Beyond Trees: The Weighted Center Problem on Gromov Hyperbolic Graphs

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Abstract

The Weighted Center problem takes as its input a graph $G = (V, E)$ together with a profile π such that every vertex v is mapped to some nonnegative multiplicative weight $\pi(v)$. Its output must be some vertex c minimizing $\max\{\pi(v)d_G(c, v) : v \in V\}$. The classic Center problem corresponds to the case where $\pi(v) = 1$ for every vertex v. In the literature, various almost linear-time algorithms have been proposed for the Center problem on some well-structured classes of graphs. By contrast, similarly efficient algorithms for the Weighted Center problem have been scarce. We investigate how the Gromov hyperbolicity, alone or in combination with other metric and geometric properties on graphs, can be used in the design of exact and approximate almost linear-time algorithms for the Weighted Center problem. In particular, we derive almost optimal algorithms for the following well-studied classes of graphs: chordal graphs, distance-hereditary graphs (both in $\mathcal{O}(m)$ time), dually chordal graphs and chordal bipartite graphs (both in $\mathcal{O}(m \log n)$ time).

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1 Introduction

Problem considered. We refer to [6] for basics in Graph Theory. Graphs considered throughout the paper will be simple, undirected, unweighted and connected. As usual, let d_G denote the shortest-path metric of a graph $G = (V, E)$, *i.e.*, for any vertices u and v their distance $d_G(u, v)$ is equal to the minimum number of edges on a (u, v)-path. The following facility location problem on graphs is considered:

Weighted Center
Input: A graph $G = (V, E)$; a profile $\pi : V \mapsto \mathbb{R}_{\geq 0}$.
Output: A vertex $c \in V$ s.t. $r_\pi(c) = \max\{\pi(v)d_G(c, v) : v \in V\}$ is minimized.

Roughly, $\pi(v)$ represents the importance for vertex v to be close to the new facility. For example, $\pi(v)$ can be the expected material flux between v and a new factory, the number of children in a building at v who will attend to a new neighborhood school, etc. Furthermore, if we set $\pi(v') = 0$ for some vertices v', then we can account for the plausible situation where the placement of a new facility is only relevant for a subset of the vertices. Note that we cannot simply remove the zero-weight vertices for it could change the distances in the graph.

In line with existing literature [33], the function r_π is called a *radius function*. The solution vertex c is called a *center*. The WEIGHTED CENTER problem can be defined more generally for any metric space. In particular, a linear-time algorithm for the WEIGHTED CENTER problem on Euclidean spaces of constant dimension was presented in [28]. For graphs, the existing literature is mostly focused on the following restricted variant:

CENTER
Input: A graph $G = (V, E)$.
Output: A vertex $c \in V$ s.t. $\mathrm{ecc}(c) = \max\{d_G(c, v) : v \in V\}$ is minimized.

The radius function ecc is also called the *eccentricity function* of G. More generally, if $\pi(v) \in \{0, 1\}$ for every vertex v, then π is called a binary profile, and r_π is called a binary radius function. The special case of binary profiles on graphs has received some attention in the literature, see [10, 14]. As a rule of thumb, most of the techniques for the CENTER problem can be also applied for binary profiles. However, such is not the case for the more general WEIGHTED CENTER problem.

Related work. The WEIGHTED CENTER problem can be reduced to All-Pairs Shortest-Paths. For n-vertex m-edge graphs, this straightforward reduction leads to an $\mathcal{O}(nm)$-time algorithm. In [1], the Hitting Sets Conjecture was introduced to prove a conditional lower bound in $\Omega(n^{2-o(1)})$ for the CENTER problem on n-vertex graphs, even if the number of edges is at most in $n^{1+o(1)}$. Therefore, breaking the quadratic barrier for the CENTER problem, and even more so for the WEIGHTED CENTER problem, is likely to require additional restrictions on the classes of graphs we consider. There is a plethora of subquadratic-time algorithms, and even better, of almost linear-time algorithms, for the CENTER problem on some well-structured classes of graphs; we refer to [1, 12, 19, 25] for a small sample of such results. By comparison, the only published linear-time algorithm for the WEIGHTED CENTER problem, to the best of our knowledge, is that of Megiddo for the class of trees [39]. Recently, a parameterized almost linear-time algorithm for the WEIGHTED CENTER problem on the graphs of bounded clique-width was presented in [25], using orthogonal range queries and the so-called convex hull trick. In particular, for the distance-hereditary graphs, the clique-width of which is at most three [29], this result leads to an $\mathcal{O}(m \log^c n)$-time algorithm for the WEIGHTED CENTER problem, for some unspecified constant $c > 1$. An $\mathcal{O}(m)$-time algorithm for the CENTER problem was presented much earlier, in [20].

The authors in [11] introduced a different approach for the WEIGHTED CENTER problem that is based on a *metric* property, hereafter called *G^p-unimodality* (see also [26] for the case $p = 1$). Roughly, a radius function r_π on a graph G is called G^p-unimodal if any vertex that minimizes r_π within its ball of radius p is a center. It was proved in [11] that for many classes of graphs studied in Metric Graph Theory, all their radius functions are G^2-unimodal. The latter was used in the design of randomized $\tilde{\mathcal{O}}(m\sqrt{n})$-time[1] local-search algorithms for the WEIGHTED CENTER problem on these classes of graphs. Although the algorithmic properties of the classes studied in Metric Graph Theory have insofar received limited attention (the results in [2, 4, 21] count among the few exceptions), we want to stress here that they are natural generalizations of better-studied classes, including: chordal graphs, chordal bipartite graphs, and dually chordal graphs. In particular, before this work, the $\tilde{\mathcal{O}}(m\sqrt{n})$-time algorithms from [11, 26] were the best ones known for the WEIGHTED CENTER problem on chordal (bipartite) graphs, and dually chordal graphs. By comparison, $\mathcal{O}(m)$-time algorithms for the CENTER problem can be found in [7, 13, 24].

[1] The $\tilde{\mathcal{O}}()$ notation suppresses poly-logarithmic factors.

One common property of chordal graphs and their relatives, but also of distance-hereditary graphs, AT-free graphs, and several other well-studied classes of graphs (*e.g.*, see [14, 43]) is that they have bounded *Gromov hyperbolicity*. Roughly, the hyperbolicity of a graph represents how close, locally, its shortest-path metric is to a tree metric (see Sec. 2 for a formal definition). The introduction of this parameter by Gromov, see [30], has revolutionized Geometric Group Theory. Later on, extensive experiments have evidenced that many complex networks also have bounded hyperbolicity [34]. There is now a rich literature of approximation algorithms on graphs whose performances depend on the hyperbolicity value [32, 37]. Of special interest to us is the main result from [14], which states that on any δ-hyperbolic graph G, for every *binary* profile π on G, a vertex that is at distance at most $\mathcal{O}(\delta)$ to the center can be computed in $\mathcal{O}(m)$ time. Note that it leads to an approximation algorithm for the *radius*: defined as $\text{rad}_\pi(G) = \min\{r_\pi(v) : v \in V\}$, with additive one-sided error in $\mathcal{O}(\delta)$. Furthermore, this result was combined with the local-search techniques from [11] to refine the runtime of their exact algorithms (only for binary profiles) from $\mathcal{O}(m\sqrt{n})$ to $\mathcal{O}(\delta m)$. The motivation for our paper was to generalize the result from [14] to arbitrary profiles.

Our contributions. For a graph G, and a profile π, let us define $C_\pi(G)$ as the set of all its centers. In what follows, the ball of center v and radius ρ is denoted by $B_\rho(v)$. We provide the following two main results for the WEIGHTED CENTER problem on δ-hyperbolic graphs.

▶ **Theorem A.** *Given a δ-hyperbolic graph G with n vertices and m edges, and an arbitrary profile π, one can compute:*
1. *a vertex c_1 such that $C_\pi(G) \subseteq B_{12\delta\log n+4\delta+11}(c_1)$, in $\mathcal{O}(m)$ time;*
2. *a vertex c_2 such that $C_\pi(G) \subseteq B_{124\delta+21}(c_2)$, in $\mathcal{O}(\delta m \log^2 n)$ time.*

In particular, for an *arbitrary* profile π, a vertex that is at distance $\mathcal{O}(\delta)$ to the center can be computed in $\tilde{\mathcal{O}}(\delta m)$ time. This weakly generalizes the result of [14] for binary profiles, but at the price of a slightly worse running time. Furthermore, if π is k-bounded, then we get an approximation of the radius with additive one-sided error in $\mathcal{O}(k\delta)$. The latter is also a multiplicative $(1 + \mathcal{O}(\varepsilon\delta))$-approximation of the radius, where $\varepsilon = \frac{\max_{v\in V}\pi(v)}{\text{rad}_\pi(G)}$.

Theorem A(1) follows from efficient algorithms for embedding a δ-hyperbolic graph into a tree with additive distortion in $\mathcal{O}(\delta\log n)$, used in combination with the aforementioned linear-time algorithm for the WEIGHTED CENTER problem on trees. By doing so, it is expected that we get a vertex c_1 such that $r_\pi(c_1)$ and $\text{rad}_\pi(G)$ are close. However, it is less immediate that c_1 and the true centers of G are also close. Theorem A(2) is based on Theorem A(1). Roughly, we search for c_2 in a ball of center c_1 and radius $\mathcal{O}(\delta\log n)$. For that, we reduce our search to $\tilde{\mathcal{O}}(\delta)$ instances of the WEIGHTED CENTER problem with *binary* profiles, which may be of independent interest. In particular, we are using the result of [14].

As our second main contribution in the paper, we present a set of new algorithms for the WEIGHTED CENTER problem on some important examples of graphs with bounded hyperbolicity. These results are reported in Table 1. They are based on Theorem A, and on additional metric properties of these graphs. Our running times are optimal, or quasi optimal, for all the classes of graphs considered.

Our result for chordal graphs builds upon the local-search techniques from [11]. Indeed, in this special case a vertex c_1 that is at distance $\mathcal{O}(1)$ to the center can be computed in $\mathcal{O}(m)$ time (the latter is a refinement of Theorem A(1) for this class of graphs). Then, starting from c_1, an improving sequence $(v_0 = c_1, v_1, \ldots, v_\ell)$ can be computed in $\mathcal{O}(\ell m)$ time, such that $r_\pi(v_i) > r_\pi(v_{i+1})$ for every i such that $0 \leq i < \ell$, and v_ℓ is a center. If π is a binary profile, then as our starting vertex c_1 is at distance $\mathcal{O}(1)$ to the center, we must have

Table 1 Results for the WEIGHTED CENTER problem on some classes of graphs with bounded Gromov hyperbolicity.

