Log-Concavity and Tunneling: Quantum Adiabatic Algorithm for Convex Functions (With a Spike)

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Abstract

Quantum tunneling is expected to provide a computational speedup in quantum computing, a phenomenon that Adiabatic Quantum Optimization (AQO) aims to leverage. While some academic proofs of concept have been studied, such as the "Hamming weight with a spike" (HWS) problem, the algorithmic gains of this effect remain underexplored. In this work we extend the analysis underlying HWS to more general potentials.

In the first half of the work, we establish (discrete) log-concavity of the ground state as a key structural property in this context. We devise a framework for establishing log-concavity of the ground state for a large family of discrete, 1-dimensional Schrödinger operators. The family includes convex potentials, but also certain potentials with local minima. In the convex case, this provides a discrete version of a continuous result by Brascamp and Lieb [10]. We demonstrate the utility of our result by establishing new spectral gap bounds, going beyond related results by Jarret and Jordan [18] for convex potentials.

In the second half of the work, we use our results on log-concavity to extend the perturbative analysis of HWS by Reichardt [33] to the larger family of potentials with log-concave ground state. As a concrete instantiation, we use our result to extend the HWS analysis from a linear potential (which is exactly solvable) to a quadratic potential (which is no longer solvable). Our result strongly suggests the broader applicability of tunneling to convex potentials with spikes.

2012 ACM Subject Classification Theory of computation → Quantum computation theory; Theory of computation → Design and analysis of algorithms; Mathematics of computing → Combinatorial optimization

Keywords and phrases Quantum adiabatic computing, Convex optimization

Digital Object Identifier 10.4230/LIPIcs.ESA.2026.110

Related Version *Full Version*: `https://arxiv.org/abs/2606.23614` [8]

Funding SA was supported in part by the European QuantERA project QOPT (ERA-NET Cofund 2022-25) and the French ANR project QUOPS (ANR-22-CE47-0003-01). SA and AB were supported by the French PEPR integrated project HQI (ANR-22-PNCQ-0002).

1 Introduction

Adiabatic quantum optimization (AQO) was introduced as a method for finding the minima of complicated cost functions $V : \{0,1\}^n \to \mathbb{R}$ on the hypercube, leveraging quantum fluctuations to escape local minima that trap classical algorithms [14, 22, 13]. Its theoretical analysis reduces to understanding the spectral gap of the interpolating Hamiltonian

$$H(s) = -(1-s)\sum_{i=1}^{n} X_i + sV \tag{1}$$

34th Annual European Symposium on Algorithms (ESA 2026).
Editors: Philip Bille, Seth Pettie, and Sabine Storandt; Article No. 110; pp. 110:1–110:19
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

for $s \in [0,1]$, where X_i is the Pauli-X operator acting on the i-th qubit, and V is the diagonal operator satisfying $V|x\rangle = V(x)|x\rangle$ for $x \in \{0,1\}^n$. If the gap remains at least inverse-polynomially large in n throughout the annealing path, then the adiabatic theorem guarantees that the target ground state can be prepared efficiently. However, establishing lower bounds on the gap is in general a hard problem, and the cases where it can be done rigorously are relatively few – see e.g. [13, 34, 1, 33, 17, 15, 9, 26] for some examples.

A large part of the results concerns the restrictive yet still interesting family of cost functions that is called "symmetric", satisfying $V(x) = V(|x|)$. A canonical such function is the Hamming weight function $V(x) = |x|$. Due to the exact solvability of the adiabatic Hamiltonian for this potential, a significant body of work was devoted to the performance of AQO for perturbations of this function [12, 25, 32, 33, 7, 5]. The resulting "Hamming weight with a spike" problem (HWS) concerns potentials of the form $V(x) = |x| + h(x)$, for some perturbation or "spike" h. While such a spike can create local minima and obstruct classical methods such as simulated annealing, the aforementioned works derive precise conditions on the shape of the spike under which the gap remains open, and hence AQO still succeeds. E.g., Reichardt [33] uses a perturbative argument based on the overlap between the unperturbed ground state and the spike to establish a spectral gap bound. Subsequent works [25, 32, 7] extended this analysis, some of them using approximation methods such as the spin-coherent path-integral instanton approach [32] and the discrete WKB method [25].

These analyses critically hinge on the fact that the unperturbed linear potential $V(x) = |x|$ yields an exactly solvable Hamiltonian, with explicit expressions for the eigenvalues and eigenvectors, and indeed there seems to be no work considering more general convex potentials with a spike.[1] In this work, we go beyond linear potentials and consider a larger family of potentials, including but not limited to the broadly relevant family of convex symmetric potentials. While exact expressions for the corresponding ground states seem elusive, we establish strong structural properties based on "log-concavity" that give us a useful handle on them. As our main proof-of-concept, we extend the HWS analysis from linear to quadratic potentials (for which already we have no closed-form ground state), showing that the tunneling mechanism persists even in this more challenging setting. Quadratic potentials are the textbook workhorse for convex optimization, and we see our work as opening up the study of quantum tunneling in the more general setting of discrete convex optimization with AQO.

1.1 Organization of the paper

We now give a brief overview on the remainder of the paper.

In Section 2, we collect preliminary facts on adiabatic quantum optimization and discrete log-concavity. In Section 3 we prove our main results on log-concave ground states. In Section 4 we use these to establish new spectral gap bounds based on Hardy's inequality. In Section 5, we use log-concavity to analyze quantum tunneling through spike perturbations. Finally, in Section 6, we analyze the quadratic HWS problem.

2 Preliminaries

2.1 Adiabatic quantum optimization

We will consider families of Hamiltonians of the form $H(s) = -(1-s)\sum_i X_i + sV$ for $s \in [0,1]$, with V a diagonal potential. Such Hamiltonians are stoquastic, and so the Perron-Frobenius theorem implies that the ground state $|\psi_0\rangle = \sum_k \psi_k |k\rangle$ can be chosen to

[1] There are some exceptions in the *continuous* setting, see e.g. [28] and references therein.

have nonnegative real amplitudes $\psi_k \geq 0$. We use $\Delta_{\min}$ to denote the minimum spectral gap of $H(s) = -(1-s)\sum_i X_i + sV$ over all $s \in [0,1]$. By the adiabatic theorem, AQO will correctly return the ground state of V granted that its runtime satisfies the bound $\Omega\Big(\frac{\|H(1)-H(0)\|^2}{\Delta_{\min}^3}\Big)$. Since we will have $\|H(0)\|, \|H(1)\| \in \mathrm{poly}(n)$, AQO will be efficient provided that $\Delta_{\min} \in \Omega(1/\mathrm{poly}(n))$. We provide more details on AQO in the comprehensive version [8].

2.2 Symmetry reduction

In this section, we describe the reduction from the bitstring space $\{0,1\}^n$ of dimension 2^n to the Hamming weight subspace of dimension $n+1$ spanned by uniform superpositions over bitstrings of a given Hamming weight.

