Small Independent Sets Versus Small Separator in Geometric Intersection Graphs

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Abstract

While most classical NP-hard graph problems cannot be solved in time $2^{o(n)}$ on general graphs under the Exponential Time Hypothesis (ETH), many exhibit the *square-root phenomenon* and admit optimal algorithms running in time $2^{O(\sqrt{n})}$ on certain geometric intersection graphs, such as planar graphs or unit disk graphs. In 2018, de Berg *et al.* developed a general algorithmic framework for such problems on intersection graphs of similarly sized fat objects in $\mathbb{R}^d$, achieving running times of the form $2^{O(n^{1-1/d})}$, along with matching lower bounds under ETH.

In this paper, we identify problems that do not exhibit the square-root phenomenon, yet still admit subexponential algorithms on intersection graphs of similarly sized fat objects in $\mathbb{R}^d$, for every fixed dimension $d \geqslant 2$. We introduce the notion of a *weak square-root phenomenon*: problems that can be solved in time $2^{\tilde{O}(n^{1-1/(d+1)})}$, and for which matching lower bounds hold under ETH. We develop both an algorithmic framework and a corresponding lower bound framework. As concrete examples, we show that the problems 2-Subcoloring and Two Sets Cut-Uncut exhibit this behavior.

Our algorithms rely on a new win-win structural theorem, which can be informally stated as follows: every such graph admits a sublinear separator whose removal leaves connected components with sublinear independence number. To facilitate the design of these algorithms, we introduce a new graph parameter, the *α-modulator number*, which generalizes both the independence number and the vertex cover number.

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1 Introduction

Many classical graph problems require running time $2^{\Omega(n)}$ on general graphs. However, the same problems often admit *subexponential time* algorithms, that is, algorithms running in time $2^{o(n)}$, when the input graph enjoys certain geometric properties. The most famous example is the *square-root phenomenon* in planar graphs: numerous problems can be solved in time $2^{O(\sqrt{n})}$ on this class. These include, among others, Maximum Independent Set, Hamiltonian Cycle, and 3-Coloring.

The main structural insight enabling these algorithms is the celebrated *Planar Separator Theorem*, which states that every planar graph admits a balanced separator of size $O(\sqrt{n})$. A direct consequence is that every planar graph has treewidth $O(\sqrt{n})$. Since many classical problems can be solved in time $2^{\mathrm{tw}(G)} \cdot n^{O(1)}$, it follows that on planar graphs these problems admit $2^{O(\sqrt{n})}$-time algorithms.

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This naturally raises the question: *which graph classes exhibit this square-root phenomenon?* One well-studied generalization is the class of H-minor-free graphs (for any fixed graph H), which generalize planar graphs via Kuratowski-Wagner's theorem. These classes also have treewidth $O(\sqrt{n})$ and therefore enjoy subexponential-time algorithms for a wide range of problems.

Another direction for generalizing planar graphs is through the lens of *geometric intersection graphs.* Given a set of objects $F \subseteq \mathbb{R}^d$, the intersection graph $G[F]$ has one vertex per object of F and edges between pairs of intersecting objects. A classical example is that of *unit disk graphs*, the intersection graphs of unit-radius disks in the plane. Despite their geometric origin, such graphs behave very differently from planar graphs: objects may pairwise intersect in large numbers, and unit disk graphs can contain arbitrarily large cliques, unlike planar graphs. Consequently, general intersection graphs do not admit sublinear treewidth, and the standard treewidth-based subexponential techniques do not apply.

In this paper, we focus on geometric objects that remain "close" to d-dimensional balls. More precisely, given a constant $\beta \geqslant 1$, a family of objects F in $\mathbb{R}^d$ is said to be *similarly sized β-fat* if for every object $O \in F$ there exist two d-dimensional balls $B_{\text{in}} \subseteq O \subset B_{\text{out}}$ such that B_{in} has diameter 1 and B_{out} has diameter β. When F is a set of balls of equal radius in $\mathbb{R}^d$, the intersection graph $G[F]$ is called a *d-dimensional unit ball graph*, or simply a *unit ball graph.*

In their seminal work, de Berg *et al.* [13] developed a powerful algorithmic framework showing that intersection graphs of similarly sized fat objects in $\mathbb{R}^d$ admit $2^{O(n^{1-1/d})}$-time algorithms for a broad range of problems. They also proposed a matching lower-bound framework, establishing ETH-tightness for all the problems they considered. Their results unify a long line of previous research [1, 18, 19, 29] and provide a robust methodology covering a wide variety of problems.

However, their framework does not cover all problems. Several classical problems exhibit a *curse of dimensionality*: there exists a dimension d such that no subexponential-time algorithm exists. A prominent example is the Maximum Clique problem on unit ball graphs: it is polynomial-time solvable in dimension $d = 2$, yet admits no subexponential-time algorithm in dimension $d = 4$ under the ETH [5]. Another example is Clique Cover, which can be solved in time $2^{O(\sqrt{n})}$ on unit disk graphs, but has no subexponential-time algorithm in dimensions $d \geqslant 5$, under the ETH [26].

In this paper, we aim to capture problems that lie "in between" the two previously described extremes: they admit subexponential-time algorithms on intersection graphs of similarly sized fat objects in $\mathbb{R}^d$ for any fixed dimension $d \geqslant 2$, but with a slightly worse running time than the one obtained via the framework of de Berg *et al.* We call this behavior the *weak square-root phenomenon.* More precisely, our framework yields algorithms with running time

$$2^{\tilde{O}\left(n^{1-1/(d+1)}\right)},$$

where the $\tilde{O}(\cdot)$ notation hides polylogarithmic factors. We also provide nearly matching lower bounds (up to these polylogarithmic factors) under the ETH.

The remainder of the introduction presents an overview of the algorithmic technique, including examples of problems it captures, as well as the corresponding lower-bound framework.

