Sparse Oriented Spanners in Metric Spaces

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Abstract

Oriented spanners were presented at ESA'23 as an extension of the well-researched geometric spanners: Given a set P of points in a metric space and an oriented graph $\overrightarrow{G}$, the oriented dilation of two points $p, q \in P$ is the length of the shortest closed walk in $\overrightarrow{G}$ containing p and q divided by the minimum perimeter triangle of p and q. $\overrightarrow{G}$ is called a t-spanner, if the maximum dilation over all pairs of points in P is at most t.

This paper presents the first constructions of sparse oriented spanners for metric spaces beyond the Euclidean space. Given an orientation of the complete graph (i.e. a tournament) with dilation t on n points that satisfies an additional short-cycle property, we show how to extract a $(t+\varepsilon)$-spanner with $\mathcal{O}(k)$ edges in $\mathcal{O}(kn^2 + T(n))$ time, for any metric space admitting a well-separated pair decomposition with k pairs computable in $T(n)$ time. We supplement this with an improved construction of tournaments for metric point sets, obtaining dilation $5/3$. This improves the previous bound of 2 and approaches the lower bound of 1.5. Combined, for n points in a metric space with constant doubling dimension d, this yields a $(5/3+\varepsilon)$-spanner with $(1/\varepsilon)^{\mathcal{O}(\mathrm{d})}n$ edges computable in $(1/\varepsilon)^{\mathcal{O}(\mathrm{d})}n^3$ time using $\mathcal{O}(n^2)$ space. This improves the dilation over the $(2+\varepsilon)$-spanner for Euclidean point sets presented at SoCG'25 while applying to more general metric spaces.

Moreover, we generalize the known $(2+\varepsilon)$-spanner to doubling spaces. In particular, an oriented $(2+\varepsilon)$-spanner with $\mathcal{O}(\varepsilon^{-\mathrm{d}}n)$ edges can be constructed in $(1/\varepsilon)^{\mathcal{O}(\mathrm{d})}n \log n$ time using $\mathcal{O}(\varepsilon^{-\mathrm{d}}n)$ space.

Since the oriented dilation can be dominated by one pair of points, we also consider the oriented average dilation, which is the sum over the oriented dilation of all pairs of points divided by the number of pairs. While oriented $(1+\varepsilon)$-spanners do not exist for every point set, we present an algorithm that computes a spanner with average dilation $1+\varepsilon$ for point sets in a metric space of constant doubling dimension d: More concretely, our algorithm computes an oriented spanner with average dilation at most $1+\mathcal{O}(1/s)+s^{\mathcal{O}(\mathrm{d})}/n$ with $s^{\mathcal{O}(\mathrm{d})}n$ edges in $s^{\mathcal{O}(\mathrm{d})}n \log n$ time using $s^{\mathcal{O}(\mathrm{d})}n$ space, where s is any sufficiently large number that may depend on n.

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1 Introduction

Geometric spanners are sparse graphs that efficiently approximate all pairwise distances of a point set. They are widely used in applications such as transportation networks, motion planning, and distributed systems, which occur in Euclidean as well as more general metric spaces. In particular, metrics with bounded doubling dimensions can be used for ad hoc and communication networks, where distances often obey doubling-like properties, and robot motion planning, where configuration spaces might be high-dimensional, but structured.

The concept of oriented spanners was introduced at ESA'23 [4] as a natural extension of (directed) geometric spanners. In many applications it is desirable to avoid bidirectional edges: for example, in road networks, one-way tracks may be necessary due to space limitations (e.g., in city centers); in motion planning, it can simplify collision avoidance; and in ad hoc networks, it can represent a transmitter-receiver relation.

Undirected spanners have been studied for decades and sparse spanners with small dilation are well known. Already in 1987, the Delaunay triangulation was shown to be a plane 5.08-spanner [11] in $\mathbb{R}^2$, which has been improved ever since, up to 1.998 at SOCG'11 [24]. (See [3] for a survey on plane spanners and [21] for a survey on geometric spanner networks in the Euclidean space.) For spanners with bounded degree and low weight, the Greedy spanner was presented in 1990 [1], giving a t-spanner of size $\mathcal{O}(n)$ for any constant t in $\mathbb{R}^d$ and later extended to doubling metrics [2, 13]. Further results on bounded-degree spanners for doubling metrics were obtained in [8], while additional constructions of light spanners appear in [16]. Using the well-separated pair decomposition (WSPD), introduced by Callahan and Kosaraju [7], a $(1+\varepsilon)$-spanner with $\mathcal{O}(n)$ edges can be obtained in $\mathcal{O}(n \log n)$ time for Euclidean point sets as well as for point sets in doubling spaces [17].

In contrast, only initial results for oriented spanners are known at this point. With their introduction at ESA'23 [4], the authors provided results for Euclidean 1-dimensional point sets and for Euclidean 2-dimensional point sets in convex position. They also present a 2-spanner for any metric point set by orienting the complete graph greedily. At SOCG'25, a first algorithm to compute a sparse spanner with constant dilation has been presented: In [5], the authors give a $(2+\varepsilon)$-spanner with a linear number of edges for Euclidean point sets.

In this paper, we introduce an algorithm that, for any metric point set that admits a WSPD, constructs a $(5/3+\varepsilon)$-spanner with a linear number of edges in $\mathcal{O}(n^3)$ time and $\mathcal{O}(n^2)$ space. Our result is based on an improved algorithm to compute *tournaments*, i.e. an orientation of a complete graph: We provide a tournament with dilation $5/3$ for metric point sets, which we sparsify to obtain a $(5/3+\varepsilon)$-approximation. Furthermore, for point sets in doubling spaces, we generalize the $(2+\varepsilon)$-spanner from [5]. While this admits a higher dilation, the algorithm runs in $\mathcal{O}(n \log n)$ time and only needs $\mathcal{O}(n)$ space.

Although in the undirected case $(1+\varepsilon)$-spanners with $\mathcal{O}(n)$ edges are known for metric spaces with constant doubling dimension, we cannot hope to obtain oriented spanners with dilation nearly 1: We prove a lower bound of 1.5 for point sets in doubling spaces. Our construction of a $(5/3+\varepsilon)$-spanner therefore significantly reduces the gap between the known worst-case dilation for any point set and the guaranteed algorithmically feasible dilation.

Since the maximum oriented dilation is not robust to outliers, we also consider average oriented dilation, defined as the mean over all pairs of points. Average dilation has been considered before in the context of undirected spanners: For points in $\mathbb{R}^d$, an undirected spanner with average dilation $1+\mathcal{O}((\log n/n)^{1/d})$ and $\mathcal{O}(n)$ edges can be computed in $\mathcal{O}(n \log n)$ time [12]. In this paper, we construct a sparse oriented spanner with average dilation $1+\varepsilon$ for constant $\varepsilon > 0$ even for point sets in doubling spaces.

1.1 Technical Overview / Our Contribution

Tournaments (Section 3). As mentioned before, there are point sets for which no oriented spanner with dilation 1 exists, not even an orientation of a complete graph. For a metric space, there is a point set where no t-spanner with $t < 1.5$ exists [6]. In [4], the authors give a cubic time construction of a tournament of dilation 2. In this paper, we present a greedy algorithm that computes, for any metric point set of size n, a tournament with dilation $5/3$ in $\mathcal{O}(n^3)$ time. This reduces the existing gap for the dilation of tournaments for metric point sets from $1/2$ to $1/6$ and bounds the worst-case dilation of every minimum oriented dilation spanner. We further show that the analysis of this algorithm is tight, i.e. there is a point set on which it computes a tournament of dilation exactly $5/3$.