Graph class	Prior works	**This work**	**Ref.**
Chordal	$\tilde{\mathcal{O}}(m\sqrt{n})$ [11]	$\mathcal{O}(m)$	Theorem 20(2)
Chordal bipartite	$\tilde{\mathcal{O}}(m\sqrt{n})$ [11]	$\mathcal{O}(m\log n)$	Theorem 25(2)
Distance-hereditary	$\mathcal{O}(m\log^c n),\ c>1$ [25]	$\mathcal{O}(m)$	Theorem 26
Dually chordal	$\tilde{\mathcal{O}}(m\sqrt{n})$ [26]	$\mathcal{O}(m\log n)$	Theorem 25(2)

$\ell = \mathcal{O}(1)$ no matter how we choose v_{i+1} from v_i. However, as already noted in [11], this does not hold anymore for arbitrary profiles. We introduce a new selection mechanism for the next improving vertex v_{i+1} that ensures that the bound $\ell = \mathcal{O}(1)$ still holds with arbitrary profiles.

For distance-hereditary graphs, we use a similar local-search approach as for chordal graphs. For that, we prove that all their radius functions are also G^2-unimodal. We also design an $\mathcal{O}(m)$-time procedure for computing the next improving vertex v_{i+1} from v_i in a distance-hereditary graph. The latter is based on *injective hulls*, which are important objects in the study of metric spaces and their optimal realizations, see [23]. By doing so, we contribute to the recent line of studies of the properties of injective hulls of graphs [31].

For chordal bipartite graphs and dually chordal graphs, we adapt a technique from [21]. Roughly, it allows us to compute, in some ball of constant radius, all the vertices v such that $r_\pi(v)$ is less than a fixed threshold τ. By doing so, all the centers can be computed with $\mathcal{O}(\log n)$ calls to this technique, using a nontrivial variation of binary search. At this point, we should mention that it is not that obvious that we can efficiently reduce the WEIGHTED CENTER problem to its decision version, using binary search, because the set of possible values for the radius is $\{k \times \pi(v) : 0 < k < n, v \in V\}$, which has size in $\mathcal{O}(n^2)$. To apply a naive binary-search strategy to this set would require first to sort all the values. The latter would result in an $\Omega(n^2 \log n)$-time lower bound. The existence of an efficient reduction from WEIGHTED CENTER to its decision version is a side contribution of the paper.

Our third contribution in the paper is a set of parameterized almost linear-time algorithms for the WEIGHTED CENTER problem, where the parameter is the hyperbolicity value δ. These results are reported in Table 2.

Table 2 Parameterized algorithms for the WEIGHTED CENTER problem on some classes of graphs, where the parameter is the hyperbolicity value δ.

Graph class	Running time	Ref.
Max. degree Δ	$\tilde{\mathcal{O}}(\Delta^{\mathcal{O}(\delta)}m + \delta m)$	Theorem 14(1)
Weakly bridged	$\tilde{\mathcal{O}}(\delta m)$	Theorem 20(1)
Helly	$\tilde{\mathcal{O}}(\delta^3 m)$	Theorem 25(1)
Bipartite Helly	$\tilde{\mathcal{O}}(\delta^3 m)$	Theorem 25(1)
Planar	$\tilde{\mathcal{O}}(2^{O(\delta)}n)$	Theorem 21

Our result for graphs with bounded maximum degree is a straightforward application of Theorem A(2). For weakly bridged graphs, Helly graphs and bipartite Helly graphs, we use the same techniques as for chordal graphs, dually chordal graphs, and chordal bipartite graphs, of which they are superclasses. The respective definitions of these classes of graphs are recalled in the technical sections of the paper. Finally, our most interesting entry in

Table 2 is that for planar graphs, for which we do not use Theorem A. Instead, we rely on a recent separator theorem for planar hyperbolic graphs, see [35], which is combined with a *VC-dimension* argument. To the best of our knowledge, this is the first combination of Gromov hyperbolicity with VC-theory in algorithm design.

Organization of the paper. The notations and terminology that are used in the paper are first recalled in Sec. 2. Furthermore, we introduce some prior results that are required for the presentation of our algorithms and their analysis. Sec. 3 is devoted to the proof of Theorem A. Then, in Sec. 4, we present all our exact algorithms for the WEIGHTED CENTER problem on some classes of graphs with bounded hyperbolicity. Section 5 concludes the paper, with some discussion about our work, and directions for future research.

Due to lack of space, we only detail in this extended abstract the proofs of our Theorem A, Theorem 20 (for chordal graphs), and Theorem 21 (for planar graphs). The remaining results, and their proofs, can be found in the full version of the paper [27].

2 Preliminaries

We start recalling a few notations. Let $G = (V, E)$ be a graph. We will always denote by n its number $|V|$ of vertices, and by m its number $|E|$ of edges. For every vertices u and v, we write $u \sim v$ if u and v are adjacent, and $u \nsim v$ if they are nonadjacent. Let $N_G(u) = \{v \in V : u \sim v\}$ and $N_G[u] = N_G(u) \cup \{u\}$ be the (open) *neighborhood* and the *closed neighborhood* of vertex u. For a subset X, let $N_G(X) = \bigcup\{N_G(x) \setminus X : x \in X\}$. Recall that a shortest (u, v)-path is one with minimum number of edges; the *distance* $d_G(u, v)$ is equal to the length of a shortest (u, v)-path. Furthermore, let $I_G(u, v) = \{w \in V : d_G(u, v) = d_G(u, w) + d_G(w, v)\}$ be the (metric) *interval* between u and v. Let also $I^o_G(u, v) = I_G(u, v) \setminus \{u, v\}$. For every k such that $0 \leq k \leq d_G(u, v)$, let $S_k(u, v, G) = \{w \in I_G(u, v) : d_G(u, w) = k\}$ be called a *slice*. Recall that for every vertex u and for every integer ρ, the set $B_\rho(u, G) = \{v \in V : d_G(u, v) \leq \rho\}$ is called the *ball* of center u and radius ρ. For any vertex u, and for any subset X, let $d_G(u, X) = \min\{d_G(u, x) : x \in X\}$. The (weak) *diameter* of a subset X is defined as $\mathrm{diam}_G(X) = \max\{d_G(x, x') : x, x' \in X\}$. A subset X is called (geodesically) *convex* if $I_G(x, x') \subseteq X$ for any two vertices $x, x' \in X$. If G is clear from the context, then it is omitted from our notations. Other notations are locally defined at appropriate places in the paper.

Functions defined on a graph. Let $G = (V, E)$ be a graph, and let $f : V \mapsto \mathbb{R}$. A *global minimum* of f is any vertex u such that $f(u)$ is minimal within V. A *local minimum* of f is any vertex u such that $f(u) \leq f(v)$ for every $v \in N(u)$. We call f *unimodal* if every local minimum is also a global minimum. More generally, for every integer $p \geq 1$, let the p^{th}*-power* of G be the graph $G^p = (V, E_p)$ such that there exists an edge $uv \in E_p$ if and only if $0 < d_G(u, v) \leq p$. We call f G^p*-unimodal* if it is unimodal on G^p. Note that f is G^p-unimodal if and only if every vertex u that minimizes $f(u)$ within $B_p(u, G)$ is also a global minimum. The function f is called *weakly peakless* if it satisfies the following condition $\mathrm{WP}(u, v)$ for every vertices u and v such that $u \nsim v$: there exists a vertex $w \in I^o(u, v)$ such that $f(w) \leq \max\{f(u), f(v)\}$, and equality holds only if $f(u) = f(w) = f(v)$. It is called *p-weakly peakless* if it satisfies $\mathrm{WP}(u, v)$ for every vertices u and v such that $d(u, v) \geq p + 1$.

▶ **Lemma 1** (see Theorem 11 in [3]). *Every p-weakly peakless function on a graph G is G^p-unimodal.*

Results on the Weighted Center problem. Let $G = (V, E)$ be a graph, and let $\pi : V \mapsto \mathbb{R}_{\geq 0}$ be a profile. The support of π, hereafter denoted by $\mathrm{Supp}(\pi)$, is the set of all vertices v such that $\pi(v) > 0$. Recall that $r_\pi : u \in V \mapsto \max\{\pi(v)d(u, v) : v \in V\}$ is called a radius function. Furthermore, for every vertex u, let $F_\pi(u) = \{v \in V : r_\pi(u) = \pi(v)d(u, v)\}$. Recall that we define the (π-)radius as $\mathrm{rad}_\pi(G) = \min\{r_\pi(u) : u \in V\}$. In the same way, let the center $C_\pi(G)$ contain all the vertices c such that $r_\pi(c)$ is minimized; each vertex $c \in C_\pi(G)$ is also called a center. As a side contribution of the paper, we present a reduction from the Weighted Center problem to its decision version. For that, although the search space for $\mathrm{rad}_\pi(G)$ is in $\mathcal{O}(n^2)$, we can use a strategy inspired by the median-of-medians algorithm [5] to quickly reduce the size of this space.

▶ **Lemma 2.** *Let $G = (V, E)$ be a graph, and let π be an arbitrary profile on G. If the problem of either computing a vertex c such that $r_\pi(c) \leq \tau$, or asserting that no such vertex exists, can be solved in $\mathcal{O}(T(n, m))$ time for any value τ, then the Weighted Center problem can be solved in $\mathcal{O}\left((T(n, m) + n) \log n\right)$ time on G, π.*

Proof. If $\mathrm{Supp}(\pi) = \emptyset$, then $C_\pi(G) = V$. Furthermore, if $\mathrm{Supp}(\pi)$ is a singleton, then $C_\pi(G) = \mathrm{Supp}(\pi)$. Thus, let us assume from now on $|\mathrm{Supp}(\pi)| > 1$. We maintain a set $S \subseteq V \times \mathbb{N} \times \mathbb{N}$ with the property that for some $(v, a_v, b_v) \in S$, there exists some k such that $a_v \leq k \leq b_v$, and $\mathrm{rad}_\pi(G) = \pi(v)k$. Initially, we set $S = \{(v, 1, n-1) : v \in \mathrm{Supp}(\pi)\}$. Then, our algorithm iteratively performs some procedure, detailed next, until the following halting condition, (HC), is met: for every $(v, a_v, b_v) \in S$, $a_v = b_v$. Note that once (HC) holds, we are left with $\mathcal{O}(|S|) = \mathcal{O}(n)$ distinct possibilities for $\mathrm{rad}_\pi(G)$. We can then sort all these values in $\mathcal{O}(n \log n)$ time, remove duplicates, and compute a center in $\mathcal{O}(T(n, m) \log n)$ time by using a standard binary search.