A win-win approach

Our goal is to capture problems that are simultaneously FPT (or XP) with respect to both the vertex cover number and the independence number of the input graph. The central ingredient is a new separator theorem which, informally speaking, asserts that every intersection graph of similarly sized fat objects in $\mathbb{R}^d$ has a small subset of vertices whose removal leaves each connected component with a small independence number.

▶ **Theorem 1.** *Let $d \geqslant 2$ and $\beta \geqslant 1$ be two constants, and let G be the intersection graph of n similarly sized β-fat objects in $\mathbb{R}^d$. There exists a subset $S \subseteq V(G)$ of vertices such that:*
- $|S| \leqslant dn^{1-1/(d+1)}$*, and*
- *for every connected component C of $G - S$, $\alpha(G[C]) \leqslant (2\beta d)^d n^{1-1/(d+1)}$.*

Moreover, when the geometric representation is given, such a set S can be computed in polynomial time.

This separator theorem naturally gives rise to a new graph parameter. For a graph G, the *α-modulator number* is the smallest integer k such that there exists a set $S \subseteq V(G)$ of size at most k for which every connected component of $G - S$ has independence number at most k.

From Theorem 1, it follows that intersection graphs of similarly sized β-fat objects in $\mathbb{R}^d$ has α-modulator number $O(n^{1-1/(d+1)})$.

On graphs of bounded α-modulator number, the algorithmic strategy is as follows:
- Solve the problem independently on each connected component of $G - S$, using an algorithm that is FPT or XP parameterized by the independence number, for every partial solution on S;
- Combine these partial solutions to obtain a global solution.

This immediately yields $2^{\tilde{O}(n^{1-1/(d+1)})}$-time algorithms for geometric intersection graphs.

We also identify two natural problems that fit within this framework, for which we provide matching lower bounds under the ETH.

2-subcoloring

A k-subcoloring of a graph is an assignment of colors to the vertices of the graph such that each color class induces a disjoint union of complete graphs. The *subchromatic number* of a graph is the minimum k such that the graph has a k-subcoloring. This notion was introduced in 1989 by Albertson *et al.* [2] as a natural generalization of the chromatic number, which may be small even for dense graphs such as complete graphs. Note that disjoint unions of cliques are sometimes referred to as *cluster graphs*, and this class is equivalent to the class of P_3-free graphs, that is, graphs with no induced path on three vertices.

2-Subcoloring

Input: Graph G
Output: Does G has a 2-subcoloring, *i.e.* a partition (A, B) of $V(G)$ such that $G[A]$ and $G[B]$ are both a disjoint union of cliques?

The special case $k = 2$ is already of particular interest. Indeed, the 2-Subcoloring problem is NP-complete [20], even on planar graphs [15, 30] and on unit disk graphs [28]. On the contrary, the problem is polynomial-time solvable on interval graphs [15] and on graphs of bounded independence number [24]. More precisely, it was shown in [24] that 2-Subcoloring is FPT when parameterized by the number of clusters in the 2-subcoloring,

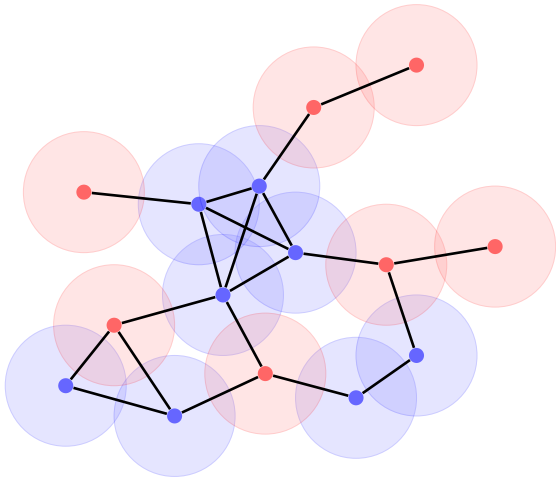

Figure 1 A 2-subcoloring of a unit disk graph.

which is a lower bound on the independence number of the input graph. In [15], an FPT algorithm parameterized by the treewidth of the input graph was provided that solves 2-Subcoloring.

Two Sets Cut-Uncut and related

Given a graph G and two disjoint sets of terminals $S, T \subseteq V(G)$, an S-T-cut of G is partition (A, B) of $V(G)$ such that $S \subseteq A$ and $T \subseteq B$. In the Two Sets Cut-Uncut problem, we are given an edge-weighted graph (G, w), where $w : E(G) \to \mathbb{N}$, and we aim for an S-T cut of minimum weight respecting some connectivity constraint. The *weight* of a cut (A, B) is the sum of the weight of edges $uv \in E(G)$ such that $u \in A$ and $v \in B$.

Two Sets Cut-Uncut

Input: A weighted graph (G, w) with $w : E(G) \to \mathbb{N}$, sets $S, T \subseteq V(G)$ with $S \cap T = \emptyset$
Output: An S-T-cut (A, B) of minimum weight such that S (resp. T) is in a connected component of $G[A]$ (resp. $G[B]$).

Bentert *et al.* [3] introduced the unweighted version of this problem. They proved that it is W[1]-hard when parameterized by $|T|$, even in the special case where $|S| = 1$ on general graphs. In contrast, they showed that the unweighted problem becomes fixed-parameter tractable on planar graphs when parameterized by $|S \cup T|$. They further established fixed-parameter tractability on planar graphs when parameterized by the minimum number of faces in a planar embedding such that each terminal is incident to at least one of these faces. Later, Bentert *et al.* [4] provided a comprehensive study of the problem under various structural parameterizations. In particular, they showed that the problem is FPT when parameterized by the treewidth of the input graph (and hence the vertex cover number), and XP when parameterized by the independence number.

This problem naturally generalizes two classical problems. The first is the 2-Disjoint Connected Subgraphs problem, which asks only for the existence of a partition (A, B) satisfying the terminal constraints and the connectivity requirements, without optimizing the number of crossing edges. This problem is already NP-complete and has been extensively studied in graph theory and computational geometry [9, 21, 33, 25, 31, 32].