The tournament of dilation 2 presented in [4] is achieved by greedily orienting the minimum perimeter triangles of all pairs of points sorted by their perimeter. Our algorithm follows a different greedy strategy: We process all pairs of points sorted by length and orient adjacent triangles in a consistent manner. This seemingly minor modification leads to a substantial change in the structure and properties of the resulting graph, allowing us to exploit short edges rather than only small triangles. The analysis proceeds by a case distinction over the orientation order, showing that for any edge pq, the previously oriented edges can be combined into a sufficient small cycle through p and q containing at most 4 edges.

Spanners of linear size (Section 4). We present a construction by sparsifying a tournament while (almost) preserving its dilation. More concretely, given any set P of n points and a tournament on P such that for every $p, q \in P$ there is an oriented cycle through p and q with $c = \mathcal{O}(1)$ edges which is at most a factor t longer than their minimum perimeter triangle, we construct an oriented $(t + \varepsilon)$-spanner for P with $\mathcal{O}(n)$ edges in $\mathcal{O}(n^3)$ time and $\mathcal{O}(n^2)$ space.

We achieve this result by extracting a linear number of cycles from the tournament, each containing at most c edges, ensuring short closed walks between all pairs of points. We identify these cycles using the WSPD. As this is the only limiting component, the algorithm applies to point sets in any metric space that admits a WSPD, such as the Euclidean space [7], the unit-disk graph metric [14], or metric spaces with constant doubling dimension [17, 23].

Plugging in our results for tournaments of dilation $5/3$, we can compute an oriented $(5/3 + \varepsilon)$-spanner with $\mathcal{O}(n)$ edges in $\mathcal{O}(n^3)$ time and $\mathcal{O}(n^2)$ space.

In [5], a spanner for Euclidean spaces is presented that requires less runtime and space while increasing the dilation by at most $1/3$. We generalize this result to doubling spaces. In particular, for n points in a metric space of constant doubling dimension, we construct a $(2 + \varepsilon)$-spanner with $\mathcal{O}(n)$ edges in $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ space. The algorithm from [5] relies on approximating the minimum perimeter triangle containing two given query points, but their solution is specific to Euclidean spaces; we adapt it to also apply to doubling spaces.

Average dilation spanners (Section 5). As the (standard notion of) dilation of a graph is not robust to outliers, considering the average dilation of a graph may provide meaningful information on the network. While the average dilation of undirected spanners has been considered before, this paper is the first to consider the oriented average dilation of a spanner. For n points in a metric space with constant doubling dimension d and a sufficiently large real number s, we provide an algorithm that computes an oriented spanner with average dilation $1 + \mathcal{O}(1/s) + s^{\mathcal{O}(\mathrm{d})}/n$ in $s^{\mathcal{O}(\mathrm{d})} n \log n$ time and $s^{\mathcal{O}(\mathrm{d})} n$ space.

Our approach again relies on a WSPD. A natural idea is to add, for each well-separated pair $\{A, B\}$, oriented edges between A and B in both directions. Although intuitive, this fails for singleton-singleton pairs. However, since only $s^{\mathcal{O}(\mathrm{d})} n$ such pairs exist, their effect on the average dilation remains insignificant once constant oriented dilation is guaranteed for them. The key task is to ensure an oriented dilation of $1+\mathcal{O}(1/s)$ across all non-singleton-singleton pairs, which our construction achieves. This yields the desired average-dilation spanner.

2 Preliminaries

Let $(P,|\cdot|)$ be a finite metric space, and let p and q be two distinct points in P. We denote by $|pq|$ the distance of p and q. The *shortest cycle* through p and q in the complete undirected graph on P is denoted by $\Delta^*(p,q)$. This cycle is the triangle Δ_{pqx} with $x = \underset{x\in P\setminus\{p,q\}}{\arg\min}\, |px|+|qx|$.

Let $\overrightarrow{G} = (P, \overrightarrow{E})$ be an *oriented graph* on P such that its underlying undirected graph is simple (in particular this means that if $(p,q) \in \overrightarrow{E}$ then $(q,p) \notin \overrightarrow{E}$). In other words, the oriented graph $\overrightarrow{G} = (P, \overrightarrow{E})$ is an *orientation* of an undirected simple graph.

For a walk C, we denote its length by $|C|$, which is the sum of the length of its edges. For two distinct points p,q, let $C_{\overrightarrow{G}}(p,q)$ be the *shortest closed walk* through p and q in $\overrightarrow{G}$. If there is no such walk, then $|C_{\overrightarrow{G}}(p,q)| = \infty$.

▶ **Definition 1** (oriented dilation). *Let $(P,|\cdot|)$ be a metric space and let $\overrightarrow{G}$ be an oriented graph with vertex set P. The* oriented dilation *of two distinct points $p,q \in P$ is defined as*

$$\mathrm{odil}_{\overrightarrow{G}}(p,q) = \frac{|C_{\overrightarrow{G}}(p,q)|}{|\Delta^*(p,q)|}.$$

The oriented dilation of $\overrightarrow{G}$ is defined as $\mathrm{odil}(\overrightarrow{G}) = \max_{p,q\in P} \mathrm{odil}_{\overrightarrow{G}}(p,q)$.

We call an oriented graph $\overrightarrow{G}$ with oriented dilation at most t an *oriented t-spanner*. An orientation $\overrightarrow{K}$ of the complete undirected graph on P is called a *tournament* on P.

▶ **Observation 2.** *Let $(P,|\cdot|)$ be a metric space and $\overrightarrow{K}$ a tournament on P. For any two distinct points $p,q \in P$, the shortest closed walk $C_{\overrightarrow{K}}(p,q)$ is a cycle.*

In this paper, we often orient triangles *consistently*, meaning that their oriented edges form a cycle. If one edge of a triangle is already oriented, there is exactly one way to complete the consistent orientation. If two edges pq and pr of a triangle Δ_{pqr} are already oriented, consistency requires that they form either the path $p \to q \to r$ or its reverse.

▶ **Definition 3** (oriented average dilation). *Let $(P,|\cdot|)$ be a metric space and let $\overrightarrow{G}$ be an oriented graph with vertex set P. The oriented average dilation of $\overrightarrow{G}$ is defined as*

$$\frac{1}{\binom{n}{2}} \sum_{p,q\in P} \mathrm{odil}_{\overrightarrow{G}}(p,q).$$

2.1 Growth-Restricted Metrics

The *doubling dimension* of a metric formalizes a packing property. Intuitively, it captures the “volume growth” of a metric space. A space has doubling dimension d if every ball of radius r can be covered by 2^{d} balls of radius $r/2$. We call it a *doubling space*, if d is constant.

Metrics of bounded doubling dimension include many spaces arising in applications, such as low-dimensional Euclidean metrics. Numerous algorithmic techniques can be tailored to run efficiently on growth-restricted metrics (see [17, 19] for applications).

2.2 Well-Separated Pair Decomposition

The *well-separated pair decomposition* (WSPD) serves as a tool to solve various distance problems on multi-dimensional point sets. Intuitively, it breaks all $\binom{n}{2}$ pairs of points down to $\mathcal{O}(n)$ subsets that approximate the complete graph. See [22] for a survey.

For a real number $s > 0$, two point sets A and B form an *s-well-separated pair*, if A and B can each be covered by balls of the same radius, say ρ, such that the distance between the balls is at least $s \cdot \rho$. An *s-well-separated pair decomposition* for a point set P is a sequence of s-well-separated pairs $\{A_i, B_i\}$, such that $A_i \cap B_i = \emptyset$ and for any two points $p, q \in P$, there is a unique index i such that $p \in A_i$ and $q \in B_i$, or $q \in A_i$ and $p \in B_i$.

Callahan and Kosaraju [7] introduced the WSPD for point sets in Euclidean spaces of constant dimension. The WSPD has been generalized to more general metric spaces, such as the unit-disk graph metric [14] and doubling spaces [17, 23].

