Towards a Universal Gateset for QMA_1

Dorian Rudolph
Department of Computer Science and Institute for Photonic Quantum Systems (PhoQS), Paderborn University, Germany

Abstract

QMA_1 is QMA with perfect completeness: in the YES-case the verifier must accept some proof with probability *exactly* 1. Whether $\mathsf{QMA}_1 = \mathsf{QMA}$ remains open, and even the definition of QMA_1 has a gateset issue, since Solovay–Kitaev only gives approximate synthesis and may destroy perfect completeness. For a gateset $\mathcal{G}$, we write $\mathsf{QMA}_1^{\mathcal{G}}$ for QMA_1 restricted to verifiers using gates from $\mathcal{G}$.

Using the gatesets $\mathcal{G}_{2^k}$ of Amy et al. (RC 2024), we prove that $\mathsf{QMA}_1^{\mathcal{G}} \subseteq \mathsf{QMA}_1^{\mathcal{G}_{2^k}}$ for every finite gateset $\mathcal{G}$ whose entries lie in the cyclotomic field $\mathbb{Q}(\zeta_{2^k})$, $\zeta_{2^k} = e^{2\pi \mathrm{i}/2^k}$. For BQP_1 (aka coRQP), the rational gateset $\mathcal{G}_2$ already suffices for all these fields. We also give complete problems for the resulting classes: quantum ℓ-SAT over $\mathbb{Q}(\zeta_{2^k})$ is complete for $\mathsf{QMA}_1^{\mathcal{G}_{2^k}}$ for all $\ell \geq 4$, and also for $\ell = 3$ when $k \geq 3$.

The main technical tool is to use linear combinations of unitaries and postselection to exactly apply operators whose entries lie in the relevant field, and then use oblivious amplitude amplification to make the postselection failure probability negligible. This also gives an exact kernel test for Hamiltonians. As a consequence, we prove the first QMA_1-complete 2-local Hamiltonian problem: for $k \geq 3$, deciding whether a 2-local Hamiltonian H over $\mathbb{Q}(\zeta_{2^k})$ has $\sigma_1(H) = 0$ or $\sigma_1(H) \geq 1/\mathrm{poly}$ is complete for $\mathsf{QMA}_1^{\mathcal{G}_{2^k}}$. The same ideas extend to sparse Hamiltonians and yield the first $\mathsf{QMA}_1(2)$-complete Hamiltonian problem.

Finally, we apply the gateset framework to clique homology. We prove that the Gapped Clique Homology problem on weighted graphs defined by King and Kohler (FOCS 2024) is $\mathsf{QMA}_1^{\mathcal{G}_2}$-complete, and the Clique Homology problem (Kaibel and Pfetsch, 2002) without promise gap is PSPACE-complete, resolving a conjecture of Crichigno and Kohler (Nat. Commun. 2024).

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1 Introduction

The complexity class QMA_1 was introduced by Bravyi in 2006 [9] as QMA with one-sided error (or *perfect completeness*), i.e., in the YES-case there exists a proof that the verifier accepts with a probability of *exactly* 1. Bravyi shows that the Quantum k-SAT problem (k-QSAT) is QMA_1-complete for $k \geq 4$ and in P for $k = 2$, where k-QSAT is the problem of deciding whether a given collection of k-local Hamiltonians has a common ground state.

Before discussing different variants of the quantum satisfiable problem, we must touch on a surprising issue in QMA_1: its definition depends on which gateset the verifier uses. We write $\mathsf{QMA}_1^{\mathcal{G}}$ to denote QMA_1 with gateset $\mathcal{G}$, where $\mathcal{G}$ is a set of unitaries. Thus, throughout this paper, hardness and completeness statements for QMA_1 are always understood relative to an explicitly specified gateset.[1] Note that QMA does not have such an issue, since we can use the Solovay–Kitaev algorithm [34, 19, 8] to synthesize gates. However, Solovay–Kitaev is only approximate and therefore the synthesized circuit does not necessarily preserve perfect completeness. Therefore, we do not in general know whether $\mathsf{QMA}_1^{\mathcal{G}} = \mathsf{QMA}_1^{\mathcal{G}'}$ for two different universal gatesets $\mathcal{G}$ and $\mathcal{G}'$. Thus, the $\mathsf{QMA}_1^{\mathcal{G}}$ complexity classes form a potentially infinite hierarchy!

k-QSAT can be seen as a quantum analogue of the classical k-SAT problem, which is NP-complete for $k \geq 3$ [14, 39, 30], and in P for $k = 2$ [49, 18, 38, 22, 6, 48]. Later works even show that 2-QSAT is solvable in linear time [5, 20], and 3-QSAT is QMA_1-complete [27]. Quantum 2-SAT on qu$d$its is QMA_1-complete [21, 46, 52], with the most recent result being the QMA_1-completeness of $(3,4)$-QSAT and $(2,5)$-QSAT [52], where in (k,l)-QSAT each local term acts on one qu-k-it and one qu-l-it. k-QSAT is a special case of the more well-known k-local Hamiltonian problem (k-LH), where the goal is to approximate $\lambda_{\min}(H)$ for a k-local Hamiltonian H. k-LH can be seen as a quantum analogue of the classical MAX-k-SAT problem (i.e. what is the maximum number of simultaneously satisfiable constraints in a 2-CNF formula?). The 2-local Hamiltonian problem is QMA-complete [35, 31, 17], just as MAX-2-SAT is NP-complete [23].

An interesting question to ask is now whether quantum k-SAT is also QMA-complete, or in other words, is $\mathsf{QMA} = \mathsf{QMA}_1$? Classically, we have $\mathsf{MA} = \mathsf{MA}_1$ [55, 26]. Quantum interactive proof systems (QIP) also have perfect completeness [36, 37]. QMA even has a two-message quantum interactive proof system with perfect completeness, i.e., $\mathsf{QMA} \in \mathsf{QIP}_1(2)$ [37]. [28] have shown that QCMA (i.e., QMA with classical proofs) has perfect completeness, i.e., $\mathsf{QCMA} \subseteq \mathsf{QMA}_1$. Despite these positive results, the question whether $\mathsf{QMA} = \mathsf{QMA}_1$ remains open to this day. Aaronson [1] gives some negative evidence in the form of a *quantum* oracle separation between QMA and QMA_1. The idea is to give the verifier access to an oracle performing a rotation by some angle θ, and the problem is to distinguish the cases $\theta \in [1,2]$ (YES-case) or $\theta = 0$ (NO-case). Aaronson uses techniques from real analysis to prove that if the verifier accepts with probability 1 for all θ in an open set, then it must accept for all θ.

Bravyi [9] defines QMA_1 as $\mathsf{QMA}_1^{\mathcal{G}}$, where $\mathcal{G}$ is the set of all 3-qubit unitaries in some subfield $\mathbb{F} \subseteq \mathbb{C}$ with exact representation. The corresponding complete 4-QSAT problem also allows all projectors with elements in $\mathbb{F}$. Thus, k-$\mathsf{QSAT} \in \mathsf{QMA}_1$ requires the verifier to be constructed specifically for the instance.[2] We cannot just give a classical description of the Hamiltonian to the quantum verifier.

Different formulations of quantum SAT make different choices about which exact amplitudes and projectors are available to the verifier. Gosset and Nagaj [27] instead only consider the gateset $\{\mathsf{H}, \mathsf{CX}, \mathsf{T}\}$ (Hadamard, CNOT, T-gate), and show completeness for k-QSAT, where each local projector Π is in $\mathbb{Z}[\frac{1}{\sqrt{2}}, \mathrm{i}]$,[3] or there exists a unitary with only these elements, such that $U\Pi U^\dagger = (\sqrt{1/3}|000\rangle - \sqrt{2/3}|001\rangle)(\sqrt{1/3}\langle 000| - \sqrt{2/3}\langle 001|)$.

[1] When we write, for instance, that a problem is $\mathsf{QMA}_1^{\mathcal{G}}$-complete, the reductions are classical polynomial-time reductions and the verifier for the target class is restricted to gates from $\mathcal{G}$. For cited results that do not display a gateset in the notation, we follow the gateset convention of the cited paper.

[2] We remark that there is a slight issue with this procedure (see [9, Lemma 5]), also mentioned by Gosset and Nagaj [27]. Decomposing general unitaries into 2-level unitaries (see [47]) does not necessarily produce unitaries inside $\mathbb{F}$ (e.g. for $\mathbb{F} = \mathbb{Q}$).

[3] We use "i" to denote the imaginary unit, disambiguating it from the index "i".

