Abstract Color Voronoi Diagrams and Circular Sequences of Color Permutations

Sang Won Bae
Division of AI Computer Science and Engineering, Kyonggi University, Suwon, South Korea

Nicolau Oliver
Faculty of Informatics, Università della Svizzera italiana, Lugano, Switzerland

Evanthia Papadopoulou
Faculty of Informatics, Università della Svizzera italiana, Lugano, Switzerland

Abstract

Abstract Voronoi diagrams are defined in terms of a given system of planar bisecting curves satisfying some simple combinatorial properties. They offer a unifying framework for a wide range of concrete Voronoi instances on generalized sites and metrics. In this paper, we formulate higher-order abstract color Voronoi diagrams of a set S of n colored abstract sites, simultaneously considering all concrete instances under their umbrella. We prove that the number of vertices in the order-k abstract color Voronoi diagram is at most $4k(n-k)-2n$, and present an iterative construction algorithm. The bound directly applies to a family of m disjoint simple polygons of total complexity n. For simple polygons the bound can further improve to $O(\min\{k(n-k),(m-k)^2n\})$. A critical ingredient of our proof is a combinatorial analysis on circular sequences of color permutations derived from the unbounded edges of these diagrams that is interesting in its own right.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases higher-order Voronoi diagrams, abstract Voronoi diagrams, color Voronoi diagrams, circular sequences, allowable sequences, generalized sites, simple polygons

Digital Object Identifier 10.4230/LIPIcs.ESA.2026.138

Related Version *Full Version*: `https://arxiv.org/pdf/2607.05383` [12]

1 Introduction

Voronoi diagrams are versatile and influential space partitioning structures. Given a set S of n sites in $\mathbb{R}^2$ and an underlying distance function, the ordinary Voronoi diagram $\mathsf{VD}(S)$ partitions the plane into maximal regions by the nearest site relation. The order-k Voronoi diagram $\mathsf{VD}_k(S)$ partitions $\mathbb{R}^2$ into regions by the k nearest sites for $1 \leqslant k \leqslant n-1$, where $\mathsf{VD}_1(S) = \mathsf{VD}(S)$. The farthest-site Voronoi diagram $\mathsf{FVD}(S)$ is equal to $\mathsf{VD}_{n-1}(S)$. Sites may often be assumed to be points in the Euclidean plane, however, generalized sites, such as disks, line segments and polygons, under generalized metrics may also constitute the set of sites. See [7, 37] for extensive information on Voronoi diagrams and applications.

Lee [32] proved the tight bound $O(k(n-k))$ on the combinatorial complexity of $\mathsf{VD}_k(S)$ of point sites in the Euclidean plane, and presented an iterative algorithm that computes the diagrams order 1 up to k in $O(k^2 n \log n)$ time. The $O(k(n-k))$ bound on the complexity of $\mathsf{VD}_k(S)$ has been extended to line segments under any L_p metric [41] and to abstract Voronoi diagrams [14]. For point sites in the L_1/L_∞ metric a better bound $O(\min\{k(n-k),(n-k)^2\})$ is known [33]. The problem of constructing the Euclidean order-k Voronoi diagram $\mathsf{VD}_k(S)$ for point sites S had been one of the most interesting open problems in computational geometry, and the first optimal $O(n \log n + k(n-k))$-time algorithm was presented recently by Chan et al. [19], after a series of algorithmic advances for over four decades [2, 3, 8, 18, 20, 36, 42]. For generalized sites, however, there is a notable scarcity of corresponding results.

34th Annual European Symposium on Algorithms (ESA 2026).
Editors: Philip Bille, Seth Pettie, and Sabine Storandt; Article No. 138; pp. 138:1–138:20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In *color Voronoi diagrams*, colors are assigned to the sites in S, modeling a common property that sites of the same color share; let K be the set of these $m \leqslant n$ colors. The color assignment aggregates simple sites, such as points, segments, or disks, into compound sites of non-constant complexity, such as simple polygons, arc polygons, or site clusters. The color, a non-spatial property, facilitates the modeling of diverse applications including facility location [1], shape matching [28], spatial databases [21], wireless sensor networks [31], nearest-neighbor classification [17], fault detection and analysis in VLSI networks [40] and references therein. Nearest, higher-order, and farthest color Voronoi diagrams can be naturally defined.

Different variants of color Voronoi diagrams have been considered in the literature, such as the *Hausdorff* (also called *cluster*) *Voronoi diagram*, e.g., [5, 24, 38–40], and the *farthest color Voronoi diagram*, e.g., [1, 10, 28, 34], as motivated by different application demands, see e.g., [27, 43]. *Higher-order color Voronoi diagrams* were recently introduced in [11]. In $\mathbb{R}^d$, *chromatic Delaunay mosaics* of colored point sets were also recently introduced in [13] as motivated by topological data analysis.

Concrete color Voronoi diagrams are based on distance-to-color functions [11]: for each color $a \in K$ and any point $x \in \mathbb{R}^2$, let $d_a(x) := \min_{s \in S_a} \delta_s(x)$ and $\bar{d}_a(x) := \max_{s \in S_a} \delta_s(x)$ be the minimal and maximal distance-to-color a from x, where $\delta_s(x)$ denotes the prescribed distance to site s from x, and $S_a \subseteq S$ is the set of sites of color a. For each $1 \leqslant k \leqslant m-1$, the *order-k minimal color Voronoi diagram* $\mathsf{CVD}_k(S)$ of colored sites S partitions $\mathbb{R}^2$ into regions by k nearest colors with respect to the minimal distance-to-color functions $\{d_a\}_{a \in K}$, while the *order-k maximal color Voronoi diagram* $\overline{\mathsf{CVD}}_k(S)$ partitions $\mathbb{R}^2$ by k farthest colors with respect to the maximal distance-to-color functions $\{\bar{d}_a\}_{a \in K}$. In [11] the authors proved the tight upper bound $4k(n-k) - 2n$ on the total number of vertices in $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$ for a range of well-behaved distance to site functions, under a set of assumptions satisfied by point sites under convex distance functions, and presented an iterative construction algorithm. The assumptions of [11], however, are not satisfied by non-point sites, thus, the derived bounds do not apply in any setting involving sites more general than points. In this paper we remove these assumptions generalizing upon [11].

The order-k color Voronoi diagrams $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$ build upon several known diagrams that have received extensive interest in the literature. For $k = 1$, the minimal diagram $\mathsf{CVD}_1(S)$ is a subset of $\mathsf{VD}(S)$, while the maximal diagram $\overline{\mathsf{CVD}}_1(S)$ is contained in $\mathsf{FVD}(S)$. If $m = n$, that is, every site in S has a distinct color, then $\mathsf{CVD}_k(S) = \mathsf{VD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S) = \mathsf{VD}_{n-k}(S)$. For $k = m-1$, $\mathsf{CVD}_{m-1}(S)$ is exactly the *farthest color Voronoi diagram* $\mathsf{FCVD}(S)$, and $\overline{\mathsf{CVD}}_{m-1}(S)$ is the *Hausdorff Voronoi diagram* $\mathsf{HVD}(S)$.

Abstract Voronoi diagrams (AVDs) [29] offer a unifying framework for a wide range of concrete Voronoi instances. Rather than sites and distance measures, AVDs are defined in terms of bisecting curves that satisfy some simple combinatorial properties. Examples of concrete diagrams under the AVD umbrella include Voronoi diagrams of (non-intersecting) line segments, or convex polygons of constant complexity, in the L_p norms; Euclidean Voronoi diagrams of (non-intersecting) disks or smooth convex objects; Voronoi diagrams of point sites in any convex distance metric; additively weighted points (with non-enclosing circles) in the Euclidean plane. Bohler et al. [14] proved the tight upper bound $2k(n-k) + k + 1 - n$ on the number of faces in the order-k abstract Voronoi diagram $\mathsf{VD}_k(S)$. Efficient construction algorithms are also known [15, 16].

Contribution. In this paper, we formulate higher-order color Voronoi diagrams under the AVD model, namely *higher-order abstract color Voronoi diagrams.* By using the abstract framework, our combinatorial results are simultaneously applicable to all the concrete instances that fall under the AVD umbrella, which are also interesting individually. The obtained upper bounds are tight and analogous to those in [11], however, they are applicable to a much wider class of problems. This includes the higher-order Voronoi diagram of m simple polygons whose structural complexity, other than its farthest-site counterpart [22], had been unknown. It also results in the first study on the Hausdorff and farthest color Voronoi diagram in the abstract setting, which encompass all the previously considered concrete cases.

We prove tight upper bounds $O(k(n-k))$ on the combinatorial complexity of order-k abstract color Voronoi diagrams $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$ for each $1 \leqslant k \leqslant m-1$. More precisely, we show that the number of vertices of the abstract $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$ is at most $4k(n-k)-2n$, which is tight. This immediately implies the worst-case bound $O(m(n-m+1))$ on the complexity of the abstract farthest color Voronoi diagram $\mathsf{FCVD}(S)$ and the abstract Hausdorff Voronoi diagram $\mathsf{HVD}(S)$, which applies to all the concrete cases under the AVD umbrella. If m is close to n, then our new bound improves the previous one ($O(mn)$) on the complexity of the farthest color Voronoi diagram of non-crossing line segments in the L_p metric [10,28]. For a family of m disjoint simple polygons of total complexity n, we show that the order-k polygon Voronoi diagram has complexity $O(\min\{k(n-k),(m-k)^2n\})$. This is derived from a more general result: if the farthest color Voronoi diagram $\mathsf{FCVD}(S')$ (resp. $\mathsf{HVD}(S')$) for any subset $S' \subseteq S$ has linear complexity, then $\mathsf{CVD}_k(S)$ (resp. $\overline{\mathsf{CVD}}_k(S)$) has complexity $O((m-k)^2n)$ for $\frac{2m}{3} \leqslant k \leqslant m-1$. Note that the farthest polygon Voronoi diagram has linear complexity [22]. The complexity of the order-k Voronoi diagram of polygons had been an open problem for a long time.

To obtain these results we first adapt the *colorful Clarkson–Shor technique* from [11,23] to abstract Voronoi diagrams, expressing the combinatorial complexity of the order-k abstract color Voronoi diagram as a function of the diagram's unbounded edges (see Lemma 5). This reduces the problem into counting the unbounded edges of the colored order-k diagrams. For $m=n$ (uncolored case) this already answers a question posed by the authors of [14] on whether the Clarkson–Shor technique [23] could be applied in deriving the complexity of the abstract order-k diagram. Deriving bounds on the unbounded edges of colored order-k diagrams is a critical part of our proof, which is a novel contribution of this paper. In [11] the involved quantities were assumed equal and these bounds could thus be bypassed, which is valid for point sites in convex metrics, but does not extend to more general cases.

An independently interesting ingredient of our combinatorial results is based on a circular sequence of permutations of *colored* elements, which correspond to the unbounded edges of the diagrams. Circular sequences of permutations, sometimes called *allowable sequences*, have been useful as a purely combinatorial tool to analyze several basic geometric structures such as k-sets and order types [4,25,26]. We consider a colored variant of circular sequences, in which each element of permutations is assigned a color, and prove tight lower and upper bounds on the number of switches, which reflect the unbounded edges of order-k abstract color Voronoi diagrams.