▶ **Lemma 4** (Har-Peled and Mendel [17]). *Let P be a set of n points in a metric space with constant doubling dimension d, and let $s > 0$ be a real number. An s-WSPD for P consisting of $s^{\mathcal{O}(\mathrm{d})}n$ pairs can be constructed in $s^{\mathcal{O}(\mathrm{d})}n + 2^{\mathcal{O}(\mathrm{d})}n \log n$ time.*

The following properties apply for any WSPD regardless of the metric.

▶ **WSPD-Property 1** (Smid [22]). *Let $s > 0$ be a real number and let $\{A, B\}$ be an s-well-separated pair. For points $a, a' \in A$ and $b, b' \in B$, it holds that*
 (i) $|aa'| \leq 2/s \cdot |ab|$, *and*
 (ii) $|a'b'| \leq (1 + 4/s) \cdot |ab|$.

In Section 5, we process the well-separated pairs in a certain order: We sort the pairs $\{A_i, B_i\}$ ascending by the distances of two arbitrary points $a \in A_i$ and $b \in B_i$. Such an ordering is common in inductive proofs for WSPD based algorithms (see [22]). These proofs make use of the subsequent property, which follows directly from WSPD-Property 1.

▶ **WSPD-Property 2.** *Let $\{A_i, B_i\}$ for $1 \leq i \leq m$ denote the pairs of an s-WSPD for a point set P, sorted by ascending distance between arbitrary points $a_i \in A_i$ and $b_i \in B_i$. For p, q two distinct points in P, let $\{A_j, B_j\}$ and $\{A_k, B_k\}$ be well-separated pairs such that $p \in A_j$ and $q \in B_j$ and $p, q \in A_k$. By construction, it holds $j < k$.*
(Note that the index j is unique, whereas multiple indices may satisfy the condition for k.)

2.3 Approximate Nearest Neighbor Queries

For a metric space $(P, |\cdot|)$, we preprocess P to support fast *approximate nearest neighbor* queries. For any query point q, the algorithm returns a point $q' \in P$ such that $|qq'| \leq (1 + \varepsilon) \cdot |qq^*|$, where q^* is closest point to q in P.

Such queries for doubling spaces have been studied in several works, including [17, 19, 20]. In this paper, we need a data structure for a *dynamic* point set:

▶ **Lemma 5** (Cole and Gottlieb [10, 15]). *Let P be a set of n points in a metric space with doubling dimension d. A data structure of size $\mathcal{O}(n)$ can be constructed in $2^{\mathcal{O}(\mathrm{d})}n \log n$ time that supports the following operations:*

- *Given a real number $\varepsilon > 0$ and a query point q, return a $(1 + \varepsilon)$-approximate nearest neighbor of q in P in $2^{\mathcal{O}(\mathrm{d})} \log n + (1/\varepsilon)^{\mathcal{O}(\mathrm{d})}$ time.*
- *Inserting a point into P and deleting a point from P in $2^{\mathcal{O}(\mathrm{d})} \log n$ time.*

3 Tournaments

In this section, we prove the following result.

▶ **Theorem 6.** *For every metric space $(P, |\cdot|)$ with n points, an orientation $\overrightarrow{K}$ of the complete graph on P can be computed such that*

- *the oriented dilation of $\overrightarrow{K}$ is at most $5/3$, and*
- *for any two distinct points p and q in P, there exists a cycle through p and q in $\overrightarrow{K}$, whose length is at most $5/3|\Delta^*(p,q)|$ and that contains at most four edges.*

Such an orientation can be computed in $\mathcal{O}(n^3)$ time.

Algorithm 1 Tournament with Dilation 5/3.

Require: Set P of n points in a metric space
Ensure: Tournament on P with dilation $5/3$
  Set $m = \binom{n}{2}$
  Sort the m pairs of distinct points in P in non-decreasing order of their distances.
  Denote the sorted sequence by p_1q_1, p_2q_2, $\dots$, p_mq_m.
  for $i = 1, 2, \dots, m$ **do**
    if the pair p_iq_i has not been oriented yet **then**
      if there is a triangle incident to p_iq_i that can be oriented consistently **then**
        Orient p_iq_i s.t. the smallest triangle $\Delta_{p_iq_ip'}$ with $p' \in P \setminus \{p_i, q_i\}$ can be oriented consistently.
      else
        Orient p_iq_i arbitrarily.
    for all $p \in P \setminus \{p_i, q_i\}$ **do**
      Orient $\Delta_{p_iq_ip}$ consistently, if possible, otherwise skip it.

The algorithm that computes such an orientation is given in Algorithm 1. We sort all $m = \binom{n}{2}$ pairs of points ascending by length. In this order, we orient the current pair and, if possible, every adjacent triangle consistently. Observe that this algorithm orients all m edges of the complete graph on P. It is clear that the running time is $\mathcal{O}(n^3)$.

The key difference between our algorithm and the 2-spanner in [4] is the sorting criterion: while [4] processes *triangles* by perimeter, we process *edges* by length. This change in priority ensures that shorter edges are oriented earlier, however, it is non-trivial how to combine these to a short oriented cycle. The following key lemma shows that we can always find a suitable set of such edges.

▶ **Lemma 7*.** *Let p and q be two distinct points in P. Let $\overrightarrow{K} = (P, \overrightarrow{E})$ be the tournament returned by Algorithm 1. It holds $\mathrm{odil}_{\overrightarrow{K}}(p,q) \leq 5/3$.*

Proof sketch. Let r be a point in $P \setminus \{p, q\}$ such that $\Delta^*(p,q) = \Delta_{pqr}$. We distinguish whether pq is the first, second, or third edge of Δ_{pqr} that gets an orientation. In the following, we sketch parts of the case in which pq is oriented as the third.

We consider the case $(q,p), (p,r), (q,r) \in \overrightarrow{E}$. Let qr be oriented as second, in iteration y, and p_yq_y be the current pair. We can show $|p_yq_y| \leq 1/3|\Delta_{pqr}|$. If exactly one of $\{p_y, q_y\}$ is in $\{q, r\}$, say p_y, then $\overrightarrow{K}$ contains the cycle $C = q \to p \to r \to q_y \to q$ (see Figure 1, left). For $p_y = q$ (and analogously for $p_y = r$), we have

$$|C| = |qp| + |pr| + |rq_y| + |q_yq| \leq |qp| + |pr| + |rq| + 2|rq_y| = |\Delta_{pqr}| + 2|p_yq_y| \leq 5/3|\Delta_{pqr}|.$$

* Due to space restrictions, proofs of results marked by $*$ are only given in the full version.

For $qr = p_y q_y$, we show that at the beginning of the y-th iteration, qr is unoriented. The first if-clause orients qr such that a triangle $\Delta_{qrp'}$ with $p' \in P \setminus \{q, r\}$ is oriented consistently (there must be such a triangle). Due to the algorithm, it holds that $|rp'| + |p'q| \leq |rp| + |pq|$. If $p' = p$, then Δ_{pqr} is oriented consistently during iteration y. For $p' \neq p$, $\overrightarrow{K}$ contains the cycle $C = q \to p \to r \to p' \to q$ (see Figure 1, right) of length

$$|C| = |qp| + |pr| + |rp'| + |p'q| \leq 2|pq| + 2|pr| \leq |\Delta_{pqr}| - 2|pr|.$$

So, if $|qr| \geq 1/6|\Delta_{pqr}|$, then $|C| \leq 5/3|\Delta_{pqr}|$. The bound $|qr| < 1/6|\Delta_{pqr}|$ is used to prove further subcases. ◀

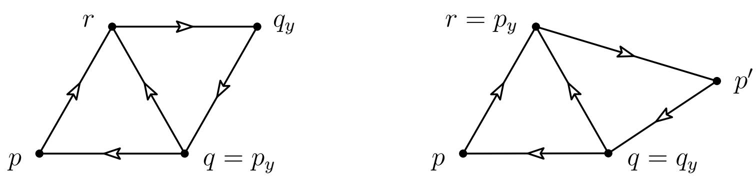

Figure 1 Visualization of the proof sketch of Lemma 7. We assume $(q,p), (p,r), (q,r) \in \overrightarrow{E}$.