In this work, we go back to Bravyi's more general subfield definition, and formally prove for a family of cyclotomic fields $\mathbb{F} \subset \mathbb{C}$ that k-QSAT over $\mathbb{F}$ is complete for $\mathsf{QMA}_1^{\mathcal{G}}$ for *finite* gatesets $\mathcal{G}$. That means, for each of the subfields $\mathbb{F}$ we consider, there does exist a universal gateset that can realize all finite gatesets in that field. The fields we focus on are the cyclotomic fields $\mathbb{Q}(\zeta_{2^k})$, obtained by adjoining a primitive 2^k-th root of unity to the rationals. They naturally contain the entries of standard exact gates such as phase gates, while still having enough algebraic structure for exact synthesis results. The restriction to powers of two is due to the exact synthesis and field-simulation tools used in this paper.[4]

The containment result for k-$\mathsf{QSAT} \in \mathsf{QMA}_1$ in [27] relies on Giles and Selinger's [25] algorithm for *exactly* synthesizing any n-qubit unitary with entries in the ring $\mathbb{Z}[1/\sqrt{2}, \mathrm{i}]$ with the "Clifford + T" gateset. Their work has been extended to further gatesets by [3]. Most recently [2] have shown that an n-qubit unitary matrix U can be exactly represented with the gateset $\mathcal{G}_{2^k}$ iff U's entries are in $\mathbb{Z}[1/2, \zeta_{2^k}]$, where $\zeta_{2^k} = e^{2\pi \mathrm{i}/2^k}$ is a primitive 2^k-th root of unity, $\mathcal{G}_2 = \{\mathsf{X}, \mathsf{CX}, \mathsf{CCX}, \mathsf{H} \otimes \mathsf{H}\}$ (X, CNOT, Toffoli, paired Hadamard), $\mathcal{G}_4 = \{\mathsf{X}, \mathsf{CX}, \mathsf{CCX}, \mathsf{S}, \zeta_8\mathsf{H}\}$, and for $k \geq 3$, $\mathcal{G}_{2^k} = \{\mathsf{H}, \mathsf{CX}, \mathsf{T}_{2^k}\}$ with $\mathsf{T}_{2^k} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \zeta_{2^k} \end{smallmatrix}\right]$ ($\mathsf{T} \equiv \mathsf{T}_8$, $\mathsf{S} \equiv \mathsf{T}_4$).

A second motivation for understanding exact gatesets comes from computational topology. Crichigno and Cade [10] proved that the k-local cohomology problem is QMA_1-hard, for a cohomology inspired by supersymmetric systems. Crichigno and Kohler [15] showed that the problem of determining whether the clique complex of a given graph has a hole of a given dimension, is QMA_1-hard. This problem was first defined by Kaibel and Pfetsch [29]. At a high level, the clique-homology reductions encode quantum constraints as boundary relations in a simplicial complex. A projector whose forbidden vector is an integer superposition can be implemented by a small graph gadget. However, arbitrary complex phases, and even coefficients such as $\sqrt{2}$, cannot simply be inserted into an unweighted clique complex, whose boundary maps have only $0, \pm 1$ entries. This is why the gateset matters here in a way that it does not matter as visibly for quantum SAT. More formally, the clique complex of a graph G is the simplicial complex obtained by declaring every $(k+1)$-clique in G to be a k-simplex. Crichigno and Kohler [15] also showed that the problem is contained in QMA when the combinatorial Laplacian has an inverse polynomial spectral gap, and conjectured QMA_1-hardness. King and Kohler [32] made progress towards this conjecture by showing that a modified version of the clique homology problem on *weighted* graphs with an inverse polynomial spectral gap, is QMA_1-hard and contained in QMA. Note that the weights are only used to lower bound the gap of the Laplacian, but do not affect the cohomology itself.

Although the results of [15, 32] show computing the sizes of cohomology groups exactly is hard in general, the quantum algorithm for topological data analysis by Lloyd, Garnerone and Zanardi [41] can approximate the number of holes in persistent homology (see also the survey [54]). Schmidhuber and Lloyd [53] proved that computing Betti numbers of a clique complex (i.e. the number of k-dimensional holes) is #P-hard, and that the decision clique homology problem is NP-hard.

The clique homology problem is therefore a natural setting where the gateset in the definition of QMA_1 truly matters. Quantum SAT and even the cohomology problem of [10] provide enough freedom to embed arbitrary local gates. This is not the case for the clique

[4] First, the exact synthesis theorem of Amy et al. that we build on is stated for the gatesets $\mathcal{G}_{2^k}$ and the rings $\mathbb{Z}[1/2, \zeta_{2^k}]$. Second, our rational simulation of cyclotomic amplitudes uses the special identity $M_a^T = M_{\overline{a}}$ for the regular representation of $\mathbb{Q}(\zeta_{2^k})$ (see Equation (16) below and [51, Eq. (23), Lemma 3.16]), which makes unitary matrices become orthogonal rational matrices. For arbitrary roots of unity this identity need not hold in the same basis, and we do not know an analogue that would give the same QMA_1 simulation.

homology problem, since the homology is completely determined by the cliques of a graph. There is no way to directly embed complex phases, or even irrational coefficients such as $\sqrt{2}$. Hence, the QMA_1-hardness proofs of [15, 32] are only able to embed quantum SAT instances with projectors of the form $|\phi\rangle\langle\phi|$, where $|\phi\rangle$ is proportional to an integer superposition of computational basis states. To show that the gapped clique homology problem is also QMA_1-complete, one therefore needs a gateset that is both powerful enough to decide the problem with perfect completeness and rational enough to be embedded into clique homology.

This motivates a more nuanced study of the gateset of the QMA_1-verifier. Along the way, we obtain a variety of novel results, including the QMA_1-completeness of the gapped clique homology problem, PSPACE-completeness of the decision clique homology problem, the first QMA_1-complete 2-local Hamiltonian problem (in contrast to Bravyi's 2-$\mathsf{QSAT} \in \mathsf{P}$), and the first $\mathsf{QMA}_1(2)$-complete Hamiltonian problem (i.e., $\mathsf{QMA}(2)$ with perfect completeness). The common obstacle behind these results is that exact amplitudes cannot simply be approximated away. We therefore separate the problem into three steps: prepare exact algebraic states with postselection, use these states to apply otherwise unavailable gates by linear combinations of implementable unitaries, and arrange every verifier so that failed postselection accepts rather than rejects. This preserves perfect completeness, while standard amplification keeps the extra acceptance probability in the NO-case negligible.

1.1 Results

The main technical contributions of this paper are the field-universal gatesets with perfect completeness (Theorem 1.1), as well as the development of a perfect-completeness algorithm for testing whether a state is in the kernel of a given sparse Hamiltonian (Lemma 1.6). Based on these techniques, we prove various Hamiltonian results, as well as the first complete problems for $\mathsf{QMA}_1(2)$. Although some of these results may appear technical, they provide the necessary tools to resolve major open questions regarding the complexity of the (gapped) clique homology problem (Theorem 1.11), a subject of significant recent interest [15, 32, 53].

Full proofs of the following theorems can be found in the full version [51]; the parenthetical theorem numbers refer to that version. Formal definitions of the involved complexity classes and problems are in Section 2.

1.1.1 Gatesets

Our first result is that for any finite gateset $\mathcal{G}$ consisting of unitaries with entries in the 2^k-th cyclotomic field, $\mathsf{QMA}_1^{\mathcal{G}}$ can be simulated using the finite gateset $\mathcal{G}_{2^k}$. We use the gatesets $\mathcal{G}_{2^k}$ defined above from [2]. Additionally, we show that $\mathsf{QMA}_1^{\mathcal{G}_4} = \mathsf{QMA}_1^{\mathcal{G}_2}$, i.e., $\mathbb{Q}(\mathrm{i})$ gatesets can be simulated with just $\mathbb{Q}$ gates. Recall that the exact synthesis of [2] can realize all gates with entries in the *ring* $\mathbb{Z}[1/2, \zeta_{2^k}]$ with the gateset $\mathcal{G}_{2^k}$. However, it is impossible to exactly *synthesize* all gates in $\mathbb{Q}(\zeta_{2^k})$ with a finite gateset (full version, Proposition 3.5). Postselection avoids this obstruction.

▶ **Theorem 1.1** (Full version, Theorems 3.2 and 3.3)**.** *For any finite gateset $\mathcal{G}$ in $\mathbb{Q}(\zeta_{2^k})$ with $k \in \mathbb{N}$, it holds that $\mathsf{QMA}_1^{\mathcal{G}} \subseteq \mathsf{QMA}_1^{\mathcal{G}_{2^k}}$. For any finite gateset $\mathcal{G}$ in $\mathbb{Q}(\mathrm{i})$, it holds that $\mathsf{QMA}_1^{\mathcal{G}} \subseteq \mathsf{QMA}_1^{\mathcal{G}_2}$.*

Note that $\mathsf{QMA}_1^{\mathcal{G}_2} = \mathsf{QMA}_1^{\mathcal{G}_4} \subseteq \mathsf{QMA}_1^{\mathcal{G}_8} \subseteq \mathsf{QMA}_1^{\mathcal{G}_{16}} \subseteq \cdots$. The witness-free case admits a stronger collapse: for BQP_1, the rational gateset $\mathcal{G}_2$ suffices to simulate all fixed cyclotomic gatesets.[5]

[5] This simulation even works for $k = O(\log n)$, which does not suffice to simulate an n-qubit quantum Fourier transform.

▶ **Theorem 1.2** (Full version, Theorem 3.4). *For any finite gateset $\mathcal{G}$ in $\mathbb{Q}(\zeta_{2^k})$ with $k \in \mathbb{N}$ it holds that* $\mathsf{BQP}_1^{\mathcal{G}} \subseteq \mathsf{BQP}_1^{\mathcal{G}_2}$.

We exhibit complete problems for the classes $\mathsf{QMA}_1^{\mathcal{G}_{2^k}}$. k-$\mathsf{QSAT}^{\mathbb{F}}$ denotes the quantum k-SAT problem with local Hamiltonians in $\mathbb{F}^{2^k \times 2^k}$. The following result is obtained from adapting the reductions of [9, 27] to our gateset framework.