Assume for what follows that (HC) does not hold. For every $v \in \mathrm{Supp}(\pi)$, we define its weight $\omega(v)$ as follows: if there is some triple (v, a_v, b_v) in S, then $\omega(v) = b_v - a_v$, and otherwise $\omega(v) = 0$. Furthermore, let $\omega(S) = \sum\{\omega(v) : v \in \mathrm{Supp}(\pi)\}$. As (HC) does not hold, $\omega(S) > 0$. Then, the following procedure is performed. For some arbitrary ordering $v_0, v_1, \ldots, v_\ell$ of all the vertices of positive weight, we construct an array whose entries are the values $\tau_i := \pi(v_i)\left\lfloor \frac{a_{v_i} + b_{v_i}}{2} \right\rfloor$, for $0 \leq i \leq \ell$. We compute the *lowest weighted median* of the array: defined as the value τ_{i^*} such that $\sum\{\omega(v_j) : \tau_j < \tau_{i^*}\} < \omega(S)/2$ and $\sum\{\omega(v_j) : \tau_j > \tau_{i^*}\} \leq \omega(S)/2$. Then, we test whether there exists a vertex c s.t. $r_\pi(c) \leq \tau_{i^*}$.

- If yes, then for each $(v, a_v, b_v) \in S$, let us define $b'_v = \left\lfloor \frac{\tau_{i^*}}{\pi(v)} \right\rfloor$. If $b'_v < a_v$, then we remove this triple from S. Otherwise, if in addition $b'_v < b_v$, then we replace (v, a_v, b_v) with (v, a_v, b'_v) in S.
- Else, for each $(v, a_v, b_v) \in S$, let us define $a'_v = \left\lfloor \frac{\tau_{i^*}}{\pi(v)} \right\rfloor + 1$. If $a'_v > b_v$, then we remove this triple from S. Otherwise, if in addition $a'_v > a_v$, then we replace (v, a_v, b_v) with (v, a'_v, b_v) in S.

As noted in [18], the lowest weighted median can be computed in $\mathcal{O}(\ell) = \mathcal{O}(n)$ time by using a variation of the median-of-medians algorithm, see [5]. Therefore, this above procedure can be done in $\mathcal{O}(T(n, m) + n)$ time. Furthermore, we claim that the total weight $\omega(S)$ is decreased by a factor at least $3/4$ after each call to the procedure. Indeed, if there is a vertex c such that $r_\pi(c) \leq \tau_{i^*}$, then the weight of every vertex v_j such that $\tau_j \geq \tau_{i^*}$ is at least halved. If there is no such vertex, then the weight of every vertex v_j such that $\tau_j \leq \tau_{i^*}$ is at least halved. Therefore, we always halve the weights for a subset of vertices such that the sum of their weights is at least $\omega(S)/2$. In particular, the total weight decreases by at least $\omega(S)/4$, thus proving our claim. Overall, as initially $\omega(S) = \mathcal{O}(n^2)$, there are $\mathcal{O}(\log n)$ calls to the procedure. ◀

δ-hyperbolic graphs and their properties. For every nodes u, v, x, y in a tree T, it holds $d_T(u,v)+d_T(x,y) \leq \max\{d_T(u,x)+d_T(v,y), d_T(u,y)+d_T(v,x)\}$. The definition of Gromov hyperbolicity can be regarded as a relaxation of this four-point condition. A graph $G = (V,E)$ is called *δ-hyperbolic* if for every $u, v, x, y \in V$, $d(u,v) + d(x,y) \leq \max\{d(u,x) + d(v,y), d(u,y) + d(v,x)\} + 2\delta$ holds. Alternatively, if $d(u,y) + d(v,x) \leq d(u,x) + d(v,y) \leq d(u,v) + d(x,y)$, then $d(u,v) + d(x,y) - d(u,x) - d(v,y) \leq 2\delta$. The *hyperbolicity* of G is the smallest δ such that it is δ-hyperbolic. In what follows, we gather some properties of δ-hyperbolic graphs.

The first result is a relaxation of a well-known Helly property for trees: every family of pairwise intersecting subtrees (and so, every family of pairwise intersecting balls in a tree) has a nonempty common intersection.

▶ **Lemma 3** (see Lemma 1 in [16]). *If G is δ-hyperbolic, then for any family of pairwise intersecting balls $B_{\rho_1}(v_1), B_{\rho_2}(v_2), \ldots, B_{\rho_k}(v_k)$, $\bigcap\{B_{\rho_i+2\delta}(v_i) : 1 \leq i \leq k\} \neq \emptyset$.*

The next result is a special case of Morse lemma [41], of which we give a direct proof. Recall that in a tree, there exists a unique path between every two nodes. Roughly, we relax this property for a δ-hyperbolic graph such that almost shortest-paths between every two vertices must stay close to each other.

▶ **Lemma 4.** *Let u, v, x, y be vertices in a δ-hyperbolic graph G. If $u \in I(x,y)$, $d(x,v) + d(v,y) \leq d(x,y) + \lambda$, and $d(v,x) \geq d(u,x)$, $d(v,y) \geq d(u,y)$, then $d(u,v) \leq \lambda + 2\delta$.*

Proof. Since G is δ-hyperbolic, $d(u,v)+d(x,y) \leq \max\{d(u,x)+d(v,y), d(u,y)+d(v,x)\}+2\delta$. Therefore, $d(u,v) + d(x,y) \leq \max\{d(u,x), d(v,x)\} + \max\{d(u,y), d(v,y)\} + 2\delta \leq d(v,x) + d(v,y) + 2\delta \leq d(x,y) + \lambda + 2\delta$. The latter implies as claimed that $d(u,v) \leq \lambda + 2\delta$. ◀

Finally, observe that for any subtree H in a tree (in particular, for any ball in a tree), the nodes out of H can be partitioned in rooted subtrees with their roots being taken in the boundary[2] of H. In particular, every node y out of H has a unique closest node v in H (namely, the root of its subtree), and the unique path from y to any node of H goes by v. We prove a relaxation of this simple property for balls in a δ-hyperbolic graph, namely:

▶ **Lemma 5.** *Let u, x, y be vertices in a δ-hyperbolic graph G. If $x \in B_r(u)$ and $y \notin B_r(u)$, then for any $v \in S_r(u,y)$, $d(x,v) + d(v,y) \leq d(x,y) + 2\delta$.*

Proof. Consider the three sums $A = d(u,y)+d(v,x)$, $B = d(u,v)+d(x,y)$, and $C = d(u,x)+d(v,y)$. Since $v \in S_r(u,y)$, we get that $d(u,v) = r \geq d(u,x)$. Furthermore, $d(y,z) \geq d(y,v)$ for any $z \in B_r(u)$. So, in particular, $d(x,y) \geq d(v,y)$. Therefore, $B \geq C$. As $v \in I(u,y)$, $d(u,y) = d(u,v) + d(v,y)$. Hence, $A = d(u,v) + d(v,y) + d(v,x) \geq d(u,v) + d(x,y) = B$. Since G is δ-hyperbolic, $A - B \leq 2\delta$. As a result, $d(v,y) + d(v,x) - d(x,y) \leq 2\delta$. ◀

3 Approximation algorithms for δ-hyperbolic graphs

This section is devoted to the proof of Theorem A. Its statements (1) and (2) are proved in Sec. 3.1 and Sec. 3.2, respectively.

[2] Recall that the boundary of H is the subset of the vertices of H with at least one neighbour out of H.

3.1 Using tree embeddings

A graph G λ-*embeds* into a tree if there exists a tree T, possibly edge-weighted, such that $V \subseteq V(T)$ and for every vertices u and v of G, $d_G(u,v) \le d_T(u,v) \le \lambda d_G(u,v)$. This part is devoted to the proof of the following result:

▶ **Theorem 6.** *If a δ-hyperbolic graph $G = (V,E)$ λ-embeds into a tree, then for any profile π, one can output in $\mathcal{O}(m)$ time a vertex c_1 such that $C_\pi(G) \subseteq B_{6\lambda+4\delta+5}(c_1)$.*

Note that Theorem A(1) follows from Theorem 6 and the following Lemma 7.

▶ **Lemma 7** (see Proposition 6.1.B in [30]). *Every δ-hyperbolic graph λ-embeds into a tree, for some $\lambda \le 2\delta \log n + 1$.*

Stronger versions of Lemma 7 are used for some of the classes of graphs considered in Sec. 4.

Roughly, the proof of Theorem 6 consists in solving approximately the Weighted Center problem in a δ-hyperbolic graph G by solving this problem exactly on a tree T in which G embeds. It is known from prior works that such a tree T with almost optimal (additive) distortion can be computed efficiently, namely:

▶ **Lemma 8** (see Corollary 4 in [15]). *If a graph $G = (V,E)$ λ-embeds into a tree, then there exists an unweighted tree $T = (V,F)$ (without Steiner points), constructible in $\mathcal{O}(m)$ time, such that $d_T(x,y) - 2 \le d_G(x,y) \le d_T(x,y) + 3\lambda$ for any vertices $x, y \in V$.*

Given G and T as above, our contribution in this part is to prove that for an arbitrary profile π, for any centers $c \in C_\pi(G)$ and $c' \in C_\pi(T)$, their distance (in G) is at most linear in the hyperbolicity of G and the additive distortion of the embedding. We prove it next:

▶ **Proposition 9.** *Given a δ-hyperbolic graph $G = (V,E)$ and a tree $T = (V,F)$ such that $d_T(x,y) - p \le d_G(x,y) \le d_T(x,y) + q$ for any vertices $x, y \in V$ $(p, q \ge 0)$, for an arbitrary profile π, $C_\pi(G) \subseteq B_{2(p+q)+4\delta+1}(c')$ holds for any $c' \in C_\pi(T)$.*

Proof. Set $\tau = p + q + 2\delta + 1$. Suppose for the sake of contradiction the existence of a center $c \in C_\pi(G)$ such that $d_G(c,c') \ge 2\tau$. Let $w \in I_G(c,c')$ such that $d_G(c,w), d_G(w,c') \ge \tau$.