▶ **Definition 1.** *The Hamming weight subspace is spanned by the vectors*

$$|k\rangle = \frac{1}{\sqrt{\binom{n}{k}}} \sum_{|x|=k} |x\rangle\,, \qquad 0 \leq k \leq n.$$

The key point is that, for a symmetric potential V satisfying $V\,|x\rangle = V(|x|)\,|x\rangle$, the adiabatic algorithm remains restricted to the Hamming weight basis. In this subspace, the mixing Hamiltonian $\sum_i X_i$ becomes the tridiagonal operator:

$$(A_{hc})_{k-1,k} = (A_{hc})_{k,k-1} := a_k := \sqrt{k(n+1-k)} \tag{2}$$

for $0 < k \leq n$. Since we assume symmetric potentials throughout this work, we will only consider symmetry-reduced Hamiltonians of the form $H(s) = -(1-s)A_{hc} + sV$. A more detailed treatment of this reduction is given in [18, Section IV].

2.3 Binomial state and Hamming weight Hamiltonian

We define the binomial state as follows.

▶ **Definition 2** (Binomial state $|\varphi_m\rangle$)**.** *For $m \in [0, n/2]$, define the binomial state $|\varphi_m\rangle$ as*

$$|\varphi_m\rangle = \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \left(\frac{m}{n}\right)^{k/2} \left(1 - \frac{m}{n}\right)^{\frac{n-k}{2}} |k\rangle\,.$$

The distribution $|\langle k|\varphi_m\rangle|^2$ describes a binomial law $\mathcal{B}(n, m/n)$, and so the mean and variance are

$$\langle\varphi_m|K|\varphi_m\rangle = m \quad \text{and} \quad \langle\varphi_m|(K-m)^2|\varphi_m\rangle = \frac{m(n-m)}{n},$$

where K denotes the linear potential, $K\,|k\rangle = k\,|k\rangle$.

Incidentally, as discussed for instance in [33], the binomial state is also the ground state of the linear Hamming weight Hamiltonian.

▶ **Lemma 3** ((Linear) Hamming weight Hamiltonian)**.** *Fix $m \in [0, n]$ and let*

$$\alpha = \frac{n-2m}{\sqrt{m(n-m)}}. \tag{3}$$

Then the Hamiltonian $H_\alpha = -A_{hc} + \alpha K$ has ground state $|\varphi_m\rangle$, ground state energy $\lambda_\alpha = \frac{n}{2}\big(\alpha - \sqrt{4+\alpha^2}\big)$ and spectral gap $\Delta_\alpha = \sqrt{4+\alpha^2}$.

2.4 Discrete log-concavity

A key concept in our work is that of (discrete) log-concavity. In this section we recall some properties of log-concave sequences.

▶ **Definition 4** (Discrete log-concavity [20]). *A non-negative sequence $\{p_k\}_{k\in\mathbb{Z}}$ is log-concave on interval $I \subseteq \mathbb{Z}$ if*

$$p_k^2 \geq p_{k+1}p_{k-1} \tag{4}$$

for all $k \in I$. If $I = \mathbb{Z}$ then we simply say that $\{p_k\}_{k\in\mathbb{Z}}$ is log-concave.

Equivalently, defining the score sequence $\{r_k\}$ by $r_k = p_{k+1}/p_k$, log-concavity at k is equivalent to $\{r_k\}$ being non-increasing from $k-1$ to k. It turns out that many classical discrete distributions belong to this class: the discrete uniform, Bernoulli, binomial, Poisson, geometric, and negative binomial distributions, as well as convolutions of Bernoulli distributions with arbitrary parameters (see [21] and references therein).

In this list, the binomial distribution will play a privileged role throughout this work. In fact, one can base a strengthened notion of log-concavity on the binomial distribution.

▶ **Definition 5** (Discrete ultra-log-concavity [27]). *Fix integer $n \geq 0$ and let $q_k = \binom{n}{k}$ for $k \in [n]$. A non-negative sequence $\{p_k\}_{k\in[n]}$ is ultra-log-concave (of order n) if*

$$\left(\frac{p_k}{q_k}\right)^2 \geq \frac{p_{k+1}}{q_{k+1}}\frac{p_{k-1}}{q_{k-1}} \qquad (0 < k < n). \tag{5}$$

We highlight some properties of log-concave sequences in Appendix A.

3 (Ultra-)Log-concavity for convex potentials (and beyond)

As our first contribution, we establish a large family of potentials V for which the corresponding ground state is discretely log-concave or even ultra-log-concave. We do so for 1-dimensional stoquastic Hamiltonians of the form

$$H = -A + V$$

where V encodes the potential on $\{0, 1, ..., n\}$, and A is defined by $\langle k-1|A|k\rangle = \langle k|A|k-1\rangle = a_k$ for a concave, positive sequence $\{a_k \in \mathbb{R}\}_{k=1}^n$. When $a_k = \sqrt{k(n+1-k)}$ this corresponds to the symmetry-reduced adjacency matrix A_{hc} of the hypercube from Section 2.2.

3.1 Prior work

In the continuous setting it was established by Brascamp and Lieb [10, Theorem 6.1] that the ground state of a continuous Schrödinger operator $-\partial^2 + V$ on the real line with a convex potential is log-concave. Stronger even, they showed that if $V(x) = \alpha^2x^2 + W(x)$ for a convex W, then the ground state ψ has the form $\psi(x) = e^{-\alpha x^2}\varphi(x)$ for some log-concave function φ. Clearly, this is a continuous analogue to discrete ultra-log-concavity (Definition 5). Apart from being a useful structural property, this was also used later to establish strong bounds on the spectral gap of these operators [2, 35].

Transferring this picture to the discrete setting turns out to be non-trivial. Indeed, a clear bottleneck when transferring the proof in [10] to the discrete setting is that discrete log-concavity loses some of the nice properties of its continuous variant. E.g., in the continuous setting, multivariate log-concavity is well-understood and enjoys strong stability properties: in particular, [10] uses that marginalization of a log-concave distribution remains log-concave. In the discrete setting, however, defining a meaningful notion of multivariate log-concavity is more delicate, and key properties such as preservation under marginalization fail [4, Chapter 6].

One exception that we found is the work by Johnson [20]. As explained in e.g. [16], Johnson proves a discrete version of the Brascamp-Lieb result, but only for one particular A matrix on the half-line $\mathbb{N}$ that corresponds to a particular queuing process on $\mathbb{N}$.

3.2 Ultra-log-concavity from convexity

As our first result in this direction, we establish a discrete version of the aforementioned result by Brascamp and Lieb [10]. Let $|\psi\rangle = \sum_{k=0}^{n} \psi_k \, |k\rangle$ denote the ground state of $H = -A + V$ with energy E_0, and define the function

$$F(k) := V(k) - E_0.$$

By Perron–Frobenius and positivity of the a_k's, we can assume $\psi_k > 0$ for all $0 \leq k \leq n$. Projecting the eigenvalue equation $H\,|\psi\rangle = E_0\,|\psi\rangle$ onto $\langle k|$ gives a key recurrence relation:

$$F(k)\psi_k = a_{k+1}\psi_{k+1} + a_k\psi_{k-1}, \qquad 0 \leq k \leq n, \tag{6}$$

with the boundary convention $\psi_{-1} = \psi_{n+1} = 0$. Given that the a_k's and ψ_k's are positive, this implies that also $F(k) > 0$ for all $0 \leq k \leq n$. We use this recurrence relation to prove our theorem.

▶ **Theorem 6** (Ultra-log-concavity for convex potentials). *If $\{a_k = \sqrt{k(n+1-k)}\}$ and V is convex, then $\{\psi_k^2\}$ is ultra-log-concave.*

Proof. We refer the reader to the comprehensive version [8]. ◀

3.3 γ-log-concavity from γ-relative monotonicity

As our second contribution in this direction, we describe a more technical framework for establishing log-concavity of the ground state. Critically, it allows us to go beyond convex potentials, showing that log-concavity can persist even in the occurrence of local minima. While somewhat technically involved, we show that the framework is tight in (and motivated by) capturing limiting cases such as the uniform and binomial ground state.

