Online Coloring for Graphs of Large Odd Girth

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Abstract

We study the problem of online coloring for graphs with large odd girth. The best previously known algorithm uses $O(n^{1/2})$ colors, which was discovered by Kierstead in 1998. This algorithm works when the odd girth is 7 or more. In this paper, we provide the following: for every $\varepsilon > 0$, there exists a constant $g' \in \{3, 5, 7, \dots\}$ such that graphs with odd girth at least g' can be deterministically colored online using $O(n^{\varepsilon})$ colors.

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1 Introduction

We study the *online coloring* problem, which concerns online algorithms for graph coloring. In this setting, the vertices are revealed one by one (along with adjacent edges to already arrived vertices), and we must assign each vertex a color immediately after it is revealed, so that its color differs from the colors of all its neighbors. The goal is to design a strategy that minimizes the number of colors used. See Figure 1 for an example.

However, since the algorithm has no knowledge of future vertices, it requires a clever strategy to color the graph effectively. For example, one strategy is to assign the least-indexed available color to each arriving vertex. This strategy is called First-Fit [10]. However, it costs $\frac{1}{2}n$ colors even for bipartite graphs (Figure 2), which shows the ineffectiveness of First-Fit.

The problem is known to be difficult for general graphs, where a deterministic algorithm that achieves a competitive ratio[1] of $O(\frac{n}{\log \log n})$ was very recently given by Kawarabayashi, Yoneda, Yoneda [14]. In contrast, the competitive ratio of any algorithm is $\Omega(\frac{n}{(\log n)^2})$, due to Halldórsson and Szegedy [11].

Given strong negative results for general graphs, effective online coloring algorithms for special graph classes have also received attention. For example, for tree, bipartite, planar, bounded-treewidth, or disk graphs, deterministic algorithms that use $O(\log n)$ colors are known [3, 20, 13, 2]. For interval graphs, a deterministic algorithm with a competitive ratio of 3 is given by Kierstead and Trotter [17].

[1] The *competitive ratio* for an algorithm is the maximum value of "the algorithm's performance" (e.g., the

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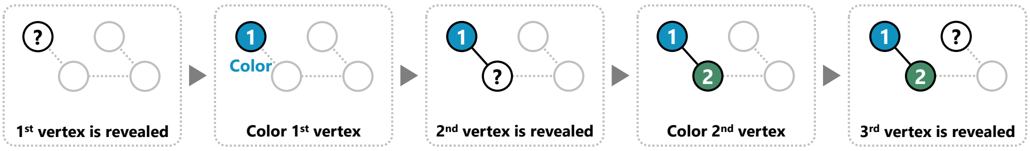

Figure 1 An example of online coloring. At each point, only black vertices and edges are revealed, and gray vertices and edges are unknown.

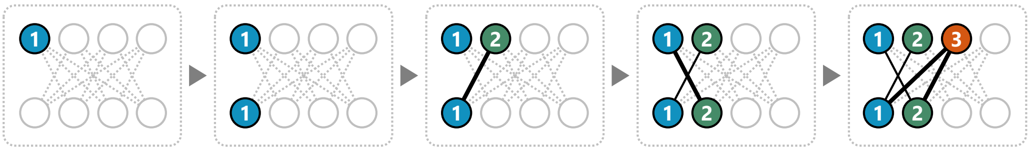

Figure 2 A worst-case input for the First-Fit algorithm, which uses $\frac{1}{2}n$ colors for bipartite graphs, where n is the number of vertices. The number written inside each vertex is the color. The gray dotted lines are the edges of the graph which we still do not know at each point.

1.1 The Odd Girth: Almost Bipartite Graphs

In graph theory, graphs with large odd girth have been widely studied. The odd girth g is the length of the shortest odd cycle in a graph. This is closely connected to the chromatic number – as suggested by the intuition that graphs with large odd girth can be interpreted as "almost bipartite graphs" (cf. bipartite graphs have no odd cycle).

The chromatic number of graphs with odd girth at least 5 (i.e., triangle-free graphs) is a classical problem in graph theory, which is also studied as the asymptotics of the Ramsey number $R(3, k)$. Mycielski (1955) [21] introduced a construction for such graphs with arbitrarily large chromatic number. Later, the maximum chromatic number of such graphs was shown to be $\Theta(\sqrt{n/\log n})$, where the lower bound is given by Ajtai, Komlós, Szemerédi (1980) [1] and the upper bound is given by Kim (1995) [18]. On the other hand, the existence of graphs with an arbitrarily large girth (not just "odd girth") and an arbitrarily large chromatic number was shown by the famous result of Erdős (1959) [7]. This was a landmark result that established the proof technique of probabilistic method.

To the best of our knowledge, the best-known bounds on the chromatic number of graphs with general constant odd girth $g \geq 5$ are $O((\frac{n}{\log n})^{\frac{2}{g-1}})$ by Denley (1994) [5], and the existence of graphs with chromatic number $\Omega(n^{\frac{1}{g-3}}/\log n)$ was given by Krivelevich (1995) [19, Theorem 1].

1.2 Previous Results

Graphs with odd girth g had been extensively studied also in online coloring.

Algorithms. For the case $g = 5$ (triangle-free graphs), the best-known deterministic algorithm by Bradač et al. (2026) [4] uses $O(\frac{n \log\log n}{\log n})$ colors. For the case $g = 7$ ((C_3, C_5)-free graphs), Kierstead (1998) [15] presented a deterministic algorithm that uses $O(\sqrt{n})$ colors. For general odd girth $g \geq 9$, Nagy-György (2010) [22] presented a deterministic algorithm that uses $O(\sqrt{\frac{n \log g}{g}})$ colors. However, when g is constant, it is still $O(\sqrt{n})$ colors.

number of colors used) divided by "the optimal value" (e.g., chromatic number) for all possible inputs.

Table 1 Comparison of our algorithm with previous results by odd girth. c $(0 < c < 1)$ is a constant.

Layers k	0	1	2	3	$\cdots$	$c\log_5 n$
Odd Girth	≥ 7	≥ 29	≥ 139	≥ 689	$\cdots$	$\geq \Omega(n^c)$
Previous [15, 22]	$O(n^{1/2})$	$O(n^{1/2})$	$O(n^{1/2})$	$O(n^{1/2})$	$\cdots$	$O(n^{(1-c)/2}\sqrt{\log n})$
Our Result	$O(n^{1/2})$	$O(n^{2/5})$	$O(n^{1/3})$	$O(n^{2/7})$	$\cdots$	$O(\log n)$

Lower bounds. It is known that any deterministic algorithm uses $\Omega(n^{1/2})$ colors for the case $g = 5$ [6] and $\Omega((\frac{n}{\log n})^{1/3})$ colors for the case $g = 7$ [9]. For $g \geq 9$, the chromatic number lower bound by Krivelevich [19] remains the best known.

