ with $|Y| \geq t$ such that for every $y \in Y$, we have $\mathrm{rank}(X \cup \{y\}) = \mathrm{rank}(X)$ (equivalently, $\mathrm{rank}(X \cup Y) = \mathrm{rank}(X)$)?

We use the following simple characterization of span in laminar matroids.

▶ **Lemma 2** (★). *Let M be a laminar matroid on ground set E, let X be an independent set, and let $e \in E \setminus X$. Then $e \in \mathrm{span}(X)$ if and only if there exists $F \in \mathcal{F}$ with $e \in F$ and $|X \cap F| = b(F)$.*

We represent the laminar family by its inclusion tree which we call as laminar tree. If necessary, we add an artificial root E with capacity $b(E) = |E|$; this does not change the matroid. The children of a node F are the maximal proper subsets of F that belong to $\mathcal{F}$. Let P_F be the number of elements of F that do not belong to any child of F. Let R_F denote the maximum size of a set that can be chosen from F while satisfying all capacity constraints $|X \cap A| \leq b(A)$ for sets $A \in \mathcal{F}$ with $A \subseteq F$. Then

$$R_F = \min\left\{b(F), P_F + \sum_{F' \text{ child of } F} R_{F'}\right\}.$$

Indeed, from the private elements of F one can choose at most P_F elements, and from each child subtree F' one can choose at most $R_{F'}$ elements; the only further restriction at node F is the capacity $b(F)$.

We compute the optimum solution for Max Span on Laminar Matroid by dynamic programming on the laminar tree. For a node F and an integer q, let $\mathrm{DP}_F[q]$ be the maximum number of elements of $F \setminus X$ that are spanned by X due to tight sets contained in F, over all sets $X \subseteq F$ of size exactly q satisfying $|X \cap A| \leq b(A)$ for every $A \in \mathcal{F}$ with $A \subseteq F$. Thus $\mathrm{DP}_F[q]$ ignores span contributions caused only by tight ancestors of F; those are accounted for when the DP reaches such an ancestor. If no such set X exists, we set $\mathrm{DP}_F[q] = -\infty$.

The recurrence is as follows. First, $\mathrm{DP}_F[q] = -\infty$ for $q > R_F$. If $q = b(F)$, then F is tight and every element of $F \setminus X$ lies in $\mathrm{span}(X)$ by Lemma 2; hence $\mathrm{DP}_F[q] = |F| - q$, provided $q \leq R_F$. If $q < b(F)$, then F itself is not tight, and only descendant sets can contribute spanned elements. Therefore

$$\mathrm{DP}_F[q] = \max_{\substack{q_0 + \sum_{F'} q_{F'} = q \\ 0 \leq q_0 \leq P_F,\ 0 \leq q_{F'} \leq R_{F'}}} \sum_{F' \text{ child of } F} \mathrm{DP}_{F'}[q_{F'}],$$

where the maximum is over all combinations of q_0 and $q_F's$ which satisfy $q_0 + \sum_{F'} q_{F'} = q$, $0 \leq q_0 \leq P_F$, $0 \leq q_{F'} \leq R_{F'}$. Finally, if $q > b(F)$, the value is infeasible.

▶ **Lemma 3** (★). *For every node F of the laminar tree and every integer q, the value $\mathrm{DP}_F[q]$ computed by the recurrence equals the maximum number of elements of $F \setminus X$ that lie in $\mathrm{span}(X)$, over all sets $X \subseteq F$ of size exactly q satisfying all capacity constraints in the subtree rooted at F, under the convention that no ancestor of F is tight.*

▶ **Theorem 4** (★). *Maximum Span on Laminar Matroids is solvable in polynomial time.*

4 Graphic Matroids

In this section, we investigate Maximum Span on graphic matroids. For a graph G, the associated graphic matroid has ground set $E(G)$, where a subset $F \subseteq E(G)$ is *independent* if and only if the subgraph $(V(G), F)$ is a forest. For any independent set F, an element $e \in E(G) \setminus F$ lies in $\mathrm{span}(F)$ precisely when the graph $(V(G), F \cup \{e\})$ contains a cycle. Equivalently, the endpoints of e lie in the same connected component of the forest $(V(G), F)$. We now define the problem formally.

MAXIMUM SPAN ON GRAPHIC MATROIDS
Input: A graph G and positive integers k, t.
Parameters: k, t, and $k + t$.
Question: Does there exist an independent set $X \subseteq E(G)$ with $|X| \leq k$ and a set $Y \subseteq E(G) \setminus X$ with $|Y| \geq t$ such that for every $y \in Y$, we have $\mathrm{rank}(X \cup \{y\}) = \mathrm{rank}(X)$? Equivalently, $\mathrm{rank}(X \cup Y) = \mathrm{rank}(X)$.

In Section 4.1, we establish the W[1]-hardness of MAXIMUM SPAN ON GRAPHIC MATROIDS when parameterized by k, t, as well as $k + t$. Furthermore, Section 4.2 shows that there is no algorithm running in time $f(k) \cdot n^{\mathcal{O}(1)}$ that approximates the maximum value of $\mathrm{span}(X) \setminus X$, taken over all independent sets X of size at most k, within a factor of $k^{o(1)}$.

4.1 W[1]-hardness parameterized by k, t, and $k + t$

▶ **Theorem 5.** *MAXIMUM SPAN ON GRAPHIC MATROIDS is* W[1]*-hard when parameterized by $k + t$. Consequently, it is also* W[1]*-hard when parameterized by k or t alone.*

Proof. We begin by observing an important structural property of graphic matroids. Let F be an independent set. An edge $e \in E(G) \setminus F$ lies in the span of F if and only if the endpoints of e lie in the same connected component of the forest $(V(G), F)$.