In the most general case, we will focus on proving a weighted log-concavity property of the ground-state, i.e.

$$\psi_k^2 \geq \gamma_k \, \psi_{k-1}\psi_{k+1} \tag{7}$$

where the weights $\{\gamma_k\}$ are defined such that: $\gamma_1, \ldots, \gamma_{n-1} > 0$ and $\gamma_0 = \gamma_n = 1$. More precisely, we will say that $\{\psi_k\}$ is γ-log-concave (γ-LC) on an interval $I \subseteq \{0, \ldots, n\}$ if Equation (7) is satisfied for all $k \in I$. As an example, ultra-log-concavity of $\{\psi_k^2\}$ (Definition 5) corresponds to the case where $\gamma_k = \sqrt{\frac{k+1}{k}\frac{n-k+1}{n-k}}$.

To prove γ-LC, we assume that $\{a_k\}_{k=1}^{n}$ is positive (while setting $a_0 = a_{n+1} = 0$) and satisfies a related log-concavity condition:

$$a_k^2\gamma_k\gamma_{k-1} \geq a_{k-1}a_{k+1}, \qquad \forall 1 \leq k \leq n. \tag{8}$$

Recalling the function $F(k) = V(k) - E_0 > 0$, we also need to define the cumulative weights

$$\Gamma_k^- := \frac{a_1}{F(0)} \prod_{\ell=0}^{k-1} \gamma_\ell, \qquad \Gamma_k^+ := \frac{a_n}{F(n)} \prod_{\ell=k+1}^{n} \gamma_\ell,$$

with the convention that empty products are equal to 1. These weights are motivated by the following bound.

▶ **Lemma 7** (Cumulative ratio)**.**

- *If $\{\psi_\ell\}$ is γ-LC on $\{0, \dots, k-1\}$ then $\frac{\psi_{k-1}}{\psi_k} \geq \Gamma_k^-$.*
- *If $\{\psi_\ell\}$ is γ-LC on $\{k+1, \dots, n\}$ then $\frac{\psi_{k+1}}{\psi_k} \geq \Gamma_k^+$.*

Proof. We refer the reader to the comprehensive version [8]. ◀

We can now define a condition linking the potential V to $\{a_k\}$ and $\{\gamma_k\}$ under which we will prove γ-LC of $\{\psi_k\}$.

▶ **Definition 8** (γ-relative monotonicity (γ-RM))**.** *The Hamiltonian $H = -A + V$ satisfies the left γ-RM condition on $\{0, \dots, k_0\}$ if, for all $1 \leq k \leq k_0$,*

$$a_k \gamma_k F(k) - a_{k+1} F(k-1) \leq \Gamma_{k-1}^- \left(a_k^2 \gamma_k \gamma_{k-1} - a_{k+1} a_{k-1}\right). \tag{9}$$

It satisfies the right γ-RM condition on $\{k_1, \dots, n-1\}$ if, for all $k_1 \leq k \leq n-1$,

$$a_{k+1} \gamma_k F(k) - a_k F(k+1) \leq \Gamma_{k+1}^+ \left(a_{k+1}^2 \gamma_{k+1} \gamma_k - a_{k+2} a_k\right). \tag{10}$$

Under this condition, we can prove weighted log-concavity of $\{\psi_k\}$ on the interval I.

▶ **Theorem 9** (γ-LC from γ-RM)**.** *Assume that $\{a_k\}$ satisfies Equation (8).*

- *If $H = -A + V$ satisfies the left γ-RM condition on $\{1, \dots, k_0\}$ then $\{\psi_k\}$ is γ-LC on $\{0, \dots, k_0\}$.*
- *If $H = -A + V$ satisfies the right γ-RM condition on $\{k_1, \dots, n-1\}$ then $\{\psi_k\}$ is γ-LC on $\{k_1, \dots, n\}$.*

In particular, if these conditions hold for some $k_1 = k_0 + 1$ then $\{\psi_k\}$ is γ-LC on $[n]$.

The full proof is available at [8], Section 3.4. In Appendix B, we details several examples that have concrete application. In particular we show how this condition also encapsulate non-convex potentials.

4 Spectral gap bounds from (ultra-)log-concavity

To demonstrate the power of log-concavity, we show that it implies useful lower bounds on the spectral gap Δ of H. The argument builds on a discrete Hardy inequality described by Miclo [30].

4.1 Hardy inequality

We again consider a 1-dimensional Hamiltonian $H = -A + V$ on $\{0, \dots, n\}$ with diagonal potential V and A band-diagonal with $\langle k-1|A|k\rangle = \langle k|A|k-1\rangle = a_k$. We will assume that $\{a_k\}$ is positive, in which case there is a unique ground state $|\psi_0\rangle = \sum_{\ell=0}^{n} \psi_\ell |\ell\rangle$ that can be chosen positive. We denote by $\Delta > 0$ the spectral gap of H.

We can connect such a stoquastic Hamiltonian to a Markov chain through the so-called ground state transformation. Indeed, the operator

$$L = -\mathrm{Diag}(\psi_\ell)^{-1}(H - E_0)\mathrm{Diag}(\psi_\ell)$$

is a valid Markov chain generator, satisfying $\langle k|L|\ell\rangle \geq 0$ for $k \neq \ell$ and $L\vec{1} = 0$. More precisely, it corresponds to a birth-and-death process on $\{0, \ldots, n\}$ with stationary probabilities

$$p_k = \psi_k^2$$

and ergodic flows

$$b_\ell = p_{\ell-1}L_{\ell-1,\ell} = p_\ell L_{\ell,\ell-1} = a_\ell\sqrt{p_\ell p_{\ell-1}}.$$

Following [30], we associate to L a family of discrete Hardy constants: for any $0 < i < n$,

$$B_+(p, i) := \max_{i<k\leq n} p([k, n]) \sum_{\ell=i+1}^{k} \frac{1}{b_\ell}, \qquad B_-(p, i) := \max_{0\leq k<i} p([0, k]) \sum_{\ell=k+1}^{i} \frac{1}{b_\ell}.$$

The following lemma shows that these Hardy constants tightly characterize the spectral gap of H.

▶ **Lemma 10** (Discrete Hardy inequality). *For any index $0 < i < n$,*

$$\frac{1}{4}\min\left(\frac{1}{B_+(p, i)}, \frac{1}{B_-(p, i)}\right) \leq \Delta \leq \min\left(\frac{1}{p([0, i])B_+(p, i)}, \frac{1}{p([i, n])B_-(p, i)}\right).$$

Proof. The lower bound follows from the lower bound in [30, Proposition 3]. The upper bound is not stated explicitly as such, but it follows from the proof of [30, Proposition 2]. Indeed, replacing the median m by i in that proof recovers the bound that we claim. ◀

▶ Remark 11. A particularly relevant case is when $i = m$ a median of p, so that $p([0, m]) \geq 1/2$ and $p([m, n]) \geq 1/2$. Then

$$\frac{1}{4B} \leq \Delta \leq \frac{2}{B}$$

with $B = \max\{B_+(p, m), B_-(p, m)\}$. This corresponds to [30, Proposition 2].