A large gap remains between the upper and lower bounds. A natural question is whether the 28-year-old square-root barrier can be broken and $O(n^{1/2-\varepsilon})$ colors can be achieved for some $\varepsilon > 0$. If so, another question is how much further the exponent can be improved.

1.3 Our Contributions

In this paper, we answer the 28-year-old question affirmatively with the following theorem:

▶ **Theorem 1.** *For every $\varepsilon > 0$, there exists a constant $g' \in \{3, 5, 7, \dots\}$ such that graphs with odd girth at least g' can be deterministically colored online using $O(n^{\varepsilon})$ colors.*

More precisely, we present a deterministic online algorithm that colors graphs with odd girth $g \geq \frac{11}{2} \cdot 5^k + \frac{3}{2}$ using $O(k \cdot n^{\frac{2}{k+4}})$ colors, for every integer $k \geq 0$ (Theorem 23). This significantly improves the previous result of $O(n^{1/2})$ colors by Kierstead (1998) [15].

Our result also achieves $O(\log n)$ colors for graphs with odd girth $g = \Omega(n^c)$, where c $(0 < c < 1)$ is a constant (Corollary 24). This substantially improves the previous result of $O(n^{(1-c)/2}\sqrt{\log n})$ colors by Nagy-György (2010) [22] and matches the $\Theta(\log n)$-color bound for bipartite graphs [20]. Table 1 shows the relationship between the odd girth and the number of colors used.

For online coloring in general, the bottleneck to achieving $O(n^{1/2-\varepsilon})$ colors in many algorithms, including Kierstead's $O(\sqrt{n})$-color algorithm for graphs of odd girth $g \geq 7$ and Vishwanathan's $O(\sqrt{n})$-color *randomized* algorithm for 3-colorable graphs [23], is that they only utilize the structure of the (first) neighborhood.

Our algorithm makes use of the second neighborhood, and as a result, we achieve $O(n^{2/5})$ colors for graphs of odd girth $g \geq 29$ (Theorem 19). In addition, we generalize this result to the third neighborhood and further neighbors, to obtain our main result (Theorem 1). The challenge is that when each vertex has degree at least Δ, the size of the k-th neighborhood may not be $\Omega(\Delta^k)$; in the worst case, the size can be as small as $O(\Delta)$. We overcome this issue by using a novel framework called *online group coloring*, which allows us to apply an approach similar to First-Fit and benefit in cases where these neighborhoods are small.

1.4 Odd Girth vs. Girth

Some readers may wonder how many colors are needed for online coloring of graphs with girth g (instead of odd girth). In this case, First-Fit performs effectively, where the following result was shown by Zaker (2007) [24]. We provide a simpler proof here:

▶ **Proposition 2** ([24]). *Let $k \geq 1$ be an integer. For a graph $G = (V, E)$ of girth $g \geq 2k + 1$, First-Fit uses only $O(k \cdot n^{1/k})$ colors.*

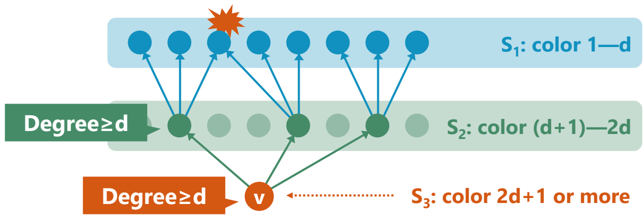

Figure 3 A sketch of our analysis for the case $k = 2$. If the endpoints of two paths clash, a cycle of length at most 4 is formed, a contradiction. Hence, there must be at least d^2 vertices in S_1.

Proof. Let $d = \lceil n^{1/k} \rceil$. For each $i = 1, \ldots, k$, let S_i be the set of vertices that are colored by colors $(i-1)d+1, \ldots, id$. Also, let $S_{k+1} = V \setminus (S_1 \cup \cdots \cup S_k)$. Then, for each $i = 2, \ldots, k+1$ and $v \in S_i$, we have $|N_{S_{i-1}}(v)| \geq d$. This is because v must be adjacent to some vertices with each of color $(i-2)d+1, \ldots, (i-1)d$ by the definition of First-Fit.

Suppose there exists a vertex v in S_{k+1}. Then, there are at least d^k paths of length k that start from v and pass through a vertex in $S_k, \ldots, S_1$ in this order. For all of these paths, the endpoints (in S_1) are distinct. Otherwise, in the two such paths with a common endpoint, a cycle of length at most $2k$ would appear, which contradicts the girth requirement. Therefore, $|S_1| \geq d^k$ $(\geq n)$ must hold (Figure 3). However, we also have $|S_1| \leq n-1$, a contradiction. Thus, S_{k+1} must be empty, meaning that First-Fit uses at most $kd = O(k \cdot n^{1/k})$ colors. ◀

We note that the above result matches the best-known upper bound $O((\frac{n}{\log n})^{1/k})$ for chromatic number of graphs with girth $g \geq 2k+1$, given by Denley [5], up to poly-logarithmic factors. Also, one can derive from the standard Moore bound argument [12, 8] that any graph of girth $g \geq 2k+1$ is an $n^{1/k}$-degenerate graph. Hence, the result by Irani [13] that First-Fit only uses $O(d \log n)$ colors for d-degenerate graphs implies the bound of $O(n^{1/k} \log n)$ colors. Proposition 2 is better by a factor of $\log n$.

In contrast, for graphs with large odd girth, First-Fit may perform much worse, since it can use $\Theta(n)$ colors even for bipartite graphs (see Figure 2). This is a fundamental difference between "girth" and "odd girth" in the context of online coloring.

1.5 Organization

Section 2 presents the preliminaries needed for the proof. Section 3 presents the online group coloring problem and its solution, which is a key tool in our algorithm. Section 4 presents an algorithm that uses $O(n^{2/5})$ colors for graphs with odd girth $g \geq 29$. Section 5 extends the idea to multiple layers and obtains our main result, Theorem 1.

2 Preliminaries

Before moving on to our algorithm, we present the notation and definitions, along with Kierstead's $O(\sqrt{n})$-color algorithm for (C_3, C_5)-free graphs.

2.1 Notation and Definitions

First, we formally define the online coloring problem.

▶ **Definition 3** ([14]). *In the online coloring problem, an undirected graph $G = (V, E)$ is initially empty; in other words, there are zero vertices. Vertices are added to G one by one, together with their incident edges. Immediately after a vertex v is added (along with its* *incident edges), we must assign a color $c(v) \in \mathbb{N}$ to v. This color must differ from that of all its neighbors. The next vertex is presented only after assigning a color. The goal is to minimize the number of colors used, that is, $|\{c(v) : v \in V\}|$.*

For notation, given a graph $G = (V, E)$, we denote the set of vertices adjacent to $v \in V$ as $N(v)$ or $N_G(v)$, and the set of vertices adjacent to at least one vertex in $S \subseteq V$ as $N(S)$ or $N_G(S)$. Note that when describing online coloring algorithms, notation such as $N(v)$ and $N(S)$ is taken from the current graph (not the final graph). We also define the *odd girth* along with its useful property.