We consider some vertex x such that $\pi(x)d_G(w,x)$ is maximized. Since G is δ-hyperbolic, $d_G(c,c')+d_G(w,x) \le \max\{d_G(c,w)+d_G(c',x), d_G(c,x)+d_G(c',w)\}+2\delta$. By the choice of vertex w, $d_G(c,c') \ge \max\{d_G(c,w), d_G(c',w)\}+\tau$, and so, $d_G(w,x) \le \max\{d_G(c',x), d_G(c,x)\}+2\delta-\tau$. Furthermore, by the hypothesis $2\delta-\tau < 0$. Since $d_G(c,x) \le d_G(w,x)$ because $c \in C_\pi(G)$, it follows that $d_G(w,x) \le d_G(c',x) + 2\delta - \tau$.

We also consider some vertex y such that $\pi(y)d_T(w,y)$ is maximized. Again since G is δ-hyperbolic, $d_G(c,c') + d_G(w,y) \le \max\{d_G(c,w) + d_G(c',y), d_G(c,y) + d_G(c',w)\} + 2\delta$. Recall that by the choice of vertex w, $d_G(c,c') \ge \max\{d_G(c,w), d_G(c',w)\} + \tau$, and so, $d_G(w,y) \le \max\{d_G(c',y), d_G(c,y)\}+2\delta-\tau$. Since $c' \in C_\pi(T)$, we obtain $d_T(c',y) \le d_T(w,y)$, and so, $d_G(c',y) \le d_T(c',y) + q \le d_T(w,y) + q \le d_G(w,y) + p + q$. Furthermore, by the hypothesis $p + q + 2\delta - \tau < 0$. Therefore, $d_G(w,y) \le d_G(c,y) + 2\delta - \tau$.

Suppose $\pi(x)d_G(w,x) \le \pi(y)d_T(w,y)$. Then, $d_T(w,y) \le d_G(w,y) + p \le d_G(c,y) + p + 2\delta - \tau < d_G(c,y)$. It implies $\pi(y)d_G(c,y) > \pi(y)d_T(w,y) \ge \pi(x)d_G(w,x)$, thus contradicting our assumption that $c \in C_\pi(G)$. Therefore, $\pi(x)d_G(w,x) > \pi(y)d_T(w,y)$. Then, $d_G(w,x) \le d_G(c',x) + 2\delta - \tau \le d_T(c',x) + q + 2\delta - \tau < d_T(c',x)$. However, it implies $\pi(x)d_T(c',x) > \pi(x)d_G(w,x) > \pi(y)d_T(w,y)$, thus contradicting our assumption that $c' \in C_\pi(T)$. ◀

Theorem 6 now follows from the combination of Lemma 7, Proposition 9, and the following known algorithmic result:

▶ **Lemma 10** (see Section 3 in [39]). *Given a tree $T = (V,E)$, and an arbitrary profile π, one can solve the Weighted Center problem in $\mathcal{O}(n)$ time.*

3.2 Getting closer to the center

The remainder of this section is devoted to the proof of statement (2) of Theorem A.

▶ **Theorem 11.** *Given a δ-hyperbolic graph $G = (V, E)$, an arbitrary profile π, and a vertex c_1 and a radius ρ such that $C_\pi(G) \subseteq B_\rho(c_1)$, one can output in $\mathcal{O}(\rho m \log n)$ time a vertex c_2 such that $C_\pi(G) \subseteq B_{124\delta+21}(c_2)$.*

Note that Theorem 11, in combination with Theorem A(1) implies Theorem A(2). To prove Theorem 11, roughly, our strategy consists in projecting the support of the profile onto the ball $B_\rho(c_1)$. Then, roughly, we reduce the computation of an approximate center c_2 for π to that of approximate centers for $\mathcal{O}(\rho \log n)$ binary profiles. For that, we will also need some results on the Weighted Center problem for binary profiles:

▶ **Lemma 12** (see Proposition 6 in [14]). *Given a δ-hyperbolic graph $G = (V, E)$, and a binary profile π, one can output in $\mathcal{O}(m)$ time a vertex c^* such that $C_\pi(G) \subseteq B_{5\delta+1}(c^*)$.*

Consider the important special case of the eccentricity function, *i.e.*, where $\pi(v) = 1$ for every vertex v. Then, it is known that for any node v in a tree T, $\mathrm{ecc}(v) = d(v, C(T)) + \mathrm{rad}(T)$. Furthermore, a relaxation of this result was proved in [22], Theorem 5, for all δ-hyperbolic graphs. We prove it next for any binary profile.

▶ **Lemma 13.** *Let $G = (V, E)$ be δ-hyperbolic, and let π be a binary profile on G. For any vertex $v \in V$, $d(v, C_\pi(G)) + \mathrm{rad}_\pi(G) - 6\delta - 1 \le r_\pi(v) \le d(v, C_\pi(G)) + \mathrm{rad}_\pi(G)$.*

Proof. The upper bound holds for any graph G and any binary profile. For the lower bound, let $c \in C_\pi(G)$ be arbitrary, and let $u, w \in \mathrm{Supp}(\pi)$ be maximizing $d(u, w)$. Since G is δ-hyperbolic, $d(v, c) + d(u, w) \le \max\{d(v, u) + d(c, w), d(v, w) + d(c, u)\} + 2\delta$, and so, $d(v, c) + d(u, w) \le r_\pi(v) + r_\pi(c) + 2\delta = r_\pi(v) + \mathrm{rad}_\pi(G) + 2\delta$. By [14], Proposition 4, $d(u, w) \ge 2\,\mathrm{rad}_\pi(G) - 4\delta - 1$. Therefore, $r_\pi(v) \ge d(v, c) + d(u, w) - \mathrm{rad}_\pi(G) - 2\delta \ge d(v, C_\pi(G)) + d(u, w) - \mathrm{rad}_\pi(G) - 2\delta \ge d(v, C_\pi(G)) + (2\,\mathrm{rad}_\pi(G) - 4\delta - 1) - \mathrm{rad}_\pi(G) - 2\delta = d(v, C_\pi(G)) + \mathrm{rad}_\pi(G) - 6\delta - 1$. ◀

For simplicity of the presentation, we assume in what follows that the hyperbolicity value δ is known to us. In practice, any approximation $\delta' \ge \delta$ can be used, but then the distance of c_2 to $C_\pi(G)$ also depends on the approximation ratio.

Sketch Proof of Theorem 11. We shall assume throughout the proof that $\rho > 124\delta + 21$, since otherwise it suffices to output $c_2 = c_1$. Furthermore, we shall also assume that $\mathrm{Supp}(\pi)$ contains at least two vertices: otherwise, either $\mathrm{Supp}(\pi) = \emptyset$, and so, $C_\pi(G) = V$ holds; or $\mathrm{Supp}(\pi) = \{v\}$ for some $v \in V$, and then, $C_\pi(G) = \{v\}$ holds. By Lemma 2 (slightly modified), we only need to present an $\mathcal{O}(\rho m)$-time algorithm such that, for any value τ: it either asserts $\mathrm{rad}_\pi(G) < \tau$; or it asserts $\mathrm{rad}_\pi(G) > \tau$; or it outputs some c^* such that $C_\pi(G) \subseteq B_{124\delta+21}(c^*)$.

For every $v \in \mathrm{Supp}(\pi)$, let $q(v)$ be a closest-to-v vertex in the ball $B_\rho(c_1)$, and let $\ell(v) = d(v, q(v))$. – Roughly, we project the support of π onto $B_\rho(c_1)$. – So, in particular, if $d(c_1, v) \le \rho$, then we set $q(v) = v$, and $\ell(v) = 0$. If $d(c_1, v) > \rho$, then we set $\ell(v) = d(v, c_1) - \rho$, and we choose an arbitrary $q(v) \in S_\rho(c_1, v)$. All the values $q(v), \ell(v)$, for $v \in \mathrm{Supp}(\pi)$, can be computed by dynamic programming on a shortest-path tree rooted at c_1.

Let τ be some value to be compared to $\mathrm{rad}_\pi(G)$. For each $v \in \mathrm{Supp}(\pi)$, we compute $\lambda_\tau(v) = \left\lfloor \frac{\tau}{\pi(v)} \right\rfloor$: the maximum distance between v and any c such that $r_\pi(c) \le \tau$. If $\lambda_\tau(v) < \ell(v)$ for some $v \in \mathrm{Supp}(\pi)$, then as $C_\pi(G) \subseteq B_\rho(c_1)$, we assert $\mathrm{rad}_\pi(G) > \tau$. If

$\lambda_\tau(v) \geq \ell(v) + 2\rho$, then as $C_\pi(G) \subseteq B_\rho(c_1) \subseteq B_{\lambda_\tau(v)}(v)$, we can discard v. Furthermore, if $\lambda_\tau(v') \geq \ell(v') + 2\rho$ for *every* $v' \in \mathrm{Supp}(\pi)$, then it implies $r_\pi(c_1) < \tau$, and so, we assert $\mathrm{rad}_\pi(G) < \tau$. Thus, we now assume $\ell(v) \leq \lambda_\tau(v) \leq \ell(v) + 2\rho - 1$ for each $v \in \mathrm{Supp}(\pi)$. To avoid some degenerate cases in our analyzes, we further assume $\lambda_\tau(v) > 0$ for each $v \in \mathrm{Supp}(\pi)$. Indeed, assume $\lambda_\tau(v) = 0$ for some v. If $v \notin B_\rho(c_1)$, then, as $\ell(v) > \lambda_\tau(v)$, we assert $\mathrm{rad}_\pi(G) > \tau$. Otherwise, either $r_\pi(v) \leq \tau$, or $r_\pi(v) > \tau$. In the former case, necessarily $C_\pi(G) = \{v\}$. In the latter case, we assert $\mathrm{rad}_\pi(G) > \tau$.