Lemma 7 implies that the oriented dilation of $\overrightarrow{K}$ is at most 5/3. Further, for any two distinct points p and q in P, there exists a cycle through p and q in $\overrightarrow{K}$, whose length is at most $5/3|\Delta^*(p,q)|$ and that contains at most 4 edges (see proof of Lemma 7 in full version). This is tight, since any tournament on four points has a pair that is not incident on any cycle of length 3. This completes the proof of Theorem 6.

We finish this section by giving an example of a set P of four points in the Euclidean plane, such that the orientation computed by Algorithm 1 has oriented dilation equal to 5/3.

Consider Figure 2. Let p_1, p_2 and p_3 be collinear and $|p_1p_2| = |p_2p_3| = 1$. Let p_2, p_3, and p_4 form an equilateral triangle with unit length. Algorithm 1 starts with p_1p_2 and orients $\Delta_{p_1p_2p_3}$ and $\Delta_{p_1p_2p_4}$. Now, only p_3p_4 remains unoriented. Since p_3p_4 is not neighboring any triangle that can be oriented consistently anymore, p_3p_4 is oriented arbitrarily. Let (p_4, p_3) be its orientation. We have

$$\begin{aligned}
\mathrm{odil}(p_2, p_4) &= \frac{|p_2p_4| + |p_4p_1| + |p_1p_2|}{|\Delta_{p_2p_3p_4}|} = \frac{2+\sqrt{3}}{3} \approx 1.244, \\
\mathrm{odil}(p_3, p_4) &= \frac{|p_4p_3| + |p_3p_1| + |p_1p_2| + |p_2p_4|}{|\Delta_{p_2p_3p_4}|} = 5/3
\end{aligned}$$

and the dilation of any other pair is 1. Therefore, dilation of the returned graph is 5/3.

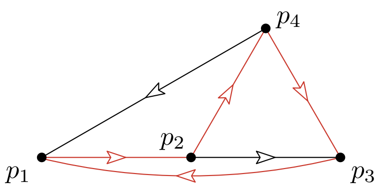

Figure 2 A point set of size four and its tournament $\overrightarrow{K}$ by Algorithm 1. The edge (p_3, p_1) is curved for clarity, though it represents the straight line segment. The dilation is $t = \mathrm{odil}(p_3, p_4) = 5/3$, and the shortest closed walk $C_{\overrightarrow{K}}(p_3, p_4)$ is shown in red.

4 Spanners of Linear Size

In this section, we construct sparse oriented spanners of linear size for general metric spaces, with special attention to doubling spaces. As shown in [4], there are point sets in the Euclidean space where no oriented t-spanner exists for $t < 2\sqrt{3} - 2 \approx 1.46$. In [6], a lower bound of 1.5 is given for any metric point set. The question arises as to which of these bounds apply to doubling spaces.

▶ **Proposition 8.** *Let $(P, |\cdot|)$ be a metric space with constant doubling dimension. The worst-case dilation t of a minimum dilation spanner on P is $1.5 \leq t \leq 5/3$.*

The upper bound follows from Theorem 6. Figure 3 shows an embedding in $(\mathbb{R}^2, L_1)$ (Manhattan distance) of the point set from [6]. It is easy to see that every disc of radius r can be covered with at most 3 discs of radius $r/2$. Thus, this point set can be embedded in a metric space with doubling dimension $\log 3$.

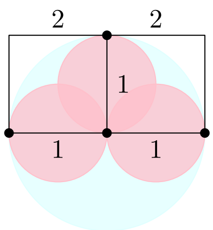

Figure 3 The point set from [6] and its embedding in $(\mathbb{R}^2, L_1)$.

4.1 Sparsification of Tournaments

Given a tournament on a point set P that contains for every two distinct points $p, q \in P$ an oriented cycle with $\mathcal{O}(1)$ edges whose length is at most $t|\Delta^*(p,q)|$, we show how to compute a sparse subgraph with oriented dilation $t + \varepsilon$. Thus, we reduce the number of edges from $\binom{n}{2}$ to $\mathcal{O}(n)$ while increasing the dilation only by a given parameter ε.

The algorithm that computes such an orientation is given in Algorithm 2. It first computes the WSPD for the given point set. For each well-separated pair, we add up to four cycles with at most c edges. Therefore, we run the Bellman-Ford algorithm for only $c-1$ iterations, to compute the shortest path in $\overrightarrow{K}$ with at most $c-1$ edges between two given points. Note that the returned graph $\overrightarrow{G}$ is a subgraph of $\overrightarrow{K}$. Since we add at most four cycles with $c = \mathcal{O}(1)$ edges for k well-separated pairs, $\overrightarrow{G}$ has $\mathcal{O}(k)$ edges.

▶ **Theorem 9***. *Let $(P, |\cdot|)$ be a metric space with n points that admits a WSPD with k pairs constructible in $T(n)$ time. Let $c \in \mathbb{N}$ be a constant and $t \geq 1$ and $0 < \varepsilon < 2$ be real numbers. Let $\overrightarrow{K}$ be a tournament on P such that for any $p, q \in P$ there is an oriented cycle through p and q with at most c edges and length at most $t|\Delta^*(p,q)|$. An oriented $(t+\varepsilon)$-spanner for P with $\mathcal{O}(ck)$ edges can be constructed in $\mathcal{O}(kcn^2 + T(n))$ time using $\mathcal{O}(n^2)$ space.*

Note that the runtime of Algorithm 2 may depend on the construction time and size of the WSPD. For all metrics where a subquadratic-size WSPD is known, it can be constructed in $o(n^2)$ time. While linear-size WSPDs are known for Euclidean [7] and doubling metrics [23], it is open whether the unit-disk graph metric admits a WSPD with $\mathcal{O}(n)$ pairs [18].

In Section 3, we showed how to orient the complete graph on P such that for any $p, q \in P$ there is an oriented cycle through p and q whose length is at most $5/3|\Delta^*(p,q)|$ and which has at most four edges. Thus, combining Theorem 6 and 9 leads to a sparse $(5/3+\varepsilon)$-spanner.

Algorithm 2 Sparse $(t+\varepsilon)$-Spanner.

Require: Set P of n points in a metric space that admits a WSPD with $\mathcal{O}(k)$ pairs, a constant $c \in \mathbb{N}$, positive real numbers $\varepsilon < 2, t \geq 1$, and a tournament $\overrightarrow{K}$ on P such that for any pair $p, q \in P$ there is an oriented cycle through p and q with at most c edges and length at most $t\Delta^*(p,q)$

Ensure: $(t+\varepsilon)$-spanner for P with $\mathcal{O}(k)$ edges

Compute an s-WSPD $\{A_i, B_i\}$ for $1 \leq i \leq m = \mathcal{O}(n)$ with $s = 6 + \frac{10t+6}{\varepsilon}$.

$\overrightarrow{E} = \emptyset$

for $i = 1$ **to** m **do**

  $\mathcal{A}$ = Pick $\min\{|A_i|, 2\}$ arbitrary points of A_i

  $\mathcal{B}$ = Pick $\min\{|B_i|, 2\}$ arbitrary points of B_i

  for all $pq \in \mathcal{A} \times \mathcal{B}$ **do**

    Depending on whether (p,q) or $(q,p) \in \overrightarrow{K}$, compute the shortest path Π in $\overrightarrow{K}$ with at most $c-1$ edges from q to p (or resp. from p to q).