4.1.1 Hardy constant for uniform and binomial distribution

We first treat some cases of special interest. Consider the uniform distribution over $\{0, \ldots, n\}$.

▶ **Lemma 12** (Uniform Hardy constant). *Let $S_n = \sum_{l=1}^{n} \frac{1}{a_l}$ and $\{p_k = \frac{1}{n+1}\}$. Then, for any i,*

$$\max\{B_+(p, i), B_-(p, i)\} \leq (n + 1)S_n.$$

Proof. We refer the reader to the comprehensive version [8]. ◀

We will also need a bound on the Hardy constants for the binomial distribution $p_k \propto c^k\binom{n}{k}$ with $\{a_k = \sqrt{k(n + 1 - k)}\}$ derived from the hypercube symmetry reduction. Rather than deriving it explicitly, we can use that the corresponding Hamiltonian is precisely the symmetry reduction of the Hamming weight Hamiltonian $-A + \alpha K$ as described in Section 2.3, for the particular choice of $\alpha = \frac{n-2\mu}{\sqrt{\mu(n-\mu)}}$ (with μ the mean of q). From Lemma 3 we know that this Hamiltonian has a spectral gap $\Delta = \sqrt{\alpha^2 + 4} \geq 2$. Combined with the upper bound on the gap from Lemma 10, we get the lemma below.

▶ **Lemma 13** (Binomial Hardy constant). *Let $a_k = \sqrt{k(n-k+1)}$, $p_k \propto c^k \binom{n}{k}$, and m_p a median of p satisfying $p([0, m_p]) \geq 1/2$ and $p([m_p, n]) \geq 1/2$.*

- *If $i \geq m_p$, then $B_+(p, i) \leq 1$.*
- *If $i \leq m_p$, then $B_-(p, i) \leq 1$.*

4.2 Hardy constant from relative log-concavity

In this section we show one way of bounding the Hardy constant: if a distribution p is "log-concave relative to" some other distribution q, then we can upper bound the Hardy constant of p using that of q. For distributions p and q, we say that p is log-concave relative to q if the distribution $\{p_k/q_k\}$ is log-concave. It is easily seen that any log-concave distribution p is relative log-concave with respect to $q_k \propto c^k$ (Definition 4). More interestingly, if $q_k \propto \binom{n}{k}$ then we recover the notion of ultra-log-concavity (Definition 5). We will be using the following technical lemma.

▶ **Lemma 14** (Single crossing lemma). *Suppose that p is log-concave relative to q, and let k^* denote the mode of $\{r_k = p_k/q_k\}$. Let $b_\ell = a_\ell\sqrt{p_{\ell-1}p_\ell}$ and $c_\ell = a_\ell\sqrt{q_{\ell-1}q_\ell}$.*

1. *For all k, ℓ such that $k^* < \ell \leq k$, we have*

$$p([k, n]) \leq r_k q([k, n]) \qquad \text{and} \qquad b_\ell \geq r_k c_\ell.$$

2. *Assume there exists $0 < i < k^*$ such that either $q([0, i]) = p([0, i])$ or $q([i, n]) = p([i, n])$. Then for all k, ℓ such that $k > i$ and $i < \ell \leq k^*$, we have*

$$p([k, n]) \leq q([k, n]) \qquad \text{and} \qquad b_\ell \geq c_\ell.$$

Proof. We refer the reader to the comprehensive version [8]. ◀

The following theorem then shows how to bound the Hardy constant of a distribution p by using the Hardy constant of a distribution q relative to which it is log-concave and with which it shares the same tail mass. Note that we can always enforce this condition by replacing $\{q_k\}$ with $\{c^k q_k\}$ for an appropriate choice of c (indeed, log-concavity relative to $\{q_k\}$ implies log-concavity relative to $\{c^k q_k\}$).

▶ **Theorem 15** (Relative Hardy bound). *Suppose that p is log-concave relative to q, and let k^* be the mode of $\{p_k/q_k\}$. Let $0 < i < n$ be such that either*

$$q([0, i]) = p([0, i]) \qquad \text{or} \qquad q([i, n]) = p([i, n]) \qquad \text{or} \qquad i = k^*.$$

Then

$$B_+(p, i) \leq B_+(q, i) \qquad \text{and} \qquad B_-(p, i) \leq B_-(q, i).$$

Proof. We refer the reader to the comprehensive version [8]. ◀

4.3 Spectral gap bounds

Combined with our criteria for log-concavity of the ground state, we get our main result of the section.

▶ **Theorem 16.** *Consider 1-dimensional $H = -A + V$ with ground state distribution p and spectral gap Δ.*

1. ***Uniform, log-concave:*** *if $\{a_\ell = 1\}$ and p log-concave then*

$$\Delta \in \Omega(1/n^2).$$

This holds e.g. when $\{a_\ell = 1\}$ and V is single-well (Appendix B).

2. ***Hypercube, log-concave:*** *if $\{a_\ell = \sqrt{\ell(n+1-\ell)}\}$ and p log-concave then*

$$\Delta \in \Omega(1/n).$$

This holds e.g. when $\{a_\ell = \sqrt{\ell(n+1-\ell)}\}$ and H satisfies the 1-RM condition (Appendix B).

3. ***Hypercube, ultra-log-concave:*** *if $\{a_\ell = \sqrt{\ell(n+1-\ell)}\}$ and p ultra-log-concave then*

$$\Delta \in \Omega(1).$$

This holds e.g. when $\{a_\ell = \sqrt{\ell(n+1-\ell)}\}$ and V is convex (Theorem 6).

Proof. We refer the reader to the comprehensive version [8]. ◀

It is interesting to compare our bounds with the results by Jarret and Jordan in [18]:

- For the 1-dimensional Hamiltonian $H = -A + V$ with uniform $\{a_\ell\}$ and V convex, they show a bound $\Omega(1/n^2)$ on the spectral gap. Point 1. in Theorem 16 generalizes this bound to the much weaker case where $-V$ is merely unimodal.
- For the hypercube Hamiltonian $H = -A_{hc} + V$ with a symmetric potential V, we can use the symmetry reduction in Section 2.2. Point 3. in Theorem 16 then shows a constant gap when V is convex, in correspondence to the findings in [18]. Point 2. in Theorem 16 goes beyond this, and shows an $\Omega(1/n)$ lower bound when H satisfies the 1-RM condition (Definition 8). This includes potentials that are non-convex and can even have local minima.

We do mention that the bounds in Jarret and Jordan [18] are tight down to the constant, in analogy to the continuous fundamental gap proof [2] which their work is based on. This implies that points 1. and 3. in Theorem 16 are tight up to a constant.

Regarding point 2. in Theorem 16, we note that this bound is also tight up to a constant. Indeed, for the aforementioned potential $V(k) = a_k + a_{k+1}$, the ground state is uniform and hence log-concave. If the a_k's are derived from the hypercube then point 2. in Theorem 16 gives a bound $\Omega(1/n)$. To see that this is tight, note that the ground state energy in this case is 0, and the first excited energy can be shown to be $O(1/n)$ by using an ansatz state proportional to $\sum_k (k - n/2)|k\rangle$, orthogonal to the uniform ground state.