For every i such that $0 \leq i \leq 2\rho - 1$, we consider $M_i = \{q(v) : \lambda_\tau(v) = \ell(v) + i\}$. Roughly, if $M_i \neq \emptyset$, then we want to construct some superset C^i of $C_\pi(G)$. More specifically, let us define a binary profile π_i such that $\pi_i(v) = 1$ if and only if $v \in M_i$. By Lemma 12, a vertex c^i such that $C_{\pi_i}(G) \subseteq B_{5\delta+1}(c^i)$ can be computed in $\mathcal{O}(m)$ time. If $r_{\pi_i}(c^i) > i + 7\delta + 1$, then we assert $\mathrm{rad}_\pi(G) > \tau$. Roughly, this is because if $\mathrm{rad}_\pi(G) \leq \tau$, then we can prove $\mathrm{rad}_{\pi_i}(G) \leq i + 2\delta$ (and so, $r_{\pi_i}(c^i) \leq i + 7\delta + 1$) using Lemma 5. See Claim 28 in the appendix. Otherwise, we set $C^i = B_{i - r_{\pi_i}(c^i) + 18\delta + 3}(c^i) \cap B_\rho(c_1)$.

Let $C^* = \bigcap\{C^i : M_i \neq \emptyset\}$. If $C^* = \emptyset$, then we assert $\mathrm{rad}_\pi(G) > \tau$. Roughly, this is because if $\mathrm{rad}_\pi(G) \leq \tau$, then we can use Lemma 13 to prove $C_\pi(G) \subseteq C^*$. See Claim 29 in the appendix. Else, let $c^* \in C^*$ be arbitrary. If $C^* \subseteq B_{124\delta+21}(c^*)$, then we output c^*. Otherwise, we assert that $\mathrm{rad}_\pi(G) < \tau$. This is the most technical part of the proof. Recall that if $\mathrm{rad}_\pi(G) \leq \tau$, then we can prove $C_\pi(G) \subseteq C^*$. So, we need to consider the case $\mathrm{rad}_\pi(G) \geq \tau$. Then, we prove that for some $u, v \in \mathrm{Supp}(\pi)$, $\lambda_\tau(u) + \lambda_\tau(v) - \mathcal{O}(\delta) \leq d(u,v) \leq \lambda_\tau(u) + \lambda_\tau(v) + \mathcal{O}(\delta)$. For that, we exploit both the fact that $C^* \neq \emptyset$ (for the upper bound) and the Helly-type property of Lemma 3 (for the lower bound). Furthermore, due the way we construct C^*, and due to the definition of $C_\pi(G)$, we observe that every vertex of $C^* \cup C_\pi(G)$ must be contained in an almost shortest (u,v)-path: at distances $\lambda_\tau(u) \pm \mathcal{O}(\delta)$ and $\lambda_\tau(v) \pm \mathcal{O}(\delta)$ of u and v. As a result, we can use Lemma 4 to upper bound the diameter of $C^* \cup C_\pi(G)$. See Claims 31 and 32 in the appendix.

As we call Lemma 12 $\mathcal{O}(\rho)$ times, the runtime is in $\mathcal{O}(\rho m)$. ◀

4 Exact algorithms

We present (almost) linear-time algorithms for the Weighted Center problem. With the notable exception of Sec. 4.3, all the results in what follows are based on our Theorem A, used in combination with other properties of the classes of graphs considered. Furthermore, for the classes of graphs with unbounded hyperbolicity, we present parameterized algorithms, where the parameter is the hyperbolicity value.

4.1 Bounded-degree graphs

The *growth function* of a graph $G = (V, E)$ is defined as $\gamma_G : \rho \in \mathbb{N} \mapsto \max\{|B_\rho(v)| : v \in V\}$. Note that $\gamma_G(\rho) = \mathcal{O}(\Delta^\rho)$ if G has maximum degree Δ. A graph G of *polynomial growth* is one such that $\gamma_G(\rho) = \mathcal{O}(\rho^\alpha)$, for some fixed exponent α. See [40], and the references therein, for previous studies on the graphs of polynomial growth. Finally, a graph G has *doubling dimension* β if every ball in G of positive radius is included in the union of at most 2^β balls with half-radius. Then, $\gamma_G(\rho) = \mathcal{O}(\rho^\beta)$.

▶ **Theorem 14.** *For any δ-hyperbolic graph G, and for any profile π, the Weighted Center problem can be solved in $\mathcal{O}((\gamma_G(\mathcal{O}(\delta)) + \delta \log^2 n)m)$ time, with γ_G the growth function. In particular, this algorithm runs*

1. *in $\mathcal{O}((\Delta^{O(\delta)} + \delta \log^2 n)m)$ time, if G has maximum degree Δ;*
2. *in $\mathcal{O}(\delta^\alpha m \log^2 n)$ time, if G has polynomial growth with exponent α;*
3. *in $\mathcal{O}(\delta^\beta m \log^2 n)$ time, if G has doubling dimension β.*

Proof. We apply Theorem A(2) to compute a vertex c_2 such that $C_\pi(G) \subseteq B_{124\delta+21}(c_2)$. To compute $C_\pi(G)$, it suffices to compute $r_\pi(x)$, for every $x \in B_{124\delta+21}(c_2)$, which requires $\mathcal{O}(m)$ time per vertex. There are at most $\gamma_G(124\delta + 21)$ vertices in the ball. ◀

4.2 Chordal graphs, and weakly bridged graphs

Recall that a *chordal graph* is one with no induced cycle of length more than three. A graph G is called *weakly bridged* if it has convex balls and if any of its induced cycles of length five is contained in the neighborhood of some vertex. Note that chordal graphs are weakly bridged. However, chordal graphs are 1-hyperbolic [9], whereas the hyperbolicity of weakly bridged graphs is unbounded [36]. The properties of weakly bridged graphs were studied in [17]. In particular, they satisfy the following *Triangle Condition*:

- (TC) If $u \sim v$ and $d(u, w) = d(v, w)$, for some vertices u, v, w, then $S_1(u, w) \cap S_1(v, w) \neq \emptyset$.

We will need the following characterization of graphs with convex balls:

▶ **Lemma 15** (see Theorem 2 in [42])**.** *All balls $B_k(v)$ ($v \in V$, $k \geq 1$) of a graph G are convex if and only if G does not contain isometric cycles of length $\ell > 5$, and for any two vertices x, y of G the slice $S_1(x, y)$ is a clique.*

The Weighted Center problem on weakly bridged graphs was studied in [11], Sec. 5. In particular, every radius function of a weakly bridged graph is 2-weakly peakless (see [11], Proposition 5.5). In what follows, we gather some more results from [11]; we provide more precise statements than the authors in [11], based on a careful analysis of their proofs. Let $G = (V, E)$ be weakly bridged, and let π be an arbitrary profile:

- (WB.1) for any $v \in V$ such that v is *not* a local minimum of r_π in G, we can compute in $\mathcal{O}(m)$ time the subset of all its neighbors $u \in N(v)$ such that $r_\pi(u) < r_\pi(v)$, and the corresponding values $r_\pi(u)$ (follows from [11], Lemma 5.9 and its proof);
- (WB.2) for any $v \in V$ such that v is a local minimum of r_π in G, but $v \notin C_\pi(G)$, a vertex $u \in B_2(v)$ such that $r_\pi(u) < r_\pi(v)$ can be computed in $\mathcal{O}(m)$ time; furthermore, for any $c \in C_\pi(G)$, there exists some $u_c^* \in S_2(v, c)$ such that $u \in N[u_c^*]$ (follows from [11], Proposition 5.11 and its proof[3]).

Starting from any vertex v_0, we can repeatedly apply either (WB.1) or (WB.2) until we find a center. By doing so, we construct an improving sequence $v_0, v_1, \ldots, v_\ell$ for which r_π is monotonically decreasing, and $v_\ell \in C_\pi(G)$. Roughly, to bound the length ℓ of the sequence, we must enforce that $d(v_{i+1}, C_\pi(G)) < d(v_i, C_\pi(G))$ at each step i of the process.

▶ **Lemma 16.** *Let $G = (V, E)$ be weakly bridged, let π be an arbitrary profile, and let $v \in V$. If v is not a local minimum of r_π in G, then, for any $c \in C_\pi(G)$, there exists some neighbor u_c of v such that $r_\pi(u_c)$ is minimum within $N[v]$, and $d(u_c, c) < d(v, c)$.*

Proof. Let u_0 be minimizing r_π within $N[v]$. As v is a not a local minimum of r_π in G, $r_\pi(u_0) < r_\pi(v)$. Suppose by contradiction $d(v, c) < d(u_0, c)$. For every $x \in \mathrm{Supp}(\pi)$, as the balls in G are convex, we get $d(w, x) \leq \max\{d(u_0, x), d(c, x)\}$ for every $w \in I(u_0, c)$. In particular, $r_\pi(w) \leq \max\{r_\pi(u_0), r_\pi(c)\} = r_\pi(u_0) < r_\pi(v)$. As $v \in I(u_0, c)$, a contradiction arises. Therefore, $d(u_0, c) \leq d(v, c)$. If $d(u_0, c) < d(v, c)$, then we set $u_c := u_0$, and we are done. Otherwise, by (TC), there exists some $u_1 \in S_1(u_0, c) \cap S_1(v, c)$. By similar convexity arguments as above, $r_\pi(u_1) \leq r_\pi(u_0)$. Due to the minimality of $r_\pi(u_0)$ within $N[v]$, we obtain $r_\pi(u_1) = r_\pi(u_0)$. We set $u_c := u_1$. ◀

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[3] More specifically, for some $u_c^* \in S_2(v, c)$ the authors in [11] proved that $r_\pi(u_c^*) < r_\pi(v)$, and that v, u, u_c^* have some common neighbor $w_{\max}$. As v is a local minimum of r_π in G, $r_\pi(u) < r_\pi(v) \leq r_\pi(w_{\max})$, and similarly $r_\pi(u_c^*) < r_\pi(w_{\max})$. It implies that for any $x \in F_\pi(w_{\max})$, $u, u_c^* \in S_1(w_{\max}, x)$ holds. By Lemma 15, either $u = u_c^*$, or $u \sim u_c^*$.

