A 0.8395-Approximation Algorithm for the EPR Problem

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Abstract

We give an efficient 0.8395-approximation algorithm for the EPR Hamiltonian. Our improvement comes from a new nonlinear monogamy-of-entanglement bound on star graphs and a refined parameterization of a shallow quantum circuit from previous works. We also prove limitations showing that current methods cannot achieve substantially better approximation ratios, indicating that further progress will require fundamentally new techniques.

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1 Motivation

Deciding the maximum (or minimum) energy of local Hamiltonians to inverse polynomial precision (LHP) is a central problem in quantum computation. Even for 2-local Hamiltonians, the LHP is complete for the class QMA [12], the quantum analogue of NP. Since exact computation is intractable, a natural question is: *how well can we approximate these extremal energies efficiently?* Approximation quality is measured by the *approximation ratio*, i.e., the fraction of optimal energy that an algorithm guarantees in the worst case. Developing approximation algorithms for classical constraint satisfaction problems (which can be viewed as specializations of the LHP) has yielded unexpected and far-reaching insights. For quantum LHPs, understanding such ratios may yield new algorithms for approximating extremal energy states, or provide insight into the quantum PCP conjecture by establishing barriers for efficient approximation.

We study a specific 2-local LHP that has recently attracted significant attention. Consider a graph $G = (V, E, w)$ on vertices $V \stackrel{\text{def}}{=} [n]$, with edges $E \subseteq V \times V$ and edge weights $w \in \mathbb{R}^E_{>0}$. The *EPR problem* is to find the maximum eigenvalue of the following Hamiltonian:

$$H(G) \stackrel{\text{def}}{=} \sum_{(i,j)\in E} w_{ij} h_{ij} \,, \qquad h_{ij} \stackrel{\text{def}}{=} \frac{1}{2}(I_iI_j + X_iX_j - Y_iY_j + Z_iZ_j)\,, \tag{1}$$

where I is the identity operator and $\{X_i, Y_i, Z_i\}$ are the Pauli operators acting on qubit i. The EPR problem was introduced by [11], who showed the problem is in StoqMA [4]. However, a matching lower bound is not known to us. In fact, the problem may even admit polynomial classical or quantum algorithms. We study α-approximation algorithms for the EPR problem. Such an algorithm $\mathcal{A}$ inputs a graph G, and outputs a value $\mathcal{A}(G)$ such that $\alpha \cdot \lambda_{\max}(H(G)) \leq \mathcal{A}(G) \leq \lambda_{\max}(H(G))$. An efficient α-approximation algorithm was first shown in [11] for $\alpha > 0.7071$. This was improved to $\alpha = 0.72$ in [9], and later to $\alpha > 0.8090$ in [2, 10].

2 Our Contributions

Our main result is a further improvement in the approximation ratio for the EPR problem:

▶ **Theorem 1.** *There is an efficient α-approximation algorithm for the EPR problem for $\alpha > 0.8395$.*

2.1 Algorithm

Our algorithm uses the same quantum circuit structure (ansatz) as previous approximation algorithms for the EPR problem [1, 11, 2]. This ansatz does the following:

1. Solve an efficient relaxation of the EPR problem to obtain values g_{ij} for each edge (i, j) in G.
2. Apply a depth-1 quantum circuit parameterized by a set of angles $\{\theta_{ij}\}_{(i,j)\in E}$. The angles are chosen as a deterministic function of $\{g_{ij}\}_{(i,j)\in E}$.

For convenience, we assume that θ_{ij} depends *only* on g_{ij}; i.e. $\theta_{ij} = \nu(g_{ij})$ for some function ν. We choose values $\{g_{ij}\}_{(i,j)\in E}$ by solving the quantum moment sum-of-squares (moment-SoS) hierarchy defined in Section 3.1. We state this ansatz formally as Algorithm 1:

Algorithm 1 EPR approximation ansatz.