5 Log-concavity and tunneling

We now demonstrate the utility of log-concavity in analyzing quantum tunneling through localized spike perturbations. Following a variation of Reichardt's argument for the HWS problem [33], we show that for log-concave ground states we can easily control the spectral gap when adding a spike perturbation. In the following we define the mean μ and variance σ^2 of a ground state $|\psi_0\rangle$ as the mean and variance of the distribution $p_k = \psi_k^2$. The key point will be that for a log-concave ground state, the mass placed on a short interval is uniformly controlled by the variance and the distance of the interval from the mean.

▶ **Theorem 17.** *Consider a Hamiltonian $H = -A + V$ with spectral gap Δ and a log-concave ground state with variance σ and mean μ. Consider a perturbation $h : \{0, 1, \ldots, n\} \to \mathbb{R}_{\geq 0}$ supported on an interval $[\ell, u-1] \subseteq [0, n]$ and satisfying an upper bound $h(k) \leq \bar{h}$. If $\alpha = \max\{\mu - u, \ell - \mu\}$ and*

$$\bar{h} \cdot \min\left\{\frac{2(u-\ell)}{\sqrt{1+4\sigma^2}}, 2\exp\left(-\frac{\alpha}{2(2\sigma+1)}\right)\right\} \leq \Delta/2$$

then the perturbed Hamiltonian $H' = H + V'$ with $V' |k\rangle = h(k) |k\rangle$ has spectral gap $\Delta' \geq \Delta/2$.

Proof. We refer the reader to the comprehensive version [8]. ◀

6 Application: Quadratic Hamming weight with a spike

In this section we develop the "quadratic Hamming weight with a spike" (quadratic HWS) problem. For this we will specialize our analysis to quadratic potentials perturbed by a spike. Specifically we consider a family of Hamiltonians

$$H_{qs}(s) = -(1-s)A_{hc} + sV_{qs}$$

for $s \in [0, 1]$, with $V_{qs} = V_q + h$ where $V_q(k) = a(k - k_0)^2$ for $k_0 \leq 0$ (ensuring that the minimizer is at $k = 0$) and h is the spike perturbation. We aim to use Theorem 17 to bound the spectral gap $\Delta_{qs}(s)$ of H_{qs}, which requires a bound on the spectral gap $\Delta_q(s)$ of the *unperturbed* quadratic Hamiltonian $H_q(s) = -(1-s)A_{hc} + sV_q$, as well as estimates of the mean and variance of its ground state $|\psi_q(s)\rangle$. While for a linear potential, as in the HWS problem, the ground state is known explicitly as a binomial distribution, this is not so for quadratic potentials. We address this key challenge by showing that $|\psi_q(s)\rangle$ does maintain a constant overlap with the binomial ground state of a linear potential. Combined with log-concavity of the ground state, this yields explicit control of the mean and variance of $|\psi_q(s)\rangle$. This allows us to invoke the perturbation bound from Section 5 and generalize the HWS problem to quadratic potentials.

6.1 Unperturbed quadratic: Overlap with the binomial ansatz

We now consider the Hamiltonian $H = -A_{hc} + V$ for a convex potential V. The following claim quantifies the overlap of the ground state of H with an ansatz state $|\varphi_m\rangle$. The overlap depends on the second difference of the potential, which is defined as $\Delta^2 V(k) = V(k+1) + V(k-1) - 2V(k)$. Smaller $\Delta^2 V(k)$ leads to higher overlaps with the binomial state. We prove the claim in Appendix A of the comprehensive version [8].

▷ Claim 18 (Overlap bound). Let $|\psi_0\rangle$ denote the ground state of $H = -A_{hc} + V$ with V a convex function with bounded second difference $\Delta^2 V(k) \leq C$. Let m be chosen so as to minimize $\langle\varphi_m|H|\varphi_m\rangle$. Then it holds that

$$|\langle\psi_0|\varphi_m\rangle|^2 \geq 1 - \left(1 - \frac{2m}{n}\right)\left((\mu - m)\left(1 - \frac{\Delta V(\lfloor m \rfloor)}{\mathbb{E}[\Delta V(Y)]}\right) + \frac{C\,m(1-m/n)}{2\mathbb{E}[\Delta V(Y)]}\right),$$

where $Y \sim \mathcal{B}(n-1, m/n)$ and $\mu = \langle\psi_0|K|\psi_0\rangle$.

In the quadratic case $V_q(k) = a(k-k_0)^2$, we have $\Delta V_q(k) = a(2(k-k_0)+1)$, and so the lower bound becomes simpler. We prove the below theorem in Appendix B of the comprehensive version [8], and we plot the optimal mean $m^*(a)$ as a function of a in Figure 1.

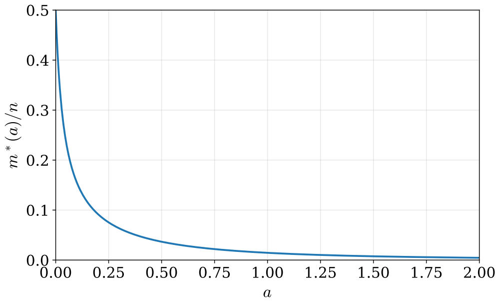

Figure 1 Plot of the function $m^*(a)/n$ obtained for $n = 40$ and $k_0 = -3$. $m^*(a)$ is obtained by solving the equation $a = \frac{n-2m^*}{\sqrt{m^*(n-m^*)}} \cdot \frac{1}{1-2k_0+2(n-1)m^*/n}$. As expected, m^* is located at $n/2$ for $a = 0$ and it decreases to 0 as a increases.

▶ **Theorem 19** (Overlap with the binomial ansatz for quadratic potentials)**.** *Let $|\psi_{q,a}\rangle$ denote the ground state of $H = -A_{hc} + V_q$ with $V_q(k) = a(k - k_0)^2$ for $k_0 \leq 0$ and $a \in \mathbb{R}_{\geq 0}$. Then there exists $m^*(a) \in [0, n/2]$ such that*

$$\left|\langle\psi_{q,a}|\varphi_{m^*(a)}\rangle\right|^2 \geq \frac{1}{2}.$$

Now denote by μ_a and σ_a^2 the mean and variance associated to $|\psi_{q,a}\rangle$, which are given by

$$\mu_a = \langle\psi_{q,a}|K|\psi_{q,a}\rangle\,, \qquad \sigma_a^2 = \langle\psi_{q,a}|(K-\mu)^2|\psi_{q,a}\rangle\,.$$

Using the fact that both $|\psi_{q,a}\rangle$ and $|\varphi_m\rangle$ are log-concave, we can relate the associated mean and variance using Lemma 29. Applying this lemma to $\{p_k = |\langle k|\varphi_{m^*(a)}\rangle|^2\}$ and $\{q_k = |\langle k|\psi_{q,a}\rangle|^2\}$, we obtain

$$\begin{aligned} \mu_a &= m^*(a) + O\big(\sqrt{m^*(a)}\big) + O(1), \\ \sigma_a &= \Theta\big(\sqrt{m^*(a)}\big) + O(1). \end{aligned} \tag{11}$$

Thus, although the quadratic ground state does not admit a closed form, the binomial ansatz provides a tractable approximation throughout the annealing path.

6.2 Unperturbed quadratic: Improved spectral gap bound

We now turn back to the interpolating, adiabatic Hamiltonian with quadratic potential,

$$H_q(s) = -(1-s)A_{hc} + sV_q.$$

From Theorem 16 (or [18]), we get a bound $\Delta(H_q(s)) \in \Omega(1-s)$ on the spectral gap of the unperturbed, quadratic Hamiltonian $H_q(s)$. Unfortunately, that bound becomes useless around the end of the annealing path, when $s \to 1$. Luckily, a finer reading shows that the bound of [18] can actually be extended until the end of the annealing path.