Finally, we show how to construct both $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$ in an iterative fashion, for orders 1 to k, in time $O(k^2 n\log n)$. Because $\overline{\mathsf{CVD}}_{k-1}(S)$ does not contain sufficient information to compute $\overline{\mathsf{CVD}}_k(S)$, we first provide a direct divide-and-conquer algorithm to compute the unbounded edges of $\overline{\mathsf{CVD}}_k(S)$ in time $O(k^2 n\log m + n\log n)$, after which $\overline{\mathsf{CVD}}_k(S)$ can be computed from $\overline{\mathsf{CVD}}_{k-1}(S)$. A byproduct of this technique leads to an $O((n-k)^2 n\log n)$-time iterative algorithm to compute the (uncolored) order-k abstract Voronoi diagram $\mathsf{VD}_k(S)$, in decreasing order of k, starting from $\mathsf{FVD}(S)$, which is efficient for large values of k. To the best of our knowledge, no such algorithm was known before.

Most proofs are omitted due to the page limit, but can be found in the full version [12].

2 Review on higher-order abstract Voronoi diagrams

For any $A \subseteq \mathbb{R}^2$, we denote by ∂A, $\operatorname{int} A$, and $\operatorname{cl} A$ the boundary, interior, and closure of A, respectively.

Let S be a set of n *abstract sites* that define a family $\mathcal{J} = \{J(p,q) \mid p,q \in S, p \neq q\}$ of bisecting curves. The bisector $J(p,q)$ of two sites $p,q \in S$ is an unbounded simple curve, homeomorphic to a line, that divides the plane into two open domains: the *dominance region* of p over q and the dominance region of q over p, denoted by $D(p,q)$ and $D(q,p)$, respectively, see [29]. The dominance region $D(p,q)$ can be regarded as the set of all points that are *nearer* to p than to q according to a relevant proximity notion.

The *nearest* and the *farthest Voronoi region* of site $p \in S$ are respectively defined as $\mathrm{VR}(p,S) := \bigcap_{q \in S \setminus \{p\}} D(p,q)$ and $\mathrm{FVR}(p,S) := \bigcap_{q \in S \setminus \{p\}} D(q,p)$. The *nearest Voronoi diagram* $\mathsf{VD}(S)$ and the *farthest Voronoi diagram* $\mathsf{FVD}(S)$ are the collection of the region boundaries: $\mathsf{VD}(S) := \bigcup_{p \in S} \partial\,\mathrm{VR}(p,S)$ and $\mathsf{FVD}(S) := \bigcup_{p \in S} \partial\,\mathrm{FVR}(p,S)$.

The family of bisecting curves $\mathcal{J}$ is called *admissible* if it satisfies the following axioms [29], for every $S' \subseteq S$:

(A1) Each nearest Voronoi region $\mathrm{VR}(p,S')$ is non-empty and pathwise connected.
(A2) Each point of the plane belongs to the closure of a nearest Voronoi region $\mathrm{VR}(p,S')$.
(A3) Each bisector $J(p,q)$ is unbounded. After stereographic projection to the sphere, it can be completed to a closed Jordan curve through the north pole.
(A4) Any two bisectors intersect transversally, in a finite number of points.

Axiom (A4) can be relaxed as shown in [30] but for technical simplicity we still require it. The verification of the axioms can be done with constant size ($\leqslant 4$) examples, see [14]. Bisectors that have a site in common are called *related*. When two related bisectors $J(p,q)$ and $J(p,r)$ intersect, bisector $J(q,r)$ intersects with them at the same point(s). If $\mathcal{J}$ is admissible, there are at most two such intersection points, which are the vertices of $\mathsf{VD}(\{p,q,r\})$. Then $\mathsf{VD}(S)$ and $\mathsf{FVD}(S)$ are plane graphs of complexity $O(n)$ [29], and $\mathsf{FVD}(S)$ is a tree [35].

The *higher-order abstract Voronoi diagram* was first introduced in [14]. For $1 \leqslant k \leqslant n-1$ and each subset $H \subset S$ with cardinality k, the *order-k Voronoi region* of H and the *order-k abstract Voronoi diagram* $\mathsf{VD}_k(S)$ are defined as

$$\mathrm{VR}_k(H,S) := \bigcap_{p \in H, q \in S \setminus H} D(p,q) \quad \text{and} \quad \mathsf{VD}_k(S) := \bigcup_{H \subseteq S, |H|=k} \partial\,\mathrm{VR}_k(H,S).$$

Note that $\mathsf{VD}_1(S) = \mathsf{VD}(S)$ and $\mathsf{VD}_{n-1}(S) = \mathsf{FVD}(S)$. The higher-order abstract Voronoi diagram $\mathsf{VD}_k(S)$ is a plane graph that encompasses classic concrete cases, including points [32] and segments [41] in the Euclidean plane, from which the tight combinatorial complexity bound $O(k(n-k))$ is also derived [14].

The dominance regions define an abstract notion of *k-th nearest site*. In particular, given $\mathcal{J}$ and a point $x \in \mathbb{R}^2$, a total order of S can be defined [14, 16]. For $p, q \in S$, we write

$$p <_x q \text{ iff } p \neq q \text{ and } x \in D(p,q), \;\; p =_x q \text{ iff } p = q \text{ or } x \in J(p,q), \;\; p \leqslant_x q \text{ iff } p <_x q \text{ or } p =_x q.$$

These relations can be interpreted as x being *nearer* to p than to q, *equidistant* to p and q, and *not farther* from p than from q, respectively. The axioms imply the transitivity of $<_x$ [29] and its reflexive closure $\leqslant_x$ [16, Lemma 2], which define a total order on S allowing us to say that p_k is the *k-th nearest site at x*. In this sense, the order-k region $\mathrm{VR}_k(H,S)$ of $H \subseteq S$ consists of all points x such that H is the set of k nearest sites at x.

Let Γ be a closed simple curve in $\mathbb{R}^2$ sufficiently large such that no pair of bisecting curves in $\mathcal{J}$ cross on or outside of Γ; furthermore, every bisecting curve intersects Γ transversally, exactly twice. By this construction, Γ encloses every vertex of $\mathsf{VD}_k(S)$, for $1 \leqslant k \leqslant n-1$, and the unbounded features of $\mathsf{VD}_k(S)$ are traversed by Γ in the same order as they appear at infinity. We consider Γ as *the closed curve at infinity.*

An admissible bisector system $\mathcal{J}$ is said to be in *general position*, if only three related bisecting curves can intersect at the same point. From now on, we assume that the given system $\mathcal{J}$ of bisecting curves is admissible and in general position.

Circular sequences of permutations. The authors of [14] proved that the number of faces in the order-k diagram $\mathsf{VD}_k(S)$ is at most $2k(n-k)+k+1-n$ and at least $n-k+1$. Their combinatorial result is based on an inductive approach extending the original method by Lee [32], which had been extended to segment sites in [41], and a careful analysis on the circular sequences induced by the unbounded edges of $\mathsf{VD}_k(S)$, for all orders $1 \leqslant k \leqslant n-1$, on the closed curve Γ at infinity.

For each $x \in \Gamma$, consider the permutation $\pi = (p_1, \ldots, p_n)$ of S induced by $\leqslant_x$. If x avoids all bisecting curves in $\mathcal{J}$, then $\leqslant_x$ induces a unique permutation $(p_1, \ldots, p_n)$ such that $p_1 <_x \cdots <_x p_n$; otherwise, there is a unique index j such that $x \in J(p_j, p_{j+1})$ and hence $p_1 <_x \cdots <_x p_j =_x p_{j+1} <_x \cdots <_x p_n$. The corresponding permutation of S changes by a *switch* of two consecutive sites when and only when we cross a bisector $J(p,q)$, while we walk along Γ. As a result, we obtain a circular sequence $\Pi(S) = (\pi_0, \ldots, \pi_{N-1}, \pi_N = \pi_0)$ of permutations π_i of sites S such that:

- (P1) π_i and π_{i+1} for any i differ by a switch of two consecutive elements; and
- (P2) each pair of two elements switches exactly twice. ($N = 2\binom{n}{2} = n(n-1)$).

In [14] the authors showed that each switch between positions k and $k+1$ in π_i and π_{i+1} corresponds to an unbounded edge of $\mathsf{VD}_k(S)$. Thus, they obtained tight lower and upper bounds on the total number of unbounded edges in the diagrams $\mathsf{VD}_1(S), \ldots, \mathsf{VD}_k(S)$ by proving tight bounds on the number of switches in the first k positions: at least $k(k+1)$ and at most $k(2n-k-1)$.

3 Defining abstract color Voronoi diagrams

Let S be a set of n sites and $\mathcal{J} = \{J(p,q) \mid p,q \in S, p \neq q\}$ be an admissible bisector system in general position. We assume that each site $p \in S$ is assigned a color from a set $K = \{1, \ldots, m\}$ of $m \leqslant n$ colors. Let $S_a \subseteq S$ be the set of sites of color $a \in K$.

Abstract color Voronoi diagrams can be seen as Voronoi diagrams of *colors* in K, where each color $a \in K$ represents its entire color class S_a. We consider two kinds of color dominance regions: for each pair of two distinct colors $a, b \in K$, define

$$D(a,b) := \operatorname{int}\left(\bigcup_{p \in S_a} \operatorname{cl} \mathrm{VR}(p, S_a \cup S_b)\right) \quad \text{and} \quad \overline{D}(a,b) := \operatorname{int}\left(\bigcup_{p \in S_a} \operatorname{cl} \mathrm{FVR}(p, S_a \cup S_b)\right)$$

called the *minimal* and *maximal color dominance regions* of color a over color b, respectively. See the red and blue regions in Figure 1.

Let $J(a,b) := \partial D(a,b)$ and $\overline{J}(a,b) := \partial \overline{D}(a,b)$ be the *minimal* and *maximal color bisector*, respectively, between two colors $a, b \in K$. Since $J(a,b)$ (resp. $\overline{J}(a,b)$) separates Voronoi regions of different color in $\mathsf{VD}(S_a \cup S_b)$ (resp. $\mathsf{FVD}(S_a \cup S_b)$), it may consist of several disjoint simple curves, either unbounded or closed; see Figure 1. In particular, the minimal bisector $J(a,b)$ consists of one or more unbounded or closed curves, while the maximal bisector $\overline{J}(a,b)$ consists of only unbounded curves and can also be an empty set. These color bisectors may contain degree-2 vertices.

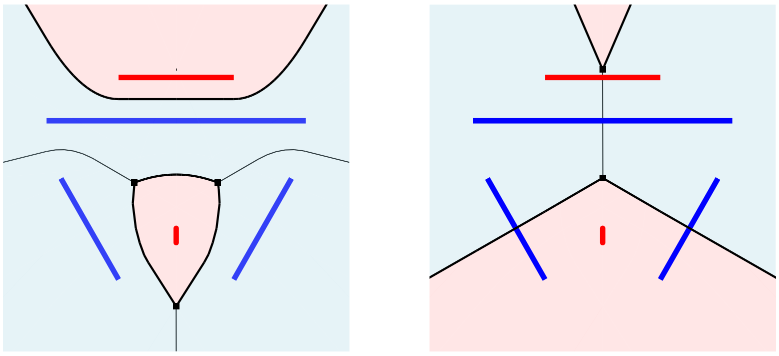


Figure 1 Minimal and maximal dominance regions. The red segments are S_a and the blue ones are S_b. The thin edges belong to $\mathsf{VD}(S_a \cup S_b)$ and $\mathsf{FVD}(S_a \cup S_b)$, the color bisectors are in bold.