Input: weighted graph $G(V, E, w)$ and function $\nu : [-1, 1] \to [0, 1]$

1 Solve the level-2 quantum moment-SoS hierarchy (Equation (10)) to obtain $\{g_{ij}\}_{(i,j)\in E}$.

2 Output the state

$$|\psi_G\rangle \stackrel{\text{def}}{=} \prod_{(i,j)\in E} \exp\left(\frac{\mathbf{i}\,\theta_{ij}}{4}(X_i - Y_i) \otimes (X_j - Y_j)\right) |0\rangle^{\otimes n}\,, \tag{2}$$

where $\mathbf{i}$ is the imaginary unit and $\theta_{ij} = \nu(g_{ij})$.

Algorithm 1 comes with a classically computable lower bound on its average energy:

▶ **Lemma 2** ([11, Lemma 9]). *Algorithm 1 prepares a state with energy at least*

$$\langle\psi_G| H(G) |\psi_G\rangle \geq \ell(G) \stackrel{\text{def}}{=} \sum_{(i,j)\in E} w_{ij} \cdot \frac{1 + A_{ij}A_{ji} + (A_{ij} + A_{ji}) \sin \nu(g_{ij})}{2}\,, \tag{3}$$

where $A_{ij} \stackrel{\text{def}}{=} \prod_{k\in N(i)\setminus\{j\}} \cos \nu(g_{ik})$ *and* $N(i)$ *is the set of neighbors of* i *in* G. *Furthermore, the inequality is an equality for all triangle-free graphs.*

To fully specify Algorithm 1, we must choose a function ν. We provide our specific choice of ν in Theorem 12. This function was obtained via numerical search, and we prove the correctness of the approximation ratio analytically in Theorem 12. We show in Section 5 that our choice of ν is essentially optimal.

2.2 Techniques

Our improved approximation ratio relies on two crucial insights. First, we prove a new *monogamy of entanglement* (MoE) statement, generalizing the bound of [13, Lemma 3] from a pair of edges to a star. As with other MoE bounds, we show this statement holds for the EPR problem *and* for its level-k semidefinite relaxation whenever $k \geq 2$. We describe these relaxations in Section 3 and prove the lemma in the appendix of the full version.

▶ **Lemma 3** (Nonlinear monogamy of entanglement on a star). *Fix any graph* $G = (V, E, w)$ *and state* $|\psi_G\rangle$. *For each edge* $(i, j) \in E$, *let* $g_{ij} = \langle\psi_G| h_{ij} |\psi_G\rangle - 1$. *Then for any vertex* $i \in V$ *with degree* $d_i \geq 2$ *and* $j \in N(i)$,

$$\sum_{k\in N(i)\setminus\{j\}} g_{ik} \leq \begin{cases} 1, & \text{if } -1 \leq g_{ij} < -\frac{1}{d_i}, \\ \frac{1}{2}\left(2 - d_i - g_{ij} + \sqrt{(d_i^2 - 1)\left(1 - g_{ij}^2\right)}\right), & \text{if } -\frac{1}{d_i} \leq g_{ij} \leq 1\,, \end{cases} \tag{4}$$

where $N(i)$ *is the set of neighbors of* i. *Moreover, Equation (4) holds for any* g *that is the solution to the level-*k *semidefinite relaxation of* $H(G)$ *(defined in Section 3), for any* $k \geq 2$.

Our second insight follows from [8]. We choose the parameterization function ν from a set $\mathcal{C}$ that dramatically simplifies the approximation ratio analysis:

▶ **Definition 4.** *Let $\mathcal{C}$ be the set of monotonically increasing functions $\Theta : [0,1] \to [0,1]$ where $\Theta(0) = 0$, and for all $x_1, x_2, \ldots, x_p \in [0,1]$ satisfying $\sum_{i=1}^{p} x_i \leq 1$, we have*

$$\prod_{i=1}^{p} (1 - \Theta(x_i)) \geq 1 - \Theta\left(\sum_{i=1}^{p} x_i\right). \tag{5}$$

We sketch how the set $\mathcal{C}$ simplifies our analysis. Suppose $\nu(x) = \arcsin\sqrt{\Theta(x^+)}$ for some $\Theta \in \mathcal{C}$, where we use the notation $x^+ \stackrel{\text{def}}{=} \max\{x, 0\}$. Then, the energy from Lemma 2 becomes

$$A_{ij} = \prod_{k \in N(i)\setminus\{j\}} \sqrt{1 - \Theta(g_{ik}^+)} \geq \sqrt{1 - \Theta\left(\sum_{k \in N(i)\setminus\{j\}} g_{ik}^+\right)}.$$

Through our parameterization of ν, we have converted a product of trigonometric functions to a sum over g^+ values on neighboring edges. We may then directly apply MoE bounds on these g values to lower bound A_{ij}.