We give a parameterized reduction from CLIQUE. In CLIQUE, given a graph and a positive integer r, the question is whether there exists a clique of size r. The problem CLIQUE is W[1]-hard when parameterized by r [8]. We may assume that $r \geq 3$. Let (G, r) be an instance of CLIQUE. We construct an instance of MAXIMUM SPAN ON GRAPHIC MATROIDS on G by setting $k := r - 1$ and $t := \binom{r}{2} - (r - 1) = \binom{r-1}{2}$.

▷ Claim 6 (★). (G, r) is a Yes-instance of CLIQUE if and only if (G, k, t) is a Yes-instance of MAXIMUM SPAN ON GRAPHIC MATROIDS.

Since $k+t$ is a function of r alone, this is a parameterized reduction. The same construction also has both k and t bounded by functions of r. Therefore the problem is W[1]-hard when parameterized by $k + t$, and also when parameterized by k or by t alone. ◀

4.2 FPT-Inapproximability parameterized by k

While Theorem 5 rules out an exact FPT algorithm parameterized by k, one might still hope for an FPT-approximation. We show that this is unlikely by giving a parameterized reduction from the perfect-completeness version of DENSEST k-SUBGRAPH. For a graph H and a vertex set $S \subseteq V(H)$, the density of the induced subgraph $H[S]$ is

$$\mathrm{Den}(H[S]) = |E(H[S])| / \binom{|S|}{2}.$$

In the perfect-completeness version of DENSEST k-SUBGRAPH, the input consists of a graph H and an integer k, together with the promise that H contains a clique of size k. The goal is to output a subset $S \subseteq V(H)$ of exactly k vertices maximizing $\mathrm{Den}(H[S])$. It is known that, under Gap-ETH, there is no algorithm running in time $f(k) \cdot n^{\mathcal{O}(1)}$ that always outputs a set of k vertices with density at least $k^{-o(1)}$ on such instances [4].

▶ **Theorem 7.** *Assuming* Gap-ETH*, there is no $f(k) \cdot n^{\mathcal{O}(1)}$-time algorithm that approximates the maximization version of MAXIMUM SPAN ON GRAPHIC MATROIDS within a factor $k^{o(1)}$.*

Proof. We give a parameterized reduction from the perfect-completeness version of DENSEST k-SUBGRAPH. Given an instance (H, k), construct a graph G by adding one universal vertex u adjacent to every vertex of H. We consider the maximization version of MAXIMUM SPAN ON GRAPHIC MATROIDS on G with budget k, where the objective is to maximize $|\operatorname{span}(X) \setminus X|$ over independent sets X of size at most k.

Completeness. Let $S \subseteq V(H)$ be a k-clique. In G, let $X^\star$ be the star centered at u with leaves exactly the vertices of S. Then $X^\star$ is a tree with k edges, hence an independent set of size k in the graphic matroid. Its vertex set is $S \cup \{u\}$. The extra edges spanned by $X^\star$ are precisely the edges induced by S in H, and therefore $\mathrm{OPT}_{\mathrm{MS}}(G, k) \geq \binom{k}{2}$.

Soundness. Assume that there exists an FPT $\rho(k)$-approximation algorithm for MAXIMUM SPAN ON GRAPHIC MATROIDS, where $\rho(k) = k^{o(1)}$. Run it on (G, k), and let X be the returned forest. Let T be the number of extra edges spanned by X. Since $\mathrm{OPT}_{\mathrm{MS}}(G, k) \geq \binom{k}{2}$, we have $T \geq \mathrm{OPT}_{\mathrm{MS}}(G, k)/\rho(k) \geq \binom{k}{2}/\rho(k)$.

Let V_X be the set of vertices incident with the forest X. Since X has at most k edges, $|V_X| \leq 2k$. Every extra edge spanned by X has both endpoints in the same connected component of X, and hence both endpoints lie in V_X. Therefore $G[V_X]$ contains at least T edges outside the forest. Among these extra edges, at most $|V_X| - 1 \leq 2k - 1$ are incident with the universal vertex u. Hence, for $W := V_X \setminus \{u\}$, the graph $H[W]$ contains at least $T - (2k - 1)$ edges and $|W| \leq 2k$.

We now extract a k-vertex set from W while losing only a constant factor in the number of induced edges. If $|W| < k$, extend W arbitrarily to a set W' of size k; then $H[W']$ contains all edges of $H[W]$. Otherwise, $k \leq |W| \leq 2k$. Among all k-subsets S of W, the average value of $|E(H[S])|$ is $|E(H[W])| \cdot \binom{|W|-2}{k-2}/\binom{|W|}{k} = |E(H[W])| \cdot k(k-1)/(|W|(|W|-1))$. Since $|W| \leq 2k$, this average is at least $|E(H[W])| \cdot (k-1)/(2(2k-1)) \geq |E(H[W])|/6$ for $k \geq 2$. Thus there exists a set $W' \subseteq W$ of size k such that $|E(H[W'])| \geq |E(H[W])|/6$, and such a set can be found by enumerating all k-subsets of W in time $\binom{2k}{k} \cdot n^{\mathcal{O}(1)} = 2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.