▶ **Definition 4.** *The odd girth of a graph $G = (V, E)$ is the length of the shortest odd-length cycle in G, or $+\infty$ if no such cycle exists. The odd girth is also equal to the length of the shortest odd-length closed walk in G.*

In the following, we define the notion of *even-diameter* in a graph, a concept that is used extensively throughout our online coloring algorithms.

▶ **Definition 5.** *Let $G = (V, E)$ be a graph. The "even-distance" between two vertices $s, t \in V$ is the length of the shortest even-length walk between s and t, or $+\infty$ if no such walk exists. The "even-diameter" of a vertex set $S \subseteq V$ is the maximum even-distance between any two vertices in S. Note that the walks are allowed to pass through vertices outside S.*

2.2 Kierstead's Algorithm

Next, we review Kierstead's algorithm for graphs without 3-cycles and 5-cycles. First, it is a standard assumption in online coloring that the final number of vertices n is known in advance [16, 14]. This is justified by the following lemma.

▶ **Lemma 6** ([14]). *Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a non-decreasing unbounded function. For a specific graph class (e.g., graphs with odd girth at least 7), suppose there exists a deterministic online coloring algorithm for n-vertex graphs that uses at most $f(n)$ colors, for every $n \in \mathbb{N}$. Then there exists a deterministic online coloring algorithm for this graph class that uses at most $4f(n)$ colors, even when the final number of vertices, n, is unknown.*

▶ **Theorem 7** ([15]). *For graphs of odd girth at least 7, there is a deterministic online coloring algorithm that uses $O(n^{1/2})$ colors.*

Proof. The idea of his algorithm is to divide vertices into "good" sets $X_1, X_2, \ldots$ $(\subseteq V)$, where for each i, the vertices of X_i are adjacent to a common vertex $w_i \in V$. The convenience of these sets is that a group of vertices adjacent to at least one vertex in X_i, say Y_i $(\subseteq V)$, can be colored with the same color (we color them with color $\lceil n^{1/2} \rceil + i$). Indeed, if two such vertices $y_1, y_2 \in Y_i$ are adjacent, this forms a closed walk of length 5: $y_1 \to x_1 \to w_i \to x_2 \to y_2 \to y_1$, where x_j is a vertex in X_i adjacent to y_j (for $j = 1, 2$). This contradicts the assumption of odd girth ≥ 7.

Next, we describe the algorithm. Let $X_1, \ldots, X_r$ be the current list of these "good" sets. Initially, $r = 0$. When a new vertex $v \in V$ arrives, we perform the following procedure:

1. If v can be colored by any of colors $1, \ldots, \lceil n^{1/2} \rceil$, color v using First-Fit (Figure 4a).
2. Otherwise, if there exists some i such that $X_i \cap N(v) \neq \emptyset$, we pick one such i and color v with color $\lceil n^{1/2} \rceil + i$ (Figure 4b). This also means that v is added to group Y_i (i.e., $Y_i \leftarrow Y_i \cup \{v\}$).
3. Otherwise, we create a new "good" set $X_{r+1} := N(w_{r+1})$ with $w_{r+1} := v$ and then we color v with color $\lceil n^{1/2} \rceil + (r + 1)$ (Figure 4c). This also means that v is added to group Y_{r+1} (i.e., $Y_{r+1} \leftarrow Y_{r+1} \cup \{v\}$). This increases r by 1.

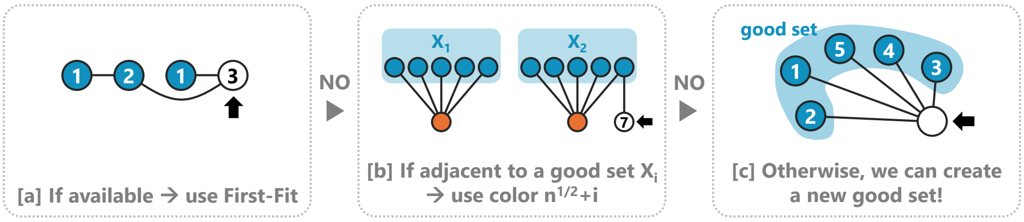

Figure 4 A sketch of Kierstead's algorithm when $\lceil n^{1/2} \rceil = 5$. The white vertex with an arrow is a newly arrived vertex v that is to be colored. The numbers inside the vertices are the assigned colors.

For each i, we have $|X_i| \geq \lceil n^{1/2} \rceil$ since w_i is not colored in Step 1 (First-Fit). Also, $X_j \cap X_i = \emptyset$ for each $j = 1, \ldots, i-1$ since w_i is not colored in Step 2. Therefore, $r \leq n^{1/2}$; we use only $\lceil n^{1/2} \rceil + r = O(n^{1/2})$ colors in total. ◀

3 Online Group Coloring

In this section, we define a novel problem, *online group coloring*, which is a key tool in our algorithm. This is a variant of online coloring where vertices are divided into groups.

▶ **Definition 8.** *The online group coloring problem is a special case of online coloring where each vertex v belongs to a group $g(v) \in \mathbb{N}$. The group to which v belongs is revealed at the time of its arrival. As in online coloring, we need to color v immediately. It is guaranteed that there are no edges between vertices in the same group.*

This problem specifies a parameter $\Delta \geq 0$. The "neighbor graph" H is defined as follows: each group forms a vertex, and there is an edge between two groups if and only if at least one edge exists between them in G. When a vertex v arrives, the degree of $g(v)$ in H is guaranteed to be at most Δ (it is allowed for the degrees of other vertices to exceed Δ as a result).

▶ **Theorem 9.** *There exists a deterministic algorithm that uses at most $\Delta^2 + 2$ colors for online group coloring.*

Proof. The strategy is to try to keep using the same color inside each group, but if this color is already used in adjacent groups (in the neighbor graph H), we switch to a new color. For each group i, let c_i be the current color for this group, and let L_i be the set of colors ever used for this group. For convenience, we initially set $c_i = 1$ and $L_i = \{1\}$.

When a vertex $v \in V$ arrives, we normally color v with color $c_{g(v)}$. However, this color may already be used in adjacent groups, i.e., when $c_{g(v)} \in \bigcup_{i \in N_H(g(v))} L_i$. In this case, we update $c_{g(v)}$ to the smallest color not in $\bigcup_{i \in N_H(g(v))} L_i$, add this new color to $L_{g(v)}$, and then color v with the new color $c_{g(v)}$. It is easy to see that the same color is not used for adjacent vertices.