The two kinds of color dominance regions define two hierarchies of higher-order abstract color Voronoi diagrams; see Figures 3(a–c) and 4(a–c). For each subset $H \subseteq K$ of k colors, the *order-k minimal* and *maximal color Voronoi regions* are defined as

$$\mathrm{R}_k(H,S) := \bigcap_{a\in H, b\in K\setminus H} D(a,b) \qquad \text{and} \qquad \overline{\mathrm{R}}_k(H,S) := \bigcap_{a\in H, b\in K\setminus H} \overline{D}(a,b).$$

The *order-k minimal color Voronoi diagram* $\mathsf{CVD}_k(S)$ and the *order-k maximal color Voronoi diagram* $\overline{\mathsf{CVD}}_k(S)$ are defined as

$$\mathsf{CVD}_k(S) := \bigcup_{H\subseteq K, |H|=k} \partial\,\mathrm{R}_k(H,S) \quad \text{and} \quad \overline{\mathsf{CVD}}_k(S) := \bigcup_{H\subseteq K, |H|=k} \partial\,\overline{\mathrm{R}}_k(H,S).$$

We consider the following color dominance relations on colors K at any point $x \in \mathbb{R}^2$, analogously to $<_x$, $=_x$, and $\leqslant_x$ on sites S, defined in [14]. For any colors $a, b \in K$, we write

$$a \prec_x b \text{ iff } a \neq b \text{ and } x \in D(a,b),\ a \sim_x b \text{ iff } a = b \text{ or } x \in J(a,b),\ a \preccurlyeq_x b \text{ iff } a \prec_x b \text{ or } a \sim_x b;$$
$$a \mathrel{\overline{\succ}_x} b \text{ iff } a \neq b \text{ and } x \in \overline{D}(a,b),\ a \mathrel{\overline{\sim}_x} b \text{ iff } a = b \text{ or } x \in \overline{J}(a,b),\ a \mathrel{\overline{\succcurlyeq}_x} b \text{ iff } a \mathrel{\overline{\succ}_x} b \text{ or } a \mathrel{\overline{\sim}_x} b.$$

▶ **Lemma 1.** *For any $x \in \mathbb{R}^2$, both $\preccurlyeq_x$ and $\overline{\succcurlyeq}_x$ are transitive.*

Lemma 1 implies (since both $\preccurlyeq_x$ and $\overline{\succcurlyeq}_x$ are reflexive) that $\preccurlyeq_x$ and $\overline{\succcurlyeq}_x$ induce two orderings of the m colors in K at any point $x \in \mathbb{R}^2$: $a_1 \preccurlyeq_x \cdots \preccurlyeq_x a_m$ and $\bar{a}_1 \mathrel{\overline{\succcurlyeq}_x} \cdots \mathrel{\overline{\succcurlyeq}_x} \bar{a}_m$. We say that a_k is the *k-th nearest color* at x with respect to $\preccurlyeq_x$, and that $\bar{a}_k$ is the *k-th farthest color* at x with respect to $\overline{\succcurlyeq}_x$. Based on transitivity, we establish the following in Lemmas 2, 3, and 4 of the full version [12]: (i) the closures of the minimal (resp. maximal) order-k regions $\mathrm{R}_k(H,S)$ (resp. $\overline{\mathrm{R}}_k(H,S)$) cover the plane; (ii) any two distinct minimal (resp. maximal) order-k regions are disjoint; and (iii) adjacent minimal (resp. maximal) order-k regions differ by exactly one color that define their common boundary.

We conclude that the order-k minimal diagram $\mathsf{CVD}_k(S)$ is a plane graph that partitions the plane into order-k minimal regions $\mathrm{R}_k(H,S)$, for $H \subseteq K$ with $|H| = k$, while the order-k maximal diagram $\overline{\mathsf{CVD}}_k(S)$ is a plane graph that partitions $\mathbb{R}^2$ into order-k maximal regions $\overline{\mathrm{R}}_k(H,S)$. The edges of $\mathsf{CVD}_k(S)$ (resp. $\overline{\mathsf{CVD}}_k(S)$), called *Voronoi edges*, are portions of minimal (resp. maximal) color bisectors that bound their incident order-k color Voronoi regions. The Voronoi vertices are intersection points of related color bisectors. See Figure 3 for a concrete example of line-segment sites.

As with ordinary order-k diagrams, the vertices of $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$ are characterized as *new* and *old*. A vertex of $\mathsf{CVD}_k(S)$ (resp. $\overline{\mathsf{CVD}}_k(S)$) is called *new* if it does not appear in $\mathsf{CVD}_{k-1}(S)$ (resp. $\overline{\mathsf{CVD}}_{k-1}(S)$); and *old*, otherwise. See Figures 3(a–c) and 4(a–c), where new vertices are marked by small squares.

▶ **Lemma 2.** *A new vertex v of $\mathsf{CVD}_{k-1}(S)$ is an old vertex of $\mathsf{CVD}_k(S)$, and v is not a vertex of $\mathsf{CVD}_{k+1}(S)$. No Voronoi edge of $\mathsf{CVD}_k(S)$ remains in $\mathsf{CVD}_{k+1}(S)$. Analogous statements hold for $\overline{\mathsf{CVD}}_k$.*

Farthest color and Hausdorff Voronoi diagrams. The case of $k = m - 1$ is of a special interest. The order-$(m-1)$ minimal diagram $\mathsf{CVD}_{m-1}(S)$ is determined by $m - 1$ nearest colors, or equivalently the farthest color, with respect to the minimal dominance $\preccurlyeq_x$, resulting in the *farthest color Voronoi diagram* $\mathsf{FCVD}(S)$; the $\overline{\mathsf{CVD}}_{m-1}(S)$ is determined by $m - 1$ farthest colors, or equivalently the nearest color, with respect to the maximal dominance $\overline{\succcurlyeq}_x$, resulting in the *Hausdorff Voronoi diagram* $\mathsf{HVD}(S)$. For a color $a \in K$, its *farthest color Voronoi region* is $\mathrm{FR}(a, S) := \bigcap_{b \in K \setminus \{a\}} D(b, a) = \mathrm{R}_{m-1}(K \setminus \{a\}, S)$; while its *Hausdorff Voronoi region* is $\mathrm{HR}(a, S) := \bigcap_{b \in K \setminus \{a\}} \overline{D}(b, a) = \overline{\mathrm{R}}_{m-1}(K \setminus \{a\}, S)$.

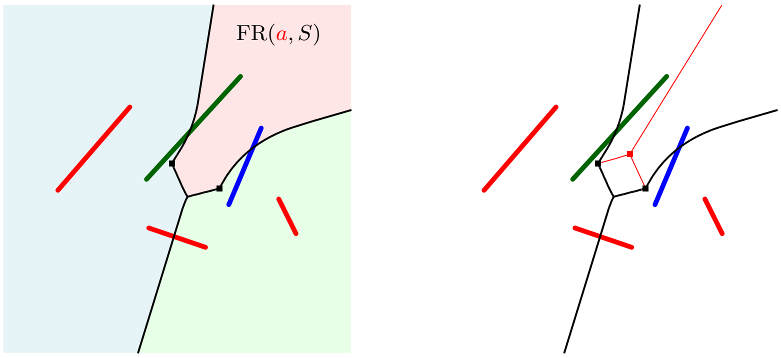
The structure of these Voronoi regions becomes evident once we overlay them with $\mathsf{VD}(S_a)$, and $\mathsf{FVD}(S_a)$, respectively; see the red regions in Figures 2(a–b). Each face f of $\mathrm{FR}(a, S) \cap \mathsf{VD}(S_a)$ is associated with a site $p \in S_a$, where color a is the farthest, with respect to the minimal dominance relation $\preccurlyeq_x$, for any point $x \in f$, and site p is the nearest at x among the sites in S_a. The resulting refined diagram is denoted by $\mathsf{FCVD}^*(S)$; see Figure 3(g). Analogously, each face $\bar{f}$ of $\mathrm{HR}(a, S) \cap \mathsf{FVD}(S_a)$ is associated with a site $p \in S$ such that the color a of p is the nearest, with respect to the maximal dominance relation $\overline{\succcurlyeq}_x$, for any point $x \in \bar{f}$, while site $p \in S_a$ is the farthest at x among the sites in S_a. The resulting refined diagram is denoted by $\mathsf{HVD}^*(S)$; see Figure 4(g) in Appendix A.

Our abstract definition of $\mathsf{FCVD}(S)$ and $\mathsf{HVD}(S)$ encompasses all concrete cases that have been studied before [9–11, 34, 39], including the farthest polygon Voronoi diagram [22], and any other case of generalized sites or metrics under the AVD model.

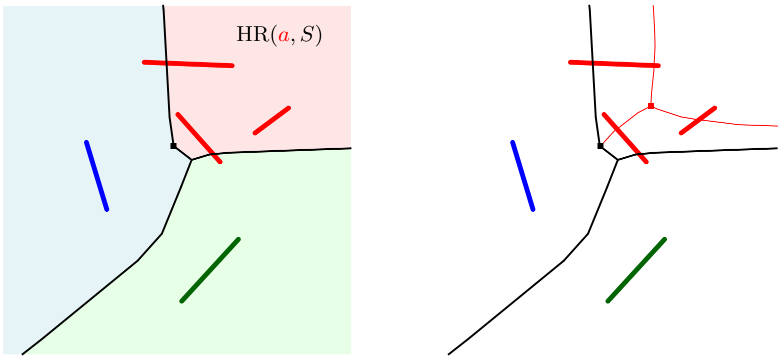
Refined order-k color diagrams. We can analogously refine the order-k diagrams $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$, denoted by $\mathsf{CVD}^*_k(S)$ and $\overline{\mathsf{CVD}}^*_k(S)$, respectively, for each $1 \leqslant k \leqslant m$, where $\mathsf{CVD}_m(S)$ and $\overline{\mathsf{CVD}}_m(S)$ consist of a single region $\mathrm{R}_m(K, S) = \overline{\mathrm{R}}_m(K, S) = \mathbb{R}^2$.

For each nonempty region $R = \mathrm{R}_k(H, S)$ of $\mathsf{CVD}_k(S)$, we overlay $\mathsf{FCVD}^*(S_H)$ inside R, where $S_H = \bigcup_{a \in H} S_a$. The resulting refined diagram is denoted $\mathsf{CVD}^*_k(S)$. Recall that $\mathsf{FCVD}(S_H) = \mathsf{CVD}_{k-1}(S_H)$. Note that each face f of $\mathsf{CVD}^*_k(S)$ is associated with a set $H \subseteq K$ of k colors and a site $p \in S_a$ for some color $a \in H$ such that, at any point $x \in f$, H is the common set of k nearest colors, color a is the k-th nearest color with respect to the minimal dominance $\preccurlyeq_x$, and $p \in S_a$ is the nearest site of color a; that is, $f \subseteq \mathrm{R}_k(H, S) \cap \mathsf{FCVD}(S_H) \cap \mathrm{VR}(p, S_a)$. See Figures 3(d–g), where faces of $\mathsf{CVD}^*_k(S)$ associated to the sites p and q are shaded purple and red, respectively.