Consequently, in FPT time we obtain a set W' of k vertices in H such that $|E(H[W'])| \geq (T - (2k-1))/6$. Since $\rho(k) = k^{o(1)}$, for all sufficiently large k we have $T \geq \binom{k}{2}/\rho(k) \gg k$, and hence $T - (2k - 1) \geq T/2$. Therefore $|E(H[W'])| \geq T/12 \geq \binom{k}{2}/(12\rho(k)) = k^{2-o(1)}$. It follows that $\mathrm{Den}(H[W']) = |E(H[W'])|/\binom{k}{2} \geq k^{-o(1)}$.

Thus an FPT $k^{o(1)}$-approximation for MAXIMUM SPAN ON GRAPHIC MATROIDS would yield an FPT algorithm for perfect-completeness DENSEST k-SUBGRAPH that outputs a k-vertex set of density at least $k^{-o(1)}$, contradicting the known Gap-ETH lower bound [4]. This completes the proof. ◀

5 Cographic Matroids

In this section, we study MAXIMUM SPAN on cographic matroids. Given a connected graph G, the corresponding cographic matroid has ground set $E(G)$, and a subset $X \subseteq E(G)$ is *independent* if and only if the graph $G - X$ remains connected.

> MAXIMUM SPAN ON COGRAPHIC MATROIDS
> **Input:** A connected graph G and positive integers k, t.
> **Parameters:** k, t, or $k + t$.
> **Question:** Does there exist an independent set $X \subseteq E(G)$ with $|X| \leq k$ and a set $Y \subseteq E(G) \setminus X$ with $|Y| \geq t$ such that for every $y \in Y$, we have $\operatorname{rank}(X \cup \{y\}) = \operatorname{rank}(X)$? Equivalently, $\operatorname{rank}(X \cup Y) = \operatorname{rank}(X)$.

For any independent set X, an element $e \in E(G) \setminus X$ belongs to $\mathrm{span}(X)$ exactly when $G - (X \cup \{e\})$ is disconnected. Since $G - X$ is connected, this is equivalent to saying that e is a bridge of $G - X$. Thus, in cographic matroids, MAXIMUM SPAN asks for an edge set X of size at most k such that $G - X$ is connected and contains at least t bridges. This viewpoint is equivalent to a bounded-size r–WAY CUT formulation, and hence we state Theorem 8.

▶ **Theorem 8** (★). *Let G be a connected graph. Then the following are equivalent:*
1. *there exists an independent set X of the cographic matroid on G with $|X| \leq k$ such that $|\mathrm{span}(X) \setminus X| \geq t$;*
2. *there exists an edge set $Z \subseteq E(G)$ with $|Z| \leq k + t$ such that $G - Z$ has $t + 1$ connected components.*

The equivalence above immediately yields both hardness and tractability results. In particular, it allows us to transfer known complexity bounds from r-WAY CUT to our setting. On one hand, it implies W[1]-hardness under the considered parameterization, ruling out the existence of an FPT algorithm unless standard complexity assumptions fail. On the other hand, it also enables the design of an FPT algorithm under a richer parameterization by exploiting the structural characterization provided by the equivalence.

5.1 W[1]-hardness parameterized by t

In this subsection, we record that MAXIMUM SPAN ON COGRAPHIC MATROIDS is W[1]-hard when parameterized by t. We reduce from the unweighted r-WAY CUT problem, which is W[1]-hard when parameterized by r [7]. Given an instance (G, r) of r-WAY CUT, we consider the cographic matroid on G and set $k := |E(G)| - (r - 1)$ and $t := r - 1$. Now, by Theorem 8, the following result follows immediately.

▶ **Corollary 9** (★). *MAXIMUM SPAN ON COGRAPHIC MATROIDS is* W[1]*-hard when parameterized by t.*

5.2 FPT algorithm parameterized by $k + t$

In this subsection, we view the problem in terms of the combined parameter $k + t$. Although the problem is W[1]-hard when parameterized solely by t, we state that it is fixed-parameter tractable when parameterized by $k + t$.

▶ **Corollary 10** (★). *MAXIMUM SPAN ON COGRAPHIC MATROIDS is fixed-parameter tractable when parameterized by $k + t$.*

6 Transversal Matroids

In this section, we study MAXIMUM SPAN on transversal matroids. Let $H = (A \uplus B, E(H))$ be a bipartite graph. The transversal matroid associated with H has ground set A, and a set $X \subseteq A$ is independent if and only if there exists a matching in H saturating X.

MAXIMUM SPAN ON TRANSVERSAL MATROIDS
Input: A bipartite graph $H = (A \uplus B, E(H))$ and positive integers k, t.
Parameters: k, t, or $k + t$.
Question: Does there exist an independent set $X \subseteq A$ with $|X| \leq k$ and a set $Y \subseteq A \setminus X$ with $|Y| \geq t$ such that for every $y \in Y$, we have $\mathrm{rank}(X \cup \{y\}) = \mathrm{rank}(X)$? Equivalently, $\mathrm{rank}(X \cup Y) = \mathrm{rank}(X)$.

We now state Hall's theorem, which will be used throughout this section.

▶ **Theorem 11** (Hall's Theorem, [17]). *Let $G = (X \uplus Y, E(G))$ be a bipartite graph. Then there exists a matching saturating A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

Observe that for an independent set $X \subseteq A$ and a vertex $y \in A \setminus X$, we have $y \in \text{span}(X)$ if and only if there is no matching in H saturating $X \cup \{y\}$. By Theorem 11, the last condition is equivalent to the existence of a subset $S \subseteq X \cup \{y\}$ with $|N(S)| < |S|$. Therefore, in transversal matroids, the span condition is governed by Hall-type obstructions. This immediately suggests a reduction from the Hall Set problem.