Next, we analyze the number of colors used. First, each L_i is updated only if, compared to the time when the previous vertex in group i arrived, either of the following holds: (1) a group newly becomes adjacent to group i (in H), or (2) L_j for an already adjacent group j (in H) newly starts to include c_i. However, case (2) means that c_j was updated to the same color as c_i after that time, but c_i was already contained in L_i, so this cannot occur. Also, L_i will not be updated after the final vertex in group i is processed. Hence, L_i is updated at most Δ times, so $|L_i \setminus \{1\}| \leq \Delta$. Thus, for each arrival of $v \in V$:

$$c_{g(v)} \leq \left| \bigcup_{i \in N_H(g(v))} L_i \right| + 1 = \left| \bigcup_{i \in N_H(g(v))} (L_i \setminus \{1\}) \right| + 2 \leq \sum_{i \in N_H(g(v))} |L_i \setminus \{1\}| + 2$$

which is at most $\Delta^2 + 2$ by the degree constraint $|N_H(g(v))| \leq \Delta$. ◀

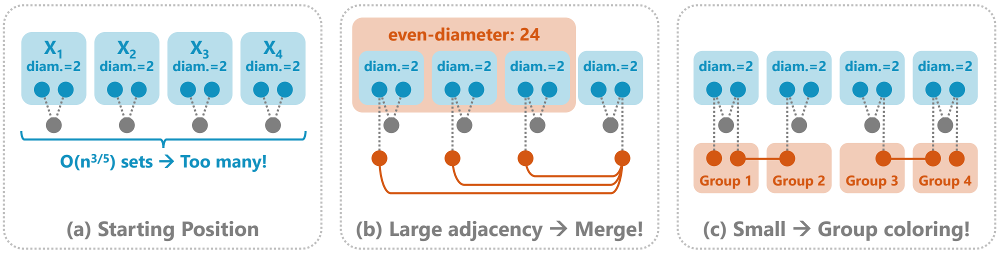

Figure 5 A sketch of our method. (a) In Kierstead's algorithm where we use $\lceil n^{2/5} \rceil$ colors for First-Fit, we need to handle $O(n^{3/5})$ sets. (b) When a group is adjacent to many groups, we merge these groups to form a large set with even-diameter ≤ 24. (c) When a group is adjacent to few groups, we use the procedure of online group coloring.

4 The Algorithm with $O(n^{2/5})$ colors

In this section, we present an algorithm that uses $O(n^{2/5})$ colors for graphs with odd girth ≥ 29. This improves the previous $O(n^{1/2})$-color bound due to Kierstead [15].

4.1 Overview

In Subsection 2.2, Kierstead's algorithm covered V with "good" sets $X_1, \dots, X_r$, where the even-diameter of each X_i is only 2. His observation was that with odd girth ≥ 7, the entire group Y_i can be colored using the same color.

In this section, as we consider graphs with odd girth ≥ 29, the requirements for "good" sets are relaxed: their even-diameters only need to be at most 24. Hence, our goal is to cover V with $O(n^{2/5})$ large sets of even-diameter ≤ 24. Below, we sketch the intuition for our algorithm to achieve this goal.

- We attempt to directly improve Kierstead's algorithm to $O(n^{2/5})$ colors. Since only $n^{2/5}$ colors can be used for First-Fit, $r = O(n^{3/5})$ sets $X_1, \dots, X_r$ of even-diameter 2 are created in total, each having size $\Omega(n^{2/5})$ (Figure 5a). However, since we need to color $Y_1, \dots, Y_r$ within $O(n^{2/5})$ colors, we must use the same color across multiple groups Y_i.
- The key observation is that when a group Y_i is adjacent to at least $\Omega(n^{1/5})$ other groups Y_j, we can "merge" these neighboring even-diameter-2 sets X_j to form a larger set of even-diameter ≤ 24, of size $\Omega(n^{3/5})$ (Figure 5b). In this way, we can cover V with $O(n^{2/5})$ sets of even-diameter ≤ 24.
- On the other hand, when each group has only $O(n^{1/5})$ adjacent groups, the coloring procedure reduces to the *online group coloring* problem with parameter $\Delta := \Theta(n^{1/5})$. This enables us to use the same color across different groups (Figure 5c).

4.2 Formalization

As a foundation for building our algorithm, we formalize the procedure we must perform for coloring non-First-Fit vertices in Kierstead's algorithm in a generalized manner. We define the subproblem "(r^*, d^*)-subroutine" as follows. Note that in Kierstead's algorithm, the procedure for coloring non-First-Fit vertices is equivalent to the $(n^{1/2}, 2)$-subroutine.

▶ **Definition 10.** *We consider the (r^*, d^*)-subroutine, where $r^* \in \mathbb{R}_{\geq 0}$ and $d^* \in \{2, 4, 6, \dots\}$ are parameters. In this subroutine, we maintain sets $X_1, \dots, X_r \subseteq V$ called "bases" and sets $Y_1, \dots, Y_r \subseteq V$ called "groups". Initially, there are no such sets, i.e., $r = 0$. The following types of events may occur in an arbitrary order (see also Figure 6):*

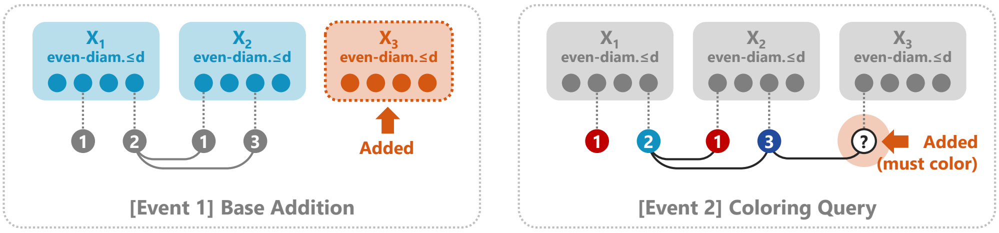

Figure 6 Two types of events in the (r^*, d^*)-subroutine. The number of bases added is guaranteed to be at most r^*. Note that the vertices in X_i can be regarded as "uncolored" because we assume that their colors are disjoint from the colors we use in $Y_1 \cup \cdots \cup Y_r$.

Base addition. *A new base $X_{r+1} \subseteq V$ is added. It is guaranteed that the even-diameter of X_{r+1} is at most d^* (in the graph G for online coloring). The corresponding group Y_{r+1} is initially empty. This increases r by 1.*

Coloring query. *A vertex $v \in V$ is added to a group Y_i. It is guaranteed that v is adjacent to at least one vertex in X_i. Then, we need to color v so that its color differs from the colors of all adjacent vertices in $Y_1 \cup \cdots \cup Y_r$. (We ignore the coloring on $V \setminus (Y_1 \cup \cdots \cup Y_r)$ because we assume that their colors are disjoint from the colors in $Y_1 \cup \cdots \cup Y_r$.)*

It is guaranteed that at most r^* bases are added, i.e., $r \leq r^*$. The objective is to minimize the number of colors used across all coloring queries. Note that the bases $X_1, \dots, X_r$ are not assumed to be disjoint, but the groups $Y_1, \dots, Y_r$ are clearly disjoint by definition.