    Add all edges of Π and (p,q) (resp. (q,p)) to $\overrightarrow{E}$.

return oriented graph $\overrightarrow{G} = (P, \overrightarrow{E})$

▶ **Corollary 10.** *Let $(P, |\cdot|)$ be a metric space with n points that admits a WSPD with $\mathcal{O}(n)$ pairs constructible in $\mathcal{O}(n^3)$ time. Given a real number $0 < \varepsilon < 2$, an oriented $(5/3+\varepsilon)$-spanner for P with $\mathcal{O}(n)$ edges can be constructed in $\mathcal{O}(n^3)$ time and $\mathcal{O}(n^2)$ space.*

4.2 Sparse $(2+\varepsilon)$-Spanner for Doubling Metrics

In this subsection, we prove the following result.

▶ **Theorem 11.** *Given a set P of n points in a metric space with constant doubling dimension d and a real number $0 < \varepsilon < 1$, an oriented $(2+\varepsilon)$-spanner for P with $(1/\varepsilon)^{\mathcal{O}(\mathrm{d})}n$ edges can be constructed in $(1/\varepsilon)^{\mathcal{O}(\mathrm{d})}n \log n$ time using $(1/\varepsilon)^{\mathcal{O}(\mathrm{d})}n$ space.*

This spanner is constructed by the algorithm from [5], which works as follows:

1. Compute the WSPD on the given point set.
2. For two points of each well-separated pair, approximate a minimum perimeter triangle containing those points (see Lemma 12).
3. Orient the undirected graph whose edge set consists of all edges of the approximated triangles, using the greedy algorithm of [5] (see Lemma 14).

Since WSPDs exist for metric spaces with low doubling dimension (see Lemma 4) and the greedy algorithm in Step 3 applies to any metric space (see Lemma 14), it remains only to extend the query in Step 2 to doubling spaces.

▶ **Lemma 12*.** *Let P be a set of n points in a metric space with constant doubling dimension d and let $0 < \varepsilon_1 < 2$ be a real number. In $2^{\mathcal{O}(\mathrm{d})}n \log n$ time, P can be preprocessed into a data structure of size $\mathcal{O}(n)$, such that, given any two distinct query points p and q in P, a $(1+\varepsilon_1)$-approximation of the triangle $\Delta^*(p,q)$ can be computed in $(1/\varepsilon_1)^{\mathcal{O}(\mathrm{d})} \log n$ time.*

We approximate the minimum perimeter triangle by the following query:

1. We compute an approximate nearest neighbor p' of p.
2. If p' is "far" from p, then $\Delta_{pqp'}$ is a suitable approximation of $\Delta^*(p,q)$.
3. Otherwise, the searched triangle lies in a ball B of radius $1/\varepsilon|pq|$ centered at p. We cover B by balls of radius $\varepsilon|pq|$.
4. For each ball, we approximate a nearest neighbor x_c of the ball center.
5. We return the smallest triangle Δ_{pqx_c} over all balls.

In Step 1, we require a point p' which is not q, therefore we first delete and later reinsert q in the point set (the same holds for p and q in Step 4). Thus, the used data structure is dynamic (see Lemma 5 [10]). Alternatively, we could also approximate the 2nd and 3rd nearest neighbor.

The query and the proof of Lemma 12 closely follow [5], but adapting the approach to doubling spaces requires additional care in Step 3. By the doubling property, every ball of radius R can be covered with $\mathcal{O}((R/r)^{\mathrm{d}})$ balls of radius r. Thus, the number of considered balls is constant. However, computing such a cover is nontrivial for doubling spaces.

▶ **Observation 13.** *Let $(P, |\cdot|)$ be a metric space with n points and doubling dimension* d*. Given a point $p \in P$ and two radii R and $r < R$, we can compute a cover of the ball B of radius R centered at p by balls of radius r in $\mathcal{O}(\log n + (R/r)^{\mathrm{d}})$.*

To compute this cover, we use the net-tree constructed in the spanner algorithm for the computation of the WSPD (see [17]). A node at level i represents a set of points contained in a ball of radius 2^{-i} centered at a representative point in P. Computing the cover reduces to identifying all nodes at level i with $2^{-i} = r$, whose representatives have at most distance $R + r$ to p. Using the point-location properties of the net-tree, we can find the relevant ancestor node in $\mathcal{O}(\log n)$ time.

▶ **Lemma 14** [5, Lemma 3]**.** *Let $(P, |\cdot|)$ be a finite metric space with n points*[1]*. Let $\varepsilon_1 > 0$ be a real number and let L be a list of m point triples $(p, q; r)$, with p, q, and r being pairwise distinct points in P. Assume that for each $(p, q; r)$ in L, $|\Delta_{pqr}| \leq (1 + \varepsilon_1)|\Delta^*(p, q)|$. Let $G = (P, E)$ be the undirected graph whose edge set consists of all edges of the m triangles Δ_{pqr} defined by the triples $(p, q; r)$ in L. In $\mathcal{O}(m(\log m + \log n))$ time, we can compute an orientation $\overrightarrow{G}$ of G, such that $\mathrm{odil}_{\overrightarrow{G}}(p, q) \leq 2 + 2\varepsilon_1$ for each $(p, q; r)$ in L. In particular, for each $(p, q; r)$ in L, there exists a closed walk C in $\overrightarrow{G}$ containing p and q which consists of at most four edges and whose length is $|C| \leq (2 + 2\varepsilon_1)|\Delta^*(p, q)|$.*[2]

As our generalized triangle query still guarantees an approximation factor of $1 + \varepsilon$, the algorithm from [5] computes a $(2 + \varepsilon)$-spanner. Plugging in the runtime and space requirements for a WSPD (Lemma 4), an approximate nearest neighbor data structure (Lemma 5) and minimum perimeter triangle approximation (Lemma 12) for doubling spaces, we obtain Theorem 11.

5 Average Dilation Spanners

This section presents the first construction of a sparse oriented spanner with average dilation $1 + \varepsilon$ for sufficiently large point sets. Our algorithm consists of the following steps:

1. Compute the WSPD on the given point set.
2. For each singleton-singleton pair $\{\{p\}, \{q\}\}$, approximate $\Delta^*(p, q)$ (see Lemma 12).
3. Orient the undirected graph whose edge set consists of all edges of the approximated triangles, using the greedy algorithm of [5] (see Lemma 14).
4. Sort all well-separated pairs ascending by the distance of two representatives per pair.
5. For each non-singleton-singleton pair, we add at most five oriented edges.

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[1] The original statement in [5] is for Euclidean spaces, but its proof applies for any metric space.
[2] The number of edges is relevant for the proof of Lemma 15. This guarantee is not in the original statement [5, Lemma 3], but can be inferred from its proof.

The novelty of our approach lies in how we adapt the framework of [5] to achieve an average dilation of $1+\varepsilon$. While Steps 1–3 follow [5] (compare to Section 4.2), the essential difference is that in that work, Step 2 was applied uniformly to all well-separated pairs, resulting in a worst-case (and average) dilation of $2+\varepsilon$. In contrast, the proposed algorithm treats singleton-singleton and non-singleton-singleton pairs separately.

For singleton-singleton pairs, a dilation of $1+\varepsilon$ is impossible once minimum perimeter triangles overlap and their consistent orientations conflict. Even when using a $(1+\varepsilon)$-approximation instead of the minimum triangles, such conflicts are unavoidable. However, since they are only a small portion of all well-separated pairs, their effect on the average dilation is negligible, as long as their dilation is constant.

For non-singleton-singleton pairs, the edges oriented in Step 5 form a closed walk connecting points from both sets, ensuring a worst-case dilation of $1+\varepsilon$. The main challenge is preserving previously constructed closed walks: orienting an edge for one pair must not alter any walk belonging to previous pairs. This is ensured by our processing order (Step 4) together with structural properties of the WSPD.