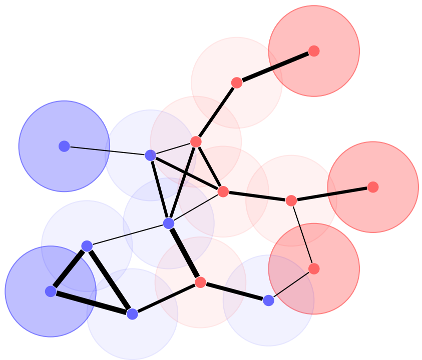

Figure 2 An example instance of the TWO SETS CUT-UNCUT problem on a unit disk graph together with a solution (A, B) represented with blue and red disks. The terminal sets S and T are shown as opaque blue and red disks, respectively. Edge widths are proportional to their weights.

The second related problem is NETWORK DIVERSION, in which S and T are singletons $\{s\}$ and $\{t\}$, respectively, and there is a designated edge $b = uv \in E(G)$ that must lie in the cut between A and B. One can solve NETWORK DIVERSION by solving TWO SETS CUT-UNCUT on the two instances $(S = \{s, u\}, T = \{t, v\})$ and $(S' = \{s, v\}, T' = \{t, u\})$. The NETWORK DIVERSION problem has attracted considerable attention in the computer networking community [7, 8, 14, 23, 27].

Lower Bound Framework

We establish matching lower bounds of the form $2^{o\left(n^{1-1/(d+1)}\right)}$ via reductions from MONOTONE NOT-ALL-EQUAL-3-SAT. The reduction embeds the input formula into a d-dimensional grid of side length $O(n^{1/d})$ and uses a breadth-first search tree T of height $O(n^{1/d})$ to control the propagation of truth assignments. Each clause is associated with a distinct grid cell and replaced by a gadget that satisfies the following properties:

1. it can be realized within a d-dimensional box of constant side length;
2. it propagates the truth values of its literals to selected adjacent cells of the grid;
3. it enforces the Not-All-Equal constraint of the clause.

A crucial feature of the construction is that all occurrences of any given literal are restricted to a single root-to-leaf path of T. As a consequence, the resulting graph has size $O(n^{(d+1)/d})$, which yields the claimed ETH-based lower bound.

Due to space constraints, some proofs (those marked with a *) are omitted and can be found in the full version of the paper.

2 Algorithmic framework

2.1 The separator theorem

This section is dedicated to the proof of Theorem 1. Let $d \geqslant 2$ and $\beta \geqslant 1$ be two constants. Let F be a set of n similarly sized β-fat objects in $\mathbb{R}^d$, and let $G := G[F]$. For every object $o \in F$, denote by $X_o \in \mathbb{R}^d$ the center of the smallest enclosing ball of o (which has diameter at most β).

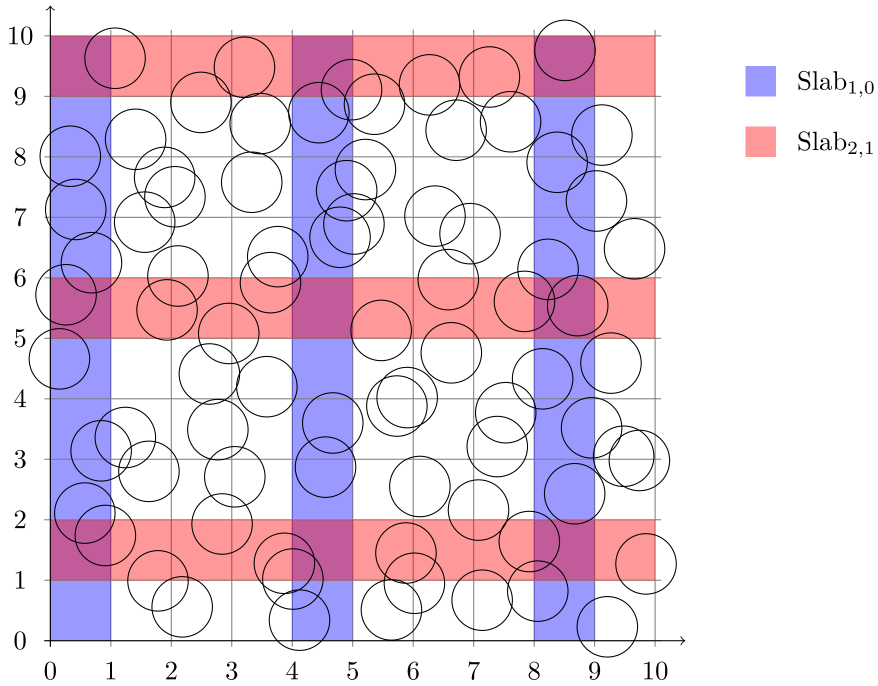

Figure 3 An example of the slabs defined in the proof of Theorem 1, in the case of $d = 2$, $\beta = 1$ and $p = 5$.

Let $p = \lceil n^{1/(d+1)} \rceil$ (so that $p > 1$). For $d' \in [d]$ and $i \in \mathbb{Z}/p\mathbb{Z}$, define

$$\mathrm{Slab}_{d',i} := \bigcup_{\substack{q \in \mathbb{Z} \\ q \equiv i \pmod p}} \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d \;\middle|\; x_{d'} \in [\beta q, \beta(q+1)] \right\}.$$

Informally, $\mathrm{Slab}_{d',i}$ consists of all slabs of width β whose index is congruent to i modulo p. An illustration in the case of unit disks ($d = 2$ and $\beta = 1$) is given on Figure 3.

For each (d', i), define $V_{d',i} \subseteq V(G)$ as the set of vertices corresponding to objects $o \in F$ such that $X_o \in \mathrm{Slab}_{d',i}$.