4.3 Dividing the Problem

We solve the online coloring problem for graphs with odd girth ≥ 29 using the following strategy. Part A and Part C can be solved similarly to Kierstead's algorithm. The key difficulty lies in Part B, where we use online group coloring and merge bases of even-diameter 2 to obtain a large set with even-diameter ≤ 24.

Part A. Show that, given an algorithm for the $(n^{3/5}, 2)$-subroutine using q colors, online coloring can be solved using $q + O(n^{2/5})$ colors.

Part B. Show that, given an algorithm for the $(n^{2/5}, 24)$-subroutine using q colors, the $(n^{3/5}, 2)$-subroutine can be solved using $q + O(n^{2/5})$ colors.

Part C. Solve the $(n^{2/5}, 24)$-subroutine using $O(n^{2/5})$ colors.

First, we solve Part A. Intuitively, if we only use $\lceil n^{2/5} \rceil$ colors for First-Fit in Kierstead's algorithm, the problem of coloring the remaining vertices falls to the $(n^{3/5}, 2)$-subroutine. Below, we formally prove this fact.

▶ **Lemma 11.** *For graphs with odd girth at least 29, if there exists a deterministic algorithm $\mathcal{A}$ that solves the $(n^{3/5}, 2)$-subroutine with $O(n^{2/5})$ colors, then there exists a deterministic online coloring algorithm with $O(n^{2/5})$ colors.*

Proof. We use an algorithm very similar to Theorem 7. To solve online coloring, we use the $(n^{3/5}, 2)$-subroutine, denoted by $\mathcal{S}_0$. We use disjoint color palettes for $\mathcal{S}_0$ and for First-Fit (i.e., we use colors $\lceil n^{2/5} \rceil + 1$ and above in $\mathcal{S}_0$). Whenever a vertex $v \in V$ arrives, we perform the following procedure:

1. If v can be colored by any of colors $1, \dots, \lceil n^{2/5} \rceil$, we color v using First-Fit.

Algorithm 1 Constructing an online coloring algorithm from an algorithm for the $(n^{3/5}, 2)$-subroutine, with $O(n^{2/5})$ additional colors.

```
1: S_0 ← the (n^{3/5}, 2)-subroutine
2: V_FF ← ∅
3: for each arrival of v ∈ V do
4:     if v can be colored by any of colors 1, ..., ⌈n^{2/5}⌉ then
5:         V_FF ← V_FF ∪ {v}, and color v with First-Fit
6:     else if v is adjacent to any existing base in S_0 then
7:         color v inside S_0 (by adding v to S_0 as a coloring query)
8:     else
9:         add N(v) as a base for S_0
10:        go to lines 6–7 to color v (because v neighbors a new base N(v))
```

2. Otherwise, if v is adjacent to any existing base in $\mathcal{S}_0$, we color v inside $\mathcal{S}_0$. Formally, we pick one such base and add v to the corresponding group for this base in $\mathcal{S}_0$ (as a coloring query). This means that we use the algorithm $\mathcal{A}$ to color it.
3. Otherwise, we add $N(v)$ as a base for $\mathcal{S}_0$. After that, we can color v by going back to Step 2, because v is adjacent to a new base $N(v)$.

The pseudocode for this algorithm is given in Algorithm 1. The input to the $(n^{3/5}, 2)$-subroutine is valid, because (1) each base added to $\mathcal{S}_0$ clearly has even-diameter ≤ 2, and (2) the number of bases created is at most $\frac{n}{\lceil n^{2/5} \rceil} \leq n^{3/5}$, shown by the same argument as Theorem 7. For number of colors used, we use $\lceil n^{2/5} \rceil$ colors for First-Fit, and another $O(n^{2/5})$ colors in $\mathcal{S}_0$ (using the algorithm $\mathcal{A}$). Thus, we use $O(n^{2/5})$ colors in total. ◀

Next, we solve Part C. Intuitively, when the odd girth is ≥ 29, we can use the same color for each group as in Step 2 of Kierstead's algorithm. Below, we formally prove this fact.

▶ **Lemma 12.** *For graphs with odd girth at least 29, there exists a deterministic algorithm that solves the $(n^{2/5}, 24)$-subroutine with $n^{2/5}$ colors.*

Proof. Given that each base X_i has even-diameter ≤ 24, no two vertices in group Y_i are adjacent. Suppose that $y_1, y_2 \in Y_i$ are adjacent. Let x_j be a vertex in X_i that is adjacent to y_j (for $j = 1, 2$). Then there exists a closed walk $y_1 \to x_1 \to \cdots \to x_2 \to y_2 \to y_1$, where $x_1 \to \cdots \to x_2$ is an even-length walk of length at most $d^* = 24$. This is an odd closed walk of length at most $d^* + 3 = 27$, which contradicts the assumption of odd girth ≥ 29.

Therefore, when a vertex v is put into group Y_i in a coloring query, we can color v with color i. We use at most $r^* = n^{2/5}$ colors in total. ◀

4.4 The Main Algorithm

It remains to solve Part B, that is, to solve the $(n^{3/5}, 2)$-subroutine. We solve this part using the $(n^{2/5}, 24)$-subroutine, which is denoted as $\mathcal{S}_1$, and the online group coloring problem where "group i" is used for a subset of vertices in Y_i. Note that we use disjoint color palettes for $\mathcal{S}_1$ and for online group coloring. To outline the strategy, when a vertex v arrives in a coloring query, we color it in the following way:

Case 1. We color v inside the $(n^{2/5}, 24)$-subroutine $\mathcal{S}_1$ whenever it is possible.

Case 2. Otherwise, we color v in either of the following ways: (a) by putting it into online group coloring, or (b) by creating new bases for the $(n^{2/5}, 24)$-subroutine.

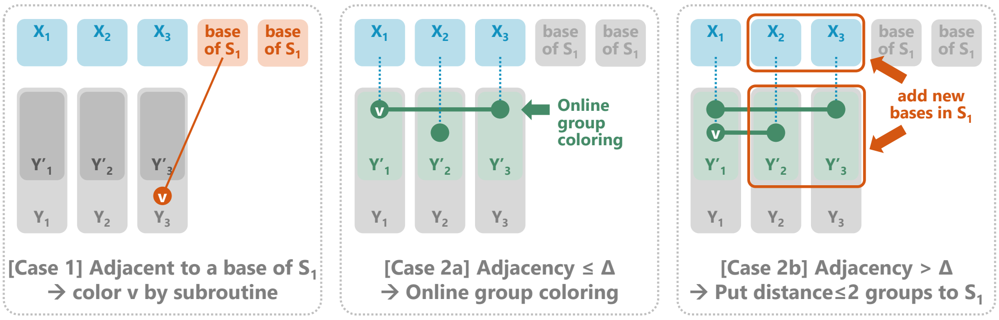

Figure 7 A sketch of the solution to Part B and the relevant variables. (a) When v is adjacent to a base in $\mathcal{S}_1$, we color v in the $(n^{2/5}, 24)$-subroutine and do not add v to Y_i'. (b) Otherwise, if $|N_{H^+}(i)| \le \Delta$, we color v using our algorithm for online group coloring. (c) Otherwise, we merge the distance-$0, 1, 2$ groups and put them into $\mathcal{S}_1$ as bases.