Finally, in Section 5, we discuss natural limitations of our ansatz and analysis. For example, one step of our analysis considers the approximation ratio on the *worst-case* edge of a graph. We show that under this worst-case edge analysis, our choice of ν is essentially optimal. To obtain better approximation ratios, each angle θ_{ij} in Algorithm 1 must depend on more than just g_{ij}, or the analysis must avoid reducing to the worst-case edge.

3 Semidefinite relaxation of the EPR problem

To prove the approximation ratio, we find two numbers $l(G), u(G)$ that depend on the input graph, such that $u(G) \geq \lambda_{\max}(H(G)) \geq \langle\psi_G|H(G)|\psi_G\rangle \geq l(G) \geq 0$. We show $\ell(G) \geq \alpha \cdot u(G)$, which in turn gives $\langle\psi_G|H(G)|\psi_G\rangle \geq \alpha \cdot \lambda_{\max}(H(G))$, proving the approximation ratio. For the lower bound, we use Lemma 2. We construct u by upper-bounding a relaxation of the EPR problem given by the *quantum moment-SoS hierarchy* [16], which we briefly introduce here. For more detailed descriptions, see [6, 11, 15]. The idea of using a semidefinite program to upper bound an objective function on graphs goes back to the work of Goemans and Williamson on MaxCut problem [7].

3.1 Defining the relaxation

Consider the Pauli monomials on n qubits with at most k non-identity terms:

$$\mathcal{P}_k \stackrel{\text{def}}{=} \left\{\sigma_{i_1}^{\alpha_1} \cdots \sigma_{i_t}^{\alpha_t} \,\middle|\, t \leq k,\ \alpha_j \in \{X, Y, Z\},\ 1 \leq i_1 < \cdots < i_t \leq n\right\}, \tag{6}$$

as well as their span with respect to real coefficients:

$$\mathcal{O}_k \stackrel{\text{def}}{=} \operatorname{span}_{\mathbb{R}}\left\{\mathcal{P}_k\right\}.$$

The kth level of the quantum moment-SoS hierarchy is defined with respect to $\mathcal{M}_k$, which is the set of real symmetric *moment matrices*, $\Gamma \in \mathbb{R}^{\mathcal{P}_k \times \mathcal{P}_k}$ (the notation $\mathbb{R}^S$ refers to a real vector indexed by elements of set S) satisfying:

$$\begin{aligned}
&\Gamma \succeq 0, && && (7)\\
&\Gamma(A,B) = \Gamma(A',B') && \forall\, A,B,A',B' \in \mathcal{P}_k : AB = A'B', && (8)\\
&\Gamma(A,B) = -\Gamma(A',B') && \forall\, A,B,A',B' \in \mathcal{P}_k : AB = -A'B', && (9)\\
&\Gamma(A,B) = 0 && \forall\, A,B \in \mathcal{P}_k : AB \text{ not Hermitian}, &&\\
&\Gamma(A,A) = 1 && \forall\, A \in \mathcal{P}_k. &&
\end{aligned}$$

For convenience, we define the real linear functional $L : \mathcal{O}_{2k} \to \mathbb{R}$ that satisfies $L(C) = \Gamma(A, B)$ whenever $C = AB$ with $A, B \in \mathcal{P}_k$. The equality constraints of $\mathcal{M}_k$ ensure that $L(C)$ is well defined, and Equation (7) is equivalent to $L(A^2) \geq 0$ for all $A \in \mathcal{O}_k$ (see [5, Lemma 1.44]). The value $L(C)$ is also called the *pseudo-expectation* of the operator $C \in \mathcal{O}_{2k}$.