▷ Claim 2. For every $d' \in [d]$, there exists $i \in \{0, \dots, p-1\}$ such that $|V_{d',i}| \leqslant n^{1-1/(d+1)}$.

Proof. For fixed d', the sets $\{V_{d',i}\}_{0 \leqslant i < p}$ are pairwise disjoint and their union has size at most n. Hence, for some i, $|V_{d',i}| \leqslant \frac{n}{p} \leqslant n^{1-1/(d+1)}$. ◁

Applying Claim 2 for each $d' \in [d]$, we obtain indices $(i_1, \dots, i_d) \in \{0, \dots, p-1\}^d$ such that

$$\left| \bigcup_{d'=1}^{d} V_{d',i_{d'}} \right| \leqslant d\, n^{1-1/(d+1)}.$$

Let

$$S := \bigcup_{d'=1}^{d} V_{d',i_{d'}}.$$

▷ Claim 3. Let $F' \subseteq F$ be the set of objects corresponding to a connected component of $G - S$. Then all centers $\{X_o : o \in F'\}$ lie in a hypercube of side length at most βp.

Proof. Fix a dimension $d' \in [d]$. Observe that if two points belong to slabs $\mathrm{Slab}_{d',i}$ and $\mathrm{Slab}_{d',i'}$ with $|i - i'| \geqslant 2$, then their d'-th coordinates differ by at least β, and hence their Euclidean distance is greater than β.

Since each object has diameter at most β, two such objects cannot intersect, and therefore the corresponding vertices are not adjacent in G.

Now consider a connected component C of $G - S$, and let F' be its corresponding objects. Since all vertices whose centers lie in $\mathrm{Slab}_{d',i_{d'}}$ have been removed, the remaining centers cannot cross this slab. Hence, along each dimension d', the centers of objects in F' must lie within at most p consecutive slabs.

Since each slab has width β, this implies that along each coordinate the range is at most βp. Therefore, all centers lie in a hypercube of side length at most βp. ◁

▷ Claim 4. Let $F' \subseteq F$ be a set of objects contained in a hypercube $H \subseteq \mathbb{R}^d$ of side length a. Then $\alpha(G[F']) \leqslant (ad)^d$.

Proof. Let $M := \alpha(G[F'])$. By the definition of β-fat objects (after scaling), each object contains a ball of radius at least $1/2$. Therefore, the M objects of a maximum independent set contain M pairwise disjoint balls of radius $1/2$.

Let V_d denote the volume of a d-dimensional ball of radius $1/2$. One can show that $V_d \geqslant \left(\frac{1}{d}\right)^d$. Since all objects lie in H, the hypercube of volume a^d contains M disjoint balls, hence $M \cdot V_d \leqslant a^d$. Using the bound on V_d, we obtain $M \leqslant a^d \cdot d^d = (ad)^d$. ◁

Let C be a connected component of $G - S$, and let F' be its corresponding objects. By the Claim 3, all centers lie in a hypercube of side length βp. Since each object has diameter at most β, all objects are contained in a hypercube of side length $\beta(p+1)$.

By Claim 4, we obtain

$$\alpha(G[C]) \leqslant (\beta(p+1)d)^d.$$

Since $p = \lceil n^{1/(d+1)} \rceil$, this yields

$$\alpha(G[C]) \leqslant (2\beta d)^d \, n^{1-1/(d+1)},$$

which concludes the proof.

2.2 An adapted width parameter

Theorem 1 motivates the definition of a new graph parameter, which we call the *α-modulator number*.

▶ **Definition 5.** *Given a graph G, the* α-modulator number of G*, denoted* $\alpha\text{-mod}(G)$*, is the minimum integer k such that there exists a subset $S \subseteq V(G)$ satisfying:*

- *$|S| \leqslant k$;*
- *for every connected component C of $G - S$, $\alpha(G[C]) \leqslant k$.*

The subset S is called the modulator.

Observe that both the vertex cover number and the independence number are natural upper bounds for this parameter. Using this definition, Theorem 1 can be reformulated as follows.

▶ **Theorem 6** (Reformulation of Theorem 1). *Let $d \geqslant 2$ and $\beta \geqslant 1$ be constants. Any intersection graph G of n similarly sized β-fat objects in $\mathbb{R}^d$ satisfies*

$$\alpha\text{-mod}(G) = O\left(n^{1-\frac{1}{d+1}}\right),$$

and the corresponding decomposition can be computed in polynomial time when the geometric representation is given as input.

Less trivially, the tree-independence number [11] is a lower bound for this parameter. Indeed, let $S \subseteq V(G)$ be a subset of size at most k such that all connected components of $G - S$ have independence number at most k. Construct a tree decomposition by creating a root bag S, and for each connected component C of $G - S$, a leaf bag $C \cup S$ adjacent only to the root. Observe that the independence number of each bag is at most $2k$, which yields the following bound.

▶ **Observation 7.** *For any graph G,* $\text{tree-}\alpha(G) \leqslant 2\,\alpha\text{-mod}(G)$.

However, there is no direct relationship between treewidth and the α-modulator number. Indeed, the α-modulator number is unbounded on paths, while cliques have bounded α-modulator number but unbounded treewidth.

The relationships between these parameters are summarized in Figure 4. In addition, we indicate in this figure the computational complexity of the 2-Subcoloring problem. Indeed, the problem is FPT when parameterized by treewidth (see [15]), while it is NP-complete even on graphs of bounded tree-independence number (see [28]).

However, in Section 2.3, we show that the problem is FPT when parameterized by the α-modulator number, provided that a corresponding modulator is given as part of the input. When it is not the case, this modulator can be computed in XP time $n^{O(\alpha\text{-mod}(G))}$ by simply guessing the modulator S and checking that each remaining connected component of $G - S$ has independence number at most α-mod(G). Unfortunately, we cannot hope for a better strategy in the general case, as a simple reduction (namely, the join of a graph with itself) allows us to inherit all known (parameterized) inapproximability from Maximum Independent Set.