We maintain, for each group Y_i, the set of vertices $Y_i' \subseteq Y_i$ that proceed to "Case 2" (which is Step 2 in the following algorithm). We also maintain a graph $H^+ = (\{1, \ldots, r\}, E(H^+))$ which represents the current adjacency between these subsets Y_i'. The set of edges is $E(H^+) := \{i - j : \text{there exist some edges between } Y_i' \text{ and } Y_j' \text{ in the original graph } G\}$. It is clear that the neighbor graph for online group coloring is a subgraph of H^+.

Now, we describe the algorithm. The parameter for online group coloring is set to $\Delta := 6n^{1/5}$. For each coloring query, supposing that vertex v arrives in group Y_i, we perform the following procedure (see also Figure 7):

1. If v is adjacent to any existing base in $\mathcal{S}_1$, we color v inside $\mathcal{S}_1$. Formally, we pick one such base, and add v to the corresponding group for this base in $\mathcal{S}_1$ (as a coloring query). This means that we use our algorithm for the $(n^{2/5}, 24)$-subroutine to color v.
2. Otherwise, after adding v to Y_i' and updating the graph H^+, we proceed as follows:
   a. If $|N_{H^+}(i)| \le \Delta$, then we put v into group i for online group coloring and color v using the algorithm in Theorem 9. This is because the degree requirement for online group coloring is satisfied as the neighbor graph is a subgraph of H^+.
   b. Otherwise, we cannot color v with online group coloring. Instead, we exploit the large degree and create large sets with even-diameter ≤ 24. Let D_0, D_1, D_2 be the sets of vertices with distance $0, 1, 2$ from vertex i in H^+. It can be shown that for $k = 0, 1, 2$, the sets $\bigcup_{j \in D_k} X_j$ and $\bigcup_{j \in D_k} Y_j'$ indeed have even-diameter ≤ 24 (Lemma 14). Hence, we add these 6 sets as bases for $\mathcal{S}_1$. After that, we can color v by going back to Step 1, since v is adjacent to a new base $\bigcup_{j \in D_0} X_j$ $(= X_i)$.

The pseudocode of this algorithm is given in Algorithm 2.

▶ Remark 13. We note that, after we perform Step 2-b with respect to D_0, D_1, D_2, all future vertices v that arrive in group Y_i $(i \in D_0 \cup D_1 \cup D_2)$ will be processed in Step 1 (and therefore will not proceed to Step 2, which makes Y_i' "frozen"). This is because if $i \in D_k$ $(k = 0, 1, 2)$, then v is adjacent to $\bigcup_{j \in D_k} X_j$, which is a base already added to $\mathcal{S}_1$.

4.5 Correctness & Analysis

Next, we prove the correctness of this algorithm. We show that the input to the online group coloring problem and the $(n^{2/5}, 24)$-subroutine $\mathcal{S}_1$ is valid, and in the resulting coloring, no two adjacent vertices share the same color (Lemma 17). After that, we analyze the number of colors used (Lemma 18).

Algorithm 2 Algorithm to solve the $(n^{3/5}, 2)$-subroutine with $O(n^{2/5})$ colors. Here, we define the graph $H^+ = (\{1, \dots, r\}, E(H^+))$ by $E(H^+) := \{i - j : \text{edges exist between } Y'_i \text{ and } Y'_j \text{ in } G\}$.

```
1:  S_1 ← the (n^{2/5}, 24)-subroutine
2:  for each coloring query (supposing that vertex v arrives in group Y_i) do
3:      if v is adjacent to any existing base in S_1 then
4:          color v inside S_1 (by adding v to S_1 as a coloring query)
5:      else
6:          Y'_i ← Y'_i ∪ {v}, and update the graph H^+
7:          if |N_{H^+}(i)| ≤ Δ (= 6n^{1/5}) then
8:              color v with our algorithm for online group coloring (Theorem 9)
9:          else
10:             D_k ← the set of vertices with distance k from vertex i in H^+, for k = 0, 1, 2
11:             add ⋃_{j∈D_k} X_j and ⋃_{j∈D_k} Y'_j as bases for S_1, for k = 0, 1, 2
12:             go to lines 3–4 to color v (because v neighbors a new base ⋃_{j∈D_0} X_j = X_i)
```

▶ **Lemma 14.** *Bases added to the $(n^{2/5}, 24)$-subroutine $\mathcal{S}_1$ have even-diameter at most 24.*

Proof. First, we show that for any $k \in \{0, 1, 2\}$, in Step 2-b of the algorithm, $\bigcup_{j \in D_k} X_j$ has even-diameter ≤ 22. That is, for any pair of vertices $v^{(s)}, v^{(e)} \in \bigcup_{j \in D_k} X_j$, there exists an even-length walk of length at most 22.

Let $X_{g^{(s)}}$ and $X_{g^{(e)}}$ be bases to which $v^{(s)}$ and $v^{(e)}$ belong, respectively. Let $g_0 \to g_1 \to \cdots \to g_{2k}$ $(g_0 = g^{(s)}, g_{2k} = g^{(e)})$ be a walk of length $2k$ between $g^{(s)}$ and $g^{(e)}$ in the graph H^+. This walk exists by the definition of D_k.

For $i = 0, \dots, 2k-1$, let $e_i - s_{i+1}$ (where $e_i \in Y'_{g_i}$ and $s_{i+1} \in Y'_{g_{i+1}}$) be an edge between Y'_{g_i} and $Y'_{g_{i+1}}$. This exists by the definition of the graph H^+. Let s'_i and e'_i be vertices in X_{g_i} that neighbor s_i and e_i, respectively. Also, let $s'_0 := v^{(s)}$ and $e'_{2k} := v^{(e)}$; note that $s'_0 \in X_{g_0}$ and $e'_{2k} \in X_{g_{2k}}$. Then, we obtain the following walk (see Figure 8 for visualization):

$$
\begin{aligned}
(v^{(s)} =)\ &[s'_0 \to \cdots \to e'_0 \to e_0] \to [s_1 \to s'_1 \to \cdots \to e'_1 \to e_1] \to \cdots \\
&\to [s_{2k-1} \to s'_{2k-1} \to \cdots \to e'_{2k-1} \to e_{2k-1}] \to [s_{2k} \to s'_{2k} \to \cdots \to e'_{2k}]\ (= v^{(e)})
\end{aligned}
$$

where each $s'_i \to \cdots \to e'_i$ is an even-length walk of length at most $d^* = 2$; this exists because each X_{g_i} has even-diameter ≤ 2. Hence, the length of this walk from $v^{(s)}$ to $v^{(e)}$ is an even number at most $(d^* + 3)(2k + 1) - 3 \leq 5d^* + 12 = 22$, since $k \leq 2$.