In Hall Set, given a bipartite graph $H = (A \uplus B, E(H))$ and a positive integer p, the question is whether there exists a set $S \subseteq A$ such that $|S| \leq p$ and $|N(S)| < |S|$. To establish the W[1]-hardness of the Hall Set problem, consider the parameterized reduction from Clique. Let (G, k) be an instance of Clique. We construct a bipartite graph with bipartition (A, B) as follows. The set A contains one vertex for each edge of G, and the set B contains one vertex for each vertex of G. For every edge $uv \in E(G)$, the corresponding vertex in A is made adjacent to the two vertices u and v in B. In addition, we add $\binom{k}{2} - k - 1$ new vertices to B, each of which is adjacent to every vertex in A. The key property of this construction is that any Hall set of size at most $\binom{k}{2}$ must in fact have size exactly $\binom{k}{2}$, and such a set corresponds precisely to the set of edges of a k-clique in G. Therefore, G contains a clique of size k if and only if the constructed bipartite graph contains a Hall set of size at most $\binom{k}{2}$. This proves that the Hall Set problem is W[1]-hard.

We are now ready to state the W[1]-hardness of Maximum Span on Transversal Matroids parameterized by k, t, and $k + t$.

▶ **Theorem 12.** *Maximum Span on Transversal Matroids is* W[1]*-hard when parameterized by $k + t$. Consequently, it is also* W[1]*-hard when parameterized by k or t alone.*

Proof. We give a parameterized reduction from Hall Set. Let $(H = (A \uplus B, E(H)), p)$ be an instance of Hall Set. Let us consider the transversal matroid associated with H on ground set A. We create an instance of Maximum Span on Transversal Matroids as $(H,\ k,\ t) := (H,\ p - 1,\ 1)$ and claim the following.

▷ Claim 13 (★). (H, p) is a `Yes` instance of Hall Set if and only if $(H, p - 1, 1)$ is a `Yes` instance of Maximum Span on Transversal Matroids.

This is a parameterized reduction, and since Hall Set is W[1]-hard parameterized by p, it follows that Maximum Span is W[1]-hard on transversal matroids parameterized by $k + t$. ◀

7 Strict Gammoids

In this section, we turn our attention to strict gammoids. We start by defining the notions of gammoids and strict gammoids.

▶ **Definition 14** (Linked from S). *Let D be a directed graph, $S \subseteq V(D)$ be a designated source set, and $T \subseteq V(D)$ be a set of destination vertices. A set $X \subseteq T$ is* linked from S *if there exist $|X|$ pairwise vertex-disjoint directed paths whose starting vertices are distinct vertices of S and whose ending vertices are exactly the vertices of X. Here, we allow a directed path of length 0; in particular, if a vertex v lies in S, then the trivial path at v may be used.*

▶ **Definition 15** (Gammoid and strict gammoid)**.** *The* gammoid *on the directed graph D, source set S, and destination set T is the matroid with ground set T whose independent sets are exactly the subsets of T that are linked from S. The* strict gammoid *associated with D and S is the gammoid when $T = V(D)$. Equivalently, the strict gammoid has ground set $V(D)$, and its independent sets are exactly the subsets of $V(D)$ that are linked from S.*

We restrict our focus to strict gammoids in this section. We now state Menger's Theorem, which we will use throughout the section.

▶ **Theorem 16** (Menger's Theorem [18])**.** *If x, y are vertices of a graph G and $xy \notin E(G)$, then the minimum size of an x–y separator equals the maximum number of pairwise internally vertex-disjoint x–y paths.*

Analogous extensions of Menger's theorem to edge-connectivity and directed graphs are well known. Moreover, a version for vertex sets A and B (in place of single vertices a and b) can be obtained by introducing new vertices a and b, and adding edges aa' for each $a' \in A$ and bb' for each $b' \in B$, thereby reducing the statement to the standard form of Menger's theorem and getting vertex-disjoint paths instead of internally vertex-disjoint paths.

▶ **Definition 17** (Rank)**.** *The rank of a set $Z \subseteq V(D)$ in $\mathcal{M}(D, S)$, denoted by* rank(Z) *equals the maximum number of pairwise vertex-disjoint directed paths from distinct vertices of S to vertices of Z. By directed Menger's theorem, this is also the minimum size of an S–Z vertex separator.*

> MAXIMUM SPAN ON STRICT GAMMOIDS
> **Input:** A directed graph D, a source set $S \subsetneq V(D)$, and positive integers k, t.
> **Parameters:** k, t, or $k + t$.
> **Question:** Does there exist an independent set $X \subseteq V(D)$ with $|X| \leq k$ and a set $Y \subseteq V(D) \setminus X$ with $|Y| \geq t$ such that for every $y \in Y$, we have $\text{rank}(X \cup \{y\}) = \text{rank}(X)$ (equivalently, $\text{rank}(X \cup Y) = \text{rank}(X)$)?

7.1 W[1]-hardness parameterized by k, t, and $k + t$

▶ **Theorem 18.** *MAXIMUM SPAN ON STRICT GAMMOIDS is* W[1]*-hard when parameterized by $k + t$. Consequently, it is also* W[1]*-hard when parameterized by k or t alone.*

Proof. We reduce from CLIQUE. Let (G, r) be an instance of CLIQUE, where $V(G) = \{u_1, \ldots, u_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$.