▶ **Lemma 20.** *Consider $H(s) = -(1-s)A_{hc} + sV$ with V convex, and define $\beta = \min_{0\leq i<j\leq n}|\frac{V(j)-V(i)}{j-i}|$. Then*

$$\Delta(H(s)) \geq \sqrt{s^2\beta^2 + 4(1-s)^2}.$$

In particular, if there exists $c > 0$ such that $\beta \geq c$, then

$$\Delta(H(s)) \geq \frac{2c}{\sqrt{c^2+4}} \qquad \textit{uniformly for all } s \in [0,1].$$

Proof. We refer the reader to the comprehensive version [8]. ◀

In particular, for V with unique minimizer at 0, we have $\beta = V(1) - V(0) > 0$ which satisfies by definition this lemma. For the quadratic potential $V_q(k) = a(k-k_0)^2$ that we consider, we have that $\Delta V_q(k) \geq a$. Assuming $a \in \Omega(1)$, this yields a uniform bound $\Delta_q(s) \in \Omega(1)$ on the spectral gap of the unperturbed quadratic Hamiltonian $H_q(s)$.

6.3 Perturbed quadratic and AQO

We will now combine the bound on the spectral gap of the quadratic Hamiltonian $H_q(s)$, as well as the estimates on the mean and variance of its ground state, with the perturbative tunneling argument in Section 5. This will allow us to establish our analysis of the quadratic HWS problem.

6.3.1 Spectral gap of quadratic HWS

We now combine the general spike bound with the overlap analysis. The key point is that the variance of the true ground state is comparable to that of the binomial ansatz.

▶ **Theorem 21** (Quadratic potential with a spike). *Consider the quadratic Hamming weight with a spike Hamiltonian*

$$H_{qs}(s) = -(1-s)A_{hc} + sV_{qs} = -(1-s)A_{hc} + s(V_q + h)$$

for $s \in [0,1]$, where the spike h is supported on $[\ell, u-1] \subseteq [0,n]$ and satisfies an upper bound $h(k) \leq \bar{h}$. If

$$\bar{h}\,\frac{u-\ell}{\sqrt{\ell}} \in O(1)$$

then the spectral gap of $H_{qs}(s)$ satisfies $\Delta_{qs}(s) \in \Omega(1)$.

Proof. We refer the reader to the comprehensive version [8]. ◀

The spike condition $\bar{h}\,\frac{u-\ell}{\sqrt{\ell}} \in O(1)$ is identical to the condition derived by Reichardt for the HWS problem [33]. When for instance $u, \ell \in \Omega(n)$, the condition admits a spike of height and width $\Omega(n^{1/4})$.

6.4 Runtime for quadratic HWS

Finalizing our analysis, we combine our spectral gap bound with the AQO algorithm. By Section 2.1, AQO approximately follows the ground state $|\psi_{qs}(s)\rangle$ of $H_{qs}(s)$ and prepares the target state $|\psi_{qs}(1)\rangle = |0\rangle$ in time

$$O\left(\frac{(n + \max_k |V(k)|)^2}{(\min_s \Delta(s))^3}\right).$$

This yields the following corollary.

▶ **Corollary 22** (Informal version of Theorem 21). *Let $V_q(k) = a(k-k_0)^2$ with $k_0 \leq 0$ and consider the perturbed Hamiltonian $H_{qs}(s) = -(1-s)A_{hc} + s(V_q + h)$, where the spike h is supported on $[\ell, u-1] \subseteq [0,n]$ and satisfies $h(k) \leq \bar{h}$. If $\bar{h}\,\frac{u-\ell}{\sqrt{\ell}} = O(1)$, then AQO prepares an approximation of $|\psi_{qs}(1)\rangle = |0\rangle$ from $|\psi_{qs}(0)\rangle$, the binomial state with mean at $n/2$, in time $O(n^4)$.*

We note that the setup is symmetric with respect to shifting the minimizer $|0\rangle$ to some other, hidden string $|z^*\rangle$, $z^* \in \{0,1\}^n$. In that case, the potential V would satisfy the symmetry condition $V(x) = V(|x \oplus z^*|)$ and the same analysis goes through.

7 Summary, related work and open questions

Our work establishes discrete log-concavity as a critical tool for analyzing discrete optimization with the quantum adiabatic algorithm. We proved log-concavity for a large family of potentials (including and beyond convex potentials), and used it to establish new spectral gap bounds. As a concrete application we proved that AQO can efficiently solve the quadratic HWS problem, yielding a direct extension to the results by Reichardt [33] and Farhi, Goldstone and Gutmann [12] on the (linear) HWS problem.

7.1 Related work

In this section we compare our results to some related works.

7.1.1 Spectral gap analysis of spike Hamiltonians

There exist more refined analyses of the linear HWS Hamiltonian. For spike position $u, \ell \approx n/4$, spike width $u - \ell = n^\alpha$ and spike height $h = n^\beta$, it was shown in [33] that $\alpha + \beta < 1/2$ implies a constant gap for linear potentials, and our result extends this bound to quadratic potentials. In later works, Brady and van Dam [7] (improving on Kong and Crosson [25]) showed that if $\alpha + \beta > 1/2$ but $2\alpha + \beta < 1$, then the gap scales as $n^{1/2-\alpha-\beta}$, while for $2\alpha + \beta > 1$, the gap becomes exponentially small. It would be interesting to extend these results to other potentials, as is suggested in [7], and our result seems to be the first one in this direction.

7.1.2 Classical benchmark algorithms

Our polynomial runtime bound for the quadratic HWS problem should be held against comparable classical algorithms (which, similarly to AQO, do not exploit any particular structure of the problem). To this end, we note that a typical approach based on classical Markov chains and simulated annealing would take exponential time to solve this problem. This can be demonstrated in an identical fashion to the arguments in [12, 11] for the linear HWS problem. Let h be the height and $w = u - \ell$ the width of the spike, and assume $h, w \geq n^c$ for $c \in \Omega(1)$. If the Markov chain flips less than w bits in its proposal, then it can only cross the spike by accepting a move that jumps into the spike, but this will be rejected with probability $1 - O(\exp(-n^c))$. Conversely, if the Markov chain flips more than w bits, then the probability of a proposal ever jumping to the minimizer is $O(\exp(-n^c))$.

Notably, it was shown by Crosson and Harrow [11] and Bergamaschi [5] that a more advanced classical Markov chain algorithm called simulated quantum annealing can solve the (linear) HWS problem under exactly the same assumptions on the spike perturbations. This algorithm bypasses the former argument by using non-local “wordline updates”. It seems reasonable to expect that this algorithm would also solve the quadratic HWS problem, and so we only claim an advantage against the simplest simulated annealing algorithms.

7.2 Open questions

Finally, the results of this work suggest many directions for future research.

7.2.1 Sufficient and necessary condition for log-concavity

The $\gamma-$RM condition is a sufficient condition. We leave the question open of whether it is also a necessary condition. We do prove that at least two important limiting cases of (ultra-)log-concavity (the uniform and the binomial distribution) satisfy the $\gamma-$RM condition with equality.