▶ **Theorem 1.3** (Full version, Theorem 4.2). $4\text{-}\mathsf{QSAT}^{\mathbb{Q}(\zeta_{2^k})}$ *is* $\mathsf{QMA}_1^{\mathcal{G}_{2^k}}$*-complete for all $k \in \mathbb{N}$.*

▶ **Theorem 1.4** (Full version, Theorem 4.3). $3\text{-}\mathsf{QSAT}^{\mathbb{Q}(\zeta_{2^k})}$ *is* $\mathsf{QMA}_1^{\mathcal{G}_{2^k}}$*-complete for all $k \geq 3$.*

Notably, we also give the first 2-local QMA_1-complete Hamiltonian problem. We define the k-local Exact Hamiltonian problem (k-ELH, see Problem 2.4) as the problem of deciding whether a k-local Hamiltonian on n qubits has an eigenvalue of exactly 0 or all eigenvalues are inverse polynomially bounded away from 0 (i.e., $\sigma_1(H) = 0$, or $\sigma_1(H) \geq 1/\mathrm{poly}(n)$ for σ_1 the smallest singular value). We have k-$\mathsf{QSAT} \subseteq k$-$\mathsf{ELH} \subseteq 2k$-$\mathsf{LH}$, but k-ELH has a weaker promise than k-QSAT: In the YES-case only a nonempty nullspace is required, but H may still be frustrated (i.e. the local Hamiltonians do not have a common ground state). k-ELH has a stronger promise than k-LH: In the YES-case not just low energy is required, but an eigenvalue of *exactly* 0.

▶ **Theorem 1.5** (Full version, Theorem 5.1). $2\text{-}\mathsf{ELH}^{\mathbb{Q}(\zeta_{2^k})}$ *is* $\mathsf{QMA}_1^{\mathcal{G}_{2^k}}$*-complete for all $k \geq 3$.*

We can generalize Theorem 1.1 to $\mathsf{QMA}_1(2)$, and show the first complete Hamiltonian problem ESSH (Exact Separable Sparse Hamiltonian problem, see Problem 2.6), which is the problem of deciding whether a sparse Hamiltonian H has a separable nullstate, or for all product states, we have $\|H|\psi_1\rangle|\psi_2\rangle\| \geq 1/\mathrm{poly}$. The key technical insight for this result is an algorithm for testing whether a given quantum state is in the kernel of a sparse Hamiltonian, which we call ESH (Exact Sparse Hamiltonian problem):

▶ **Lemma 1.6** (Full version, Lemma 6.1). $\mathsf{ESH}^{\mathbb{Q}(\mathrm{i})} \in \mathsf{QMA}_1^{\mathcal{G}_2}$.

▶ **Theorem 1.7** (Full version, Theorem 6.3). *For any finite gateset $\mathcal{G}$ in $\mathbb{Q}(\zeta_{2^k})$ with $k \in \mathbb{N}$, it holds that* $\mathsf{QMA}_1^{\mathcal{G}}(2) \subseteq \mathsf{QMA}_1^{\mathcal{G}_{2^k}}(2)$.

▶ **Theorem 1.8** (Full version, Theorem 6.4). *For any finite gateset $\mathcal{G}$ in $\mathbb{Q}(\mathrm{i})$, it holds that* $\mathsf{QMA}_1^{\mathcal{G}}(2) \subseteq \mathsf{QMA}_1^{\mathcal{G}_2}(2)$.

▶ **Theorem 1.9** (Full version, Theorem 6.6). $\mathsf{ESSH}^{\mathbb{Q}(\zeta_{2^k})}$ *is* $\mathsf{QMA}_1^{\mathcal{G}_{2^k}}(2)$*-complete for all $k \in \mathbb{N}$.*

We also give the first complete problem, AESSH (Almost Exact Separable Sparse Hamiltonian), for "high precision" $\mathsf{QMA}(2)$ (i.e. with a promise gap of less than $1/\mathrm{poly}$), which is an approximate version of ESSH where in the YES-case the existence of a product state with negligible $\|H|\psi_1\rangle|\psi_2\rangle\|$ suffices. This solves an open problem of [24].

▶ **Theorem 1.10** (Full version, Theorem 6.7). $\mathsf{AESSH}_\varepsilon$ *is* ε-$\mathsf{QMA}(2)$*-complete for $\varepsilon \in n^{-\Omega(1)}$, where* ε-$\mathsf{QMA}(2) := \bigcup_{c \in 1-\varepsilon^{\omega(1)}, s \in 1-\varepsilon^{O(1)}} \mathsf{QMA}_{c,s}(2)$.

1.1.2 Clique homology

CH is the problem of determining whether the clique complex of a given graph has a hole of a given dimension. GCH additionally promises that in the NO-case, the combinatorial Laplacian has a minimum eigenvalue of $1/\mathrm{poly}(n)$, where n is the number of vertices. The

membership $\mathsf{GCH} \in \mathsf{QMA}_1^{\mathcal{G}_2}$ follows from the sparse-kernel test (see Lemma 3.10) and the rational gateset simulation, while hardness plugs $\mathcal{G}_2$ into the machinery of [32][6]. The PSPACE result for CH uses the same reduction without the inverse-polynomial promise gap, together with QMA_1 with exponentially small promise gap equaling PSPACE [40].

▶ **Theorem 1.11** (Full version, Theorems 6.8 and 6.9)**.** GCH *is* $\mathsf{QMA}_1^{\mathcal{G}_2}$*-complete.*
CH *is* PSPACE*-complete.*

This improves the previous state of the art, which was QMA_1-hardness and membership in QMA for GCH [32], and QMA_1-hardness for CH [15]. GCH is particularly nice as a QMA_1-complete problem since the "perfect completeness" is inherent to the problem: there exists a hole iff the kernel of the Laplacian is nonempty. For k-QSAT, there is (almost) no physical difference between $\lambda_{\min}(H) = 0$ and $\lambda_{\min}(H) < 1/\exp(n)$, whereas even $\lambda_{\min}(\Delta) > 1/\exp(n)$ still just implies there is no hole, where Δ is the combinatorial Laplacian.

Schmidhuber and Lloyd [53] show that computing the Betti number of a clique complex is #P-hard, which is not directly comparable since that is not a decision problem. However, note that the k-th Betti number β_k denotes the number of k-dimensional holes, and so we prove that only deciding whether $\beta_k \neq 0$ is already PSPACE-hard. On its face, the PSPACE-completeness of CH is a completely classical result, which we remarkably obtain using techniques from quantum complexity theory.

Relation to prior work. Several hardness reductions in this paper deliberately build on existing frameworks. The 4- and 3-QSAT hardness results adapt the constructions of [9, 27]; the point here is to place them in a finite-gateset cyclotomic-field framework and to give matching containment results. Similarly, [2] already gives exact synthesis over the ring $\mathbb{Z}[1/2, \zeta_{2^k}]$, while our contribution is to handle arbitrary finite gatesets over the field $\mathbb{Q}(\zeta_{2^k})$, which finite exact synthesis alone cannot do. The new Hamiltonian ingredient is the exact kernel test based on postselected LCU, which proves membership for the exact local and sparse Hamiltonian problems in the relevant QMA_1 classes and leads to the 2-local QMA_1-complete Hamiltonian problem. For $\mathsf{QMA}_1(2)$, the hardness direction follows the separable sparse Hamiltonian framework of [11], while the exact sparse formulation and the membership proof for ESSH are new here. For clique homology, we use the topological gadget framework of [32], but the new $\mathcal{G}_2$ gateset gadgets and the exact-kernel membership proof are what yield $\mathsf{QMA}_1^{\mathcal{G}_2}$-completeness for GCH and PSPACE-completeness for CH without a promise gap.

1.2 Techniques

Linear combinations of unitaries (LCU). Our main technical contribution is as follows. Suppose we wish to apply some unitary U to our state, but we do not know how to do that with our gateset. Instead, we can write U as a linear combination of unitaries that we *can* efficiently implement. We follow the idea of Childs and Wiebe [12] in the context of Hamiltonian simulation, which allows us to apply U exactly after a successful measurement. Conveniently, the Pauli matrices $\{\mathsf{I}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ form an orthonormal basis, and thus we can write any unitary as $U = \sum_x a_x P_x$, where the P_x are tensor products of Paulis. If we can prepare $|U\rangle = \sum_x a_x |x\rangle$, then we can conditionally apply P_x to an input state $|\psi\rangle$ to obtain $\sum_x a_x |x\rangle \otimes P_x|\psi\rangle$. Projecting the first register onto the "all $|+\rangle$" state then gives

[6] The PSPACE-completeness result could already be obtained using the earlier framework of [15], but the gadgets developed here are for $\mathcal{G}_2$ in the framework of [32].

$\sum_x a_x P_x|\psi\rangle = U|\psi\rangle$ in the second register. Note that if U is in $\mathbb{Q}(\zeta_{2^k})$ ($k \geq 2$), then $|U\rangle$ is also in $\mathbb{Q}(\zeta_{2^k})$. We show how to efficiently prepare such states with postselection. In the context of QMA_1, postselection means that the verifier accepts if postselection fails to maintain perfect completeness. Although the success probability of applying U in this way is only 2^{-2n} for an n-qubit operator, we can boost it exponentially close to 1 using the *oblivious amplitude amplification* technique of [7].