For each nonempty region $\overline{R} = \overline{\mathrm{R}}_k(H, S)$ of $\overline{\mathsf{CVD}}_k(S)$, we overlay $\mathsf{HVD}^*(S_H)$ inside $\overline{R}$. The resulting refined diagram is denoted by $\overline{\mathsf{CVD}}^*_k(S)$. Similarly as above, each face $\bar{f}$ of $\overline{\mathsf{CVD}}^*_k(S)$ is associated with a subset $H \subseteq K$ of k colors and a site $p \in S_a$ for some $a \in H$ such that, at any point $x \in \bar{f}$, H is the common set of k farthest colors, color a is the k-th



(a) $\mathsf{FCVD}(S)$ and $\mathsf{FCVD}^*(S)$



(b) $\mathsf{HVD}(S)$ and $\mathsf{HVD}^*(S)$

Figure 2 The farthest color and Hausdorff Voronoi diagrams next to their refined counterparts. In (a) the red edges are $\mathsf{VD}(S_a) \cap \mathrm{FR}(a, S)$. In (b) the red edges are $\mathsf{FVD}(S_a) \cap \mathrm{HR}(a, S)$.

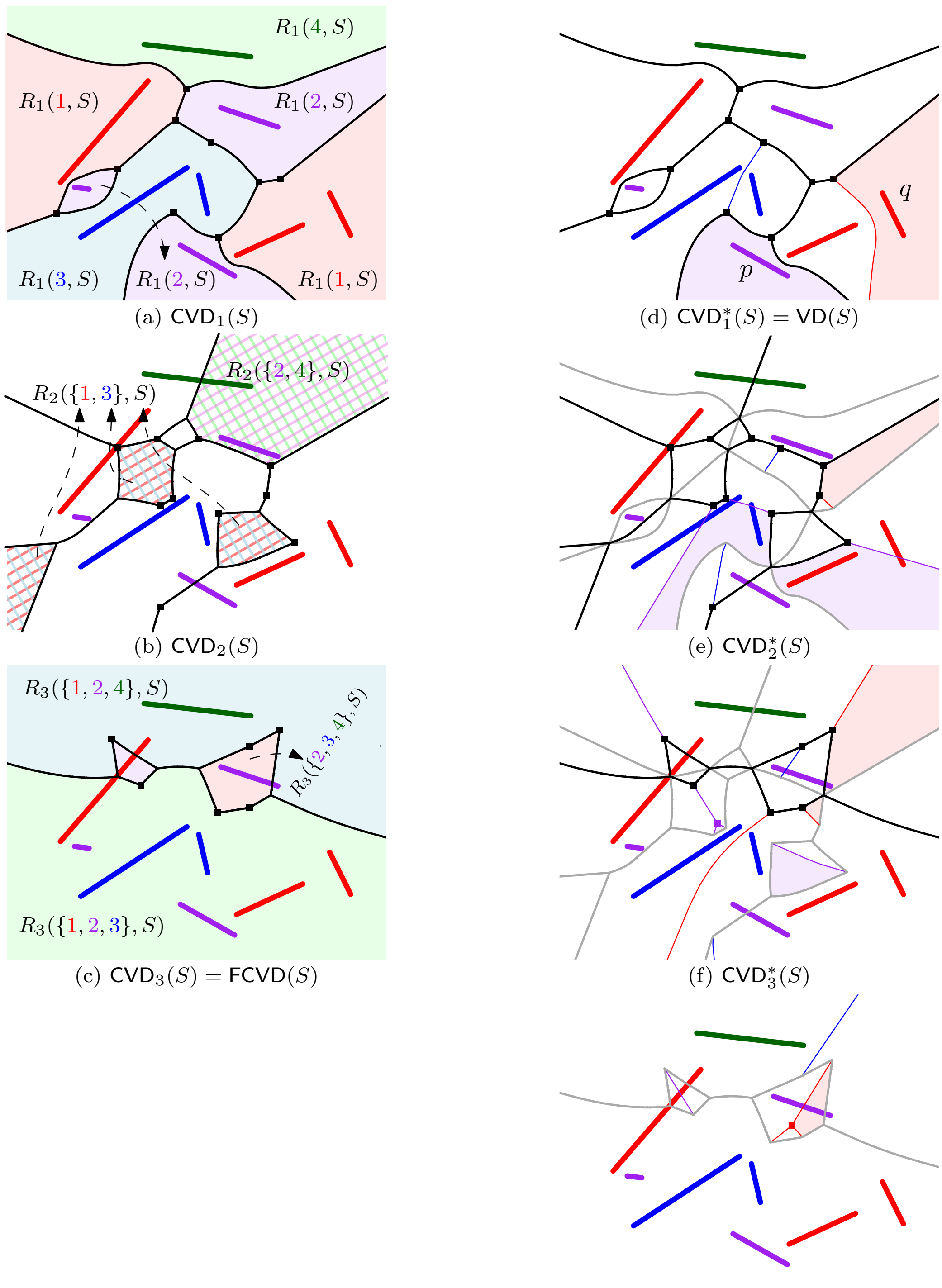
farthest color with respect to the maximal dominance $\overline{\succ}_x$, and $p \in S_a$ is the farthest site among those in S_a; that is, $\bar{f} \subseteq \overline{\mathrm{R}}_k(H, S) \cap \mathsf{HVD}(S_H) \cap \mathrm{FVR}(p, S_a)$. See Figures 4(d–g), where faces of $\overline{\mathsf{CVD}}^*_k(S)$ associated to the sites p and q are also shaded purple and red.

Note $\mathsf{CVD}^*_m(S) = \mathsf{FCVD}^*(S)$ and $\overline{\mathsf{CVD}}^*_m(S) = \mathsf{HVD}^*(S)$, see Figures 3(g) and 4(g), whereas $\mathsf{CVD}_{m-1}(S) = \mathsf{FCVD}(S)$ and $\overline{\mathsf{CVD}}_{m-1}(S) = \mathsf{HVD}(S)$, see Figures 3(c) and 4(c).

Distinguishing chromatic features. We can distinguish different features of the color diagrams by the number of their defining colors. Any edge or vertex of a color diagram is called *c-chromatic*, where $c \in \{1, 2, 3\}$ is the number of distinct colors of the sites defining it. In $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$, Voronoi edges and their degree-2 vertices are bichromatic (2-chromatic), while the Voronoi vertices are all trichromatic (3-chromatic). No monochromatic (1-chromatic) features exist in $\mathsf{CVD}_k(S)$ or $\overline{\mathsf{CVD}}_k(S)$.

As with ordinary order-k diagrams, the following properties are derived from definitions.

▶ **Lemma 3.** *For each face $R \subseteq \mathrm{R}_k(H, S)$ of $\mathsf{CVD}_k(S)$, it holds: $\mathsf{FCVD}(S_H) \cap R = \mathsf{CVD}_{k-1}(S) \cap R$. For each face $\overline{R} \subseteq \overline{\mathrm{R}}_k(H, S)$ of $\overline{\mathsf{CVD}}_k(S)$: $\mathsf{HVD}(S_H) \cap \overline{R} = \overline{\mathsf{CVD}}_{k-1}(S) \cap \overline{R}$. Thus, superimposing $\mathsf{CVD}_k(S)$ and $\mathsf{CVD}_{k-1}(S)$ results in $\mathsf{CVD}^*_k(S)$ with its monochromatic vertices and edges removed. Analogous statements hold for the maximal diagrams.*



(a) $\mathsf{CVD}_1(S)$
(d) $\mathsf{CVD}^*_1(S) = \mathsf{VD}(S)$
(b) $\mathsf{CVD}_2(S)$
(e) $\mathsf{CVD}^*_2(S)$
(c) $\mathsf{CVD}_3(S) = \mathsf{FCVD}(S)$
(f) $\mathsf{CVD}^*_3(S)$
(g) $\mathsf{CVD}^*_4(S) = \mathsf{FCVD}^*(S)$

Figure 3 The minimal color Voronoi diagrams of all orders. The colors $K = \{1, 2, 3, 4\}$ are red, purple, blue and green. New 2-chromatic edges are in black, old 2-chromatic edges in gray, 1-chromatic edges in their own color, and new vertices are small squares.

▶ **Lemma 4.** *A c-chromatic vertex or edge appears in the diagrams* $\mathsf{CVD}^*_k(S)$ *or* $\overline{\mathsf{CVD}}^*_k(S)$ *for c consecutive orders, and in* $\mathsf{CVD}_k(S)$ *or* $\overline{\mathsf{CVD}}_k(S)$ *for* $c-1$ *consecutive orders.*

A full characterization of the c-chromatic features of $\mathsf{CVD}^*_k(S)$ and $\overline{\mathsf{CVD}}^*_k(S)$ can be established based on the ordering of sites in S with respect to $\leqslant_x$ and the ordering of colors in K with respect to $\preccurlyeq_x$ and $\overline{\succcurlyeq}_x$; see Lemmas 8 and 9 in [12].

A concrete example: the order-k polygon Voronoi diagram. Let $\mathcal{P}$ be a set of m disjoint simple polygons with a total of n vertices. Each polygon induces a distinct color in set K. We can cast the order-k polygon Voronoi diagram as a concrete case of an order-k abstract color (minimal) Voronoi diagram, $\mathsf{CVD}_k(S)$, by selecting the set S of sites so that the underlying bisector system is admissible. There are two ways to do so, each resulting in a slightly different (but equally valid) minimal refined diagram: (1) let S be a set of points and open line segments derived from the vertices and edges of the polygons in $\mathcal{P}$; and (2) let S be a set of line segments derived from the polygonal edges in $\mathcal{P}$. In the latter case, if two segments p and q have a common endpoint, then replace the two-dimensional portion of their bisector by the piece of their angular bisector contained in their equidistant area [6], see Figure 5. In both cases, the resulting families of bisecting curves are admissible as they satisfy the required axioms. Both definitions yield an identical $\mathsf{CVD}_k(S)$, however, the corresponding refined diagrams $\mathsf{CVD}^*_k(S)$ can have different monochromatic features.

The maximal diagram $\overline{\mathsf{CVD}}_k(S)$ depends on which convention underlies the choice of S. Under convention (1) $\overline{\mathsf{CVD}}_k(S)$ reduces to a point maximal diagram $\overline{\mathsf{CVD}}_k(S')$, where $S' \subset S$ consists of the vertices in $\mathcal{P}$; this is not the case under convention (2).

4 Combinatorial complexity

In this section, we establish the upper bound $4k(n-k)-2n$, which is tight, for both minimal and maximal order-k abstract color Voronoi diagrams. Our proof is based on two techniques: (1) a random sampling technique, known as the *Clarkson–Shor technique* [23] with a *colorful* extension in [11]; and (2) a combinatorial analysis on the circular sequences of permutations induced by the unbounded edges of the order-k color Voronoi diagrams. Part (1) adapts the technique from [11] to abstract Voronoi diagrams and results in Lemma 5. Part (2) is detailed in Section 5 and its results are summarized in Lemma 6.

We first introduce some notation. Consider the *new* vertices and the *new* unbounded edges of $\mathsf{CVD}^*_k(S)$ and $\overline{\mathsf{CVD}}^*_k(S)$, $1 \leqslant k \leqslant m$. Recall that a feature of an order-k diagram is called *new* if it does not appear in the diagram of order-$(k-1)$; and *old*, otherwise. Their numbers are denoted as follows.