Construction. We construct a directed graph D as follows. For every vertex $u_i \in V(G)$, create a source vertex s_i, and let $S := \{s_1, \ldots, s_n\}$. For every vertex $u_i \in V(G)$, create a vertex p_i and add the arc $s_i \to p_i$; let $P := \{p_1, \ldots, p_n\}$. For every edge $e = u_iu_j \in E(G)$, create a vertex q_e and add the arcs $s_i \to q_e$ and $s_j \to q_e$; let $Q := \{q_e : e \in E(G)\}$. Thus $V(D) = S \cup P \cup Q$. Finally, set $k := r$ and $t := r + \binom{r}{2}$.

We prove the following equivalence.

▶ **Lemma 19.** *(G, r) is a* `Yes`*-instance of CLIQUE if and only if (D, S, k, t) is a* `Yes`*-instance of MAXIMUM SPAN ON STRICT GAMMOIDS.*

Since $k + t$ is a function of r alone, this is a parameterized reduction. The same construction also has both k and t bounded by functions of r. Hence the theorem follows from Lemma 19. ◀

Proof of Lemma 19.

Forward direction. Suppose that G contains a clique K of size r. Define $X \coloneqq \{s_i : u_i \in K\}$ and $Y \coloneqq \{p_i : u_i \in K\} \cup \{q_e : e \in E(G[K])\}$. Then $|X| = r = k$ and $|Y| = r + \binom{r}{2} = t$. The set X is independent in $\mathcal{M}(D, S)$, since for each $s_i \in X$ we may use the length-0 path at s_i, and these paths are pairwise vertex-disjoint. Thus $\mathrm{rank}(X) = |X| = r$.

Let $y \in Y$. If $y = p_i$, then every directed path from S to p_i starts at s_i, and since $s_i \in X$, any family of pairwise vertex-disjoint directed paths from S to $X \cup \{p_i\}$ has size at most $|X|$. Hence $\mathrm{rank}(X \cup \{p_i\}) = \mathrm{rank}(X)$. If $y = q_e$ for some edge $e = u_iu_j \in E(G[K])$, then every directed path from S to q_e starts at either s_i or s_j. Since both s_i and s_j lie in X, any family of pairwise vertex-disjoint directed paths from S to $X \cup \{q_e\}$ has size at most $|X|$. Hence $\mathrm{rank}(X \cup \{q_e\}) = \mathrm{rank}(X)$. Therefore (D, S, k, t) is a `Yes`-instance.

Reverse direction. Suppose that (D, S, k, t) is a `Yes`-instance. Then there are disjoint sets $X, Y \subseteq V(D)$ such that X is independent, $|X| \leq k = r$, $|Y| \geq r + \binom{r}{2}$, and $\mathrm{rank}(X \cup \{y\}) = \mathrm{rank}(X)$ for every $y \in Y$. Equivalently, $\mathrm{rank}(X \cup Y) = \mathrm{rank}(X)$. Since X is independent, $\mathrm{rank}(X) = |X| \leq r$. Let $W \coloneqq X \cup Y$. Then $\mathrm{rank}(W) = \mathrm{rank}(X \cup Y) = \mathrm{rank}(X) = |X|$.

We define a bipartite graph B_W with left side a copy of W and right side a copy of S. We identify the copies with the original labels. If $s_i \in W \cap S$, then the left vertex s_i is adjacent only to the right vertex s_i. If $p_i \in W \cap P$, then the left vertex p_i is adjacent only to the right vertex s_i. If $q_e \in W \cap Q$ and $e = u_iu_j$, then the left vertex q_e is adjacent to the right vertices s_i and s_j.

▷ Claim 20 (★). In the matroid $\mathcal{M}(D, S)$, we have $\mathrm{rank}(W) = \nu(B_W)$, where $\nu(B_W)$ denotes the size of a maximum matching in B_W.

Let C be a minimum vertex cover of B_W. By Kőnig's Theorem and Claim 20, we have $|C| = \nu(B_W) = \mathrm{rank}(W) = |X|$. Define $C_W \coloneqq C \cap W$ and $C_S \coloneqq C \cap S$.

▷ Claim 21 (★). $|W \setminus C_W| \leq 2|C_S| + \binom{|C_S|}{2}$.

▷ Claim 22 (★). We have $|C_S| = r$, $|Y| = r + \binom{r}{2}$, and $|W \setminus C_W| = 2|C_S| + \binom{|C_S|}{2}$.

▷ Claim 23 (★). $|(W \setminus C_W) \cap Q| = \binom{|C_S|}{2}$.

We now extract a clique in G. Define $K \coloneqq \{u_i \in V(G) : s_i \in C_S\}$. By Claim 22, $|C_S| = r$, and hence $|K| = r$. Consider any vertex $q_e \in (W \setminus C_W) \cap Q$, where $e = u_iu_j \in E(G)$. In B_W, the left vertex q_e is adjacent exactly to the right vertices s_i and s_j. Since $q_e \notin C_W$ and $C = C_W \cup C_S$ is a vertex cover, both s_i and s_j belong to C_S. Hence both u_i and u_j lie in K. Therefore every vertex of $(W \setminus C_W) \cap Q$ corresponds to an edge of $G[K]$, and so $|E(G[K])| \geq |(W \setminus C_W) \cap Q| = \binom{|C_S|}{2} = \binom{r}{2}$. Since K has exactly r vertices, $G[K]$ has at most $\binom{r}{2}$ edges. Thus $|E(G[K])| = \binom{r}{2}$, and K is a clique of size r in G. ◀

7.2 XP algorithm parameterized by t

We begin with two simple reductions. The first shows that, although source vertices belong to the ground set of a strict gammoid, the counted set of spanned elements may be assumed to avoid the source set.