7.2.2 Can we extend the quadratic HWS problem to general convex potentials?

We specialized to quadratic potentials with minimizer at 0 because we can prove a strong overlap bound between the corresponding ground states and our binomial ansatz states. A natural question is whether this approach extends to other convex potentials. First, note that if the minimizer is not at 0 then the ansatz is not well suited. We show this in Figure 2 where we plot the overlap with the best binomial ansatz for a shifted quadratic potential $V(k) = (k - k_*)^2$ with minimizer at $k_* = n/3$. More unexpectedly, the overlap can decrease even if the minimizer is at 0. We show this in the figure for a piecewise-linear potential.

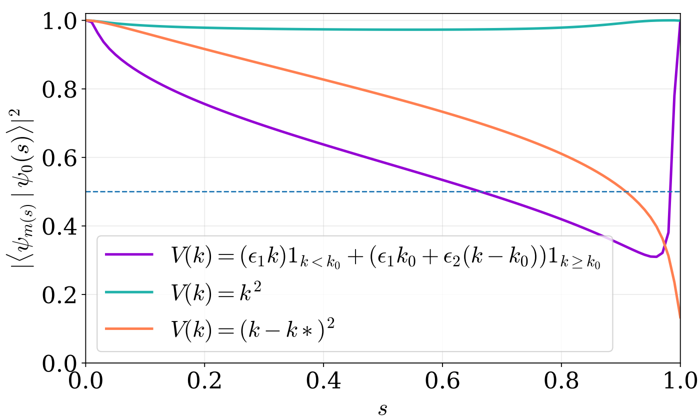

Figure 2 Overlap between the optimal binomial ansatz state $|\varphi_m\rangle$ and the ground state $|\psi_0(s)\rangle$ of $H(s) = -(1-s)A + sV$ for $n = 40$ and different potentials: the quadratic potential $V(k) = k^2$ (used in Section 6), a shifted quadratic potential $V(k) = (k - k_*)^2$ with minimizer at $k_* = n/3$, and a piecewise-linear potential $V(k) = \epsilon_1 k\, \mathbf{1}_{k<k_0} + (\epsilon_1 k_0 + \epsilon_2(k - k_0))\, \mathbf{1}_{k\geq k_0}$, with $n = 40$, $\epsilon_1 = 1/n$, $\epsilon_2 = 2n$, and $k_0 = n/4$.

7.2.3 Can this framework be applied to study tunneling for local Hamiltonians?

A key point is that a localized spike perturbation corresponds to a highly non-local Hamiltonian (coupling many qubits together). It would be interesting to extend our discussion to local target Hamiltonians. E.g., the Curie-Weiss model considers target Hamiltonian $H_1 = -\sum_{i,j\in[n]} Z_i Z_j$. As discussed for instance in [3], the spectral gap of $H(s) = -(1-s)\sum_i X_i + sH_1$ as a function of s undergoes a phase transition from constant to exponentially small in n. Such a phase transition makes the Curie-Weiss model a natural benchmark for our techniques. We leave it as an open question whether the log-concavity framework developed here can be adapted to handle such local models.

7.2.4 Can this framework be extended to non-permutation-symmetric potentials?

A key ingredient of our analysis is the reduction from the hypercube to the $(n+1)$-dimensional Hamming weight subspace, which allows us to work on a weighted path graph. Extending our analysis to more general potentials on the hypercube is a clear open question. It raises interesting points such as what the correct notions of convexity and log-concavity are on the hypercube. We expect this to be non-trivial. E.g., it was shown in [19] that single-basin potentials on general graphs, a condition slightly weaker than convexity, can already yield exponentially small spectral gaps. This contrasts with the 1-dimensional case where monotonicity of the potential on a uniform path was enough (Section 3.3). Additionally, multivariate log-concavity is less trivial to define in the discrete case than in the continuous case [4, 24], and identifying the right structural substitute in higher dimensions should be interesting.

Finally, a slightly different approach was taken by Montanaro and Zhou [31], who argued that certain quantum algorithms (QAOA in particular) for symmetric functions can be shown to be robust against bounded, asymmetric perturbations.

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A Some properties of log-concave sequences

A.1 Useful facts and bounds

We will use the following two elementary facts.

▶ **Lemma 23.** *Fix an interval $I \subseteq \mathbb{Z}$ and a non-negative sequence $\{p_k\}_{k\in\mathbb{Z}}$.*

- *If $\{p_k\}$ is concave on I, then it is log-concave on I.*
- *If $\{p_k\}$ is log-concave on I, then so is $\{c^k p_k\}$ for any $c \geq 0$.*

Proof. We refer the reader to the comprehensive version [8]. ◀

▶ Remark 24. A consequence of the first bullet is that the hypercube coefficients $a_k = \sqrt{k(n+1-k)}$ from Equation (2) are log-concave, since they are concave:

$$a_{k+1} + a_{k-1} - 2a_k = \sqrt{(k+1)(n-k)} + \sqrt{(k-1)(n+2-k)} - 2\sqrt{k(n+1-k)} < 0$$

We continue with some crucial yet less elementary claims. A first key property of a log-concave distribution is that it is *unimodal*, meaning that there exists a mode k^* such that $p_{k-1} \leq p_k$ for $k \leq k^*$ and $p_k \geq p_{k+1}$ for $k \geq k^*$, and so in particular $\max_k p_k = p_{k^*}$.

▷ Claim 25 ([23, Theorem 3]). A log-concave distribution $\{p_k\}$ is unimodal.

Moreover, the mode of a log-concave distribution can be related to its variance.

▷ Claim 26 ([6, Theorem 1.1]). Let σ^2 be the variance of a log-concave distribution $\{p_k\}$. Then

$$\frac{1}{\sqrt{1+12\sigma^2}} \leq p_{k^*} \leq \frac{2}{\sqrt{1+4\sigma^2}}.$$

Another important property of log-concave distributions is that they exhibit geometric decay away from their mode. This is captured by the following tail bound.

▷ Claim 27 ([29, Lemma 2.3]). If μ is the mean of a log-concave distribution $\{p_k\}$, then for all $\alpha \geq 0$ it holds that

$$\sum_{|k-\mu|\geq\alpha} p_k \;\leq\; 2\exp\left(-\frac{\alpha}{2\left(2\sum_k |k-\mu|\, p_k + 1\right)}\right).$$

▶ Remark 28. From this bound we can derive the more convenient bound

$$\sum_{|k-\mu|\geq\alpha} p_k \leq 2\exp\left(-\frac{\alpha}{2(2\sigma+1)}\right).$$

This follows from Jensen's inequality applied to the concave function $\sqrt{\cdot}$:

$$\sum_k |k-\mu| p_k = \sum_k \sqrt{(k-\mu)^2} p_k \leq \sqrt{\sum_k (k-\mu)^2 p_k} = \sigma.$$

A.2 Overlapping log-concave distributions

We will prove a final bound that relates the mean and variance of two log-concave distributions that have a large overlap.

▶ **Lemma 29.** *Let $\{p_k\}, \{q_k\}$ be two log-concave distributions on $\mathbb{Z}$ with means μ_p and μ_q and variances σ_p^2 and σ_q^2 such that $\left(\sum_k \sqrt{p_k q_k}\right)^2 \geq \frac{1}{2}$. Then*

$$\mu_q = \mu_p + O(\sigma_p) + O(1), \qquad \sigma_q = \Theta(\sigma_p) + O(1).$$

Proof. We refer the reader to the comprehensive version [8]. ◀

B Examples of γ-LC from γ-RM

To better grasp the full reach of this framework, we make explicit some particularly interesting cases.