Algorithm 3 Sparse Average Dilation Spanner.

Require: Set P of n points in a metric space with constant doubling dimension d, sufficiently large real number s, $\varepsilon_1 < 2$
Ensure: Oriented spanner on P with average dilation $1+\mathcal{O}(1/s)+s^{\mathcal{O}(\mathrm{d})}/n$ and $s^{\mathcal{O}(\mathrm{d})}n$ edges
  Compute an s-WSPD $\{A_i, B_i\}$ for $1 \le i \le m$; assume $|A_i| \le |B_i|$ for all i.
  Compute a data structure on P to approximate nearest neighbors [10].
  $L = \emptyset$
  for all singleton-singleton pairs $\{\{p\},\{q\}\}$ **do**
    $\Delta_{pqr} = (1+\varepsilon_1)$-approximation of $\Delta^*(p,q)$ using approx. nearest neighbors (Lemma 12)
    Add the point triple $(p,q;r)$ to L.
  Orient the edges of the triangles in L by the greedy algorithm of [5] (Lemma 14).
  Sort all well-separated pairs $\{A_i, B_i\}$ by ascending distance of any $a_i \in A_i$ and $b_i \in B_i$.
  for all $\{A_i, B_i\}$ with $|B_i| \ge 2$ (in sorted order) **do**
    Choose two arbitrary points of B_i. Let b, b' be those points.
    Choose $\min(|A_i|, 2)$ arbitrary points of A_i. Let a (and a', if $|A_i| > 1$) be those points.
    Note: The following may change orientations of edges, set by the greedy algorithm.
    if $|B_i| = 2$ **then**
      if bb' is not oriented yet **then** Orient bb' arbitrarily.
      Orient $\Delta_{abb'}$ consistently preserving the orientation of bb' (and adjusting the orientation of ab and ab' if necessary).
      if $|A_i| = 2$ **then** Orient $\Delta_{a'bb'}$ consistently preserving the orientation of bb'.
    else
      Set the following orientations: (a,b), (b,a'), (a',b'), (b',a)

To show that Algorithm 3 constructs a sparse oriented spanner with average dilation $1+\varepsilon$, first we bound the oriented dilation of two distinct points p,q. Let $\{A_i, B_i\}$ be the unique pair such that $p \in A_i$ and $q \in B_i$ or $p \in B_i$ and $q \in A_i$. We bound the dilation of p and q according to whether $\{A_i, B_i\}$ is a singleton-singleton pair or not. Later, we sum up the dilation of these types to get an upper bound on the average dilation.

▶ **Lemma 15.** *Let p and q be two distinct points in P. Let $\overrightarrow{G}$ be the graph returned by Algorithm 3. For a sufficiently large number $s \ge 30$, it holds*
- $\mathrm{odil}(p,q) \le 16$, *if $\{\{p\},\{q\}\}$ is a well-separated pair, and*
- $\mathrm{odil}(p,q) = 1+\mathcal{O}(1/s)$, *otherwise.*

Proof. Let $\{A_i, B_i\}$ be the well-separated pair with $p \in A_i$ and $q \in B_i$ or $p \in B_i$ and $q \in A_i$. W.l.o.g., let $p \in A_i$ and $q \in B_i$ and $|A_i| \leq |B_i|$. Our proof uses induction on the sorted order of the well-separated pairs.

Base case. The pair $\{A_1, B_1\}$ is a closest pair in P. Therefore, $\{A_1, B_1\}$ is the singleton-singleton pair $\{\{p\}, \{q\}\}$.

Let $\overrightarrow{G'} = (P, \overrightarrow{E'})$ be the oriented graph obtained by orienting the triangles in L. So, $\overrightarrow{G'}$ is an orientation of the undirected graph whose edge set consists of all edges of the approximated triangles of all singleton-singleton pairs of the WSPD. By construction, the shortest closed walk $C_{\overrightarrow{G'}}(p, q)$ in $\overrightarrow{G'}$ containing p and q has at most 4 edges (due to Lemma 14). W.l.o.g., let

$$C_{\overrightarrow{G'}}(p, q) = p \to q \to x \to y \to p,$$

for $x, y \in P \setminus \{p, q\}$. Further, due to Lemma 14 and $\varepsilon_1 < 2$, we have

$$|C_{\overrightarrow{G'}}(p, q)| = |pq| + |qx| + |xy| + |yp| \leq (2 + 2\varepsilon_1)|\Delta^*(p, q)| < 4|\Delta^*(p, q)|.$$

Since there is a unique pair $\{A_i, B_i\}$ such that $p \in A_i$ and $q \in B_i$, the edge pq is oriented by the first loop and is not changed later. But the orientation of edges qx, xy, and py might have flipped (at most once) in the second loop (see Figure 4).

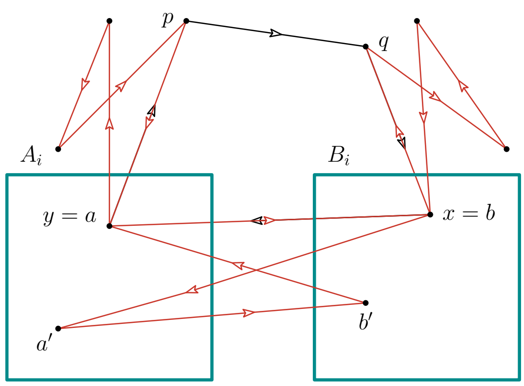

Figure 4 Illustration of the base case of Lemma 15.

Assume the orientation of xy is flipped during the second for-loop, i.e. $(x, y) \in \overrightarrow{E'}$ and $(y, x) \in \overrightarrow{E}$. We show that $\overrightarrow{G}$ contains a path from x to y whose length is at most $(3+8/s)|xy|$.

Let $\{A_j, B_j\}$ be the well-separated pair such that $y \in A_j$ and $x \in B_j$ or $x \in A_j$ and $y \in B_j$. We assume $|A_j| \leq |B_j|$. If the orientation of xy is flipped, then both x and y must have been chosen while processing the pair $\{A_j, B_j\}$. Moreover, this can only happen during the second loop, implying that $|B_j| \geq 2$. W.l.o.g., let y be chosen for A_j, say $y = a$, and x be chosen for B_j, say $x = b$. Let b' be the other point chosen for B_j and a' for A_j, if $|A_j| > 1$.

If $|B_j| = 2$, then the triangle $\Delta_{xyb'}$ is consistently oriented in $\overrightarrow{G}$. Applying WSPD-Property 1, there is a path from x to y in $\overrightarrow{G}$ of length

$$|xb'| + |b'y| \leq 2/s \cdot |xy| + (1 + 4/s) \cdot |xy| = (1 + 6/s) \cdot |xy|.$$

If $|B_j| > 2$, then $(y, x), (x, a'), (a', b'), (b', y) \in \overrightarrow{E}$. Applying WSPD-Property 1, $\overrightarrow{G}$ contains a path from x to y of length

$$|xa'| + |a'b'| + |b'y| \leq (1 + 2/s)|xy| + (1 + 4/s)|xy| + (1 + 2/s)|xy| = (3 + 8/s)|xy|.$$

We conclude that, if $(x,y) \in \overrightarrow{E'}$ and $(y,x) \in \overrightarrow{E}$, then $\overrightarrow{G}$ contains a path from x to y whose length is at most $(3+8/s)|xy|$.