The kth level of the quantum moment-SoS hierarchy is then given by the following semidefinite program (SDP)

$$\begin{aligned} \max \quad & L(H(G)), \\ \text{s.t.} \quad & \Gamma \in \mathcal{M}_k. \end{aligned} \tag{10}$$

From the output of this SDP, we obtain the values

$$g_{ij} \overset{\text{def}}{=} \frac{-1 + L(X_iX_j) - L(Y_iY_j) + L(Z_iZ_j)}{2}, \quad g_{ij}^+ \overset{\text{def}}{=} \max\{g_{ij}, 0\}\,, \tag{11}$$

where $1 + g_{ij}$ is the relaxed objective value on edge (i, j). It then holds that

$$u(G) \overset{\text{def}}{=} \sum_{(i,j)\in E} w_{ij}(1 + g_{ij}), \tag{12}$$

is an upper bound for $\lambda_{\max}\{H(G)\}$.

3.2 Monogamy of entanglement

A key element for designing approximation algorithms for the EPR problem is *monogamy of entanglement* (MoE) [1, 17, 11]. These statements bound the sum of energies (from either the original problem or its relaxation) on neighboring edges. Most research concerns MoE for the Quantum MaxCut (QMC) problem [1, 17, 18], which is distinct from the EPR problem. Specifically, given an edge weighted graph $G = (V, E, w)$, the QMC problem is to find the maximum eigenvalue of the Hamiltonian

$$H^{\mathrm{QMC}}(G) \overset{\text{def}}{=} \sum_{(i,j)\in E} w_{ij} h_{ij}^{\mathrm{QMC}}, \text{ for } h_{ij}^{\mathrm{QMC}} \overset{\text{def}}{=} \frac{1}{2}\left(I_iI_j - X_iX_j - Y_iY_j - Z_iZ_j\right),$$

similar to Equation (1). For bipartite graphs, the QMC and EPR problems are equivalent [11]. Here, we show that on bipartite graphs, the SDP relaxations of QMC and EPR are also equivalent. To show this, note that

$$q_{ij} \overset{\text{def}}{=} \frac{-1 - L(X_iX_j) - L(Y_iY_j) - L(Z_iZ_j)}{2} \tag{13}$$

is such that $1 + q_{ij}$ is the relaxed objective value on edge (i, j) for the QMC problem. That is, q_{ij} is the QMC analogue of g_{ij}, see Equation (11).

▶ **Lemma 5.** *Let E be the edge set of a bipartite graph and let $k \in \mathbb{N}$. For g and q as in Equation* (11) *and Equation* (13) *respectively, define $g(\Gamma)$ and $q(\Gamma)$ as the g and q values induced by some $\Gamma \in \mathcal{M}_k$. We have that*

$$\left\{ \{g(\Gamma)_{ij}\}_{(i,j)\in E} \,\middle|\, \Gamma \in \mathcal{M}_k \right\} = \left\{ \{q(\Gamma)_{ij}\}_{(i,j)\in E} \,\middle|\, \Gamma \in \mathcal{M}_k \right\}. \tag{14}$$

The proof is deferred to the appendix of the full version. Lemma 5 allows us to take existing MoE bounds for the QMC problem and apply them to the EPR problem. For instance, Lemma 3 provides a new MoE bound for the quantum moment-SoS hierarchy for the EPR problem. We show the following simpler corollary of it:

▶ **Corollary 6.** *Fix any graph $G = (V, E, w)$. Then the output $\{g_{ij}\}_{(i,j)\in E}$ in Equation* (11) *from the 2^{nd} level of the quantum moment-SoS hierarchy obeys the following bound for all edges $(i, j) \in E$:*

$$\sum_{k\in N(i)\setminus\{j\}} g^+_{ik} \leq Q(g^+_{ij}) \,, \tag{15}$$

where

$$Q(x) \stackrel{\text{def}}{=} \begin{cases} 1 - x, & \textit{if } 0 \leq x \leq 1/2, \\ \frac{1}{2}\left(\sqrt{3(1-x^2)} - x\right), & \textit{if } 1/2 < x \leq \sqrt{3}/2, \\ 0, & \textit{if } \sqrt{3}/2 < x \leq 1\,. \end{cases} \tag{16}$$

Proof. If $g^+_{ij} \leq \frac{1}{2}$, Equation (15) follows from [17] (see also [13, Lemma 1]) together with Lemma 5. Thus, we assume that $g^+_{ij} > \frac{1}{2}$.