- Let $\boldsymbol{v_{c,k}} = v_{c,k}(S)$ and $\bar{v}_{c,k} = \bar{v}_{c,k}(S)$ denote the number of new c-chromatic vertices of $\mathsf{CVD}^*_k(S)$ and of $\overline{\mathsf{CVD}}^*_k(S)$, respectively.
- Let $\boldsymbol{u_{c,k}} = u_{c,k}(S)$ and $\bar{u}_{c,k} = \bar{u}_{c,k}(S)$ denote the number of new c-chromatic unbounded edges of $\mathsf{CVD}^*_k(S)$ and of $\overline{\mathsf{CVD}}^*_k(S)$, respectively. Note that an edge that is unbounded in both directions is counted as two distinct unbounded edges. Also $u_{c,m} = \bar{u}_{c,m} = 0$.

We next consider the first part of our proof, which adapts the technique from [11] to abstract Voronoi diagrams, and results to the equations summarized in the following lemma.

▶ **Lemma 5.** *For each* $1 \leqslant k \leqslant m-1$*, let* $U_k := \sum_{1\leqslant j\leqslant k}(u_{2,j} + (k-j+1)u_{1,j})$ *and* $\overline{U}_k := \sum_{1\leqslant j\leqslant k}(\bar{u}_{2,j} + (k-j+1)\bar{u}_{1,j})$*. The total number of vertices in* $\mathsf{CVD}_k(S)$ *is exactly*

$$2k(2n-k) - 2n - U_k - U_{k-1} - 2\sum_{j=1}^{k-1} v_{2,j} - \sum_{j=1}^{k}(2k-2j+1)v_{1,j}$$

and the number of vertices in $\overline{\mathsf{CVD}}_k(S)$ *is exactly*

$$\overline{U}_k + \overline{U}_{k-1} - 2k^2 - 2\sum_{j=1}^{k-1} \bar{v}_{2,j} - \sum_{j=1}^{k}(2k-2j+1)\bar{v}_{1,j}.$$

Lemma 5 reduces the problem of bounding the combinatorial complexity of $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$ to the problem of bounding two quantities U_k and $\overline{U}_k$ regarding the number of new unbounded edges in the refined diagrams of orders 1 to k. The equations are identical to those in [11]. In [11] the two quantities U_k and $\overline{U}_k$ were then assumed equal, a property that holds in the case of point sites under convex distances. Equality, however, does not hold in general for other types of sites or metrics. In this paper we remove this assumption by establishing the following lemma whose proof is deferred to Section 5.

▶ **Lemma 6.** *For each* $1 \leqslant k \leqslant m-1$*,* $U_k \geqslant k(k+1)$ *and* $\overline{U}_k \leqslant k(2n-k-1)$*. Both bounds are tight.*

By combining Lemma 5 and Lemma 6 we obtain the following result.

▶ **Theorem 7.** *Let* S *be a set of* n *colored sites with* $m \leqslant n$ *colors. Given an admissible system* $\mathcal{J}$ *of bisecting curves for* S *and an integer* k *with* $1 \leqslant k \leqslant m-1$*, the total number of vertices in the order-*k *abstract color Voronoi diagram* $\mathsf{CVD}_k(S)$ *or* $\overline{\mathsf{CVD}}_k(S)$ *is at most* $4k(n-k)-2n$*. Moreover, this bound is tight.*

Proof. By Lemma 5, the number of vertices in $\mathsf{CVD}_k(S)$ is

$$2k(2n-k) - 2n - U_k - U_{k-1} - 2\sum_{j=1}^{k-1} v_{2,j} - \sum_{j=1}^{k}(2k-2j+1)v_{1,j} \leqslant 4k(n-k) - 2n$$

since $U_k \geqslant k(k+1)$ as shown in Lemma 6. The number of vertices in $\overline{\mathsf{CVD}}_k(S)$ is

$$\overline{U}_k + \overline{U}_{k-1} - 2k^2 - 2\sum_{j=1}^{k-1} v_{2,j} - \sum_{j=1}^{k}(2k-2j+1)v_{1,j} \leqslant 4k(n-k) - 2n$$

since $\overline{U}_k \leqslant k(2n-k-1)$ as shown in Lemma 6.

The tightness of the upper bound $4k(n-k)-2n$ follows from the tightness of the bounds in Lemma 6 and the fact that the number of c-chromatic vertices, $c < 3$, is zero if $n = m$. ◀

The bound extends to the refined diagrams $\mathsf{CVD}^*_k(S)$ and $\overline{\mathsf{CVD}}^*_k(S)$, as $O(k(n-k))$, by Lemma 3 and the fact that the monochromatic vertices are in total $O(n)$.

Although the bound of Theorem 7 is tight, the $\mathsf{FCVD}(S)$ or $\mathsf{HVD}(S)$ may as well have linear complexity in special cases. In such cases the $O(k(n-k))$ bound is not tight for large k. We obtain a sharper asymptotic bound for $m - k \in O(\sqrt{m})$ in the following theorem; see Theorem 14 of the full version [12] for its proof.

▶ **Theorem 8.** *Given a set S of n colored sites with $m \leqslant n$ colors and an admissible system $\mathcal{J}$ of bisecting curves for S, if $\mathsf{FCVD}(S')$ (resp. $\mathsf{HVD}(S')$) has linear complexity for any $S' \subseteq S$, then for each $\lceil \frac{2m}{3} \rceil + 1 \leqslant k \leqslant m-1$ the total combinatorial complexity of $\mathsf{CVD}^*_k(S), \ldots, \mathsf{CVD}^*_m(S)$ (resp. $\overline{\mathsf{CVD}}^*_k(S), \ldots, \overline{\mathsf{CVD}}^*_m(S)$) is $O((m-k)^2 n)$.*

A concrete example is the order-k Voronoi diagram of m disjoint simple polygons of total complexity n, where the farthest polygon Voronoi diagram has complexity $O(n)$ [22].

▶ **Corollary 9.** *Given a collection of m disjoint simple polygons of n total vertices and an integer k with $1 \leqslant k \leqslant m-1$, the order-$k$ polygon Voronoi diagram has combinatorial complexity $O(\min\{k(n-k), (m-k)^2 n\})$.*

5 Tight bounds on U_k and $\overline{U}_k$ – Proving Lemma 6

Recall the circular sequence $\Pi(S) = (\pi_0, \ldots, \pi_{N-1}, \pi_N = \pi_0)$ of permutations π_i of sites S, discussed in Section 2. In our setting, each site $p \in S$ is assigned a color $\kappa(p) \in K$. For each color $a \in K$ and $0 \leqslant i \leqslant N-1$, let $\lambda_i(a) \in S_a$ be the first site of color a in permutation π_i, called the *leader* of color a in π_i. By a slight abuse of notation, we also use λ_i to denote the subsequence of π_i that consists of the leaders of all colors. It is not hard to see that the induced circular sequence $\Lambda(\Pi(S)) = (\lambda_0, \ldots, \lambda_N = \lambda_0)$ of color-leader sequences satisfies exactly one of the following, if $\lambda_{i+1} \neq \lambda_i$:

- (L1) there is a replacement of the leader of a color $a \in K$, that is, $\lambda_{i+1}(a) \neq \lambda_i(a)$, or
- (L2) there is a switch of two consecutive leaders from λ_i to λ_{i+1},

A change of the first kind (L1) is called a *monochromatic switch* as it corresponds to a switch in $\Pi(S)$ between two sites of the same color; a change of the second kind (L2) is called a *bichromatic switch*. We say that a monochromatic switch happens *at position* j, if the leader $\lambda_i(a)$ of a color $a \in K$ is at position j in λ_i, and that a bichromatic switch happens *at positions* j, if two leaders $\lambda_i(a)$ and $\lambda_i(b)$ (to be switched) lie at positions j and $j+1$ in λ_i. We then observe the following correspondence between unbounded edges and switches.

▶ **Lemma 10.** *Let $\overline{\Pi}(S) := (\bar{\pi}_0, \ldots, \bar{\pi}_N = \bar{\pi}_0)$, where $\bar{\pi}_i$ denotes the reverse of permutation π_i. For each $1 \leqslant k \leqslant m$, the following hold for the c-chromatic unbounded edges, $c \in \{1, 2\}$:*

- *$u_{c,k}$ is equal to the number of c-chromatic switches at position k in $\Lambda(\Pi(S))$.*
- *$\bar{u}_{c,k}$ is equal to the number of c-chromatic switches at position k in $\Lambda(\overline{\Pi}(S))$.*

Thus, to obtain the bounds of Lemma 6, it is enough to count monochromatic and bichromatic switches in the circular sequences $\Lambda(\Pi(S))$ and $\Lambda(\overline{\Pi}(S))$ of color-leaders. We do the counting in a purely combinatorial setting in the next subsection deriving the bounds of Theorem 11. As a corollary to Theorem 11 we get: $U_k \geqslant k(k+1)$ and $\overline{U}_k \leqslant k(2n-k-1)$. These bounds are tight for the unbounded edges of the diagrams by the fact that any circular sequence Π of permutations satisfying conditions (P1) and (P2) can be realized by an admissible system of bisecting curves [14, Lemma 10]. This completes the proof of Lemma 6.

5.1 Switches in a sequence of permutations of colored elements

In the following, we consider a purely combinatorial setting for circular sequences as follows. Let $\Pi = (\pi_0, \ldots, \pi_N = \pi_0)$ be a given circular sequence of permutations π_i of n elements $\{1, \ldots, n\}$ that satisfies conditions (P1) and (P2), and assume that each element in $\{1, \ldots, n\}$ is assigned a color from $K = \{1, \ldots, m\}$. As above, we have the induced sequence $\Lambda = \Lambda(\Pi) = (\lambda_0, \ldots, \lambda_{N-1}, \lambda_N = \lambda_0)$ of sequences λ_i of m leaders of colors, which satisfy conditions (L1) and (L2). For each $1 \leqslant k \leqslant m$, define $\boldsymbol{g_{1,k}} = g_{1,k}(\Pi)$ to be the number of monochromatic switches at position k in Λ and $\boldsymbol{g_{2,k}} = g_{2,k}(\Pi)$ to be the number of bichromatic switches at position k in Λ. We are interested in the following quantity:

$$G_k = G_k(\Pi) := \sum_{j=1}^{k}(g_{2,j} + (k-j+1)g_{1,j}).$$

▶ **Theorem 11.** *For $1 \leqslant k \leqslant m-1$, $k(k+1) \leqslant G_k \leqslant k(2n-k-1)$. Both bounds are tight.*

The bounds are shown to be tight by constructions, see Lemma 25 in the full version [12].

Deriving the lower bound. See Appendix B for missing proofs. We charge each monochromatic or bichromatic switch to a unique color $a \in K$ as follows:

- Each monochromatic switch, replacing the leader of color $a \in K$, is charged to the corresponding color a. Let $\boldsymbol{\mu_a(j)}$ be the number of monochromatic switches at position j in Λ charged to a.
- Each bichromatic switch involving two colors $a, a' \in K$ with $a < a'$ is charged to color a with the smaller index. Let $\boldsymbol{\beta_a(j)}$ be the number of bichromatic switches at position j in Λ charged to a.