▶ **Observation 8.** *For any graph graph G, $\frac{1}{2}\alpha(G) \leqslant \alpha\text{-mod}(G+G) \leqslant \alpha(G)$.*

Proof. Let G_1 and G_2 be two distinct copies of G. Their *join*, denoted by $G_1 + G_2$, is the graph obtained from the disjoint union of G_1 and G_2 by adding all possible edges between every vertex of G_1 and every vertex of G_2. Denote by G' the graph $G_1 + G_2$, and note that $\alpha(G') = \alpha(G)$. The inequality $\alpha\text{-mod}(G') \leqslant \alpha(G)$ is trivial by considering an empty modulator. Then, by contradiction, suppose that $\alpha\text{-mod}(G') < \frac{1}{2}\alpha(G)$, and let S be the corresponding modulator of G'. Since $|S| < \frac{1}{2}\alpha(G)$, S does not fully contain $V(G_1)$ or $V(G_2)$, and thus $G' - S$ contains a unique connected component C. Since $\alpha(G'[C]) < \frac{1}{2}\alpha(G)$, we obtain $\alpha(G) \leqslant |S| + \alpha(G'[C]) < \alpha(G)$, which is a contradiction. Therefore, $\alpha\text{-mod}(G') \geqslant \frac{1}{2}\alpha(G)$. ◀

In particular, assuming GAP-ETH, for any computable function $g(k) > k$, there is no algorithm running in time $f(k) \cdot n^{o(k)}$, for any computable function f, that, given an n-vertex graph G and an integer k, distinguishes between the cases $\alpha\text{-mod}(G) \leq k$ and $\alpha\text{-mod}(G) \geq g(k)$ [6]. Note that the same result holds for the tree independence number [10].

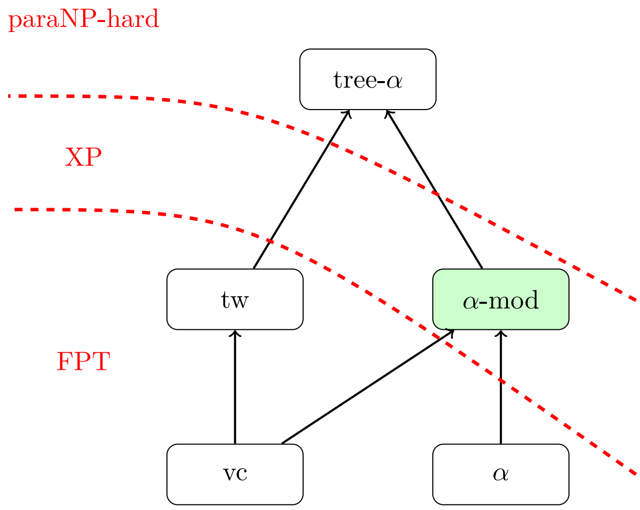

Figure 4 Hierarchy of graph parameters. An arrow from parameter p_1 to parameter p_2 implies that if p_1 is bounded, then p_2 is also bounded. The parameter in green is newly introduced. The parameterized complexity landscape of the 2-Subcoloring problem is depicted by the red dots.

Link with $\mathcal{G} \triangleright \mathcal{H}$ modulators

In [17], Fomin *et al.* introduced the notion of $\mathcal{G} \triangleright \mathcal{H}$ in order to unify several width parameters based on the concept of a *modulator*, such as $\mathcal{H}$-modulator number, $\mathcal{H}$-treewidth, and related notions.

Let $\mathcal{G}$ and $\mathcal{H}$ be two graph classes. The class $\mathcal{G} \triangleright \mathcal{H}$ consists of all graphs G that contain a subset $X \subseteq V(G)$ (called a *$\mathcal{G} \triangleright \mathcal{H}$-modulator*) such that $\mathsf{torso}(G, X) \in \mathcal{G}$ and, for every connected component C of $G - X$, we have $C \in \mathcal{H}$. Here, $\mathsf{torso}(G, X)$ is the subgraph induced by X where we also add edges between pairs of vertices with neighbors in a same connected component of $G - X$. In particular, for each connected component C of $G - X$, the neighborhood $N_G(C)$ of C induces a clique in $\mathsf{torso}(G, X)$.

For an integer $k \in \mathbb{N}$, let $\mathcal{S}_k$ denote the class of graphs with at most k vertices and $\mathcal{I}_k$ the class of graphs with independence number at most k. For instance, $\mathcal{S}_k \triangleright \mathcal{S}_1$ is precisely the class of graphs with a vertex cover of size at most k. We refer to [17] for more details.

The α-modulator number naturally fits into this framework (see Figure 5). More precisely, for a graph G, the minimum integer k such that $G \in \mathcal{S}_k \triangleright \mathcal{I}_k$ is exactly the α-modulator number.

▶ **Observation 9.** *For every graph G, α-mod$(G) = \min\{k \mid G \in \mathcal{S}_k \triangleright \mathcal{I}_k\}$.*

2.3 Example of applications

In Section 2, we introduced a parameter, the α-modulator number, which is sublinear in intersection graphs of similarly sized β-fat objects in $\mathbb{R}^d$.

In this section, we prove that the 2-Subcoloring problem is FPT when parameterized by the α-modulator number, provided that a corresponding modulator is given as input. We also show that the Two Sets Cut-Uncut problem is in XP with respect to this parameter.