Let P be the RHS of Equation (4), i.e. $\sum_{k\in N(i)\setminus\{j\}} g_{ik} \leq P(g_{ij}, d_i)$. We use N^+ to describe the subset of $N(i)\setminus\{j\}$ with positive values of g; i.e. $N^+ \stackrel{\text{def}}{=} \{k \in N(i) \setminus \{j\} \,|\, g_{ik} > 0\}$.

- Suppose $|N^+| = 0$. Then all values of g are non-positive. Since Q is non-negative,

$$\sum_{k\in N(i)\setminus\{j\}} g_{ik} \leq \sum_{k\in N(i)\setminus\{j\}} g^+_{ik} = 0 \leq Q(g^+_{ij}) \,.$$

- Otherwise, $|N^+| \geq 1$. We apply Lemma 3 to a star graph centered at i that is a subgraph of G, where i is adjacent to j and to all $k \in N^+$:

$$\sum_{k\in N(i)\setminus\{j\}} g^+_{ik} = \sum_{k\in N^+} g_{ik} \leq P(g_{ij}, |N^+| + 1) \leq \max_{d\in\mathbb{N}, d\geq 2} P(g_{ij}, d) \,.$$

Recall that we assumed $g^+_{ij} \geq \frac{1}{2}$. When $d \geq 2$ and $x \geq \frac{1}{2}$,

$$\frac{\partial}{\partial d} P(x, d) = \frac{1}{2}\left(-1 + \frac{d\sqrt{1-x^2}}{\sqrt{d^2-1}}\right) \leq \frac{1}{2}\left(-1 + \frac{\sqrt{3}}{2} \cdot \frac{d}{\sqrt{d^2-1}}\right) \,,$$

which is ≤ 0 when $d \geq 2$. So in this case $\max_{d\in\mathbb{N}, d\geq 2} P(g_{ij}, d) = P(g_{ij}, 2) \leq Q(g^+_{ij})$. ◀

Furthermore, due to the equivalence of the optimal values of the moment-SoS relaxations for the EPR problem and Quantum MaxCut (QMC) on bipartite graphs, our bounds also apply when g instead refers to the SDP edge value for the moment-SoS relaxation of QMC.

4 Analysis

We use this section to prove Claim 11, which lower-bounds the approximation ratio α of Algorithm 1 depending on some parameters. We then choose explicit parameters in Theorem 12 that gives $\alpha > 0.8395$, and prove this in the appendix of the full version.

Consider ν of the following form, given $\beta > \frac{1}{2}$, function $\Theta \in \mathcal{C}$, and function $\Lambda : [0, 1] \to [0, 1]$:

$$\nu(x) \stackrel{\text{def}}{=} \arcsin\sqrt{\tilde{\nu}(x)} \,, \qquad \tilde{\nu}(x) \stackrel{\text{def}}{=} \begin{cases} \Theta(x^+), & \text{if } x \leq \beta \,, \\ \Lambda(x^+), & \text{if } x > \beta \,. \end{cases} \tag{17}$$

We analyze the approximation ratio achieved by Algorithm 1 for this choice of ν. As described in the introduction, we will show the algorithm is an α-approximation by proving

$$\langle\psi_G|\, H(G)\,|\psi_G\rangle \geq \ell(G) \geq \alpha \cdot u(G) \geq \alpha \cdot \lambda_{\max}\left\{H(G)\right\} .$$

We use ℓ from Lemma 2 and u from Equation (12). Expanding these expressions, we get

$$\alpha \stackrel{\text{def}}{=} \min_G \frac{\ell(G)}{u(G)} = \min_G \frac{\sum_{(i,j)\in E} \frac{w_{ij}}{2}\left(1 + A_{ij}A_{ji} + (A_{ij}+A_{ji})\sin\nu(g_{ij})\right)}{\sum_{(i,j)\in E} w_{ij}\left(1+g_{ij}\right)} \tag{18}$$

$$\geq \min_G \min_{\substack{(i,j)\in E\\ 1+g_{ij}>0}} \frac{1 + A_{ij}A_{ji} + (A_{ij}+A_{ji})\sin\nu(g_{ij})}{2(1+g_{ij})} . \tag{19}$$

The right-hand side expression only depends on the values $\{g_{ij}\}_{(i,j)\in E}$, which obey Corollary 6. In fact, the expression only depends on $g_{k\ell}$ incident to i or j. We thus minimize this expression over values $\{g_{ij}\} \cup \{g_{ik}\}_{k\in K_i} \cup \{g_{kj}\}_{k\in K_j}$ obeying Corollary 6, given nodes i and j, and arbitrary-size sets of "other" neighbors $K_i \stackrel{\text{def}}{=} N(i)\setminus\{j\}$ and $K_j \stackrel{\text{def}}{=} N(j)\setminus\{i\}$.