It is obvious that $g_{1,j} = \sum_{a \in K} \mu_a(j)$ and $g_{2,j} = \sum_{a \in K} \beta_a(j)$.

Let $\rho_i(a)$ for $a \in K$ be the position, or the rank, of $\lambda_i(a)$ in permutation λ_i, and let $\rho^*(a) := \max_i \rho_i(a)$ be the maximum rank of a over the whole sequence Λ. Note that for any color $a \in K$ having a single element, we have $\mu_a(j) = 0$ for every j, since there is no change of its leader $\lambda_i(a)$. We observe the following for colors having at least two elements.

▶ **Lemma 12.** *Let $a \in K$ be any color having at least two elements. For $1 \leqslant k \leqslant m-1$,*

$$\sum_{j=1}^{k}(k-j+1)\mu_a(j) \geqslant \max\{2(k+1-\rho^*(a)), 0\}.$$

Next, we consider the quantity $\beta_a(j)$ about bichromatic switches. Note that $\beta_a(m) = 0$ for any $a \in K$ and $\beta_m(k) = 0$ for any k by definition. From here onward, we assume without loss of generality that $\pi_0 = (1, 2, \ldots, n)$ and the colors in $K = \{1, \ldots, m\}$ appear in the order in the initial permutation λ_0 of leaders, that is, $\lambda_0 = (\lambda_0(1), \lambda_0(2), \ldots, \lambda_0(m))$, so $\rho_0(a) = a$ for $a \in K = \{1, \ldots, m\}$.

▶ **Lemma 13.** *For $1 \leqslant k \leqslant m-1$ and $a \in K$, it holds that $\sum_{j=1}^{k} \beta_a(j) \geqslant \max\{2(\hat{k}_a+1-a), 0\}$, where $\hat{k}_a = \min\{k, \rho^*(a) - 1\}$.*

For colors $a \in K$ that have only a single element, we show a better lower bound on β_a in the next lemma. By combining we derive the inequality in Lemma 15.

▶ **Lemma 14.** *Let $a \in K$ be a color having only one element. For $1 \leqslant k \leqslant m-1$, it holds that $\sum_{j=1}^{k} \beta_a(j) \geqslant \max\{2(k+1-a), 0\}$.*

▶ **Lemma 15.** *For $1 \leqslant k \leqslant m-1$ and $a \in K$, it holds that*

$$\sum_{j=1}^{k}\beta_a(j) + \sum_{j=1}^{k}(k-j+1)\mu_a(j) \geqslant \max\{2(k+1-a)), 0\}.$$

Summing the inequality of Lemma 15 over all colors $a \in K$, we obtain the claimed lower bound of Theorem 11, as $\sum_{a \in K} \mu_a(j) = g_{1,j}$ and $\sum_{a \in K} \beta_a(j) = g_{2,j}$.

Deriving the upper bound. Recall our assumption that $\pi_0 = (1, 2, \ldots, n)$ and $\lambda_0 = (\lambda_0(1), \ldots, \lambda_0(m))$. We start with the following observation.

▶ **Lemma 16.** *For $a \in K$, the largest element t of color a can be the leader of a at most once in π_i over all $0 \leqslant i \leqslant N$. Thus, t is involved in exactly zero or two monochromatic switches.*

To prove the upper bound of G_k, we introduce a different charging scheme, in which each monochromatic and bichromatic switch is charged to the element t involved in the switch that has a larger index. So if t is the leader of color a in λ_0, then it is charged zero monochromatic switches. We show that the weighed sum of all switches in G_k, which are charged to t, is at most $2k$, and give a tighter bound, if t is a leader in λ_0. The derivation can be found in Section 5.1.2 of the full version [12].

6 Algorithm

The minimal color Voronoi diagram $\mathsf{CVD}_k(S)$ can be computed iteratively from order 1 up to k in $O(k^2 n + n \log n)$ expected or $O(k^2 n \log n)$ worst-case time [11]. However, this does not apply to the maximal diagram because the order $k-1$ diagram $\overline{\mathsf{CVD}}_{k-1}(S)$ does not contain sufficient information to determine $\overline{\mathsf{CVD}}_k(S)$. For point sites under convex distance functions the following property holds: *the sequence of sites defining unbounded regions of the minimal and the maximal diagram is the same.* In that case $\overline{\mathsf{CVD}}_k(S)$ can be obtained iteratively by computing $\mathsf{CVD}_k(S)$ at the same time [11]. However, this property does not hold for general sites nor for abstract Voronoi diagrams.

In this paper we devise a direct divide-and-conquer algorithm to compute the unbounded edges of $\overline{\mathsf{CVD}}_k(S)$, after which $\overline{\mathsf{CVD}}_k(S)$ can be computed from $\overline{\mathsf{CVD}}_{k-1}(S)$. This results in the first algorithm to compute the $\overline{\mathsf{CVD}}_k(S)$ for generalized sites, such as segments or disks, including the order k simple polygon Voronoi diagram, and the first iterative algorithm we know of to compute the ordinary diagrams in reverse order $\mathsf{VD}_{n-1}(S), \ldots, \mathsf{VD}_k(S)$, which is efficient for large values of k. Let the diagrams $\mathsf{CVD}^*_1(S), \ldots, \mathsf{CVD}^*_k(S)$ and $\overline{\mathsf{CVD}}^*_1(S), \ldots, \overline{\mathsf{CVD}}^*_k(S)$ be abbreviated as $\mathsf{CVD}^*_{\leqslant k}(S)$ and $\overline{\mathsf{CVD}}^*_{\leqslant k}(S)$ respectively.

Our divide-and-conquer algorithm splits the set of colors K at each step. For the base case, if $k \geqslant m/2$, then we obtain the unbounded edges in $\overline{\mathsf{CVD}}^*_{\leqslant k}(S)$ by walking along Γ and computing the unbounded edges of $\overline{\mathsf{CVD}}^*_{\leqslant m}(S)$; see Lemma 17 that details the conquer step. If $k < m/2$, we divide the m colors in K into two disjoint subsets, K_1 and K_2, of roughly equal size; let S_1 and S_2 be the sites of the colors in K_1 and K_2 respectively; see Lemma 18.

▶ **Lemma 17.** *The unbounded edges in $\overline{\mathsf{CVD}}^*_{\leqslant m}(S)$ can be computed in $O(m^2(2n-m)+n \log n)$ time.*

▶ **Lemma 18.** *Given the unbounded edges in $\overline{\mathsf{CVD}}^*_{\leqslant k}(S_1)$ and $\overline{\mathsf{CVD}}^*_{\leqslant k}(S_2)$, where $S_1 \cup S_2 = S$, the unbounded edges in $\overline{\mathsf{CVD}}^*_{\leqslant k}(S)$ can be computed in $O(k^2(2n-k))$ time.*

Using the conquer step in Lemma 17 and the divide step in Lemma 18, we obtain the divide-and-conquer algorithm.

▶ **Lemma 19.** *For $1 \leqslant k \leqslant m$, the unbounded edges in $\overline{\mathsf{CVD}}^*_{\leqslant k}(S)$ can be computed in $O(k^2 n \log m + n \log n)$ time.*

Using Lemma 19 as a subroutine to the iterative algorithm [11, Theorem 27], we can compute $\overline{\mathsf{CVD}}^*_k(S)$ for abstract Voronoi diagrams without any additional assumption.

▶ **Theorem 20.** *Let S be a set of n colored sites with $m \leqslant n$ colors. Given an admissible system $\mathcal{J}$ of bisecting curves for S and an integer k with $1 \leqslant k \leqslant m-1$, we can compute $\mathsf{CVD}^*_1(S), \ldots, \mathsf{CVD}^*_k(S)$ and $\overline{\mathsf{CVD}}^*_1(S), \ldots, \overline{\mathsf{CVD}}^*_k(S)$ in $O(k^2 n \log n)$ time. $\mathsf{CVD}^*_1(S), \ldots, \mathsf{CVD}^*_k(S)$ can also be computed in expected $O(k^2 n + n \log n)$ time.*

Recall that if $n = m$, then $\overline{\mathsf{CVD}}_k(S) = \mathsf{VD}_{n-k}(S)$. Thus, using Theorem 20 we can iteratively compute higher-order Voronoi diagrams starting from the $\mathsf{FVD}(S)$.

▶ **Corollary 21.** *Let S be a set of n uncolored sites. Given an admissible system $\mathcal{J}$ of bisecting curves for S and an integer k with $1 \leqslant k \leqslant n-1$, we can compute $\mathsf{VD}_{n-1}(S), \ldots, \mathsf{VD}_k(S)$ in $O((n-k)^2 n \log n)$ time.*

7 Concluding remarks

We have proved the exact maximum number $4k(n-k) - 2n$ of vertices in the order-k abstract color Voronoi diagrams $\mathsf{CVD}_k(S)$ and $\overline{\mathsf{CVD}}_k(S)$. The main ingredients of our proof are the colorful Clarkson–Shor technique [11] and new tight bounds on circular sequences of permutations of colored elements. The notion of permutations of colored elements is of independent interest as a purely combinatorial concept, which might find more applications in analyzing abstract or concrete geometric structures defined by a set of colored objects.

As a concrete case of abstract color Voronoi diagrams, we showed that the complexity of the order-k polygon Voronoi diagram for m disjoint simple polygons with n total vertices is $O(\min\{k(n-k), (m-k)^2 n\}$. Prior to our result, there was no known reasonable upper bound except for the case $k = m-1$, that is, the farthest polygon Voronoi diagram [22]. Nonetheless, we do not believe that this bound is tight for *every* $1 \leqslant k \leqslant m-1$. It would be interesting to obtain tight bounds for the complexity of the order-k polygon Voronoi diagram. More specifically, how many vertices can there be when $k = m/2$?

Our iterative algorithms to construct abstract color Voronoi diagrams of orders 1 to k in $O(k^2 n \log n)$ time compares to the algorithm of [11], and even to the first algorithm by Lee [32] that computes Euclidean Voronoi diagrams. For constructing the Euclidean order-k diagram, there was a recent breakthrough, achieving the optimal time $O(kn + n \log n)$ [19]. Is it possible to apply such advanced algorithmic techniques for faster construction of order-k color Voronoi diagrams, either abstract or concrete? For abstract order-k diagrams of non-colored sites, it was shown that a randomized incremental construction can be implemented in $O(k(n-k)\log^2 n + n \log^3 n)$ expected time [15].

References

1 Manuel Abellanas, Ferran Hurtado, Christian Icking, Rolf Klein, Elmar Langetepe, Lihong Ma, Belén Palop, and Vera Sacristán. The farthest color Voronoi diagram and related problems. Technical Report 002, Rheinische Friedrich–Wilhelms–Universität Bonn, 2006.

2 Pankaj K. Agarwal, Mark de Berg, Jiří Matoušek, and Otfried Schwarzkopf. Constructing levels in arrangements and higher order Voronoi diagrams. *SIAM J. Comput.*, 27(3):654–667, 1998. doi:10.1137/S0097539795281840.

3 Alok Aggarwal, Leonidas J. Guibas, James B. Saxe, and Peter W. Shor. A linear-time algorithm for computing the Voronoi diagram of a convex polygon. *Discrete Comput. Geom.*, 4(1):591–604, 1989. doi:10.1007/BF02187749.