Simulating cyclotomic fields. To implement gates $\mathcal{G}_{2^k}$ with $\mathcal{G}_2$, we extend a technique of McKague for simulating complex gates with real gates [43, 44] to cyclotomic gates. We can write any $a \in \mathbb{Q}(\zeta_{2^k})$ as $a = \sum_{i=0}^{2^{k-1}-1} a_i \zeta_{2^k}^i$ with all $a_i \in \mathbb{Q}$. Then we represent a as a vector $|v(a)\rangle = \sum_i a_i|i\rangle$, and multiplication by b in $\mathbb{Q}(\zeta_{2^k})$ becomes matrix multiplication by some M_b in $\mathbb{Q}$ so that $M_b|v(a)\rangle = |v(ab)\rangle$. Formally, there exists a field isomorphism from $\mathbb{Q}(\zeta_{2^k})$ to a subfield of invertible $\mathbb{Q}^{2^{k-1}\times 2^{k-1}}$ matrices. Applying this field isomorphism elementwise to a unitary results in an orthogonal matrix. Formally, we have a group homomorphism $\Psi : \mathrm{U}(N, \mathbb{Q}(\zeta_{2^k})) \to \mathrm{O}(2^{k-1}N, \mathbb{Q})$.[7]

Nullspace testing. We observe that the LCU technique also applies to non-unitary matrices. We can directly apply the Hamiltonian to a quantum state (i.e., not $e^{\mathrm{i}H}$ as in Hamiltonian simulation). However, the success probability is proportional to $\|H|\psi\rangle\|$. Thus, if $|\psi\rangle \in \mathcal{N}(H)$, then multiplying $|\psi\rangle$ with H fails with probability 1. Hence, we can check in QMA_1 whether a frustrated Hamiltonian has a nonempty nullspace. We further extend this technique to sparse Hamiltonians by applying tools from Hamiltonian simulation [7, 33] (modified for cyclotomic fields), to write a sparse Hamiltonian as a linear combination of 1-sparse unitaries.

Hamiltonians. We construct our Hamiltonians via the Nullspace Connection Lemma of [52], which effectively gives a blueprint for circuit-to-Hamiltonian constructions and only requires local gadget checks. These checks are performed computationally [50]. Nevertheless, our 2-local QMA_1-complete Hamiltonian requires some new tricks, since the 2-local QMA-complete circuit-to-Hamiltonian construction of [31] does not have the history state as eigenstate. We follow the idea of splitting the computation path inside the history state, conditioned on a computational register [21, 27, 52]. The new idea is to split the computation path conditioned on the eigenstates of a single-qubit gate (not just on $|0\rangle, |1\rangle$). Then we can apply a single-qubit gate as a "zero-qubit" gate (i.e. a global phase), which we can implement with a 2-local transition by using a "one-hot encoding" for the clock register.

1.3 Future work

Recent follow-up work gives another indication that $\mathcal{G}_2$ is a natural exact gateset for QMA_1. Ma and Natarajan [42] show that 6-QSAT is $\mathsf{QMA}_1^{\mathcal{G}_2}$-complete with the constraint that every projector is diagonal in either the Pauli-X basis or the Pauli-Z basis. This structural property is also shared by the code Hamiltonians of CSS codes, crucially used in the proof of the NLTS theorem [4]. Together with our $\mathsf{QMA}_1^{\mathcal{G}_2}$-completeness result of the gapped clique homology problem, [42] further supports that $\mathcal{G}_2$ is a natural choice for the gateset of QMA_1.

[7] This only works for the 2^k-th roots of unity, since they have the unique property $M_b^T = M_{\overline{b}}$, which preserves orthogonality when applied to a unitary.

Open problems. The central question whether QMA_1 equals QMA remains wide open. Does there exist a classical oracle separation? There are also more specific open questions. Can Theorem 1.2 be extended to QMA_1, i.e., does a single gateset suffice to simulate all 2^k-th cyclotomic gatesets for QMA_1? The issue here is that our encoding allows a malicious prover to send an effective "zero-state", as for $k > 2$ the powers of ζ_{2^k} are not linearly independent in the complex plane. More generally, can we extend these results to other roots of unity, i.e., not just powers of two? The main barriers are the lack of a comparable finite exact synthesis theorem and the missing orthogonality-preserving regular representation used in [51, Lemma 3.16].

The results above show that QMA_1 has natural complete problems beyond frustration-free local Hamiltonians (i.e. quantum SAT). It remains open whether the techniques extend to further QMA_1-completeness results. A few Hamiltonian completeness results are still open for $k < 3$ (3-QSAT, 2-ELH), where our constructions need the eighths root of unity to implement CNOT and Hadamard gates, respectively.

Finally, the QMA_1-completeness of GCH without vertex weights remains open.

Organization. Due to space constraints, the remainder of this proceedings version contains the formal definitions and the main technical ingredients for the results. Section 2 gives the definitions of the gatesets, complexity classes, Hamiltonian promise problems, and clique homology. Section 3 gives the ingredients for the gate-simulation theorem and the exact kernel tests, and their application to clique homology. The proofs for our Hamiltonian results (Section 2.1) and the clique homology gadgets are only in the full version.

2 Preliminaries

We recall the definitions needed to make the main statements precise. All promise gaps in this paper are inverse polynomial unless an explicit function is displayed.

For a quantum verifier circuit Q with n_1 ancilla qubits and n_2 proof qubits, define the acceptance probability on input $|\psi\rangle \in \mathbb{C}^{2^{n_2}}$ as

$$p_{\mathrm{acc}}(Q,\psi) = \mathrm{Tr}\left[(|1\rangle\langle 1|_1 \otimes I_{2,\ldots,n_1+n_2})\, Q\, (|0\rangle\langle 0|^{\otimes n_1} \otimes |\psi\rangle\langle\psi|)\, Q^\dagger\right]. \tag{1}$$

For a BQP circuit, we write $p_{\mathrm{acc}}(Q)$.

▶ **Definition 2.1** (QMA_1). *A promise problem $A = (A_{\mathrm{yes}}, A_{\mathrm{no}})$ is in $\mathsf{QMA}_1^{\mathcal{G}}$ if there is a polynomial-time uniform family of verifier circuits $\{Q_x\}$ using only gates from $\mathcal{G}$, with polynomially many ancilla and proof qubits, such that if $x \in A_{\mathrm{yes}}$ then some proof is accepted with probability exactly 1, while if $x \in A_{\mathrm{no}}$ then every proof is accepted with probability at most $1 - 1/\mathrm{poly}(|x|)$. More generally, $\mathsf{QMA}_{c,s}^{\mathcal{G}}$ denotes the same class with completeness at least c and soundness at most s, where c and s may depend on the input length; thus $\mathsf{QMA}_1^{\mathcal{G}} = \mathsf{QMA}_1^{\mathcal{G}} = \mathsf{QMA}_{1,1-1/\mathrm{poly}}^{\mathcal{G}}$. The class $\mathsf{QMA}_1^{\mathcal{G}}(2)$ is defined similarly, except that the proof must be a product state $|\psi_1\rangle \otimes |\psi_2\rangle$.*

▶ **Definition 2.2** (BQP_1, also known as coRQP). *A promise problem $A = (A_{\mathrm{yes}}, A_{\mathrm{no}})$ is in $\mathsf{BQP}_1^{\mathcal{G}}$ if there is a polynomial-time uniform family of circuits $\{Q_x\}$ using only gates from $\mathcal{G}$ such that $p_{\mathrm{acc}}(Q_x) = 1$ for $x \in A_{\mathrm{yes}}$ and $p_{\mathrm{acc}}(Q_x) \le 1 - 1/\mathrm{poly}(|x|)$ for $x \in A_{\mathrm{no}}$.*

We use the exact gatesets of Amy et al. [2] with $\mathsf{T}_{2^k} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \zeta_{2^k} \end{smallmatrix}\right)$:

$$\mathcal{G}_2 = \{\mathsf{X}, \mathsf{CX}, \mathsf{CCX}, \mathsf{H}\otimes\mathsf{H}\}, \quad \mathcal{G}_4 = \{\mathsf{X}, \mathsf{CX}, \mathsf{CCX}, \mathsf{S}, \zeta_8\mathsf{H}\}, \quad \mathcal{G}_{2^k} = \{\mathsf{H}, \mathsf{CX}, \mathsf{T}_{2^k}\}\ (k \ge 3). \tag{2}$$

These gates support the standard perfect-completeness amplification routines used below; in particular, soundness can be made exponentially small for the classes considered here.

2.1 Hamiltonian Problems

For a field $\mathbb{F} \subseteq \mathbb{C}$, a Hamiltonian over $\mathbb{F}$ means that its matrix entries are exactly represented elements of $\mathbb{F}$. The Hamiltonian problems below are promise problems: the input is guaranteed to satisfy one of the displayed YES/NO conditions.

▶ **Problem 2.3** (Quantum k-SAT (k-QSAT$^{\mathbb{F}}_{\varepsilon}$)). Given a classical description of a k-local Hamiltonian $H = \sum_S H_S \otimes I_{[n]\setminus S}$ on n qubits with $H_S \succeq 0$, $\|H_S\| \leq 1$, and entries in $\mathbb{F}$, decide:

- (YES) $\lambda_{\min}(H) = 0$.
- (NO) $\lambda_{\min}(H) \geq \varepsilon(n)$.

Positive semidefinite local terms are allowed rather than only projectors. This does not change the $k \geq 4$ completeness result used here; for $k = 3$ this convention matches the Gosset–Nagaj construction embedded into 3-QSAT$^{\mathbb{Q}(\zeta_8)}$ [51, Theorems 4.2 and 4.3].