Next, we prove the claim for $\bigcup_{j \in D_k} Y'_j$. Consider any $v^{(s)}, v^{(e)} \in \bigcup_{j \in D_k} Y'_j$; since both $v^{(s)}$ and $v^{(e)}$ are adjacent to $\bigcup_{j \in D_k} X_j$, there is a walk between $v^{(s)}$ and $v^{(e)}$ whose length is 2 longer than in the case above. Therefore, $\bigcup_{j \in D_k} Y'_j$ has even-diameter $5d^* + 14 \leq 24$. ◀

▶ **Lemma 15.** *Let $Z = [z_1, z_2, \dots]$ be the list, where z_i is the (sole) element of D_0 in the i-th occurrence of Step 2-b. Then, any pair of elements in Z are at least distance 3 apart in the final graph H^+.*

Proof. Consider D_0, D_1, D_2 in the i-th occurrence of Step 2-b (i.e., at the time of z_i). Then, D_0, D_1, D_2 are equal to the set of vertices with distance $0, 1, 2$ from z_i in the *final graph* H^+. In other words, the set of vertices with distance $0, 1, 2$ from z_i remains unchanged.

Suppose, for contradiction, that this is not the case. Then, an edge incident to D_k ($k = 0, 1, 2$) is newly added to H^+, after the occurrence of D_k (i.e., after the time of z_i). Let v' be a vertex that caused this edge addition. Then, v' proceeds to Step 2, and by Remark 13,

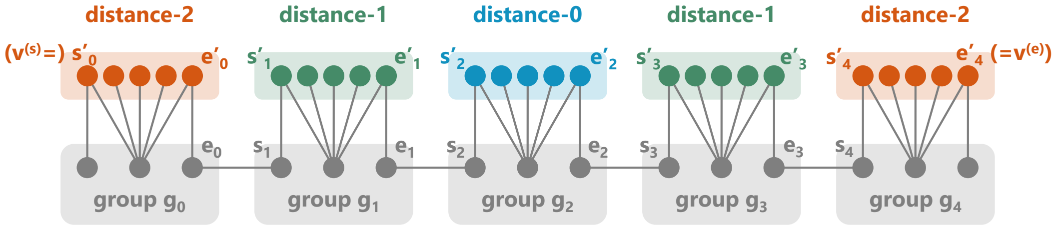

Figure 8 A sketch of the proof that $\bigcup_{j \in D_k} X_j$ has even-diameter at most 22, for $k = 2$. Indeed, the length of the walk from $v^{(s)}$ to $v^{(e)}$ is exactly 22, which is the worst case.

v' is not in groups of $D_0 \cup D_1 \cup D_2$. Thus, v' is adjacent to some Y'_j ($j \in D_k$). This implies that v' is also adjacent to $\bigcup_{j \in D_k} Y'_j$, which is a base already added to $\mathcal{S}_1$. Thus, v' should have been processed in Step 1, a contradiction.

For each $j = i+1, i+2, \ldots$, we have $z_j \notin D_0 \cup D_1 \cup D_2$ by Remark 13. Hence, with the observation above, z_j is at least distance 3 apart from z_i in the final graph H^+ as well. ◀

▶ **Lemma 16.** *At most $n^{2/5}$ bases are added to the $(n^{2/5}, 24)$-subroutine $\mathcal{S}_1$.*

Proof. Let $Z = [z_1, z_2, \ldots]$ be the list defined in Lemma 15. Then, by Lemma 15, the sets $N_{H^+}(z_1), N_{H^+}(z_2), \ldots$ are disjoint for the final graph H^+. Moreover, we have $|N_{H^+}(z_i)| > 6n^{1/5}$ $(= \Delta)$ because they were not processed in Step 2-a. Since H^+ has at most $r^* = n^{3/5}$ vertices, we have $|Z| \le \frac{n^{3/5}}{6n^{1/5}} = \frac{1}{6}n^{2/5}$. Thus, at most $6|Z| \le n^{2/5}$ bases are added to $\mathcal{S}_1$. ◀

▶ **Lemma 17.** *For graphs with odd girth at least 29, the inputs for (1) online group coloring and (2) the $(n^{2/5}, 24)$-subroutine $\mathcal{S}_1$ are valid.*

Proof. For (1) online group coloring, the degree conditions are met due to Step 2-a. Also, there are no edges connecting vertices in the same group, because there are no edges inside each Y_i. Otherwise, given that each base X_i has even-diameter at most $d^* = 2$, there would exist an odd closed walk of length at most $d^* + 3 = 5$, by the same argument as in the proof of Lemma 12. This contradicts that the odd girth is at least 29.

For (2) the $(n^{2/5}, 24)$-subroutine $\mathcal{S}_1$, the added bases have even-diameter ≤ 24, due to Lemma 14. When the vertices are added to $\mathcal{S}_1$, they are adjacent to one of its bases, due to Step 1 of the algorithm. At most $n^{2/5}$ bases are added, due to Lemma 16. ◀

▶ **Lemma 18.** *For graphs with odd girth at least 29, Algorithm 2 solves the $(n^{3/5}, 2)$-subroutine deterministically using $O(n^{2/5})$ colors.*

Proof. Lemma 17 shows that the algorithm functions properly. Therefore, we use at most $\Delta^2 + 2 = 36n^{2/5} + 2$ colors for online group coloring due to Theorem 9 and $\Delta = 6n^{1/5}$, and we use at most $n^{2/5}$ colors in the $(n^{2/5}, 24)$-subroutine $\mathcal{S}_1$ due to Lemma 12. In total, we use $O(n^{2/5})$ colors. ◀

▶ **Theorem 19.** *For graphs of odd girth at least 29, there is a deterministic online coloring algorithm that uses $O(n^{2/5})$ colors.*

Proof. By combining Lemma 11 and Lemma 18, we obtain a deterministic online coloring algorithm that uses $O(n^{2/5})$ colors for graphs with odd girth at least 29. ◀

5 The Algorithm with $O(n^{\varepsilon})$ colors

In this section, we generalize the idea from Section 4 to multiple layers. We reduce the exponent – from $\frac{1}{2}$ in Kierstead's algorithm and $\frac{2}{5}$ in Section 4 – to an arbitrarily small positive value for graphs with sufficiently large odd girth. In other words, we prove Theorem 1.

5.1 The Generalized Lemmas

We revisit three key steps in constructing our $O(n^{2/5})$-color algorithm: Lemma 11 (Part A), Lemma 18 (Part B), and Lemma 12 (Part C). Below, we generalize these lemmas from the specific $O(n^{2/5})$ setting to one with general parameters.