We proved the statement for the edge xy. The same argumentation applies for the edges qx and yp. Therefore, the shortest closed walk $C_{\overrightarrow{G}}(p,q)$ in $\overrightarrow{G}$ containing p and q has length

$$|C_{\overrightarrow{G}}(p,q)| \le |pq| + (3+8/s)\cdot(|qx|+|xy|+|yp|) \le (3+8/s)|C_{\overrightarrow{G'}}(p,q)| \le (3+8/s)\cdot 4|\Delta^*(p,q)|.$$

For $s \ge 8$, we have

$$\text{odil}(p,q) \le (3+8/s)\cdot 4 \le 4\cdot 4 = 16.$$

Induction step. Let b (and b', if $|B_i|>1$) be chosen for B_i and let a (and a', if $|A_i|>1$) be chosen for A_i. Note that for $|B_i| \ge 2$, it holds either $(a,b),(b',a) \in \overrightarrow{E}$ or $(b,a),(a,b') \in \overrightarrow{E}$. (The latter is only possible for $|A_i| \le 2$.) Note that, since there is a unique well-separated pair $\{A_i, B_i\}$ with $p \in A_i$ and $q \in B_i$, the orientations of ab and $b'a$ are not changed in any later iteration. We use these edges in a closed walk between A_i and B_i. We distinguish cases based on whether p and/or q are chosen by the algorithm while processing the pair $\{A_i, B_i\}$ (compare to Figure 5).

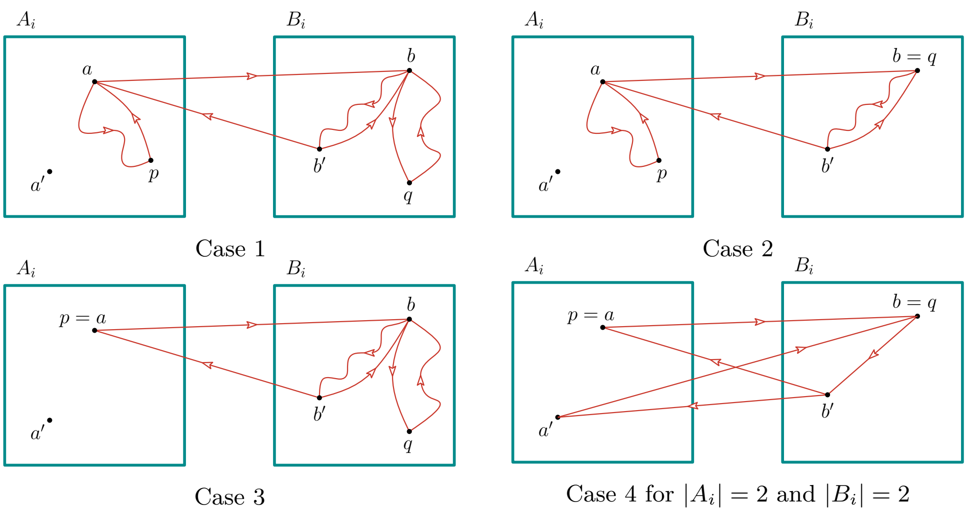

Figure 5 Illustration of cases in the inductions step of Lemma 15.

Case 1. $p \notin \{a,a'\}$ and $q \notin \{b,b'\}$

In this case, it holds that $|A_i| \ge 3$ and $|B_i| \ge 3$. We show

$$|C_{\overrightarrow{G}}(p,q)| \le |C_{\overrightarrow{G}}(p,a)| + |ab| + |C_{\overrightarrow{G}}(b,q)| + |C_{\overrightarrow{G}}(b,b')| + |b'a| \le (1+148/s)|\Delta^*(p,q)|.$$

Due to the order of the pairs (see WSPD-Property 2), we apply induction to bound $|C_{\overrightarrow{G}}(p,a)|$. Since $1+148/s \le 16$ for $s \ge 30$, we bound

$$|C_{\overrightarrow{G}}(p,a)| \le 16|\Delta^*(p,a)|$$

regardless whether p,a is a singleton-singleton pair or not. Analogously, we bound $|C_{\overrightarrow{G}}(b,q)|$ and $|C_{\overrightarrow{G}}(b,b')|$. We apply WSPD-Property 1 to bound $|ab|$ and $|b'a|$. So, we have

$$|C_{\overrightarrow{G}}(p,q)| \le 16|\Delta^*(p,a)| + (1+4/s)|pq| + 16|\Delta^*(b,q)| + 16|\Delta^*(b,b')| + (1+4/s)|pq|.$$

Since $|A_i| \geq 3$, $|\Delta^*(p,a)|$ is at most the perimeter of any triangle with vertices p, a, and one other point in A_i. Each of the three edges of such a triangle can be bounded using WSPD-Property 1. It holds that

$$|\Delta^*(p,a)| \leq 3 \cdot 2/s \cdot |pq| \leq 3/s \cdot |\Delta^*(p,q)|,$$

where the last inequality $|\Delta^*(p,q)| \geq 2|pq|$ follows from the triangle inequality. Analogously, we bound $|\Delta^*(b,q)|$ and $|\Delta^*(b,b')|$ each by $3/s|\Delta^*(p,q)|$. We conclude

$$|C_{\overrightarrow{G}}(p,q)| \leq 3 \cdot 16 \cdot 3/s \cdot |\Delta^*(p,q)| + (1+4/s)|\Delta^*(p,q)| = (1+148/s)|\Delta^*(p,q)|.$$

Case 2. $p \notin \{a,a'\}$ and $q \in \{b,b'\}$
In this case, it holds that $|A_i| \geq 3$ and therefore $|B_i| \geq 3$. W.l.o.g., let $q = b$. We show

$$|C_{\overrightarrow{G}}(p,q)| \leq |C_{\overrightarrow{G}}(p,a)| + |ab| + |C_{\overrightarrow{G}}(b,b')| + |b'a| \leq (1+148/s)|\Delta^*(p,q)|.$$

Since $|A_i| \geq 3$ and $|B_i| \geq 3$, we are in the set up as in Case 1. Therefore, length of $|C_{\overrightarrow{G}}(p,q)|$ in Case 1 is an upper bound for this case.

Case 3. $p \in \{a,a'\}$ and $q \notin \{b,b'\}$
In this case, it holds that $|A_i| \geq 1$ and $|B_i| \geq 3$. W.l.o.g., let $p = a$. We show

$$|C_{\overrightarrow{G}}(p,q)| \leq |ab| + |C_{\overrightarrow{G}}(b,q)| + |C_{\overrightarrow{G}}(b,b')| + |b'a| \leq (1+148/s)|\Delta^*(p,q)|.$$

Since $q \notin \{b,b'\}$, it holds $|B_i| \geq 3$. So, analogously to Case 1, we bound $|C_{\overrightarrow{G}}(b,q)|$ and $|C_{\overrightarrow{G}}(b,b')|$ each by $16 \cdot 3/s|\Delta^*(p,q)|$. We conclude

$$|C_{\overrightarrow{G}}(p,q)| \leq 2 \cdot 16 \cdot 3/s|\Delta^*(p,q)| + (1+4/s)|\Delta^*(p,q)| \leq (1+148/s)|\Delta^*(p,q)|.$$

Case 4. $p \in \{a,a'\}$ and $q \in \{b,b'\}$
In this case, it holds that $|A_i| \geq 1$ and $|B_i| \geq 1$. W.l.o.g., let $p = a$ and $q = b$.
If $|B_i| = 1$ and thus $|A_i| = 1$, then $\{\{p\},\{q\}\}$ is a singleton-singleton pair. Note that for the proof of the base case we only used the fact that $\{\{p\},\{q\}\}$ is a singleton-singleton pair (not that pq is the closest pair). Therefore, the same argumentation as in the base case applies here. Hence, it holds $\text{odil}(p,q) \leq 16$.
For $|B_i| = 2$, the triangle $\Delta_{abb'}$ (and the triangle $\Delta_{a'bb'}$, if $|A_i| = 2$) is oriented consistently when processing the pair $\{A_i, B_i\}$. Note that, if bb' is oriented already, this does not change its orientation. Let $\{A_j, B_j\}$ be the pair such that $b \in A_j$ and $b' \in B_j$ or $b' \in A_j$ and $b \in B_j$. Due to the order of the pairs, it holds $j < i$ (WSPD-Property 2). Therefore, the orientation of bb' is not changed in any later iteration and thus $\Delta_{abb'}$, consistently oriented in $\overrightarrow{G}$, is a closed walk in $\overrightarrow{G}$ containing p and q. Applying WSPD-Property 1, we have

$$\begin{aligned}|C_{\overrightarrow{G}}(p,q)| &\leq |\Delta_{abb'}| = |ab| + |bb'| + |b'a| \\ &\leq (1+4/s)|pq| + 2/s|pq| + (1+2/s)|pq| \leq (1+4/s)|\Delta^*(p,q)|.\end{aligned}$$