▶ **Problem 2.4** (k-local Exact Hamiltonian problem (k-ELH$^{\mathbb{F}}_{\varepsilon}$)). Given a classical description of a k-local Hamiltonian $H = \sum_S H_S \otimes I_{[n]\setminus S}$ on n qubits with entries in $\mathbb{F}$, decide:

- (YES) $\sigma_1(H) = 0$.
- (NO) $\sigma_1(H) \geq \varepsilon(n)$.

Here σ_1 is the smallest singular value.

Equivalently, one may ask whether H has a specified eigenvalue $\alpha \in \mathbb{F}$, by replacing H with $H - \alpha I$.

We say that an operator on n qubits is sparse if a polynomial-size classical circuit, given a row or column index, computes all nonzero entries in that row or column.

▶ **Problem 2.5** (Exact Sparse Hamiltonian problem (ESH$^{\mathbb{F}}_{\varepsilon}$)). Given a sparse Hamiltonian H on n qubits with entries in $\mathbb{F}$ and $\|H\| \leq 1$, decide:

- (YES) $\sigma_1(H) = 0$.
- (NO) $\sigma_1(H) \geq \varepsilon(n)$.

▶ **Problem 2.6** (Exact Separable Sparse Hamiltonian problem (ESSH$^{\mathbb{F}}_{\varepsilon}$)). Given a sparse Hamiltonian H on $2n$ qubits with entries in $\mathbb{F}$ and $\|H\| \leq 1$, decide:

- (YES) There exists a product state $|\psi_1\rangle \otimes |\psi_2\rangle$ with $H|\psi_1\rangle|\psi_2\rangle = 0$.
- (NO) All product states satisfy $\|H|\psi_1\rangle|\psi_2\rangle\| \geq \varepsilon(n)$.

The almost-exact version AESSH$^{\mathbb{F}}_{\varepsilon}$ relaxes the YES-case to $\|H|\psi_1\rangle|\psi_2\rangle\| \leq \varepsilon^{\omega(1)}$.

▶ **Definition 2.7** ($\mathbb{F} = \mathbb{Q}(\zeta_{2^k})$). *For the Hamiltonian problems over $\mathbb{Q}(\zeta_{2^k})$, the numerators and denominators of the rational coefficients of all entries are bounded by* $\mathrm{poly}(\varepsilon^{-1}, n)$ *in absolute value. For sparse problems, we additionally require a common denominator bounded by* $\mathrm{poly}(\varepsilon^{-1}, n)$*, i.e., $H = H'/h$ with $H' \in \mathbb{Z}[\zeta_{2^k}]$ and $h \leq \mathrm{poly}(\varepsilon^{-1}, n)$.*

The dependence on ε^{-1} is needed because the cyclotomic state-preparation routine has success probability polynomially related to the coefficient size [51, Lemma 3.10].

2.2 Clique Homology

For a graph G, the clique complex $\mathrm{Cl}(G)$ is the simplicial complex whose simplices are the cliques of G. Let $\mathcal{C}^k$ be the complex vector space spanned by oriented k-simplices, with the usual identification under odd/even permutations. The coboundary maps $d^k : \mathcal{C}^k \to \mathcal{C}^{k+1}$ form a chain complex, the cohomology group is $H^k = \mathrm{Ker}\, d^k / \mathrm{Im}\, d^{k-1}$, and the k-th Betti number is $\beta_k = \dim H^k$. With the standard inner product, let $\partial^k = (d^{k-1})^\dagger$ and

$$\Delta^k = d^{k-1}\partial^k + \partial^{k+1}d^k. \tag{3}$$

The Hodge theorem for finite complexes identifies H^k with $\operatorname{Ker}\Delta^k$ [10, 32]. For the fully formal conventions used here, including orientations and weighted chain complexes, see [51, Section 2].

▶ **Problem 2.8** (Clique homology problem (CH) [29, 10, 15])**.** Given a graph G and integer k, decide:

- (YES) $H^k(\mathrm{Cl}(G)) \neq 0$, equivalently $\lambda_{\min}(\Delta^k) = 0$.
- (NO) $H^k(\mathrm{Cl}(G)) = 0$, equivalently $\lambda_{\min}(\Delta^k) \neq 0$.

King and Kohler also allow vertex weights, which change the inner product and hence the Laplacian but not the underlying homology groups [32]. Their weighted clique Laplacian remains sparse and has rational entries in the instances we use. Concretely, if $w(\sigma) = \prod_{v\in\sigma} w(v)$ and $|\sigma'\rangle = w(\sigma)^{-1}|\sigma\rangle$ is the corresponding orthonormal basis vector, then for $\sigma = [v_0, \ldots, v_k]$,

$$d^k|\sigma'\rangle = \sum_{v\in\mathrm{up}(\sigma)} w(v)|([v]+\sigma)'\rangle, \qquad \partial^k|\sigma'\rangle = \sum_{j=0}^{k}(-1)^j w(v_j)|\sigma'_{-j}\rangle, \tag{4}$$

where $\sigma_{-j} = [v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k]$ and $\mathrm{up}(\sigma)$ is the set of vertices v such that $\sigma \cup \{v\}$ is a $(k+1)$-simplex. Hence Δ^k only couples k-simplices that either share a common lower simplex or are upper adjacent, so the clique Laplacian is sparse whenever the relevant clique-neighborhood queries are efficient [32, Fact 7.4]. More explicitly, if $\sigma, \tau \in \mathcal{K}^k$, then [32, Fact 7.4]

$$\langle\sigma'|\Delta^k|\tau'\rangle = \begin{cases} \sum_{u\in\mathrm{up}(\sigma)} w(u)^2 + \sum_{v\in\sigma} w(v)^2, & \sigma = \tau, \\ \pm w(v_\sigma)w(v_\tau), & \sigma,\tau \text{ share a lower face} \\ & \text{and are not upper adjacent}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where, in the off-diagonal case, $v_\sigma \in \sigma$ and $v_\tau \in \tau$ are the vertices whose removal gives the common lower face, and the sign is determined by the relative orientations of that face.

▶ **Problem 2.9** (Gapped clique homology (GCH$_\varepsilon$) [15, 32])**.** Given a vertex-weighted graph G on n vertices and integer k, decide:

- (YES) $H^k(\mathrm{Cl}(G)) \neq 0$.
- (NO) $H^k(\mathrm{Cl}(G)) = 0$ and $\lambda_{\min}(\Delta^k) \geq \varepsilon(n)$.

We write GCH for the union over inverse-polynomial ε.

3 Exact Gate Simulation and Kernel Tests

This section presents the core mechanisms behind the results, and how they apply to the clique homology results.

3.1 Postselected Exact Gates

Amy et al. [2] prove that the gatesets $\mathcal{G}_{2^k}$ exactly synthesize all unitaries with entries in the ring $\mathbb{Z}[1/2, \zeta_{2^k}]$:

▶ **Theorem 3.1** ([2])**.** *Let $k, m \in \mathbb{N}$. A $2^m \times 2^m$ unitary U can be exactly represented by an m-qubit circuit over $\mathcal{G}_{2^k}$ if and only if $U \in \mathrm{U}(2^m, \mathbb{D}[\zeta_{2^k}])$, where $\mathbb{D}[\zeta_{2^k}] = \mathbb{Z}[1/2, \zeta_{2^k}]$.*

This does not by itself synthesize arbitrary field entries: for any fixed finite gateset in $\mathbb{Q}(\zeta_n)$, even rational one-qubit Pythagorean rotations cannot all be synthesized exactly (full version, Proposition 3.5 [51]). We therefore use postselection.

▶ **Lemma 3.2** (Integer and cyclotomic state preparation; full version, Lemmas 3.6, 3.8, and 3.10)**.** *Let $|\psi\rangle \propto \sum_{i=0}^{d-1} a_i|i\rangle$. If all $a_i \in \mathbb{Z}$, then $|\psi\rangle$ can be prepared with gateset $\mathcal{G}_2$ with probability at least $1/(4d)$ in time $\mathrm{poly}(d, \log A)$ and space $O(\log d + \log A)$, where $A = \sum_i |a_i|$. If $a_i \in \mathbb{Z}[\mathrm{i}]$, the analogous statement holds over $\mathcal{G}_4$ with probability at least $1/(16d)$. If $a_i = \sum_j b_{ij}\zeta_{2^k}^j \in \mathbb{Z}[\zeta_{2^k}]$ for fixed k, the analogous statement holds over $\mathcal{G}_{2^k}$ with probability at least $1/(d \cdot \mathrm{poly}(A))$, where $A = \sum_{i,j}|b_{ij}|$.*

▶ **Lemma 3.3** (Postselected Pauli LCU; full version, Lemmas 3.11 and 3.12)**.** *Let U be an n-qubit unitary with $sU \in \mathbb{Z}[\mathrm{i}]^{2^n \times 2^n}$ for $s^2 \in \mathbb{N}$. Then U can be applied with success probability $2^{-2n} - 2^{-2^{\mathrm{poly}(n)}}$ in time and space $\mathrm{poly}(2^n, \log s)$ using $\mathcal{G}_4$. For $k > 2$ and $sU \in \mathbb{Z}[\zeta_{2^k}]^{2^n \times 2^n}$, the same holds using $\mathcal{G}_{2^k}$ with success probability $1/\mathrm{poly}(2^{2n}, s)$.*

Proof. Here we only prove the $sU \in \mathbb{Z}[\mathrm{i}]^{2^n \times 2^n}$ case. We begin by describing a procedure to apply U with a success probability of 2^{-2n}, provided that prior integer state preparation by Lemma 3.2 succeeds. Decompose $U = \sum_{x \in \{0,1\}^{2n}} a_x P_x$ in the Pauli basis with $P_x = \bigotimes_{j=1}^n \sigma_{x_{2j-1}x_{2j}}$, where $\sigma_{00} = \mathsf{I}, \sigma_{01} = \mathsf{X}, \sigma_{10} = \mathsf{Y}, \sigma_{11} = \mathsf{Z}$. We have $\sum_{x \in \{0,1\}^{2n}} |a_x|^2 = 1$, as

$$2^n = \|U\|_{\mathrm{F}}^2 = \mathrm{Tr}\, UU^\dagger = \sum_{x,y} a_x a_y^\dagger \,\mathrm{Tr}\, P_x P_y = 2^n \sum_x |a_x|^2, \tag{6}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Since sU has a unique representation in the Pauli basis as a vector in $\mathbb{Q}(\mathrm{i})^{2^{2n}}$, we have for all x, $sa_x \in \mathbb{Q}(\mathrm{i})$.