▶ **Lemma 20.** *Let $c \geq 1$ be an integer parameter. The online coloring problem with n vertices can be reduced to the $(\frac{n}{c}, 2)$-subroutine, deterministically, using at most c extra colors.*

Proof. This follows from the same argument as Lemma 11, where we use c colors for First-Fit (instead of $\lceil n^{2/5} \rceil$ colors). Then, the number of bases created is at most $\frac{n}{c}$. ◀

▶ **Lemma 21.** *Let $\Delta \geq 1$ be a real parameter. For graphs with odd girth at least $d^* + 5$, the (r^*, d^*)-subroutine can be reduced to the $(\frac{6r^*}{\Delta}, 5d^* + 14)$-subroutine, deterministically, using at most $\Delta^2 + 2$ extra colors.*

Proof. This follows from the same argument as Lemma 18, where we use Δ as the parameter for online group coloring. The even-diameter bound $5d^* + 14$ comes from the proof of Lemma 14. The maximum number of bases created, $\frac{6r^*}{\Delta}$, comes from the proof of Lemma 16. The number of extra colors used, $\Delta^2 + 2$, comes from Theorem 9. We need the odd girth requirement to guarantee that there is no edge inside the same group in online group coloring; see the proof of Lemma 17. ◀

▶ **Lemma 22.** *For graphs with odd girth at least $d^* + 5$, the (r^*, d^*)-subroutine can be solved deterministically using at most r^* colors.*

Proof. This follows from the same argument as Lemma 12. ◀

5.2 The Generalized Algorithm

Reflecting on the previous algorithms, Kierstead's algorithm for graphs with odd girth ≥ 7 reduced the online coloring problem via Lemma 20 and Lemma 22, achieving $O(n^{1/2})$ colors. Our algorithm for graphs with odd girth ≥ 29 reduced the online coloring problem via Lemma 20, Lemma 21, and Lemma 22, achieving $O(n^{2/5})$ colors.

The good news is that the reduction via Lemma 21 can be applied multiple times – this is key to constructing a further improved algorithm. More specifically, our strategy is to apply this reduction through k layers to achieve $O(n^{\frac{2}{k+4}})$ colors.

▶ **Theorem 23.** *Let $k \geq 0$ be an integer. For graphs of odd girth at least $\frac{11}{2} \cdot 5^k + \frac{3}{2}$, there exists a deterministic online coloring algorithm that uses $O(k \cdot n^{\frac{2}{k+4}})$ colors.*

Proof. First, we reduce the online coloring problem to the $(n^{\frac{k+2}{k+4}}, 2)$-subroutine using Lemma 20 with parameter $c := \lceil n^{\frac{2}{k+4}} \rceil$. This uses at most $\lceil n^{\frac{2}{k+4}} \rceil$ colors.

Next, we apply the reduction from Lemma 21, k times, each with parameter $\Delta := 6n^{\frac{1}{k+4}}$. In the ℓ-th reduction ($\ell = 1, \ldots, k$), we reduce the $(n^{\frac{k-\ell+3}{k+4}}, a_{\ell-1})$-subroutine to the $(n^{\frac{k-\ell+2}{k+4}}, a_\ell)$-subroutine, where a_ℓ is defined by $a_0 = 2$ and $a_\ell = 5a_{\ell-1} + 14$ ($\ell = 1, 2, \ldots$). This uses at most $k \cdot (\Delta^2 + 2) = k \cdot (36n^{\frac{2}{k+4}} + 2)$ colors in total.

Finally, we solve the $(n^{\frac{2}{k+4}}, a_k)$-subroutine by Lemma 22, using $n^{\frac{2}{k+4}}$ colors.

Overall, the number of colors used is $O(k \cdot n^{\frac{2}{k+4}})$. The odd girth requirement is needed to use Lemma 21 and Lemma 22, but odd girth $\geq a_k + 5$ is sufficient for all steps of the algorithm, which equals $\frac{11}{2} \cdot 5^k + \frac{3}{2}$. This follows from the closed form $a_k = \frac{11}{2} \cdot 5^k - \frac{7}{2}$. ◀

In conclusion, with Theorem 23 in hand, we claim our main theorem (Theorem 1): for every $\varepsilon > 0$, there exists a constant $g' \in \{3, 5, 7, \dots\}$ such that graphs with odd girth at least g' can be deterministically colored online using $O(n^{\varepsilon})$ colors. Table 1 shows how the number of colors used depends on the odd girth. The algorithm clearly runs in polynomial time.

5.3 Performance with Non-Constant Odd Girth

When the number of layers is $k = \Theta(\log n)$, it follows from Theorem 23 that the number of colors used in the algorithm is just $O(\log n)$. Hence, we obtain the following corollary.

▶ **Corollary 24.** *Let c be a constant such that $0 < c < 1$. For graphs with odd girth $\Omega(n^c)$, there exists a deterministic online coloring algorithm that uses $O(\log n)$ colors.*

Proof. Applying Theorem 23 with $k = (c - \varepsilon)\log_5 n$ for some ε $(0 < \varepsilon < c)$, we obtain a deterministic online algorithm that colors graphs with odd girth at least $\frac{11}{2} \cdot n^{c-\varepsilon} + \frac{3}{2}$, using only $O(\log n)$ colors. Therefore, Corollary 24 holds. ◀

This substantially improves the result of $O(n^{(1-c)/2}\sqrt{\log n})$ colors by Nagy-György [22] and matches the bipartite graph bound of $\Theta(\log n)$ colors due to Lovász, Saks, Trotter [20].

6 Conclusion

In this paper, we showed a deterministic online coloring algorithm that uses $O(k \cdot n^{\frac{2}{k+4}})$ colors for graphs with odd girth $g \geq \frac{11}{2} \cdot 5^k + \frac{3}{2}$, for every integer $k \geq 0$. This gives the first improvement in 28 years for online coloring of graphs with large constant odd girth, improving upon the $O(n^{1/2})$-color algorithm of Kierstead (1998) [15].

It is known that any online algorithm requires $\Omega(n^{\frac{1}{g-3}} / \log n)$ colors for graphs with general odd girth $g \geq 5$, as implied by the chromatic number bound of Krivelevich (1995) [19]. Therefore, for every constant g, there exists an instance requiring $\Omega(n^{\varepsilon})$ colors for some $\varepsilon > 0$. In this sense, since we achieve $O(n^{\varepsilon})$ colors for sufficiently large odd girth, our algorithm is in the same ballpark as this lower bound.

However, in terms of the speed of convergence, our exponent is approximately $\frac{2}{\log_5 g}$, which is much slower than the exponent $\frac{1}{g-3}$ in the lower bound. Therefore, it is natural to ask whether the following conjecture holds.

▶ **Conjecture 25.** *For some constant $c \geq 1$, there exists an online coloring algorithm that achieves $O(n^{c/g})$ colors for graphs with constant odd girth g.*

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