If $|B_i| \geq 3$, then, analogously to Case 3, we bound

$$\begin{aligned}|C_{\overrightarrow{G}}(p,q)| &\leq |ab| + |C_{\overrightarrow{G}}(b,b')| + |b'a| \\ &\leq 16 \cdot 3/s|\Delta^*(p,q)| + (1+4/s)|\Delta^*(p,q)| \leq (1+148/s)|\Delta^*(p,q)|. \quad \blacktriangleleft\end{aligned}$$

▶ **Theorem 16.** *Let $(P, |\cdot|)$ be a metric space with n points and constant doubling dimension* d. *Given a sufficiently large number s, an oriented graph for P with average dilation $1+\mathcal{O}(1/s)+s^{\mathcal{O}(\mathrm{d})}/n$ and $s^{\mathcal{O}(\mathrm{d})}n$ edges can be constructed in $s^{\mathcal{O}(\mathrm{d})}n\log n$ time using $s^{\mathcal{O}(\mathrm{d})}n$ space.*

Proof. Let $\overrightarrow{G}$ be the graph returned by Algorithm 3. Due to Lemma 15, we have $\mathrm{odil}_{\overrightarrow{G}}(p,q) \leq 16$ for a singleton-singleton pair and $\mathrm{odil}_{\overrightarrow{G}}(p,q) \leq 1+\mathcal{O}(1/s)$ otherwise. Let $m = s^{\mathcal{O}(\mathrm{d})}n$ be the total number of pairs and $m' \leq m$ the number of singleton-singleton pairs. It holds

$$\begin{aligned}\sum_{p\neq q}\mathrm{odil}(p,q) &\leq 16m' + \left(\binom{n}{2} - m'\right)(1+\mathcal{O}(1/s)) \leq 16m + \binom{n}{2}(1+\mathcal{O}(1/s))\\ &= s^{\mathcal{O}(\mathrm{d})}n + (1+\mathcal{O}(1/s))\binom{n}{2}.\end{aligned}$$

Therefore, the average dilation of $\overrightarrow{G}$ is

$$\frac{1}{\binom{n}{2}}\sum_{p\neq q}\mathrm{odil}(p,q) \leq 1+\mathcal{O}(1/s)+s^{\mathcal{O}(\mathrm{d})}/n.$$

Computing a WSPD costs $s^{\mathcal{O}(\mathrm{d})}n + 2^{\mathcal{O}(\mathrm{d})}n\log n$ time and $s^{\mathcal{O}(\mathrm{d})}n$ space [17]. We use the data structure of Cole and Gottlieb [10] to approximate nearest neighbors. Its construction takes $2^{\mathcal{O}(\mathrm{d})}n\log n$ time and it uses $\mathcal{O}(n)$ space. For each of the $s^{\mathcal{O}(\mathrm{d})}n$ singleton-singleton pairs, we approximate $\Delta^*(p,q)$ in $s^{\mathcal{O}(\mathrm{d})}\log n$ time (Lemma 12). Therefore, computing the list L of $s^{\mathcal{O}(\mathrm{d})}n$ triples costs $s^{\mathcal{O}(\mathrm{d})}n\log n$ time and orienting those costs $s^{\mathcal{O}(\mathrm{d})}n\log n$ time (Lemma 14). We sort the $s^{\mathcal{O}(\mathrm{d})}n$ pairs of the WSPD in $s^{\mathcal{O}(\mathrm{d})}n\log n$ time. For each of the $s^{\mathcal{O}(\mathrm{d})}n$ non-singleton-singleton pairs, we add oriented edges in $\mathcal{O}(1)$ time. Therefore, Algorithm 3 takes $s^{\mathcal{O}(\mathrm{d})}n\log n$ time and $s^{\mathcal{O}(\mathrm{d})}n$ space. ◀

Depending on the choice of s, Theorem 16 offers different guarantees. For example, if ε is a constant and $s = \Theta(1/\varepsilon)$, the average dilation is $1+\varepsilon+\mathcal{O}(1/n)$ and the number of edges is $\mathcal{O}(n)$. If $s = n^{1/(\mathrm{d}+1)}$, the average dilation is $1+\mathcal{O}(1/n^{1/(\mathrm{d}+1)})$ and the number of edges is $\mathcal{O}(n^{(2\mathrm{d}+1)/(\mathrm{d}+1)}) = o(n^2)$. If $s = (\log\log n)^{1/\mathrm{d}}$, the average dilation is $1+\mathcal{O}(1/(\log\log n)^{1/\mathrm{d}}) = 1+o(n)$ and the number of edges is $\mathcal{O}(n\log\log n)$.

6 Discussion

Tournaments. We presented an algorithm that returns a tournament on a given metric point set with dilation $5/3$ in cubic time.

This improves the known bounds on the worst-case dilation t of the minimum dilation spanner to $2\sqrt{3}-2 \leq t \leq 5/3$ for Euclidean space (compare to [4]) and $3/2 \leq t \leq 5/3$ for any metric and doubling space (compare to [6]). It remains open whether this gap can be closed.

The authors in [5] prove that, given an arbitrary undirected graph, it is NP-hard to compute a minimum-dilation orientation, but this proof does not apply to complete graphs. The hardness of computing a minimum dilation tournament is an open problem.

In the tournament returned by our algorithm, for any two points p, q there exists an oriented cycle through p and q, whose length is at most $5/3|\Delta^*(p,q)|$ and which has at most 4 edges. The latter is tight, since there are point sets where every minimum dilation tournament has a pair whose shortest closed walk has four edges. The following question arises: Is there a minimum dilation tournament on every point set such that the shortest oriented cycle for every pair has either 3 or 4 edges?

Spanners of linear size. To obtain sparse oriented spanners with dilation $5/3+\varepsilon$, we showed how to sparsify a tournament with dilation t while increasing the dilation to at most $t+\varepsilon$. This makes computing minimum dilation tournaments even more relevant.

However, relying on tournaments inherently leads to algorithms that use quadratic space and at least quadratic running time. A natural question is whether this can be avoided. Can we compute a sparse oriented spanner with dilation less than 2 using subquadratic space? Or even using subquadratic (or for now, merely subcubic) running time?

We generalized the $(2+\varepsilon)$-spanner from [5] to doubling spaces. More generally, the algorithm applies to metric spaces that admit a WSPD, support approximate nearest-neighbor queries on a dynamic point set, and have restricted growth. It would be interesting to study oriented spanners in other metrics with these properties.

Our construction of a spanner with average dilation $1+\varepsilon$ shows that the main obstacle for the $(2+\varepsilon)$-spanner lies in singleton-singleton pairs. Our 5/3-tournament is not applicable, because our proof of its dilation critically depends on the orientation of a complete graph.

Average dilation spanners. We present an algorithm for computing a sparse graph with oriented average dilation $1+\varepsilon$ for sufficiently large point sets in doubling spaces. Besides computing oriented spanners with small average dilation, there are other intriguing problems related to average dilation. For example, as done in [9] for undirected cycles and plane graphs, an interesting open problem is to compute or approximate the average dilation of a given (oriented) graph.

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