B.1 Uniform γ, uniform a_k

In this case, γ-LC (cf. Equation (7)) reduces to normal log-concavity, and the γ-RM conditions (cf. Definition 8) elegantly reduce to

$$V(k) \leq V(k-1) \qquad (1 \leq k \leq k_0), \qquad\qquad V(k) \leq V(k+1) \qquad (k_1 \leq k \leq n-1).$$

In particular, if V is a single-well potential (equivalently, $-V$ is unimodal) then the ground state of H is log-concave on the full interval $[n]$.

B.2 Uniform γ, log-concave a_k

For uniform $\{\gamma_k\}$, the condition Equation (8) reduces to log-concavity of the $\{a_k\}$. By Remark 24 this setting includes as a particular case of interest the symmetry-reduced hypercube Hamiltonian, in which case $\{a_k = \sqrt{k(n+1-k)}\}$. We will see that such non-trivial $\{a_k\}$ can yield log-concave ground states even for potentials with multiple wells.

A simple yet important limit case is when $V(k) = a_k + a_{k+1}$ (where we set $a_0 = a_{n+1} = 0$). This makes the Hamiltonian H correspond to the Laplacian of the weighted path, the ground state of which we know to be the uniform state with energy $E_0 = 0$. In some sense this is the limit case of log-concavity, but it is also the limit case of our theorem because it saturates the γ-RM conditions with equality.

Going beyond this canonical example, we observe that the conditions are satisfied if

$$\frac{F(k)}{F(k-1)} \leq \frac{a_{k+1}}{a_k} \qquad (1 \leq k \leq k_0), \qquad\qquad \frac{F(k+1)}{F(k)} \geq \frac{a_{k+1}}{a_k} \qquad (k_1 \leq k \leq n-1), \quad (12)$$

and so F has to satisfy a sort of relative monotonicity condition as compared to $\{a_k\}$ (hence the term γ-RM). It is this additional slack that allows us to prove log-concavity of the ground state for numerous non-monotone potentials. The following lemma captures one such family, of which we display an instance in Figure 3.

▶ **Lemma 30.** *If $\{a_k = \sqrt{k(n+1-k)}\}$ and $\langle \varphi_{n/2} | V | \varphi_{n/2} \rangle \leq n$, the conditions*

$$\frac{V(k)}{V(k-1)} \leq \frac{a_{k+1}}{a_k} \qquad (1 \leq k \leq n/2), \qquad\qquad \frac{V(k+1)}{V(k)} \geq \frac{a_{k+1}}{a_k} \qquad (n/2 < k \leq n),$$

are sufficient for the corresponding H to satisfy Equation (12) *and hence to have a log-concave ground state.*

▶ Remark 31. The state $|\varphi_{n/2}\rangle$ is the binomial state with mean at $n/2$ (see Section 2.3). The condition on the expected value of V at $n/2$ is implied e.g. when $\max_k V(k) \leq n$. In the extreme case, V increases from $k=0$ to $k=n/2$. This only imposes that $V(0) \leq 2\sqrt{n}$ because $\max V(k) = V(n/2) \leq \frac{a_{n/2+1}}{a_1} V(0) \simeq \frac{\sqrt{n}}{2} V(0)$. We get the same argument on the right side, i.e. $V(n) \leq 2\sqrt{n}$.

Proof. We refer the reader to the comprehensive version [8]. ◀

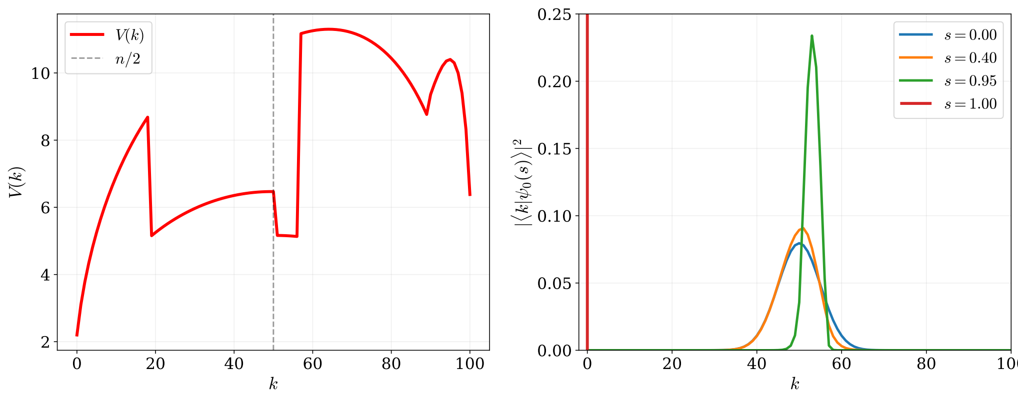

Figure 3 Example of a non-monotone potential satisfying the RM condition for $n = 100$ with $a_k = \sqrt{k(n+1-k)}$ and $\max V(k) \leq n$. Panel (a) shows the potential, which satisfies the one-sided ratio bounds $\frac{V(k)}{V(k-1)} \leq \frac{a_{k+1}}{a_k}$ for $k \leq n/2$ and $\frac{V(k+1)}{V(k)} \geq \frac{a_{k+1}}{a_k}$ for $k \geq n/2+1$, where E_0 is the ground-state energy of $H = -A + V$. Panel (b) shows the shape of the log-concave ground state of $H(s) = -(1-s)A + sV$ for different values of s.

B.3 Binomial γ, binomial a_k

As a final example of our framework, we consider the case where $\left\{\gamma_k = \sqrt{\frac{(k+1)(n-k+1)}{k(n-k)}}\right\}$. In this case, γ-LC corresponds to ultra-log-concavity (ULC) of $\{\psi_k^2\}$ (Definition 5).

We will argue that our framework again tightly captures the limiting case, which is the binomial distribution. Indeed, in Section 2.3 we already saw that the squared ground state of the Hamming weight Hamiltonian (with linear potential $V(k) = \alpha k$) is a binomial distribution. In the following we show that this example satisfies our γ-RM conditions with equality.

We first observe that $\gamma_k \frac{a_k}{a_{k+1}} = 1 + \frac{1}{n-k}$ and $\gamma_k \frac{a_{k+1}}{a_k} = 1 + \frac{1}{k}$, so the $\{a_k\}$ satisfy Equation (8). We then use a telescoping product to rewrite the product

$$\prod_{\ell=1}^{k-1} \gamma_\ell = \sqrt{\frac{kn}{n-k+1}}, \qquad \prod_{\ell=k+1}^{n-1} \gamma_\ell = \sqrt{\frac{n(n-k)}{k+1}}.$$

This simplifies the γ-RM conditions to

$$(n-k)\,[V(k) - V(k-1)] + V(k) - E_0 \leq \frac{n^2}{V(0) - E_0} \qquad (1 \leq k \leq k_0),$$

$$k[V(k) - V(k+1)] + V(k) - E_0 \leq \frac{n^2}{V(n) - E_0} \qquad (k_1 \leq k \leq n-1).$$

Now, for the linear potential $V(k) = \alpha k$ we have the exact expression $E_0 = \frac{n}{2}(\alpha - \sqrt{\alpha^2+4})$ (cf. Lemma 3), and this is easily checked to satisfy both inequalities with equality.