▶ **Lemma 24.** *If (D, S, k, t) is a `Yes`-instance of* Maximum Span on Strict Gammoids, *then it has a witness (X, Y) such that X is independent, $|X| \leq k$, $|Y| \geq t$, $X \cap Y = \emptyset$, $Y \subseteq \mathrm{span}(X)$, and $Y \cap S = \emptyset$.*

Proof. Let (X,Y) be a witness with $|Y \cap S|$ minimum among all witnesses. We show that $Y \cap S = \emptyset$. Suppose not, and let $y \in Y \cap S$. Since $y \in \mathrm{span}(X) \setminus X$ and X is independent, the set $X \cup \{y\}$ contains a unique circuit C_y, and this circuit contains y.

Every subset of S is independent in the strict gammoid, since each source can be linked to itself by a length-0 path. Hence C_y is not contained in S. As $y \in S$, there exists a vertex $x \in C_y \setminus \{y\}$ with $x \notin S$. In particular, $x \in X \setminus S$.

By the circuit-exchange property, $X' := (X \setminus \{x\}) \cup \{y\}$ is independent. Moreover, $X' \subseteq \mathrm{span}(X)$ and $|X'| = |X|$, so $\mathrm{span}(X') = \mathrm{span}(X)$. Now define $Y' := (Y \setminus \{y\}) \cup \{x\}$. Then $X' \cap Y' = \emptyset$, $|X'| = |X| \leq k$, and $|Y'| = |Y| \geq t$. Also, since $\mathrm{span}(X') = \mathrm{span}(X)$, every element of $Y \setminus \{y\}$ remains in $\mathrm{span}(X')$, and $x \in \mathrm{span}(X')$. Thus (X', Y') is again a valid witness.

But $x \notin S$ and $y \in S$, so $|Y' \cap S| = |Y \cap S| - 1$, contradicting the minimality of $|Y \cap S|$. Therefore $Y \cap S = \emptyset$. ◀

For a set $Y \subseteq V(D) \setminus S$, we call a set $C \subseteq V(D) \setminus Y$ a directed S–Y vertex separator *avoiding* Y if every directed path from a vertex of S to a vertex of Y intersects C. Thus vertices of S are allowed to belong to C, whereas vertices of Y are not.

▶ **Lemma 25.** *An instance (D,S,k,t) is a* `Yes`*-instance if and only if there exists a set $Y \subseteq V(D) \setminus S$ with $|Y| = t$ such that the minimum size of a directed S–Y vertex separator avoiding Y is at most k.*

Proof. Suppose first that (D,S,k,t) is a `Yes`-instance. By Lemma 24, there is a witness (X,Y') with $Y' \cap S = \emptyset$. Choose any subset $Y \subseteq Y'$ of size exactly t. Since $Y \subseteq \mathrm{span}(X)$, we have $\mathrm{rank}(Y) \leq \mathrm{rank}(X \cup Y) = \mathrm{rank}(X) = |X| \leq k$. By directed Menger's theorem in its vertex-set form, with the target set Y forbidden to be deleted, the minimum size of a directed S–Y vertex separator avoiding Y equals the maximum number of pairwise vertex-disjoint directed paths from S to Y, namely $\mathrm{rank}(Y)$. Hence such a separator has size at most k.

Conversely, suppose that there exists a set $Y \subseteq V(D) \setminus S$ of size t such that the minimum size of a directed S–Y vertex separator avoiding Y is $q \leq k$. Let $C \subseteq V(D) \setminus Y$ be such a minimum separator, with $|C| = q$. By directed Menger's theorem, there exist q pairwise vertex-disjoint directed paths from S to Y. Since C is an S–Y separator avoiding Y of size q, each of these paths meets C, and they meet C in distinct vertices. Taking the prefix of each path up to its first vertex in C gives q pairwise vertex-disjoint directed paths from S to the vertices of C. Hence C is independent in the strict gammoid and $\mathrm{rank}(C) = |C| = q$.

We claim that $\mathrm{rank}(C \cup Y) = \mathrm{rank}(C)$. Since C separates S from Y, every directed path from S to a vertex of Y intersects C. Therefore any family of pairwise vertex-disjoint directed paths from S to $C \cup Y$ has size at most $|C|$: each path ending in Y must use a vertex of C, and each path ending in C also uses a vertex of C. Since C is independent, there are already $|C|$ pairwise vertex-disjoint paths from S to C. Hence $\mathrm{rank}(C \cup Y) = |C| = \mathrm{rank}(C)$. Thus (C,Y) is a valid witness, and the instance is a `Yes`-instance. ◀

Using Lemma 25, it suffices to check whether there exists a set $Y \subseteq V(D) \setminus S$ of size t such that the minimum directed $S - Y$ separator avoiding Y has size at most k. We enumerate all such Y (at most n^t choices) and for each use a node-splitting max-flow construction to test the separator condition in polynomial time, yielding an $n^{\mathcal{O}(t)}$ XP algorithm.

▶ **Theorem 26 (★).** *Maximum Span on Strict Gammoids belongs to* XP *when parameterized by t.*

7.3 FPT-approximations in the undirected and directed case

We first recall the notion of important separators.