▶ **Lemma 17.** *Let $G = (V, E)$ be weakly bridged, let π be an arbitrary profile, and let $v \in V$. If $v \notin C_\pi(G)$, then a vertex $u \in B_3(v)$ such that $r_\pi(u) < r_\pi(v)$, and $d(u, C_\pi(G)) < d(v, C_\pi(G))$, can be computed in $\mathcal{O}(m)$ time.*

Proof. We start with the following claim:

▷ Claim 18. If (WB.2) applies to v, then it returns a vertex $u \in B_2(v)$ such that $r_\pi(u) < r_\pi(v)$, and $d(u, C_\pi(G)) < d(v, C_\pi(G))$.

Proof. Indeed, let $c \in C_\pi(G)$ be closest to v. Recall that there exists some $u_c^* \in S_2(v, c)$ such that $u \in N[u_c^*]$. Hence, $d(u, C_\pi(G)) \leq d(u, c) \leq 1 + d(u_c^*, c) = d(v, c) - 1$. ◁

We now describe our algorithm for an arbitrary vertex v. If v is a local minimum of r_π in G, then we are done by Claim 18. Otherwise, we apply (WB.1) to v. By doing so, we compute the subset $X(v)$ of all its neighbors minimizing r_π. As $X(v) \subseteq S_1(v, z)$ for each $z \in F_\pi(v)$, by Lemma 15, $X(v)$ is a clique. Furthermore, we observe that $d(u', C_\pi(G)) \leq d(v, C_\pi(G))$ for every $u' \in X(v)$. Indeed, this is because $X(v)$ is a clique and, by Lemma 16 applied for any $c \in C_\pi(G)$ closest to v, some unknown vertex in $X(v)$ is at distance $d(v, C_\pi(G)) - 1$ to the center. We pick some arbitrary $u_0 \in X(v)$. If (WB.1), (WB.2) cannot be applied to u_0, then necessarily $u_0 \in C_\pi(G)$. In this situation, we return $u := u_0$. Assume now that (WB.2) applies to u_0. By Claim 18, the latter returns some $u_1 \in B_2(u_0) \subseteq B_3(v)$ such that: $r_\pi(u_1) < r_\pi(u_0) < r_\pi(v)$; and $d(u_1, C_\pi(G)) < d(u_0, C_\pi(G)) \leq d(v, C_\pi(G))$. In this situation, we return $u := u_1$. Otherwise, we apply (WB.1) to u_0 to compute the subset $X(u_0)$ of all its neighbours minimizing r_π. By similar convexity arguments as above, $X(u_0)$ is a clique. Furthermore, $d(u'', C_\pi(G)) \leq d(u_0, C_\pi(G)) \leq d(v, C_\pi(G))$ for every $u'' \in X(u_0)$. Let u_2 be maximizing $|N(u_2) \cap X(v)|$ within $X(u_0)$. In this situation, we return $u := u_2$. Indeed, we first note that $r_\pi(u_2) < r_\pi(u_0) < r_\pi(v)$. Suppose now by contradiction $d(u_2, C_\pi(G)) \geq d(v, C_\pi(G))$. Then, $d(u_2, C_\pi(G)) = d(u_0, C_\pi(G)) = d(v, C_\pi(G)) = k$. Let $c_0 \in C_\pi(G)$ be closest to v. By Lemma 16 applied to both v, u_0, there exist $x_0 \in X(v)$, $x_2 \in X(u_0)$ such that $d(x_0, c_0) = d(x_2, c_0) = k - 1$. As $x_0, x_2 \in S_1(u_0, c_0)$, by Lemma 15, we get $x_0 \sim x_2$. However, as G has convex balls, there is no induced C_4. It implies that the subsets $N(u'') \cap X(v)$, for $u'' \in X(u_0)$, must be comparable for inclusion. Therefore, due to the maximality of $|N(u_2) \cap X(v)|$, $u_2 \sim x_0$ holds. It implies that $x_0 \in S_1(u_2, c_0)$. But then, for every $y \in F_\pi(u_2)$, as G has convex balls, $d(w, y) \leq \max\{d(u_2, y), d(c_0, y)\}$ holds for every $w \in I(u_2, c_0)$. So, $r_\pi(w) \leq \max\{r_\pi(u_2), r_\pi(c_0)\} = r_\pi(u_2)$ for every $w \in I(u_2, c_0)$. As $x_0 \in I(u_2, c_0)$, and $r_\pi(x_0) = r_\pi(u_0) > r_\pi(u_2)$, a contradiction arises. ◀

We can now state the main result of this section. Theorem A(1) is used for selecting the start vertex of our local search. To improve the runtime for chordal graphs, we use a stronger version of this result (Theorem 6), in combination with the following property of this class of graphs:

▶ **Lemma 19** (see Theorem 1 in [8])**.** *Every chordal graph 3-embeds into a tree.*

▶ **Theorem 20.** *For every weakly bridged graph G, for any profile π, the* WEIGHTED CENTER *problem can be solved*
1. *in $\mathcal{O}(\delta m \log n)$ time if G is δ-hyperbolic;*
2. *and in $\mathcal{O}(m)$ time if G is chordal.*

Proof. We compute some vertex c_1 such that $d(c_1, C_\pi(G)) \in \mathcal{O}(\delta \log n)$, whose existence follows from Theorem A(1). If G is chordal, then by the combination of Lemma 19 with Theorem 6, the vertex c_1 can be chosen such that $d(c_1, C_\pi(G)) \in \mathcal{O}(1)$. Then, starting from $v := c_1$, we repeatedly apply Lemma 17 until we find a center. The number of steps of the local search is at most $d(c_1, C_\pi(G))$. Furthermore, each step can be done in $\mathcal{O}(m)$ time. ◀

4.3 Planar graphs

▶ **Theorem 21.** *There is an $2^{\mathcal{O}(\delta)} n \log^{10} n$-time algorithm for the* Weighted Center *problem on δ-hyperbolic planar graphs.*

Recall that a *separator* of a graph G is a subset S such that $G \setminus S$ is disconnected. The *balance* of a separator S is the largest α such that every connected component of $G \setminus S$ contains at most $(1-\alpha)n$ vertices. Finally, a separator S is called *isometric* if it induces an isometric (or distance-preserving) subgraph of G. Note that if S is an isometric separator, and C is any connected component of $G \setminus S$, then $C \cup S$ also induces an isometric subgraph.

▶ **Lemma 22** (see Theorem 1 in [35]). *If G is a planar δ-hyperbolic graph, then we can compute in $\mathcal{O}(\delta^2 n \log^4 n)$ time an isometric separator S such that*
1. *either S is a path of length $\mathcal{O}(\delta^2 \log n)$, and it has constant balance;*
2. *or S is a cycle of length $\mathcal{O}(\delta)$, and it has balance at least $\frac{1}{2^{\mathcal{O}(\delta)} \log n}$.*

As it can be expected, our strategy for proving Theorem 21 consists in recursively disconnecting a planar hyperbolic graph using Lemma 22. After each separation, for any connected component C, we need to compute, for each vertex of C, its maximum weighted distance to the vertices in the other connected components. For that, we use the algorithmic toolkit from [38], which is based on VC-theory. In what follows, for any graph G, the *boundary* of an induced subgraph H is defined as the subset of all the vertices of H with at least one neighbor in $V(G) \setminus V(H)$. We denote the boundary of H by ∂H. Furthermore, we assume for what follows that ∂H is totally ordered: $\partial H = \langle s_0, s_1, \ldots, s_{|\partial H|-1} \rangle$ (the ordering is arbitrary). Then, for any vertex v, the *pattern* of v with respect to ∂H is the $|\partial H|$-dimensional array $\mathtt{p}_v$ such that, for each i, $\mathtt{p}_v[i] = d(v, s_i) - d(v, s_0)$. The following result is proved in [38] for K_h-minor-free graphs, but we restate it only for planar graphs:

▶ **Lemma 23** (see Lemma 4 in [38]). *Let H be a connected induced graph of some planar graph G. The number of distinct patterns with respect to ∂H is in $\mathcal{O}(|\partial H||V(H)|^4)$.*

Since we are dealing with arbitrary profiles (not necessarily binary), in what follows we will need to combine Lemma 23 with the following convex hull trick:

▶ **Lemma 24** (see Lemma 4.2 in [25]). *Let F be a set of n linear functions $f_i : t \mapsto a_i \cdot t + b_i$, where $a_i, b_i \geq 0$. Then after an $\mathcal{O}(n \log n)$-time pre-processing, for any $x \geq 0$ we can compute $\max\{a_i \cdot x + b_i : 1 \leq i \leq n\}$ in $\mathcal{O}(\log n)$ time.*

Proof of Theorem 21. Assume for what follows $n \geq 2^{\Theta(\delta)}$ (otherwise, we solve the Weighted Center problem by brute-force in $\mathcal{O}(n^2) = 2^{\mathcal{O}(\delta)}$ time). We apply Lemma 22 to compute an isometric separator S in $\mathcal{O}(\delta^2 n \log^4 n)$ time. Furthermore, as $|S| = \mathcal{O}(\delta^2 \log n)$, we can afford to compute $r_\pi(s)$, for each $s \in S$; it takes $\mathcal{O}(\delta^2 n \log n)$ time in total. For the other vertices, let us consider the connected components $C_1, C_2, \ldots, C_q$, $q \geq 2$, of $G \setminus S$. For each i such that $1 \leq i \leq q$, let π_i denote the restriction of π to $C_i \cup S$. Then, for any vertex v of C_i, $r_\pi(v) = \max\{r_{\pi_i}(v)\} \cup \{\pi(u)d(v,u) : u \notin C_i \cup S\}$. As $C_i \cup S$ induces an isometric subgraph G_i of G, the graph G_i is also planar and δ-hyperbolic. In particular, all the values $r_{\pi_i}(v)$, for $v \in C_i$, can be computed by applying our algorithm recursively on G_i.

For each i such that $1 \leq i \leq q$, for each $v \in C_i$, define $\lambda(v) = \max\{\pi(u)d(v,u) : u \notin C_i \cup S\}$. By the above, $r_\pi(v) = \max\{r_{\pi_i}(v), \lambda(v)\}$. So, we are left to compute the values $\lambda(v)$, for each $v \in V \setminus S$. For that, for each j with $1 \leq j \leq \lceil \log q \rceil$, let X_j, Y_j be such that:
- X_j contains all components C_i such that the j^{th} bit of i is equal to 0;
- Y_j contains all components C_i such that the j^{th} bit of i is equal to 1.