4 Noga Alon and E. Györi. The number of small semispaces of a finite set of points in the plane. *J. Combinat. Theory, Series A*, 41(1):154–157, 1986. doi:10.1016/0097-3165(86)90122-6.

5 Elena Arseneva and Evanthia Papadopoulou. Randomized incremental construction for the Hausdorff Voronoi diagram revisited and extended. *Journal of Combinatorial Optimization*, 37(2):579–600, 2019. doi:10.1007/s10878-018-0347-x.

6 Franz Aurenhammer, Robert L.S. Drysdale, and Hannes Krasser. Farthest line segment Voronoi diagrams. *Infor. Process. Lett.*, 100(6):220–225, 2006. doi:10.1016/j.ipl.2006.07.008.

7 Franz Aurenhammer, Rolf Klein, and Der-Tsai Lee. *Voronoi Diagrams and Delaunay Triangulations.* World Scientific, Singapore, 2013. doi:10.1142/8685.

8 Franz Aurenhammer and Otfried Schwarzkopf. A simple on-line randomized incremental algorithm for computing higher order Voronoi diagram. *Int. J. Comput. Geom. Appl.*, 2(4):363–381, 1992. doi:10.1142/S0218195992000214.

9 Sang Won Bae. On linear-sized farthest-color Voronoi diagrams. *IEICE Trans. Inf. Syst.*, 95-D(3):731–736, 2012. doi:10.1587/transinf.E95.D.731.

10 Sang Won Bae. Tight bound and improved algorithm for farthest-color Voronoi diagrams of line segments. *Comput. Geom.: Theory Appl.*, 47(8):779–788, 2014. doi:10.1016/j.comgeo.2014.04.005.

11 Sang Won Bae, Nicolau Oliver, and Evanthia Papadopoulou. Higher-order color Voronoi diagrams and the colorful Clarkson–Shor framework. In *Proc. 41st International Symposium on Computational Geometry (SoCG 2025)*, volume 332 of *LIPIcs*, pages 12:1–10:19, 2025. doi:10.4230/LIPIcs.SOCG.2025.12.

12 Sang Won Bae, Nicolau Oliver, and Evanthia Papadopoulou. Abstract Color Voronoi Diagrams and Circular Sequences of Color Permutations, July 2026. arXiv:2607.05383 [cs.CG]. doi:10.48550/arXiv.2607.05383.

13 Ranita Biswas, Sebastiano Cultrera di Montesano, Ondřej Draganov, Herbert Edelsbrunner, and Morteza Saghafian. On the Size of Chromatic Delaunay Mosaics. *Discrete & Computational Geometry*, 75(1):24–47, January 2026. doi:10.1007/s00454-025-00778-7.

14 Cecilia Bohler, Panagiotis Cheilaris, Rolf Klein, Chih-Hung Liu, Evanthia Papadopoulou, and Maksym Zavershynskyi. On the complexity of higher order abstract Voronoi diagrams. *Comput. Geom.: Theory Appl.*, 48(8):539–551, 2015. doi:10.1016/j.comgeo.2015.04.008.

15 Cecilia Bohler, Rolf Klein, and Chih-Hung Liu. An efficient randomized algorithm for higher-order abstract Voronoi diagrams. *Algorithmica*, 81(6):2317–2345, 2019. doi:10.1007/s00453-018-00536-7.

16 Cecilia Bohler, Chih-Hung Liu, Evanthia Papadopoulou, and Maksym Zavershynskyi. A randomized divide and conquer algorithm for higher-order abstract Voronoi diagrams. *Comput. Geom.: Theory Appl.*, 59:26–38, 2016. doi:10.1016/j.comgeo.2016.08.004.

17 David Bremner, Erik Demaine, Jeff Erickson, John Iacono, Stefan Langerman, Pat Morin, and Godfried Toussaint. Output-sensitive algorithms for computing nearest-neighbour decision boundaries. *Discrete & Computational Geometry*, 33(4):593–604, 2005. doi:10.1007/s00454-004-1152-0.

18 Timothy M. Chan. Random sampling, halfspace range reporting, and construction of ($\leq k$)-levels in three dimensions. *SIAM J. Comput.*, 30(2):561–575, 2000. doi:10.1137/S0097539798349188.

19 Timothy M. Chan, Pingan Cheng, and Da Wei Zheng. An optimal algorithm for higher-order Voronoi diagrams in the plane: The usefulness of nondeterminism. In David P. Woodruff, editor, *Proceedings of the 2024 ACM-SIAM Symposium on Discrete Algorithms, SODA 2024, Alexandria, VA, USA, January 7-10, 2024*, pages 4451–4463. SIAM, 2024. doi:10.1137/1.9781611977912.156.

20 Timothy M. Chan and Konstantinos Tsakalidis. Optimal deterministic algorithms for 2-d and 3-d shallow cuttings. *Discrete Comput. Geom.*, 56(4):866–881, 2016. doi:10.1007/s00454-016-9784-4.

21 Lisi Chen, Shuo Shang, Chengcheng Yang, and Jing Li. Spatial keyword search: a survey. *GeoInformatica*, 24(1):85–106, 2020. doi:10.1007/s10707-019-00373-y.

22 Otfried Cheong, Hazel Everett, Marc Glisse, Joachim Gudmundsson, Samuel Hornus, Sylvain Lazard, Mira Lee, and Hyeon-Suk Na. Farthest-polygon Voronoi diagrams. *Comput. Geom.: Theory and Appl.*, 44(4):234–247, 2011. doi:10.1016/j.comgeo.2010.11.004.

23 Kenneth L. Clarkson and Peter W. Shor. Application of random sampling in computational geometry, II. *Discrete Comput. Geom.*, 4:387–421, 1989. doi:10.1007/BF02187740.

24 Herbert Edelsbrunner, Leonidas J. Guibas, and Micha Sharir. The upper envelope of piecewise linear functions: Algorithms and applications. *Discrete Comput. Geom.*, 4(4):311–336, 1989. doi:10.1007/BF02187733.

25 Jacob E. Goodman and Richard Pollack. On the number of k-subsets of a set of n points in the plane. *J. Combinat. Theory, Series A*, 36(1):101–104, 1984. `doi:10.1016/0097-3165(84)90081-5`.

26 Jacob E. Goodman and Richard Pollack. Allowable sequences and order types in discrete and computational geometry. In *New Trends in Discrete and Computational Geometry*, pages 103–134. Springer-Verlag, Berlin, Heidelberg, New York, 1993. `doi:10.1007/978-3-642-58043-7_6`.

27 P. Gupta and E. Papadopoulou. Yield analysis and optimization. In C.J. Alpert, D.P. Mehta, and S.S. Sapatnekar, editors, *The Handbook of Algorithms for VLSI Physical Design Automation*, chapter 7.3. Taylor & Francis CRC Press, 2008.

28 Daniel P. Huttenlocher, Klara Kedem, and Micha Sharir. The upper envelope of Voronoi surfaces and its applications. *Discrete Comput. Geom.*, 9:267–291, 1993. `doi:10.1007/BF02189323`.

29 Rolf Klein. *Concrete and Abstract Voronoi Diagrams*, volume 400 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, Heidelberg, New York, 1989. `doi:10.1007/3-540-52055-4`.

30 Rolf Klein, Elmar Langetepe, and Zahra Nilforoushan. Abstract Voronoi diagrams revisited. *Comput. Geom.*, 42(9):885–902, 2009. `doi:10.1016/J.COMGEO.2009.03.002`.

31 Chunseok Lee, Donghoon Shin, Sang Won Bae, and Sunghee Choi. Best and worst-case coverage problems for arbitrary paths in wireless sensor networks. *Ad Hoc Networks*, 11(6):1699–1714, 2013. `doi:10.1016/j.adhoc.2013.03.005`.

32 Der-Tsai Lee. On k-nearest neighbor Voronoi diagrams in the plane. *IEEE Trans. Computers*, 31(6):478–487, 1982. `doi:10.1109/TC.1982.1676031`.

33 Chih-Hung Liu, Evanthia Papadopoulou, and Der-Tsai Lee. The k-nearest-neighbor Voronoi diagram revisited. *Algorithmica*, 71(2):429–449, 2015. `doi:10.1007/s00453-013-9809-9`.

34 Ioannis Mantas, Evanthia Papadopoulou, Rodrigo I. Silveira, and Zeyu Wang. The farthest color Voronoi diagram in the plane. *Algorithmica*, 87(10):1393–1419, 2025. `doi:10.1007/s00453-025-01311-1`.

35 Kurt Mehlhorn, Stefan Meiser, and Ronald Rasch. Furthest site abstract Voronoi diagrams. *Int. J. Comput. Geom. Appl.*, 11(6):583–616, 2001. `doi:10.1142/S0218195901000663`.

36 Ketan Mulmuley. On levels in arrangements and Voronoi diagrams. *Discrete Comput. Geom.*, 6(1):307–338, 1991. `doi:10.1007/BF02574692`.

37 Atsuyuki Okabe, Barry Boots, Kokichi Sugihara, Sung Nok Chiu, and David G. Kendall. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, Second Edition*. John Wiley and Sons, New York, 2000. `doi:10.1002/9780470317013`.

38 Evanthia Papadopoulou. Critical area computation for missing material defects in VLSI circuits. *IEEE Trans. Comput. Aided Des. Integr. Circuits Syst.*, 20(5):583–597, 2001. `doi:10.1109/43.920683`.

39 Evanthia Papadopoulou. The Hausdorff Voronoi diagram of point clusters in the plane. *Algorithmica*, 40(2):63–82, 2004. `doi:10.1007/s00453-004-1095-0`.

40 Evanthia Papadopoulou. Net-aware critical area extraction for opens in VLSI circuits via higher-order Voronoi diagrams. *IEEE T. Comput. Aid. D.*, 30(5):704–716, 2011. `doi:10.1109/TCAD.2010.2100550`.

41 Evanthia Papadopoulou and Maksym Zavershynskyi. The higher-order Voronoi diagram of line segments. *Algorithmica*, 74(1):415–439, 2016. `doi:10.1007/s00453-014-9950-0`.

42 Edgar A. Ramos. On range reporting, ray shooting, and k-level construction. In *Proceedings of the Fifteenth Annual Symposium on Computational Geometry, Miami Beach, Florida, USA, June 13-16, 1999*, pages 390–399. ACM, 1999. `doi:10.1145/304893.304993`.

43 Voronoi CAA: Voronoi Critical Area Analysis. IBM CAD Tool, Department of Electronic Design Automation, IBM Microelectronics Division, Burlington, VT. Patents: US6178539, US6317859, US7240306, US7240306, US20080168414, US20090031263.