▶ **Definition 27** (Definition 8.49 in [6]). *Let G be a directed or undirected graph, let $A, B \subseteq V(G)$, and let $V^\infty(G)$ denote the set of undeletable vertices. Let $Z \subseteq V(G) \setminus V^\infty(G)$ be an (A, B)-separator, and let R_Z be the set of vertices reachable from $A \setminus Z$ in $G - Z$. We say that Z is an* important *(A, B)-separator if it is inclusion-wise minimal and there is no (A, B)-separator $Z' \subseteq V(G) \setminus V^\infty(G)$ with $|Z'| \leq |Z|$ such that $R_Z \subset R_{Z'}$.*

▶ **Proposition 28** (Proposition 8.50 in [6]). *Let G be a directed or undirected graph and let $A, B \subseteq V(G)$. Let $Z \subseteq V(G) \setminus V^\infty(G)$ be an (A, B)-separator, and let R_Z be the set of vertices reachable from $A \setminus Z$ in $G - Z$. Then there is an important (A, B)-separator Z' such that $|Z'| \leq |Z|$ and $R_Z \subseteq R_{Z'}$.*

▶ **Theorem 29** (Theorem 8.51 in [6]). *Let $A, B \subseteq V(G)$ be two sets of vertices in a directed or undirected graph G with n vertices and m edges, and let $k \geq 0$ be an integer. Let $\mathcal{S}_k$ be the set of all important (A, B)-separators of size at most k. Then $|\mathcal{S}_k| \leq 4^k$, and $\mathcal{S}_k$ can be constructed in time $\mathcal{O}(|\mathcal{S}_k| \cdot k^2 \cdot (n + m))$.*

We will use these statements with suitable vertices declared undeletable. In particular, when enumerating r–S separators avoiding r, we put $r \in V^\infty(G)$.

7.3.1 An FPT-approximation in the undirected case

In this subsection we consider the undirected version of strict gammoids. Thus the input consists of an undirected graph G, a source set $S \subseteq V(G)$, and an integer k. The optimization version asks for an independent set $X \subseteq V(G)$ of size at most k maximizing $|\operatorname{span}(X) \setminus X|$. We denote the optimum value by OPT. For a set $X \subseteq V(G)$, define its shadow by

$$\texttt{shadow}(X) \coloneqq \{v \in V(G) \setminus X : \text{ there is no path from } v \text{ to } S \text{ in } G - X\}.$$

Thus $\texttt{shadow}(X)$ is the union of all connected components of $G - X$ that do not contain a vertex of S. If C is a connected component of $G[\texttt{shadow}(X)]$, we define its type with respect to X by $\texttt{type}_X(C) \coloneqq N_G(C) \cap X$. There are at most $2^{|X|}$ possible types.

For a vertex $r \notin S$ and two r–S separators Z, Z' avoiding r, we say that Z' dominates Z if $|Z'| \leq |Z|$ and $R_{G-Z}(r) \subseteq R_{G-Z'}(r)$, where $R_{G-Z}(r)$ is the set of vertices reachable from r in $G - Z$.

▶ **Lemma 30** (★). *Let $X \subseteq V(G)$, let $C^\star$ be a connected component of $G[\texttt{shadow}(X)]$, and let $r \in C^\star$. Suppose that X' is an r–S separator avoiding r that dominates X. Then every connected component C of $G[\texttt{shadow}(X)]$ with $\texttt{type}_X(C) = \texttt{type}_X(C^\star)$ is contained in $\texttt{shadow}(X')$.*

▶ **Theorem 31.** *There is an* FPT *algorithm parameterized by k that, given an undirected strict-gammoid instance (G, S, k), outputs disjoint sets $X, Y \subseteq V(G)$ such that X is independent, $|X| \leq k$, $\operatorname{rank}(X \cup Y) = \operatorname{rank}(X)$, and $|Y| \geq \mathrm{OPT}/2^k$. In particular, the optimization version admits an* FPT *2^k-approximation parameterized by k.*

Proof. The algorithm is as follows. For every vertex $r \in V(G) \setminus S$, enumerate all important r–S separators X' of size at most k avoiding r. For each such separator, let $Y \coloneqq \texttt{shadow}(X')$. Compute a minimum S–Y separator C avoiding Y. If $|C| \leq k$, keep the pair (C, Y). Among all kept pairs, return one maximizing $|Y|$.

We first justify feasibility. Whenever the algorithm keeps a pair (C, Y), the set C is an S–Y separator avoiding Y of size at most k. By the undirected version of Lemma 25, the set C is independent and $\mathrm{rank}(C \cup Y) = \mathrm{rank}(C)$. Hence every returned pair is feasible.

It remains to prove the approximation guarantee. Let $Y^\star$ be an optimal counted set, so $|Y^\star| = \mathrm{OPT}$. By Lemma 24, we may assume that $Y^\star \cap S = \emptyset$. By Lemma 25, there exists an S–$Y^\star$ separator $X^\star$ avoiding $Y^\star$ with $|X^\star| \leq k$. Then $Y^\star \subseteq \texttt{shadow}(X^\star)$.

Consider the connected components of $G[\texttt{shadow}(X^\star)]$ and partition them by their types with respect to $X^\star$. Since there are at most $2^{|X^\star|} \leq 2^k$ types, there is a type class whose components together contain at least $\mathrm{OPT}/2^k$ vertices of $Y^\star$. Fix such a heavy type class, choose a component $C^\star$ in this class with $C^\star \cap Y^\star \neq \emptyset$, and pick a vertex $r \in C^\star \cap Y^\star$.