Note that for each pair of vertices u, v that are in different components of $G \setminus S$, there is at least one j such that either $u \in X_j, v \in Y_j$ or $u \in Y_j, v \in X_j$. Therefore, to compute all values $\lambda(v)$, it suffices to compute the following values $\lambda_j(v)$ for any j: if $v \in X_j$, then $\lambda_j(v) = \max\{\pi(u)d(v,u) : u \in Y_j\}$; and if $v \in Y_j$, then $\lambda_j(v) = \max\{\pi(u)d(v,u) : u \in X_j\}$. By symmetry, it suffices to present an algorithm for computing all values $\lambda_j(v)$, for $v \in X_j$.

For that, let $\partial S = s_0, s_1, \ldots, s_{|\partial S|-1}$. We compute all profiles $\mathtt{p}_u$, for $u \in Y_j$, with respect to ∂S. This can be done in $\mathcal{O}(|\partial S|n) = \mathcal{O}(\delta^2 n \log n)$ by performing a BFS for each $s_i \in \partial S$. Denote by P_S the set of all distinct profiles with respect to ∂S. Note that P_S can be computed in $\mathcal{O}(|\partial S| n \log n) = \mathcal{O}(\delta^2 n \log^2 n)$ time, by sorting. Furthermore, by Lemma 23, it holds $|P_S| = \mathcal{O}(|\partial S||S|^4) = \mathcal{O}(|S|^5) = \mathcal{O}(\delta^{10} \log^5 n)$. For each $\mathtt{p} \in P_S$, let $Y_{j,\mathtt{p}}$ contain all the vertices $u \in Y_j$ such that $\mathtt{p}_u = \mathtt{p}$. For any $v \in X_j$, let $\lambda_{j,\mathtt{p}}(v) = \max\{\pi(u)d(v,u) : u \in Y_{j,\mathtt{p}}\}$. To compute $\lambda_j(v)$, it suffices to compute $\lambda_{j,\mathtt{p}}$ for each $\mathtt{p} \in P_S$.

So, let $\mathtt{p} \in P_S$ be fixed. For each k with $0 \leq k \leq |\partial S| - 1$, we define the family of functions $F_{j,\mathtt{p},k} = \{t \mapsto \pi(u) \cdot t + \pi(u)d(s_k, u) : u \in Y_{j,\mathtt{p}}\}$, for which we apply Lemma 24. It takes $\mathcal{O}(|\partial S| n \log n) = \mathcal{O}(\delta^2 n \log^2 n)$ time in total. Then, we consider each $v \in X_j$ sequentially. Let k be such that $d(v, s_k) + \mathtt{p}[k]$ is minimum. We claim that for any $u \in Y_{j,\mathtt{p}}$, $s_k \in I(v,u)$. To prove this claim, as any shortest (v,u)-path contains a vertex of ∂S, it suffices to prove that $d(v, s_k) + d(s_k, u)$ is minimum. That is indeed the case because for any k' with $0 \leq k' \leq |\partial S| - 1$, $d(v, s_k) + d(s_k, u) - d(s_0, u) = d(v, s_k) + \mathtt{p}[k] \leq d(v, s_{k'}) + \mathtt{p}[k'] = d(v, s_{k'}) + d(s_{k'}, u) - d(s_0, u)$. Consequently, by Lemma 24 applied for $F_{j,\mathtt{p},k}$ and $x = d(v, s_k)$, we can compute $\lambda_{j,\mathtt{p}}(v)$ in $\mathcal{O}(\log n)$ time.

Overall, the values $\lambda_{j,\mathtt{p}}(v)$, for $v \in X_j$, can be computed in $\mathcal{O}(\delta^2 n \log^2 n)$ time. It implies that the values $\lambda_j(v)$, for $v \in X_j$, can be computed in $\mathcal{O}(|P_S|\delta^2 n \log^2 n) = \mathcal{O}(\delta^{12} n \log^7 n)$ time. Hence, the values $\lambda(v)$, for $v \in V \setminus S$, can be computed in $\mathcal{O}(\delta^{12} n \log^8 n)$ time.

Finally, by Lemma 22, the balance of S is at least $1/(2^{\mathcal{O}(\delta)} \log n)$. Furthermore, as we assume $n \geq 2^{\Theta(\delta)}$, the inclusion of S in all the G_i's has a marginal impact on the size of these subgraphs. Therefore, the recursion depth of our algorithm is at most $2^{\mathcal{O}(\delta)} \log^2 n$. ◀

4.4 More classes of graphs

We obtain almost optimal algorithms for the WEIGHTED CENTER problem on dually chordal graphs, and chordal bipartite graphs. A graph is *dually chordal* if it is the intersection graph of the maximal cliques of some chordal graph. The same as chordal graphs can be characterized by the existence of a perfect elimination ordering, the dually chordal graphs can be also characterized by the existence of a special type of dismantling ordering [7]. A *chordal bipartite graph* is a bipartite graph with no induced cycle of length more than four. Both the dually chordal graphs and the chordal bipartite graphs are 1-hyperbolic. We also consider some of their metric generalizations. Namely, a graph is called *Helly* if every family of pairwise intersecting balls has a nonempty common intersection. A *half-ball* in a bipartite graph is the intersection of a ball with one of its two partite sets. A bipartite graph is called *bipartite Helly* if and only if every family of pairwise intersecting half-balls has a nonempty common intersection. Interestingly, every (bipartite) graph G can be isometrically embedded in a smallest (bipartite) Helly graph $\mathcal{H}(G)$, which is called its *injective hull*. Furthermore, dually chordal graphs and chordal bipartite graphs are Helly and bipartite Helly, respectively.

▶ **Theorem 25.** *For a (bipartite) Helly graph, the* WEIGHTED CENTER *problem can be solved*

1. *in $\mathcal{O}(\delta^3 m \log^2 n)$ time, if it is δ-hyperbolic;*
2. *in $\mathcal{O}(m \log n)$ time, if it is dually chordal or chordal bipartite.*

Roughly, we use Theorem A to restrict the search for a center to some ball of radius $\mathcal{O}(\delta)$. Then, for any profile π, for any value τ, we can identify the vertices u of this ball such that $r_\pi(u) \leq \tau$ using a nontrivial combination of prior results for binary profiles on these classes of graphs and a generalization of the techniques from [21], Section 4.1.

Finally, a graph G is called *distance-hereditary* if every induced path is a shortest-path. Note that the distance-hereditary graphs are 1-hyperbolic [43].

▶ **Theorem 26.** *The* Weighted Center *problem can be solved in $\mathcal{O}(m)$ time for distance-hereditary graphs.*

We prove new metric properties of distance-hereditary graphs, which we use in the design of our $\mathcal{O}(m)$-time algorithm. In particular, radius functions are 2-weakly peakless. Therefore, a local-search algorithm for computing a center can be derived. To implement each step of the local search in $\mathcal{O}(m)$ time, we use the injective hull. In fact, we need the result from [31] that the injective hull of a distance-hereditary graph is both Helly and distance-hereditary. Furthermore, [31] also provides an $\mathcal{O}(m)$-time algorithm for computing the injective hull of a distance-hereditary graph.

5 Perspectives

The main contribution of the paper is an algorithm for computing on any δ-hyperbolic graph with m edges, for any profile π, some vertex at distance $\mathcal{O}(\delta)$ to the center in $\tilde{\mathcal{O}}(\delta m)$ time (Theorem 11). In doing so, we answer one of the questions left open in [11] (Question 10.1). Our dependency on δ is linear, but it is quite large. Improving this constant could be an interesting challenge.

Based on this main result, a framework for solving the Weighted Center problem exactly on some classes of hyperbolic graphs is proposed. It implies optimal, or almost optimal, algorithms for the Weighted Center problem on chordal graphs, chordal bipartite graphs, dually chordal graphs and distance-hereditary graphs. However, the inclusion to our framework of AT-free graphs, graphs of bounded asteroidal number, and other classic examples of hyperbolic graphs, remains to be done.

Finally, we present parameterized almost linear-time algorithms for the Weighted Center problem on planar graphs and several other classes of graphs, where the parameter is the hyperbolicity. We leave for future work whether similar hyperbolicity-based parameterizations can be also achieved for some other distance-related problems on these classes of graphs.

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A Missing proofs of Section 3

The following claims complete the correctness proof of the algorithm of Theorem 11. We put all the claims together at the end of the section.

▷ Claim 27. If $\lambda_\tau(v) \geq \ell(v) + 2\rho$ for every $v \in \mathrm{Supp}(\pi)$, then $\mathrm{rad}_\pi(G) < \tau$.

Proof. Indeed, for any $v \in \mathrm{Supp}(\pi)$, $d(c_1, v) \leq \ell(v) + \rho \leq \ell(v) + 2\rho - 1$. So, in particular, $\bigcap\{B_{\lambda_\tau(v)-1}(v) : v \in \mathrm{Supp}(\pi)\} \neq \emptyset$. ◁

▷ Claim 28. If $M_i \neq \emptyset$, and z is any vertex of $B_\rho(c_1)$ such that $r_\pi(z) \leq \tau$, then $r_{\pi_i}(z) \leq i+2\delta$. Furthermore, if $\mathrm{rad}_\pi(G) \leq \tau$, then $\mathrm{rad}_{\pi_i}(G) \leq i + 2\delta$.