A Figures

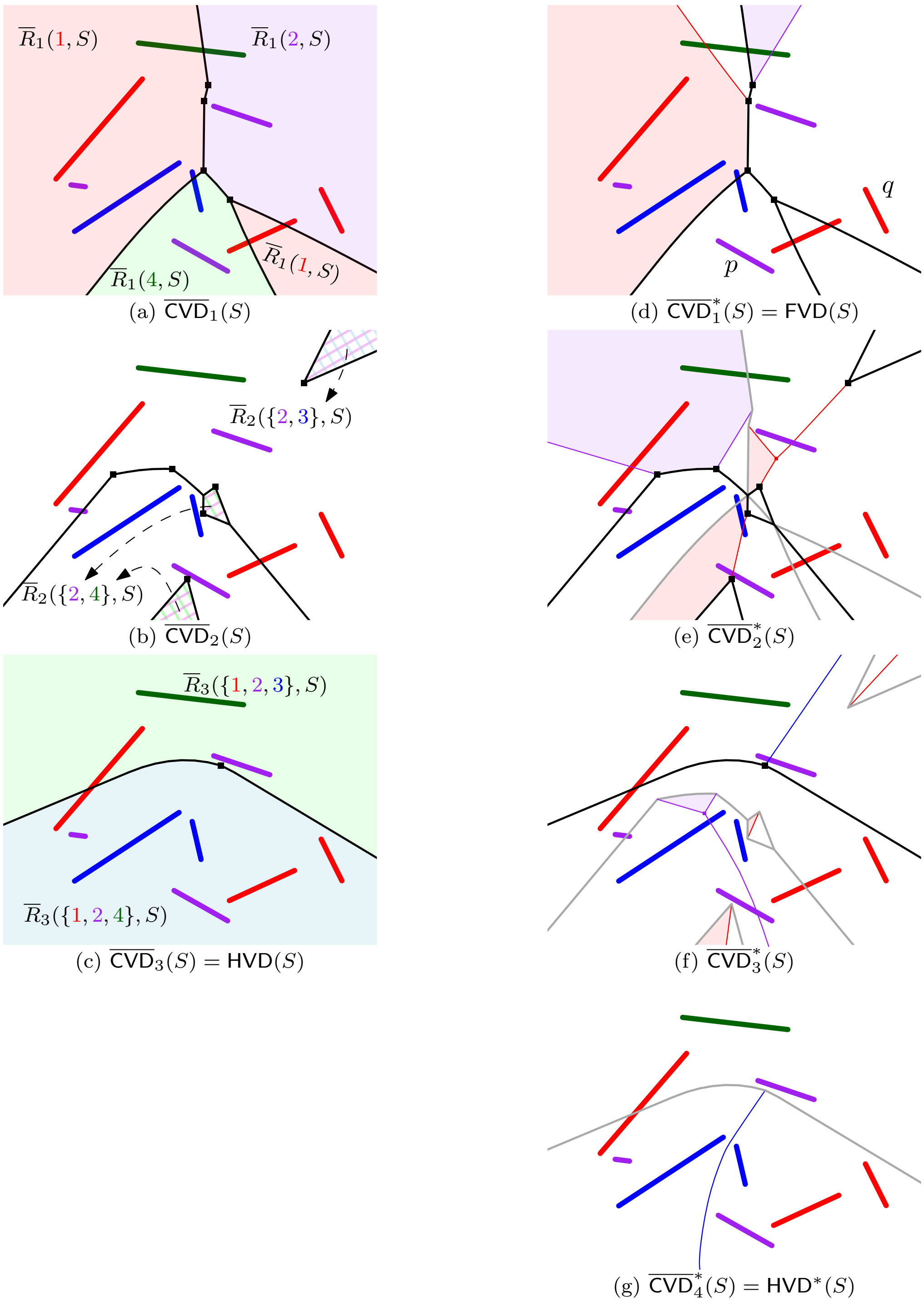


Figure 4 The maximal color Voronoi diagrams $\overline{\mathsf{CVD}}_k(S)$ and $\overline{\mathsf{CVD}}^*_k(S)$ of the same segment sites as Figure 3, using the same colors to draw the c-chromatic features of the diagrams. Selected regions of $\overline{\mathsf{CVD}}_k(S)$ are labeled and shaded correspondingly. The faces in $\overline{\mathsf{CVD}}^*_k(S)$ associated to sites p and q are shaded purple and red.

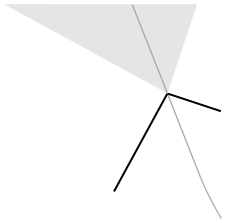


Figure 5 The angular bisector of two segments over their equidistant area shown in gray.

B Deriving the lower bound in Section 5.1

▶ **Lemma 12.** *Let $a \in K$ be any color having at least two elements. For $1 \leqslant k \leqslant m-1$,*

$$\sum_{j=1}^{k}(k-j+1)\mu_a(j) \geqslant \max\{2(k+1-\rho^*(a)), 0\}.$$

Proof. If $k < \rho^*(a)$, then $\max\{k+1-\rho^*(a), 0\} = 0$ and the inequality holds trivially, so assume $k \geqslant \rho^*(a)$.

We first observe that there are at least two changes of $\lambda_i(a)$ over $0 \leqslant i \leqslant N-1$. This is obvious from the fact that $\lambda_i(a)$ is the first element of the subsequence of π_i that contains those elements in $\{1, \ldots, n\}$ whose color is a and the property (P2) that every pair of elements switches exactly twice.

Consider any two such changes happened at $0 \leqslant i_1 < i_2 < N$, and let $j_1 := \rho_{i_1}(a)$ and $j_2 := \rho_{i_2}(a)$ be two positions of the leader of a in λ_{i_1} and λ_{i_2}, respectively. By definition, we have $j_1, j_2 \leqslant \rho^*(a) \leqslant k$. Thus, we obtain

$$\sum_{j=1}^{k}(k-j+1)\mu_a(j) \geqslant (k-j_1+1) + (k-j_2+1) \geqslant 2(k-\rho^*(a)+1)$$

by the existence of two changes of $\lambda_i(a)$. ◀

▶ **Lemma 13.** *For $1 \leqslant k \leqslant m-1$ and $a \in K$, it holds that $\sum_{j=1}^{k}\beta_a(j) \geqslant \max\{2(\hat{k}_a+1-a), 0\}$, where $\hat{k}_a = \min\{k, \rho^*(a)-1\}$.*

Proof. Recall our assumption that the leader of color a is placed at position a in λ_0, that is, $\rho_0(a) = a$. If $\hat{k}_a < a-1$, then the inequality holds trivially, so assume that $\hat{k}_a \geqslant a$.

Let z (resp., z') be the first (resp. last) index such that $\rho_z(a) = \hat{k}_a + 1 = \rho_{z'}(a)$, and let

$$A := \{a' \in K \mid a' > a \text{ and } \rho_z(a') < \rho_z(a) = \hat{k}_a + 1\}$$

be the set of colors $a' > a$ preceding a in λ_z. Since the order between colors a and each $a' \in A$ is reversed between λ_0 and λ_z, there must be at least $|A|$ many bichromatic switches while we move from λ_0 to λ_z, counted in $\sum_{j=1}^{\hat{k}_a}\beta_a(j)$ because $\rho_i(a) < \hat{k}_a+1$ for all $i < z$. On the other hand, since there are at most $a-1$ colors smaller than a, there must be at least $\hat{k}_a-(a-1)$ colors greater than a that precede a in $\lambda_{z'}$. Thus, there must be at least $\hat{k}_a-(a-1)$ many bichromatic switches while we move from $\lambda_{z'}$ to λ_N, counted in $\sum_{j=1}^{\hat{k}_a}\beta_a(j)$, because $\rho_i(a) < \hat{k}_a+1$ for all $i > z'$. This implies that

$$\sum_{j=1}^{k}\beta_a(j) \geqslant \sum_{j=1}^{\hat{k}_a}\beta_a(j) \geqslant |A| + \hat{k}_a - (a-1).$$

To bound $|A|$, let $B := \{a' \in K \mid a' < a \text{ and } \rho_z(a') < \rho_z(a)\}$. Obviously, $|B| \leqslant a-1$ and $A \cap B = \emptyset$. Since $A \cup B$ consists of all colors a' such that $\rho_z(a') < \rho_z(a) = \hat{k}_a + 1$, we have $|A \cup B| = \hat{k}_a$ and hence

$$|A| = |A \cup B| - |B| \geqslant \hat{k}_a - (a-1) = \hat{k}_a + 1 - a,$$

implying the inequality. ◀

▶ **Lemma 14.** *Let $a \in K$ be a color having only one element. For $1 \leqslant k \leqslant m-1$, it holds that $\sum_{j=1}^{k} \beta_a(j) \geqslant \max\{2(k+1-a), 0\}$.*

Proof. Let $t \in \{1, \ldots, n\}$ be the only element of color a, so t is always the leader of a: $\lambda_i(a) = t$ for all i. Since every pair of elements switch exactly twice in the sequence Π (condition (P2)), t is involved in at least two bichromatic switches with each of the other colors. Among them, those switches with $a' > a$ are counted in $\sum_j \beta_a(j)$, so we have

$$\sum_j \beta_a(j) = \sum_{j=1}^{\rho^*(a)} \beta_a(j) \geqslant 2(m-a).$$

We have two cases: whether $k \geqslant \rho^*(a)$ or $k < \rho^*(a)$. If $k \geqslant \rho^*(a)$, then we have

$$\sum_{j=1}^{k} \beta_a(j) \geqslant \sum_{j=1}^{\rho^*(a)} \beta_a(j) \geqslant 2(m-a) \geqslant \max\{2(k+1-a), 0\}$$

since $k \leqslant m-1$. Otherwise, if $k < \rho^*(a)$, then Lemma 13 is strong enough to conclude the lemma:

$$\sum_{j=1}^{k} \beta_a(j) \geqslant \max\{2(\hat{k}_a + 1 - a), 0\} \geqslant \max\{2(k+1-a), 0\}$$

since $\hat{k}_a = \min\{k, \rho^*(a) - 1\} = k$. ◀

▶ **Lemma 15.** *For $1 \leqslant k \leqslant m-1$ and $a \in K$, it holds that*

$$\sum_{j=1}^{k} \beta_a(j) + \sum_{j=1}^{k} (k-j+1)\mu_a(j) \geqslant \max\{2(k+1-a)), 0\}.$$

Proof. If there is only one element in $\{1, \ldots, n\}$ whose color is a, then $\mu_a(j) = 0$ for all j, so Lemma 14 implies the lemma. So, in the following, we assume that there are at least two elements whose color is a.

Adding the inequalities of Lemmas 12 and 13 yields

$$\sum_{j=1}^{k} \beta_a(j) + \sum_{j=1}^{k} (k-j+1)\mu_a(j) \geqslant \max\{2(\hat{k}_a + 1 - a), 0\} + \max\{2(k+1-\rho^*(a)), 0\}.$$

We simplify the right-hand side by a case analysis on $\hat{k} = \min\{k, \rho^*(a) - 1\}$. If $\rho^*(a) \leqslant k$, then $\hat{k}_a = \rho^*(a) - 1$ and $\hat{k}_a + 1 - a = \rho^*(a) - a$. Note also the trivial fact that $\rho^*(a) - a = \rho^*(a) - \rho_0(a) \geqslant 0$. So, the right-hand side is simplified to

$$2(\rho^*(a) - a) + \max\{2(k - \rho^*(a) + 1), 0\} = \max\{2(k+1-a), 0\}.$$

Otherwise, if $\rho^*(a) > k$, then $\hat{k}_a = k$ and $\max\{2(k+1-\rho^*(a)), 0\} = 0$. So, we have

$$\max\{2(\hat{k}_a + 1 - a), 0\} + \max\{2(k+1-\rho^*(a)), 0\} = \max\{2(k+1-a), 0\},$$

as claimed. ◀