To implement U, first prepare $|U\rangle_{\mathcal{A}} = \sum_{x \in \{0,1\}^{2n}} a_x|x\rangle$ using Lemma 3.2. Then, conditioned on $|x\rangle_{\mathcal{A}}$, apply $(P_x)_{\mathcal{B}}$, and measure $\mathcal{A}$ in the Hadamard basis. On input $|\phi\rangle_{\mathcal{B}}$, we thus get

$$|U\rangle_{\mathcal{A}}|\phi\rangle_{\mathcal{B}} = \sum_{x \in \{0,1\}^{2n}} a_x|x\rangle_{\mathcal{A}}|\phi\rangle_{\mathcal{B}} \mapsto \sum_{x \in \{0,1\}^{2n}} a_x|x\rangle_{\mathcal{A}} \otimes P_x|\phi\rangle_{\mathcal{B}} =: |\psi'\rangle. \tag{7}$$

If we get outcome $y \in \{0,1\}^{2n}$ for the Hadamard measurement, we project register $\mathcal{A}$ onto

$$|y_H\rangle := H^{\otimes 2n}|y\rangle = 2^{-n} \sum_{x \in \{0,1\}^{2n}} (-1)^{x \cdot y}|x\rangle, \tag{8}$$

where $x \cdot y$ denotes the inner product between x and y as vectors. Hence, the post-measurement state is given by

$$(|y_H\rangle\langle y_H|_{\mathcal{A}} \otimes I_{\mathcal{B}})|\psi'\rangle \propto |y_H\rangle_{\mathcal{A}} \otimes \sum_{x \in \{0,1\}^{2n}} (-1)^{x \cdot y} a_x P_x|\phi\rangle_{\mathcal{B}}. \tag{9}$$

Note, if $y = 0^{2n}$, then $(-1)^{x \cdot y} = 1$ for all x, and thus we get $U|\phi\rangle_{\mathcal{B}}$. Otherwise, the following operator is applied:

$$U_y = \sum_{x \in \{0,1\}^{2n}} (-1)^{x \cdot y} a_x P_x = \sum_{x \in \{0,1\}^{2n}} a_x \bigotimes_{i=1}^n (-1)^{x_{2i-1}y_{2i-1} + x_{2i}y_{2i}} \sigma_{x_{2i-1}x_{2i}} \tag{10}$$

We argue that it is of the form $P_z U P_z$ for some $z \in \{0,1\}^{2n}$. For a fixed $y \in \{0,1\}^2$, the individual terms in the tensor product are of the form $\tilde{\sigma}_x = (-1)^{x \cdot y}\sigma_x$ for all $x \in \{0,1\}^2$.

(a) If $y = 00$, we have $\tilde{\sigma}_x = \sigma_x$.
(b) If $y = 01$, the phase -1 is injected for $x \in \{01, 11\}$ (i.e. $\sigma_x \in \{\mathsf{X}, \mathsf{Z}\}$) and so $\tilde{\sigma}_x = \mathsf{Y}\sigma_x\mathsf{Y}$ as $\mathsf{YIY} = \mathsf{I}, \mathsf{YXY} = -\mathsf{X}, \mathsf{YYY} = \mathsf{Y}, \mathsf{YZY} = -\mathsf{Z}$.
(c) If $y = 10$, the phase -1 is injected for $x \in \{10, 11\}$ (i.e. $\sigma_x \in \{\mathsf{Y}, \mathsf{Z}\}$) and so $\tilde{\sigma}_x = \mathsf{X}\sigma_x\mathsf{X}$ as $\mathsf{XIX} = \mathsf{I}, \mathsf{XXX} = \mathsf{X}, \mathsf{XYX} = -\mathsf{Y}, \mathsf{XZX} = -\mathsf{Z}$.
(d) If $y = 11$, the phase -1 is injected for $x \in \{01, 10\}$ (i.e. $\sigma_x \in \{\mathsf{X}, \mathsf{Y}\}$) and so $\tilde{\sigma}_x = \mathsf{Z}\sigma_x\mathsf{Z}$ as $\mathsf{ZIZ} = \mathsf{I}, \mathsf{ZXZ} = -\mathsf{X}, \mathsf{ZYZ} = -\mathsf{Y}, \mathsf{ZZZ} = \mathsf{Z}$.

Hence, $\tilde{\sigma}_x = \sigma_{\tilde{y}}\sigma_x\sigma_{\tilde{y}}$, where $\tilde{y} = y_2y_1$ (swap the bits of y). Since the correction terms do not depend on x, they can be factored out and we have $U_y = P_{\tilde{y}}UP_{\tilde{y}}$, where now $\tilde{y} = y_2y_1y_4y_3 \cdots y_{2n}y_{2n-1}$.

The probability of measuring y is given by

$$\begin{aligned} \|(|y_H\rangle\langle y_H|_{\mathcal{A}} \otimes I_{\mathcal{B}})|\psi'\rangle\|^2 &= \Big\| |y_H\rangle_{\mathcal{A}} \otimes 2^{-n} \sum_{x\in\{0,1\}^{2n}} (-1)^{x\cdot y} a_x P_x |\phi\rangle_{\mathcal{B}} \Big\|^2 \\ &= \big\| 2^{-n} U_y |\phi\rangle_{\mathcal{B}} \big\|^2 = 2^{-2n}, \end{aligned} \tag{11}$$

as U_y is unitary if U is unitary, where $\|\cdot\|$ denotes the Euclidean norm.

Complexity analysis. We can compute the Pauli coefficients efficiently with the Hilbert-Schmidt product $a_x = 2^{-n}\operatorname{Tr}(P_xU)$. By Lemma 3.2, we can prepare $|U\rangle$ in time $\operatorname{poly}(2^n, \log s)$ with success probability $p \geq 2^{-2n-4}$. If we run the state preparation r/p times, then the probability that all fail is at most $(1-p)^{r/p} \leq e^{-r}$. ◀

Now if we want to use Lemma 3.3 to implement an entire circuit consisting of m gates, then success probability becomes an issue, since the procedure would need to succeed m times in a row, which has probability $2^{-\Omega(m)}$. Fortunately, we can amplify the success probability of Lemma 3.3 exponentially close to 1, which also brings the overall success probability exponentially close to 1.

As noted in the proof, even if the final Hadamard measurement fails by giving $y \neq 0^{2n}$, we still know that some unitary U_y was applied.[8] So we can repeat the procedure with $U' = UU_y^\dagger$, which again succeeds with probability 2^{-2n}. Unfortunately, the bit complexity of U' grows exponentially in the worst case, and therefore we only have a logarithmic number of attempts before the correction terms grow too big. Still, if we treat n as a constant, then the success probability is also a constant and $O(\log(m))$ attempts suffice for a success probability of $1 - 1/\operatorname{poly}(m)$. Otherwise, the runtime would be doubly exponential in n.

The increasing bit complexity prohibits this approach in the $k \geq 3$ setting (see Lemma 3.3). Thus, we instead use the *oblivious amplitude amplification* technique of [7, Lemma 3.6].

▶ **Lemma 3.4** (Oblivious amplitude amplification [7, Lemma 3.6][9]). *Let U and V be unitary matrices on $\mu + n$ qubits and n qubits, respectively, and let $\theta \in (0, \pi/2)$. Suppose that for any n-qubit state $|\psi\rangle$,*

$$U|0^\mu\rangle|\psi\rangle = \sin(\theta)|0^\mu\rangle V|\psi\rangle + \cos(\theta)|\Phi^\perp\rangle, \tag{12}$$

where $|\Phi^\perp\rangle$ is an $(\mu+n)$-qubit state that depends on $|\psi\rangle$ and satisfies $\Pi|\Phi^\perp\rangle = 0$, where $\Pi := |0^\mu\rangle\langle 0^\mu| \otimes I$. Let $R := 2\Pi - I$ and $S := -URU^\dagger R$. Then for any $\ell \in \mathbb{Z}$,

$$S^\ell U|0^\mu\rangle|\psi\rangle = \sin\big((2\ell+1)\theta\big)|0^\mu\rangle V|\psi\rangle + \cos\big((2\ell+1)\theta\big)|\Phi^\perp\rangle. \tag{13}$$

[8] If U is Clifford, then a Pauli correction suffices and we can succeed with probability 1 (ignoring state preparation).
[9] We use the lemma as stated in the updated arXiv version of the paper.