Now $X^\star$ is an r–S separator avoiding r. By Proposition 28, there exists an important r–S separator X' avoiding r such that $|X'| \leq |X^\star| \leq k$ and X' dominates $X^\star$. By Lemma 30, every component in the chosen heavy type class is contained in $\texttt{shadow}(X')$. Therefore $|\texttt{shadow}(X')| \geq \mathrm{OPT}/2^k$.

When the algorithm considers this vertex r and this important separator X', it sets $Y = \texttt{shadow}(X')$. Since X' itself is an S–Y separator avoiding Y of size at most k, the minimum such separator C has size at most k. Hence the algorithm keeps a feasible pair (C, Y) with $|Y| = |\texttt{shadow}(X')| \geq \mathrm{OPT}/2^k$. Therefore the returned solution has value at least $\mathrm{OPT}/2^k$.

For each fixed r, since the number of important r–S separators of size at most k is at most 4^k which can be enumerated in $\mathcal{O}^*(4^k)$ time, we try all choices of r. All other computations are polynomial-time flow computations. Thus, the total running time is $\mathcal{O}^*(4^k)$. ◀

7.3.2 An FPT-approximation in the directed case

In this subsection we return to directed strict gammoids. Let (D, S, k) be an instance of the optimization version, and let OPT denote the maximum value of $|\mathrm{span}(X) \setminus X|$ over all independent sets $X \subseteq V(D)$ with $|X| \leq k$.

For a set $X \subseteq V(D)$, define its directed shadow by $\texttt{shadow}(X) := \{v \in V(D) \setminus (S \cup X) :$ there is no directed path from S to v in $D - X\}$. Thus $\texttt{shadow}(X)$ is the set of non-source vertices, outside X, that become unreachable from S after deleting X.

We shall use the following reverse domination notion. For a vertex $r \notin S$ and two S–r separators Z, Z' avoiding r, we say that Z' *reverse-dominates* Z if $|Z'| \leq |Z|$ and $R^-_{D-Z}(r) \subseteq R^-_{D-Z'}(r)$, where $R^-_{D-Z}(r)$ is the set of vertices that can reach r in $D - Z$. Such separators can be enumerated by applying the standard important-separator enumeration to the reverse digraph D^{rev}, with r declared undeletable. We first record a simple closure property of directed shadows.

▶ **Lemma 32** (★). *Let $X \subseteq V(D)$ and let $Y := \texttt{shadow}(X)$. If $v \in Y$ and $(u, v) \in A(D)$ with $u \notin X$, then $u \in Y$.*

Let $X \subseteq V(D)$ and let $Y := \texttt{shadow}(X)$. For a strongly connected component C of $D[Y]$, define its type with respect to X by

$$\texttt{type}_X(C) := \{x \in X : \text{ there is an } x\text{–}C \text{ directed path with internal vertices in } Y\}.$$

Since $\texttt{type}_X(C) \subseteq X$, there are at most $2^{|X|}$ possible types.

▶ **Lemma 33** (★). *Let $X \subseteq V(D)$, let $Y := \texttt{shadow}(X)$, let $C^\star$ be a strongly connected component of $D[Y]$, and let $r \in C^\star$. Suppose that X' is an S–r separator avoiding r that reverse-dominates X. Then every strongly connected component C of $D[Y]$ with $\texttt{type}_X(C) = \texttt{type}_X(C^\star)$ is contained in $\texttt{shadow}(X')$.*

▶ **Theorem 34** (★). *There is an* FPT *algorithm parameterized by k that, given a directed strict-gammoid instance (D, S, k), outputs disjoint sets $X, Y \subseteq V(D)$ such that X is independent, $|X| \leq k$, $\mathrm{rank}(X \cup Y) = \mathrm{rank}(X)$, and $|Y| \geq \mathrm{OPT}/2^k$. In particular, the optimization problem on directed strict gammoids admits an* FPT 2^k*-approximation parameterized by k.*

8 Gammoids

We now turn to gammoids. We refer to Section 7 for the definition of gammoids and strict gammoids. Recall that strict gammoids are a special case of gammoids. The Maximum Span problem on gammoids is defined exactly as before.

> Maximum Span on Gammoids
> **Input:** A gammoid (D, S, U) and integers k, t.
> **Parameters:** k, t, or $k + t$.
> **Question:** Does there exist disjoint sets $X, Y \subseteq U$ such that X is independent, $|X| \leq k$, $|Y| \geq t$, and $\mathrm{rank}(X \cup Y) = \mathrm{rank}(X)$?

8.1 W[1]-hardness parameterized by k, t, and $k + t$

Since every strict gammoid is a gammoid, Theorem 18 for strict gammoids immediately extends to the larger class of all gammoids.

▶ **Corollary 35** (★). *Maximum Span on Gammoids is* W[1]*-hard when parameterized by $k + t$. Consequently, it is also* W[1]*-hard when parameterized by k or t alone.*

We can also rule out XP algorithms parameterized only by t.

▶ **Corollary 36** (★). *Maximum Span on Gammoids is* NP*-hard already for the fixed value $t = 1$. Consequently, unless* P = NP*, the problem does not admit an XP algorithm parameterized by t.*

8.2 An XP algorithm parameterized by k

In this subsection, we present an XP algorithm that enumerates all sets of size at most k, tests independence via standard max-flow computations, and determines the number of additional vectors they span.

▶ **Theorem 37** (★). *Maximum Span on Gammoids is in* XP *when parameterized by k.*

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