To design an FPT/XP algorithm parameterized by α-mod(G), two conditions are necessary. First, the problem must be FPT/XP parameterized by the independence number, in order to handle the connected components after removing the modulator. Second, it must be

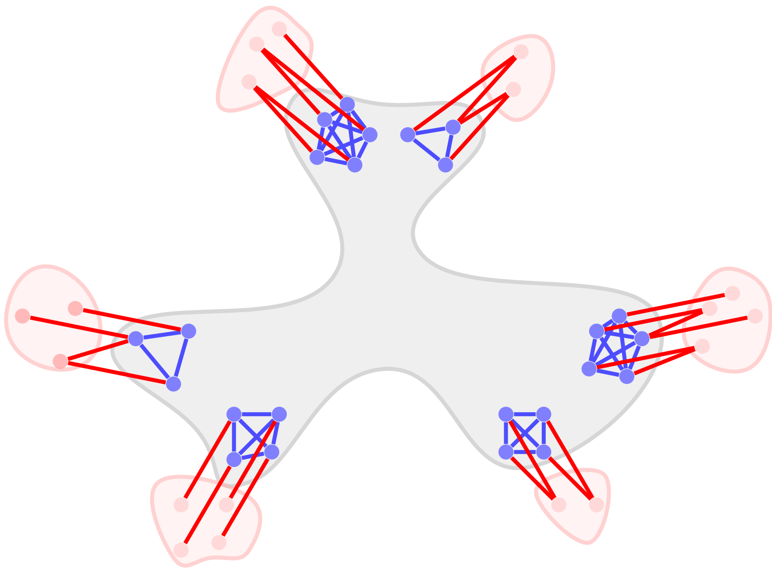

Figure 5 An illustration of the parameter $\min\{k \mid G \in \mathcal{S}_k \triangleright \mathcal{I}_k\}$, which is equal to the α-modulator number. The middle part (in gray) represents the torso $\mathsf{torso}(G, X)$, which has size at most k. The red parts correspond to the connected components of $G - X$, each having independence number at most k.

FPT/XP parameterized by the vertex cover number, to support the dynamic programming approach on the modulator. The main difficulty lies in combining these two algorithmic components coherently.

2-Subcoloring

Recall that a 2-subcoloring of a graph G is partition of the vertex set $V(G)$ into two sets A and B such that both $G[A]$ and $G[B]$ are cluster graphs. An equivalent formulation, which will be useful in the following, is that a 2-subcoloring of a graph is a 2-coloring $\mu : V(G) \to \{0, 1\}$ of $V(G)$ with no monochromatic induced path on three vertices.

In [15], it was shown that 2-Subcoloring can be solved in time $2^{O(\mathrm{tw}(G))} \cdot n$ using a standard dynamic programming algorithm over a nice tree decomposition, with one additional technical feature: for each partial coloring of a bag, the algorithm associates a Boolean flag to each clique in the bag, indicating whether some vertices of that clique have been forgotten in the decomposition.

In [24], Kanj *et al.* proved that 2-Subcoloring is FPT when parameterized by the total number of clusters in the resulting subcoloring. It is easy to observe that this number is at most twice the independence number $\alpha(G)$ of the input graph, which directly yields an FPT algorithm parameterized by $\alpha(G)$. They were even able to deal with a partial version of the problem.

▶ **Theorem 10** (Theorem 1.4 [24]). *Let G be a graph, c an integer, and (A', B') a partial 2-subcoloring of G. It is possible to decide whether there exists a 2-subcoloring (A, B) of G such that $A' \subseteq A$, $B' \subseteq B$, and the total number of clusters in (A, B) is at most c, in time $2^{O(c)} \cdot n^2$.*

We are now equipped to present the main algorithm, which borrows some ideas of the dynamic programming on tree decompositions presented in [15].

▶ **Theorem 11 (*).** *The 2-*Subcoloring *problem can be solved in time $2^{O(\alpha\text{-mod}(G))} \cdot n^{O(1)}$, when the corresponding modulator is given in input.*

As a corollary from Theorem 6 and the result above, we obtain directly a subexponential algorithm for intersection graphs of similarly-sized fat objects.

▶ **Theorem 12.** *For every constants $d \geqslant 2$ and $\beta \geqslant 1$, there is an algorithm solving 2-*Subcoloring *in time $2^{O(n^{1-1/(d+1)})}$ on intersection graphs of similarly sized β-fat objects in $\mathbb{R}^d$.*

Two Sets Cut-Uncut

Recall that the Two Sets Cut-Uncut problem asks, given an edge-weighted graph (G, w) and two sets of distinct terminal sets S and T, to find an S-T-cut (A, B) of minimum weight such that S (resp. T) is connected in $G[A]$ (resp. $G[B]$). In [4] it was proved that this problem can be solved in time $n^{O(\alpha(G)^2)}$ on n-vertex graph G. We start by observing that it is possible to drop the quadratic dependency when the input graph is the intersection graph of similarly sized fat objects. We rely on two structural results from [13]. A *κ-partition* of a graph is a partition of its vertices into parts, each of which can be covered by at most κ cliques. Any intersection graph of similarly sized fat objects admits, in polynomial time, a greedy κ-partition whose contraction has bounded maximum degree. Moreover, under such a partition, any minimal Steiner tree contains only a constant number of non-terminal vertices in each part.

This property allows us to handle connected components of small independence number by guessing a Steiner tree of each monochromatic connected component, and then solving a minimum cut problem on the remaining vertices.

The approach is similar to the one used for the 2-Subcoloring problem: we solve the problem on connected components after the removal of the modulator, and then recombine the solutions.

▶ **Theorem 13 (*).** *Let $d \geqslant 2$ and $\beta \geqslant 1$ be two constants. There exists an algorithm which, given an intersection graph G of n similarly sized β-fat objects in $\mathbb{R}^d$, solves the* Two Sets Cut-Uncut *problem on G in time $n^{O(\alpha\text{-mod}(G))}$.*

As a direct consequence, we obtain a subexponential algorithm in the case of geometric intersection graphs.