▶ **Lemma 3.5** (Full version, Lemma 3.14). *Let $\mathcal{G}$ be a fixed finite gateset in $\mathbb{Q}(\zeta_{2^k})$ with $k \geq 2$. Each gate of $\mathcal{G}$ can be simulated using $\mathcal{G}_{2^k}$ with success probability $1 - 1/\exp(m)$ in time* $\mathrm{poly}(m)$.

Proof. The idea is to apply the n-qubit gate $V \in \mathcal{G}$ using Lemma 3.3 or Lemma 3.3. Note that both in integer state preparation (Lemma 3.2), and integer gate application (Lemma 3.3), all ancillas are measured in such a way that their state is fully determined in the case of success. Hence, we can construct a unitary U that implements V as in Equation (12) with $\sin(\theta) = \sqrt{p}$ for success probability p. In order to achieve exponentially small failure probability, we need to adjust the success probability so that $(2\ell+1)\theta \approx \pi/2$.

Let $c = 2^{-r}c'$ with $c' \in [2^r]$ and even r such that $c\sqrt{p} = \sin((\pi/2)/(2\ell+1)) + O(1/\exp(m))$ for some $\ell, r \in \mathrm{poly}(m)$. Note that we can efficiently compute p and c since we are dealing with fixed-size circuits. We can build a circuit U' such that $U'|0^{r+1}\rangle = c|0^{r+1}\rangle + \sqrt{1-c^2}|\phi^\perp\rangle$ with $\langle 0^{r+1}|\phi^\perp\rangle = 0$. U' first applies $\mathsf{H}^{\otimes r}$ to the first r qubits, then flips the last qubit conditioned on the first r-qubits representing an integer $\geq c'$, and then applies $\mathsf{H}^{\otimes r}$ again. Thus,

$$\langle 0^{r+1}|V'|0^{r+1}\rangle = \langle +|^r \frac{1}{\sqrt{2^r}} \sum_{j=0}^{2^r-1} |j\rangle\langle 0|j \geq c'\rangle = \frac{1}{2^r} \sum_{i=0}^{2^r-1}\sum_{j=0}^{2^r-1} \langle i|j\rangle\langle 0|j \geq c'\rangle = \frac{c'}{2^r} = c, \quad (14)$$

where $|j \geq c'\rangle \equiv |1\rangle$ for $j \geq c'$ and $|0\rangle$ otherwise. Hence, we can use $\widetilde{U} \coloneqq U' \otimes U$ to satisfy Equation (12) with $\sin((2\ell+1)\theta) \geq 1 - 1/\exp(m)$. We can implement the unitary $S = -\widetilde{U}R\widetilde{U}^\dagger R$ from Lemma 3.4 using $\mathcal{G}_{2^k}$ since U can be implemented with $\mathcal{G}_{2^k}$ by Lemma 3.3, and U', R with $\mathcal{G}_2$. Thus, $S^\ell\widetilde{U}$ implements V with success probability $1 - 1/\exp(m)$ in time $\mathrm{poly}(m)$. ◀

Together with soundness amplification, this proves the inclusion $\mathsf{QMA}_1^{\mathcal{G}} \subseteq \mathsf{QMA}_1^{\mathcal{G}_{2^k}}$ from Theorem 1.1; the special $\mathbb{Q}(\mathrm{i})$-to-$\mathbb{Q}$ inclusion uses the rational encoding below.

▶ Remark 3.6. Lemma 3.3 only requires U to be unitary to have success probability 2^{-2n} for all outcomes y in Equation (11) independent of the input state. If U is not unitary, then the success probability on input is given by $\frac{2^{-n}}{\|U\|_{\mathrm{F}}^2} \cdot \left\|U_y|\phi\rangle\right\|^2$, where $|\phi\rangle$ is the input. Notably, the success probability is 0 if $U|\phi\rangle = 0$.

3.2 Cyclotomic Gates as Rational Gates

The rational simulation is a higher-degree version of McKague's real simulation of complex gates [43, 44]. Let $\mathbb{K} = \mathbb{Q}(\zeta_n)$ with $n = 2^k$ and $d = 2^{k-1}$. Encode

$$a = \sum_{i=0}^{d-1} a_i \zeta_n^i \in \mathbb{K} \qquad \text{as} \qquad |v(a)\rangle = \sum_{i=0}^{d-1} a_i |i\rangle, \quad (15)$$

and encode vectors entrywise. For fixed $a \in \mathbb{K}$, multiplication by a is a $\mathbb{Q}$-linear map on this basis; let M_a be its matrix, i.e., $M_a v_b = v_{ab}$. The key identity is

$$M_a^T = M_{\overline{a}}. \quad (16)$$

Indeed, $\zeta_n^d = -1$ and $\overline{\zeta_n} = \zeta_n^{-1} = -\zeta_n^{d-1}$, so

$$M_{\zeta_n} = \begin{pmatrix} 0 & & & -1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \qquad M_{\zeta_n}^T = M_{\overline{\zeta_n}}, \quad (17)$$

and linearity gives Equation (16). Therefore, applying $a \mapsto M_a$ entrywise sends a unitary over $\mathbb{K}$ to an orthogonal rational matrix. Indeed, for $U = (u_{ij}) \in \mathrm{U}(N, \mathbb{K})$, the (i, j) block of $\Psi(U)^T\Psi(U)$ is

$$\sum_{r=1}^{N} M_{u_{ri}}^T M_{u_{rj}} = \sum_{r=1}^{N} M_{\overline{u_{ri}}} M_{u_{rj}} = M_{\sum_r \overline{u_{ri}} u_{rj}} = M_{\delta_{ij}}, \tag{18}$$

so $\Psi(U)^T\Psi(U) = I$.

▶ **Lemma 3.7** (Full version, Lemma 3.16)**.** *There is a group homomorphism*

$$\Psi : \mathrm{U}(N, \mathbb{Q}(\zeta_{2^k})) \to \mathrm{O}(2^{k-1}N, \mathbb{Q})$$

such that $\Psi(U)|v(\psi)\rangle = |v(U|\psi\rangle)\rangle$ *for all* $|\psi\rangle \in \mathbb{Q}(\zeta_{2^k})^N$.

This directly proves the BQP_1 simulation. Indeed, the initial state is rational and there is no proof register, so this encoding preserves perfect completeness and gives Theorem 1.2. The soundness loss is only polynomial. The linear map $\Lambda = \sum_{i=0}^{d-1} I \otimes \zeta_n^i \langle i|_\zeta$ maps encoded vectors back to cyclotomic amplitudes. Then $\Lambda|v(\phi)\rangle = |\phi\rangle$ and $\|\Lambda\|^2 \le d$. Thus if the decoded rejecting branch has squared norm, equivalently rejection probability, at least $1 - \varepsilon$, then the corresponding encoded branch has squared norm at least $(1 - \varepsilon)/d$. For the special case $k = 2$, the above idea also works for QMA_1 as shown in [43, 44]. Applying Lemma 3.7 to a QMA_1-verifier V creates a verifier V' that "expects" a proof with real amplitudes. With standard error reduction techniques, we can transform V' to V'' with soundness ε on proofs with real amplitudes. Then V'' still has soundness 4ε against proofs with complex amplitudes. For QMA_1, the same rational-encoding argument does not give a reduction to $\mathcal{G}_2$ when $k > 2$: a dishonest prover could send a vector in $\operatorname{Ker}\Lambda$ with no corresponding nonzero cyclotomic witness, so the decoded soundness analysis would no longer apply.

3.3 Exact Kernel Tests

The LCU routine also applies to non-unitary operators. If the target operator is H, then the successful branch has norm proportional to $\|H|\psi\rangle\|$. A verifier can therefore reject exactly on successful application of H and accept when postselection fails.

▶ **Lemma 3.8** (Local exact Hamiltonian test; full version, Lemma 4.1)**.** $\ell\text{-}\mathsf{ELH}^{\mathbb{Q}(\mathrm{i})} \in \mathsf{QMA}_1^{\mathcal{G}_2}$ *and* $\ell\text{-}\mathsf{ELH}_\varepsilon^{\mathbb{Q}(\zeta_{2^k})} \in \mathsf{QMA}_{1,1-\varepsilon'}^{\mathcal{G}_{2^k}}$ *for* $\ell \in O(\log n)$,[10] $\varepsilon \le n^{-O(1)}$ *and* $\varepsilon' \in \varepsilon^{O(1)}$*, where* n *is the number of qubits of the instance.*

Proof sketch. For the cyclotomic case, the verifier is as follows. Let $H = \sum_{z\in\{0,1\}^s} H_{S_z}$, where 2^s upper-bounds the number of nonzero local terms, and write each local term as

$$H_{S_z} = \sum_{x\in\{0,1\}^{2\ell}} a_{z,x} P_x. \tag{19}$$

The verifier prepares a normalized multiple $\eta|H\rangle$ of

$$|H\rangle_{\mathcal{A}} = \sum_{z\in\{0,1\}^s} \sum_{x\in\{0,1\}^{2\ell}} a_{z,x} |z\rangle_{\mathcal{A}_1} |x\rangle_{\mathcal{A}_2}, \tag{20}$$

[10] For $\ell \in \omega(1)$, we assume that there are only $\mathrm{poly}(n)$ nonzero H_S terms in $H = \sum_S H_S$.