Proof. Suppose, for the sake of contradiction, $r_{\pi_i}(z) > i + 2\delta$. Let $v \in \mathrm{Supp}(\pi)$ be chosen such that $q(v) \in F_{\pi_i}(z)$. Assume first that $v \in B_\rho(c_1)$. As $d(z, v) > i = \lambda_\tau(v)$, we get $r_\pi(z) \geq \pi(v)d(z, v) > \tau$, thus contradicting our assumption that $r_\pi(z) \leq \tau$. Thus, $v \notin B_\rho(c_1)$. As we assume $z \in B_\rho(c_1)$, by Lemma 5, we obtain $d(v, q(v)) + d(q(v), z) \leq d(v, z) + 2\delta$. Therefore, $d(v, z) \geq \ell(v) + d(q(v), z) - 2\delta > \ell(v) + i = \lambda_\tau(v)$. Again, this contradicts our assumption that $r_\pi(z) \leq \tau$. So, $\mathrm{rad}_{\pi_i}(G) \leq r_{\pi_i}(z) \leq i+2\delta$. Finally, note that if $\mathrm{rad}_\pi(G) \leq \tau$, then as we assume $C_\pi(G) \subseteq B_\rho(c_1)$, there exists a $z \in B_\rho(c_1)$ such that $r_\pi(z) \leq \tau$. ◁

▷ Claim 29. For any vertex $z \in B_\rho(c_1)$ such that $r_\pi(z) \leq \tau$, $z \in C^*$.

Proof. It suffices to prove that $z \in C^i$ for any i such that $M_i \neq \emptyset$. By Claim 28, $r_{\pi_i}(z) \leq i+2\delta$. Therefore, by Lemma 13, $d(z, C_{\pi_i}(G)) \leq r_{\pi_i}(z) - \mathrm{rad}_{\pi_i}(G) + 6\delta + 1 \leq i - \mathrm{rad}_{\pi_i}(G) + 8\delta + 1$. As $C_{\pi_i}(G) \subseteq B_{5\delta+1}(c^i)$, it implies that

$$\begin{aligned} d(z, c^i) &\leq d(z, C_{\pi_i}(G)) + 5\delta + 1 \leq i - \mathrm{rad}_{\pi_i}(G) + 13\delta + 2 \\ &\leq i - (r_{\pi_i}(c^i) - 5\delta - 1) + 13\delta + 2 \\ &\leq i - r_{\pi_i}(c^i) + 18\delta + 3. \end{aligned}$$

Therefore, $z \in C^i$. ◁

▷ Claim 30. If $z \in C^*$, and $v \in \mathrm{Supp}(\pi)$, then $d(z, v) \leq \lambda_\tau(v) + 18\delta + 3$.

Proof. As $C^* \neq \emptyset$, we get $\lambda_\tau(v) > 0$. If $\lambda_\tau(v) \geq \ell(v) + 2\rho$, then it holds $C^* \subseteq B_\rho(c_1) \subseteq B_{\lambda_\tau(v)}(v)$, and so, we are done. Thus, from now on, let i be such that $0 \leq i \leq 2\rho - 1$, and $q(v) \in M_i$. Then, $d(z, v) \leq d(z, c^i) + d(c^i, q(v)) + d(q(v), v)$. Recall that $d(q(v), v) = \ell(v) = \lambda_\tau(v) - i$. Furthermore, as $q(v) \in M_i$, $d(c^i, q(v)) \leq r_{\pi_i}(c^i)$. As $z \in C^* \subseteq C^i$, we get $d(z, c^i) \leq i - r_{\pi_i}(c^i) + 18\delta + 3$. As a result, $d(z, v) \leq (i - r_{\pi_i}(c^i) + 18\delta + 3) + r_{\pi_i}(c^i) + (\lambda_\tau(v) - i) \leq \lambda_\tau(v) + 18\delta + 3$. ◁

▷ Claim 31. If $\mathrm{rad}_\pi(G) \geq \tau$, and if $C^* \neq \emptyset$, then $\mathrm{diam}(C^*) \leq 124\delta + 21$.

Proof. Assume first $\lambda_\tau(v) \leq 2\delta+1$ for some $v \in \mathrm{Supp}(\pi)$. By Claim 30, $C^* \subseteq B_{20\delta+4}(v)$ holds. Therefore, it holds $\mathrm{diam}(C^*) \leq 40\delta + 8$. From now on, we assume that $\lambda_\tau(v) > 2\delta + 1$ for every $v \in \mathrm{Supp}(\pi)$. As we assume $\mathrm{rad}_\pi(G) \geq \tau$, we obtain $\bigcap\{B_{\lambda_\tau(v)-1}(v) : v \in \mathrm{Supp}(\pi)\} = \emptyset$. Therefore, by the contrapositive of Lemma 3, there exist $u, v \in \mathrm{Supp}(\pi)$ such that $B_{\lambda_\tau(u)-1-2\delta}(u) \cap B_{\lambda_\tau(v)-1-2\delta}(v) = \emptyset$. In this situation, $d(u, v) \geq \lambda_\tau(u) + \lambda_\tau(v) - 4\delta - 1$. By Claim 30, $C^* \subseteq B_{\lambda_\tau(u)+18\delta+3}(u)$. Similarly, $C^* \subseteq B_{\lambda_\tau(v)+18\delta+3}(v)$.

We fix some shortest (u, v)-path P. Then, let Q be the subpath of P that is made of all the vertices w such that $\lambda_\tau(u) - 22\delta - 4 \leq d(u, w) \leq \lambda_\tau(u) + 18\delta + 3$. To prove the claim, we will prove that $d(z, Q) \leq 42\delta+7$ for every $z \in C^*$. By doing so, $\mathrm{diam}(C^*) \leq \mathrm{diam}(Q)+84\delta+14 \leq 124\delta + 21$. In what follows, let $z \in C^*$ be arbitrary. As $d(u, v) \leq d(u, z) + d(z, v)$, there must be a vertex w_z of P such that $d(u, w_z) \leq d(u, z)$, $d(v, w_z) \leq d(v, z)$. Recall that $d(u, z) \leq \lambda_\tau(u) + 18\delta + 3$, $d(v, z) \leq \lambda_\tau(v) + 18\delta + 3$, and $d(u, v) \geq \lambda_\tau(u) + \lambda_\tau(v) - 4\delta - 1$. Consequently, $d(u, z) + d(z, v) \leq d(u, v) + \mu$, where $\mu = 40\delta + 7$. By Lemma 4, $d(z, w_z) \leq \mu + 2\delta = 42\delta + 7$. So, it remains to show that $w_z \in Q$. As $d(u, w_z) \leq d(u, z)$, we get $d(u, w_z) \leq \lambda_\tau(u) + 18\delta + 3$. Finally,

$$\begin{aligned} d(u, w_z) &= d(u, v) - d(v, w_z) \geq d(u, v) - d(v, z) \\ &\geq (\lambda_\tau(u) + \lambda_\tau(v) - 4\delta - 1) - (\lambda_\tau(v) + 18\delta + 3) \\ &\geq \lambda_\tau(u) - 22\delta - 4. \end{aligned}$$

Therefore, $w_z \in Q$. ◁

▷ Claim 32. If $C^* \neq \emptyset$, and $C^* \subseteq B_{124\delta+21}(c^*)$ for some $c^* \in C^*$, then also $C_\pi(G) \subseteq B_{124\delta+21}(c^*)$.

Proof. The result follows from Claim 29 if $\mathrm{rad}_\pi(G) \leq \tau$. Thus from now on, let us assume $\mathrm{rad}_\pi(G) = \gamma > \tau$. Note that $\lambda_\gamma(v) \geq \lambda_\tau(v)$ for any $v \in \mathrm{Supp}(\pi)$. Assume first $\lambda_\gamma(v) \leq 2\delta+1$ for some $v \in \mathrm{Supp}(\pi)$. Then, $C_\pi(G) \subseteq B_{2\delta+1}(v) \subseteq B_{22\delta+5}(c^*)$ holds, where the first inclusion follows from the definition of $\lambda_\gamma(v)$ and the second inclusion follows from Claim 30. From now on, we assume $\lambda_\gamma(v) > 2\delta + 1$ for every $v \in \mathrm{Supp}(\pi)$. Furthermore, as $\mathrm{rad}_\pi(G) = \gamma$, $\bigcap\{B_{\lambda_\gamma(v)-1}(v) : v \in \mathrm{Supp}(\pi)\} = \emptyset$. Hence, by the contrapositive of Lemma 3, there exist $u, v \in \mathrm{Supp}(\pi)$ such that $d(u,v) \geq \lambda_\gamma(u) + \lambda_\gamma(v) - 4\delta - 1$. Recall that as $c^* \in C^*$, by Claim 30, $d(u,c^*) \leq \lambda_\tau(u) + 18\delta + 3 \leq \lambda_\gamma(u) + 18\delta + 3$. Similarly, $d(v,c^*) \leq \lambda_\gamma(v) + 18\delta + 3$.

Let $c \in C_\pi(G)$ be arbitrary. In particular, $d(u,c) \leq \lambda_\gamma(u)$, and in the same way $d(v,c) \leq \lambda_\gamma(v)$. Since G is δ-hyperbolic,

$$\begin{aligned} d(u,v) + d(c,c^*) &\leq \max\{d(u,c) + d(v,c^*), d(u,c^*) + d(v,c)\} + 2\delta \\ &\leq \lambda_\gamma(u) + \lambda_\gamma(v) + 20\delta + 3 \\ &\leq d(u,v) + 24\delta + 4. \end{aligned}$$

As a result, $d(c,c^*) \leq 24\delta + 4 < 124\delta + 21$. ◁

We are now ready to prove the correctness of the algorithm of Theorem 11. For any i such that $M_i \neq \emptyset$, as $C_{\pi_i}(G) \subseteq B_{5\delta+1}(c^i)$, we get $\mathrm{rad}_{\pi_i}(G) \geq r_{\pi_i}(c^i) - 5\delta - 1$. Thus, if $r_{\pi_i}(c^i) > i + 7\delta + 1$, then we get $\mathrm{rad}_{\pi_i}(G) > i + 2\delta$. By the contrapositive of Claim 28, $\mathrm{rad}_\pi(G) > \tau$. Assume now $r_{\pi_i}(c^i) \leq i+7\delta+1$ for each i. If $C^* = \emptyset$, then, by the contrapositive of Claim 29, $\mathrm{rad}_\pi(G) > \tau$. Otherwise, let $c^* \in C^*$ be arbitrary. If $C^* \not\subseteq B_{124\delta+21}(c^*)$, then, we obtain $\mathrm{diam}(C^*) > 124\delta + 21$; by the contrapositive of Claim 31, $\mathrm{rad}_\pi(G) < \tau$. However, if $C^* \subseteq B_{124\delta+21}(c^*)$, then by Claim 32, $C_\pi(G) \subseteq B_{124\delta+21}(c^*)$.