▶ **Theorem 14.** *For every constants $d \geqslant 2$ and $\beta \geqslant 1$, there is an algorithm solving* Two Sets Cut-Uncut *in time $2^{O\left(n^{1-1/(d+1)}\ln(n)\right)}$ on intersection graphs of similarly sized β-fat objects in $\mathbb{R}^d$.*

3 A framework for matching lower bound

Overview

This section establishes a matching lower bound under the Exponential Time Hypothesis (ETH) for the running time of the form $2^{o(n^{1-1/(d+1)})}$. We provide a high-level sketch of the reduction, which starts from the Monotone Not-All-Equal-3-SAT problem, defined as follows.

Monotone Not-All-Equal-3-SAT

Input: A negation-free 3-CNF formula $\Phi = \bigwedge_{i=1}^{m}(\ell_1^j \vee \ell_2^j \vee \ell_3^j)$ with m clauses over n variables $x_1, ..., x_n$, where each variable appears exactly four times in Φ.
Output: A truth assignment $\tau\{x_1, ..., x_n\} \to \{\mathsf{false}, \mathsf{true}\}$ such that every clause contains at least one true and one false literal.

By the Sparsification Lemma [22] and the reduction of [12], this problem cannot be solved in time $2^{o(n)}$, where n is the number of variables, unless the ETH fails.

The reduction is constructed as follows. Consider a d-dimensional grid of side length $n^{1/d}$, consisting of n cells, each of side length 1. We fix a BFS tree T on the corresponding d-dimensional grid graph, where each vertex represents a cell, and edges connect vertices if their corresponding cells are adjacent. Since the grid graph has diameter $O(n^{1/d})$, the same bound holds for T.

Each clause of the formula is mapped to a cell in the grid (and thus to a node in T), and a *gadget* is placed at each node. The gadget's role is to verify whether the corresponding clause contains at least one true and one false literal, and to propagate the truth values of each literal. Each vertex in a gadget is labeled with a literal from the original formula, subject to the following constraints:

- For any literal ℓ, each clause gadget contains $O(1)$ vertices labeled with ℓ.
- If a literal ℓ appears in a clause gadget corresponding to a node $t \in V(T)$, then ℓ must belong to a clause assigned to a descendant of t in T.

This property ensures that vertices labeled with a given literal ℓ appear only along a single branch of T. Since there are at most $4n$ literals, the total number of vertices in the graph is bounded by $O(n \cdot n^{1/d}) = O(n^{(d+1)/d})$.

At the end of the construction, the resulting graph is a **yes-instance** of the target problem if and only if the original formula is a **yes-instance** of Monotone Not-All-Equal-3-SAT. This construction can be realized using unit balls in $\mathbb{R}^d$ for any $d \geqslant 3$, and with similarly sized fat objects in $\mathbb{R}^2$. Whether the lower bound also holds for unit disk graphs remains an open question.

Lower bound for 2-Subcoloring

Applying the framework described above, we prove that the running time of Theorem 12 is almost tight under the ETH, up to the logarithmic factor in the exponent.

▶ **Theorem 15.** *2-Subcoloring cannot be solved in time $2^{o\left(n^{1-1/(d+1)}\right)}$ on n-vertex intersection graphs of similarly sized fat objects in $\mathbb{R}^d$ for any $d \geqslant 2$, and even in unit ball graphs in $\mathbb{R}^d$ for any $d \geqslant 3$.*

Sketch of the proof. We reduce from *Monotone Not-All-Equal* 3-*Sat* where each variable appears in exactly four clauses. Let ϕ be such a formula with n variables and $m = 4n$ clauses.

We embed the construction into a d-dimensional grid $G_{d,p}$ of side length $p = \Theta(n^{1/d})$, and consider a BFS tree $T_{d,p}$ rooted at the center. Each clause is mapped injectively to a distinct non-central grid vertex.

To each node t of the grid, we associate a *cell gadget* H_t, built from a constant-size skeleton (represented in Figure 6) where each vertex is replaced by a clique indexed by a set $g(t)$ of literals. Intuitively, $g(t)$ contains the literals appearing in clauses located in the subtree rooted at t. These gadgets are connected along the BFS tree so as to propagate information about literals across the construction.

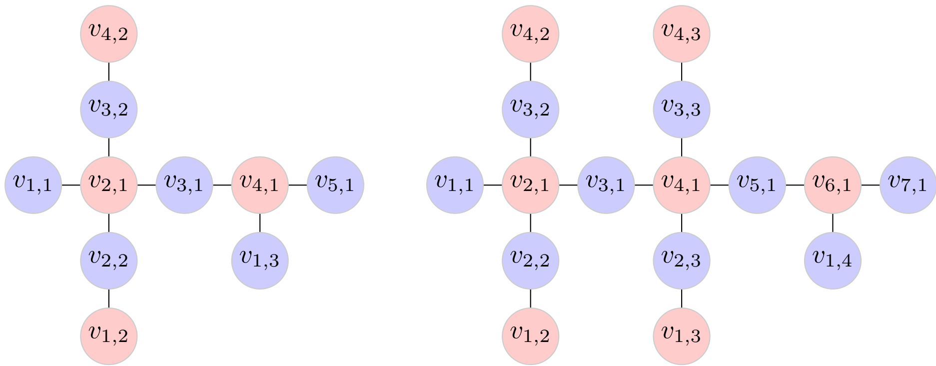

Figure 6 Skeleton of the clause gadget for $d = 2$ (on the left) and $d = 3$ (on the right) before replacing vertices with cliques.

Propagation is enforced using matchings between cliques. A key property is that in any 2-subcoloring, endpoints of such matchings must receive opposite colors. This ensures that once the coloring is fixed at the root gadget H_o, it propagates consistently throughout all gadgets.

At the root, we introduce a clique $U = u_1, \ldots, u_n$ encoding variable assignments: the color of u_i represents the truth value of x_i. The construction forces each literal $\ell = x_i$ in H_o to receive the opposite color of u_i, thereby encoding the assignment into the gadgets.

Each clause gadget is equipped with three additional vertices forming a path, connected to the three corresponding literals. This enforces the clause constraint: if all three literals receive the same color, the path becomes monochromatic, which is forbidden in a 2-subcoloring.