with success probability at least $\varepsilon^{O(1)}$ by the coefficient-size promise in Definition 2.7; if preparation fails, it accepts. Then it conditionally applies $(P_x)_{S_z}$ to the proof register $\mathcal{B}$. On proof $|\psi\rangle$, the premeasurement state is

$$|\phi\rangle = \sum_{z,x} \eta a_{z,x} |z\rangle_{\mathcal{A}_1} |x\rangle_{\mathcal{A}_2} \otimes ((P_x)_{S_z} \otimes I_{[n]\setminus S_z})|\psi\rangle_{\mathcal{B}}. \tag{21}$$

The verifier rejects exactly when the coefficient register is measured in the Hadamard-basis vector $|0_H\rangle = 2^{-s/2-\ell} \sum_{z,x} |z\rangle|x\rangle$. For $\Pi_{\mathrm{rej}} = |0_H\rangle\langle 0_H|_{\mathcal{A}} \otimes I_{\mathcal{B}}$, the rejecting branch is

$$\Pi_{\mathrm{rej}}|\phi\rangle = |0_H\rangle_{\mathcal{A}} \otimes 2^{-s/2-\ell} \sum_{z,x} \eta a_{z,x} ((P_x)_{S_z} \otimes I)|\psi\rangle_{\mathcal{B}} = |0_H\rangle_{\mathcal{A}} \otimes 2^{-s/2-\ell} \eta H |\psi\rangle_{\mathcal{B}}. \tag{22}$$

Thus the conditional rejection probability is $p_{\mathrm{rej}} = 2^{-s-2\ell}\eta^2 \|H|\psi\rangle\|^2$. Hence a null vector is accepted with probability 1, while in the NO-case every proof is rejected with inverse-polynomial probability. Including the state-preparation success probability $p_{\mathrm{prep}} \geq \varepsilon^{O(1)}$, the NO-case rejection probability is at least $p_{\mathrm{prep}}\, 2^{-s-2\ell}\eta^2 \sigma_1(H)^2 \geq \varepsilon^{O(1)}$. This proves the membership side for k-ELH, and the same kernel test gives the membership side for k-QSAT. The hardness reductions for 4-QSAT and 3-QSAT are adapted from Bravyi and from Gosset–Nagaj [9, 27, 51]. ◀

For sparse Hamiltonians, one first decomposes the sparse matrix into one-sparse Hamiltonians and then into one-sparse unitaries, following the Hamiltonian-simulation decomposition of [7, 33] but keeping exact cyclotomic coefficients.

▶ **Lemma 3.9** (One-sparse cyclotomic decomposition; full version, Lemma 6.2)**.** *Let $H \in \mathbb{Z}[\zeta_{2^k}]^{2^n \times 2^n}$ be a 1-sparse n-qubit Hamiltonian with coefficients of at most L bits. Then*

$$H = \sum_{\ell=0}^{L-1} \sum_{i=1}^{2^{k+1}} 2^{\ell-1} U^{\ell,i}, \tag{23}$$

where the $U^{\ell,i}$ are efficiently implementable 1-sparse unitaries over $\mathcal{G}_{2^k}$.

Proof sketch. The proof extends the sparse-Hamiltonian simulation decomposition of Berry et al. and Kimmel–Lin [7, 33]. First decompose each cyclotomic coefficient into binary powers and powers of ζ_{2^k}. The off-diagonal part is split into Hermitian 1-sparse matrices whose nonzero entries are $\pm\zeta_{2^k}^m$ above the diagonal and their conjugates below the diagonal; the diagonal part is split similarly. Each partial matrix becomes a unitary by replacing missing entries with ± 1. For a d-sparse Hamiltonian, one first edge-colors the graph of nonzero entries into d^2 one-sparse Hamiltonians as in [7, Lemma 4.4]. Thus, for a d-sparse H with L-bit coefficients, $H = \sum_{j=1}^{d^2} \sum_{\ell=0}^{L-1} \sum_{i=1}^{2^{k+1}} 2^{\ell-1} U_j^{\ell,i}$, where each $U_j^{\ell,i}$ is 1-sparse and exactly implementable. ◀

▶ **Lemma 3.10** (Full version, Lemma 6.1)**.** $\mathsf{ESH}^{\mathbb{Q}(\mathrm{i})} \in \mathsf{QMA}_1^{\mathcal{G}_2}$ *and* $\mathsf{ESH}_{\varepsilon}^{\mathbb{Q}(\zeta_{2^k})} \in \mathsf{QMA}_{1,1-\varepsilon'}^{\mathcal{G}_{2^k}}$ *with $\varepsilon \leq n^{-O(1)}$ and $\varepsilon' \in \varepsilon^{O(1)}$ for all $k \in \mathbb{N}$, where n is the number of qubits of the instance.*

Proof. Let H be a d-sparse Hamiltonian on n qubits ($d = \mathrm{poly}(n)$) in $\mathbb{Z}[\zeta_{2^k}]$ with coefficients of at most L bits, i.e., H has at most d non-zero entries in each row and column. By [7, Lemma 4.4], we can write $H = \sum_{j=1}^{d^2} H_j$, where each H_j is 1-sparse and a query to any H_j can be simulated with $O(1)$ queries to H. This decomposition assumes that the graph of H is bipartite, which holds without loss of generality because $\mathsf{X} \otimes H$ has the same singular

values as H. The idea is to construct a d^2-coloring of the edges of the graph of G, and letting each H_j consist of the edges of color j. Hence, each H_j is also in $\mathbb{Z}[\zeta_{2^k}]$ with coefficients of at most L bits.

Next, we prepare the QMA_1-verifier. By Lemma 3.9, $H = \sum_{j=1}^{d^2} \sum_{l=0}^{L-1} \sum_{i=1}^{2^{k+1}} 2^{l-1} U_j^{l,i}$. As in Lemma 3.8, we prepare $|H\rangle_{\mathcal{A}} \propto \sum_{l=0}^{L-1} \sum_{j=1}^{d^2} \sum_{i=1}^{2^{k+1}} 2^{l-1} |i,j,l\rangle$, and then apply the unitary $U_{\mathcal{AB}} = \sum_{ijl} |i,j,l\rangle\langle i,j,l|_{\mathcal{A}} \otimes (U_j^{l,i})_{\mathcal{B}}$ with the proof in register $\mathcal{B}$. Soundness follows analogously to Lemma 3.8, noting $L = O(\log(n))$ (see Definition 2.7) and that flipping signs increases the norm by at most $\mathrm{poly}(n)$ due to sparsity. ◀

The same verifier, with the witness restricted to product states, proves the membership direction for the separable sparse problem: $\mathsf{ESSH}^{\mathbb{Q}(\zeta_{2^k})} \in \mathsf{QMA}_1^{\mathcal{G}_{2^k}}(2)$. The matching hardness reduction is obtained from the separable sparse Hamiltonian framework of Chailloux and Sattath [11], replacing phase-estimation-based verification by the exact kernel test above.

3.4 Clique Homology

▶ **Theorem 3.11.** GCH *is* $\mathsf{QMA}_1^{\mathcal{G}_2}$*-complete.*

Proof. Containment follows by viewing the clique Laplacian as an instance of exact sparse Hamiltonian testing. By Equation (5), the weighted clique Laplacian is sparse and has rational entries, and by Lemma 3.10 the verifier can test exactly whether the witness is in its kernel while preserving perfect completeness.

Hardness follows from a slight modification of the hardness proof of King and Kohler [32]. They show $\mathsf{GCH} \in \mathsf{QMA}$ and GCH is $\mathsf{QMA}_1^{\mathcal{G}_{\mathrm{Pyth}}}$-hard, where $\mathcal{G}_{\mathrm{Pyth}} = \{\mathsf{CX}, U_{\mathrm{Pyth}}\}$ with the Pythagorean gate $U_{\mathrm{Pyth}} = \frac{1}{5}\left(\begin{smallmatrix} 3 & 4 \\ -4 & 3 \end{smallmatrix}\right)$. By Theorem 1.1, $\mathsf{QMA}_1^{\mathcal{G}_{\mathrm{Pyth}}} \subseteq \mathsf{QMA}_1^{\mathcal{G}_2}$, but we are not aware of nontrivial lower bounds for $\mathsf{QMA}_1^{\mathcal{G}_{\mathrm{Pyth}}}$, since it is unclear to us how to perform classical computation with perfect completeness using $\mathcal{G}_{\mathrm{Pyth}}$, although the gateset is certainly universal. It is straightforward to modify their construction to work with the gateset $\mathcal{G}_2$. The full version gives the corresponding 4-QSAT construction and its embedding into clique homology [51, Theorem 4.2 and Section D.1]. ◀

▶ **Theorem 3.12.** CH *is* PSPACE*-complete.*

Proof. The containment $\mathsf{CH} \in \mathsf{PSPACE}$ holds since we can decide whether the determinant of the Laplacian is zero in $\mathsf{NC}(\mathrm{poly}) = \mathsf{PSPACE}$ [16].[11] Hardness follows from the fact that QMA_1 with exponentially small promise gap, denoted $\mathsf{QMA}_{1,1-1/\exp(n)}$, is equal to PSPACE [40]. The $\mathsf{QMA}_{1,1-1/\exp(n)}$ verifier for PSPACE in [40, Algorithm 8] effectively just verifies the history state for a classical computation, and can therefore be implemented with $\mathcal{G}_2$. Then we can just embed the corresponding 4-QSAT instance into the clique homology problem as in Theorem 3.11. By [32, Theorem 10.1], we have for the Laplacian Δ^{2n-1}, $\lambda_{\min}(\Delta^{2n-1}) = 0$ iff $\lambda_{\min}(H) = 0$, where n is the number of qubits of the Hamiltonian H. ◀

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