We prove that any valid 2-subcoloring induces a truth assignment where each clause contains at least one true and one false literal. Conversely, given such an assignment, we construct a consistent 2-subcoloring by setting variable vertices accordingly and propagating colors through the gadgets.

The size of the construction is $O(n^{(d+1)/d})$, yielding the desired reduction and establishing the lower bound.

An illustration of the 2-dimensional representation of the clause gadget is given in Figure 7, where the objects are fat polygons. For $d \geqslant 3$, the gadget can be represented as the intersection graph of unit d-dimensional balls. This construction is more involved: one dimension is used to encode the offset between vertices of the same clique, and the main challenge is to add matchings between cliques in order to propagate colors along this dimension. An illustration of this embedding is given in Figure 8. ◀

Lower Bound for Two Sets Cut-Uncut

We prove the same lower bound for the Two Sets Cut–Uncut problem. The construction is very similar to that of Section 3.

▶ **Theorem 16** (*). *Two Sets Cut-Uncut cannot be solved in time $2^{o\left(n^{1-1/(d+1)}\right)}$ on n-vertex intersection graphs of similarly sized of fat objects in $\mathbb{R}^d$ for any $d \geqslant 2$, and even in unit ball graphs in $\mathbb{R}^d$ for any $d \geqslant 3$.*

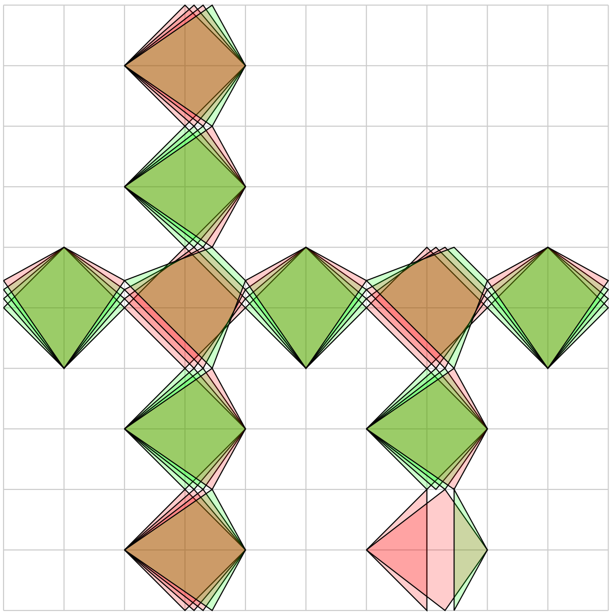

Figure 7 2D representation of $H_{L',2}$ with similarly sized fat objects.

4 Conclusion and further directions

We introduced the *weak square-root phenomenon*, revealing a new class of problems that admit $2^{\tilde{O}(n^{1-1/(d+1)})}$-time algorithms on geometric intersection graphs of similarly sized β-fat objects in $\mathbb{R}^d$, together with tight ETH lower bounds of $2^{\Omega(n^{1-1/(d+1)})}$, even on restricted subclasses. We established this phenomenon via a unified algorithmic and lower bound framework, and showed that it captures natural problems, including 2-Subcoloring and Two Sets Cut-Uncut. Along the way, we introduced a new graph parameter called α-modulator number, which may be of independent interest.

We conclude with three open problems related to our lower-bound framework. First, the case of *unit disk graphs* is not covered by our construction. Thus, we ask the following.

▶ **Problem 1.** *What is the complexity of 2-Subcoloring and Two Sets Cut–Uncut on unit disk graphs?*

▶ **Problem 2.** *What is the complexity of Two Sets Cut–Uncut on intersection graphs of similarly sized β-fat objects in $\mathbb{R}^d$ in the unweighted case?*

Then, Theorem 1 is not *robust*, as it requires the geometric representation as part of the input, leading to the following problem.

▶ **Problem 3.** *Can we obtain a robust version of Theorem 1 that does not rely on the geometric representation as input?*

We distinguish three possible outcomes for the first two problems: either they are polynomial-time solvable, or they admit a better subexponential algorithm than the one provided here, typically with running time $2^{O(\sqrt{n})}$, or it may be possible to successfully adapt the lower-bound framework to unit disk graphs.

Finally, we outline several directions for future research rather than concrete open problems:

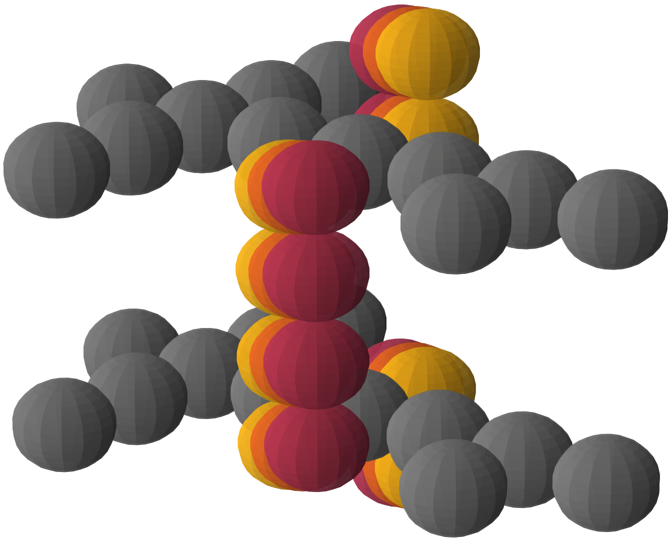

Figure 8 Representation of two adjacent clause gadgets with unit ball in 3 dimensions.

- Explore other problems that fit within this framework. In particular, it would be interesting to systematically consider the independence number $\alpha(G)$ as a natural parameter for FPT and XP algorithms. This direction has already received some attention [16].
- Investigate other classes of geometric graphs for which this framework could be applied.
- Continue the study of α-modulator number in general graphs as a structural parameter.

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