Using our parameterization of ν in Equation (17), we can simplify some of the above expressions:

$$\sin\nu(g_{ij}) = \sqrt{\tilde\nu(g_{ij}^+)}\,, \qquad A_{ij} = \prod_{k\in K_i}\sqrt{1-\tilde\nu(g_{ik}^+)}\,.$$

The reason to use the set $\mathcal{C}$ from Definition 4 is demonstrated by the following two lemmas:

▶ **Lemma 7.** *Suppose $g_{ik} \leq \beta$ for all $k \in K_i$. Then $A_{ij} \geq \sqrt{1-\Theta\left(Q(g_{ij}^+)\right)}$.*

Proof. Observe that

$$A_{ij} = \prod_{k\in K_i}\sqrt{1-\tilde\nu(g_{ik}^+)} = \prod_{k\in K_i}\sqrt{1-\Theta(g_{ik}^+)} \geq \sqrt{1-\Theta\Big(\sum_{k\in K_i} g_{ik}^+\Big)} \geq \sqrt{1-\Theta\left(Q(g_{ij}^+)\right)}\,.$$

The equalities hold by Equation (17), the first inequality holds by Definition 4, and the last inequality holds by Corollary 6. ◀

▶ **Lemma 8.** *Suppose $g_{ik'} > \beta$ for some $k' \in K_i$. Then $A_{ij} \geq f(g_{ij}, g_{ik'})$, where $f : [-1,1]\times[-1,1] \to \mathbb{R}$ is the function*

$$f(x,y) \stackrel{\text{def}}{=} \sqrt{\left(1-\Lambda(y^+)\right)\left(1-\Theta\left(Q(y^+)-x^+\right)\right)}\,.$$

Proof. Let $K_i^- \stackrel{\text{def}}{=} K_i \setminus \{k'\}$. Since $g_{ik'} > \beta$, Corollary 6 implies $g_{ik} \leq (1-\beta)$ for every $k \in K_i^-$. Since $\beta > \frac{1}{2}$, $1-\beta < \beta$, and so $g_{ik} < \beta$ for all $k \in K_i^-$. We then observe that

$$A_{ij} = \prod_{k\in K_i}\sqrt{1-\tilde\nu(g_{ik}^+)} = \sqrt{1-\Lambda(g_{ik'}^+)}\cdot\prod_{k\in K_i^-}\sqrt{1-\Theta(g_{ik}^+)} \geq \sqrt{1-\Lambda(g_{ik'}^+)}\cdot\sqrt{1-\Theta\Big(\sum_{k\in K_i^-} g_{ik}^+\Big)}\,.$$

The equalities hold by Equation (17) and the inequality holds by Definition 4. By Corollary 6, the sum $\sum_{k\in K_i^-} g_{ik}^+ \leq Q(g_{ik'}^+) - g_{ik'}^+$. By Definition 4, Θ is monotonically increasing. ◀

We use Lemmas 7 and 8 in the following case-wise analysis of the ratio in Equation (18):

Case 1

Suppose $g_{ij} \leq \beta$ on edge (i,j) and on all neighboring edges. In this case, we may apply Lemma 7 on both A_{ij} and A_{ji}, to derive that the ratio Equation (18) is lower bounded by

$$r_1(g_{ij}),\ \text{for } r_1(g) \stackrel{\text{def}}{=} \frac{2 - \Theta\big(Q(g^+)\big) + 2\sqrt{\Theta(g^+)\left(1 - \Theta\big(Q(g^+)\big)\right)}}{2\,(1+g)}. \tag{20}$$

When $g_{ij} \leq 0$, the numerator of $r_1(g)$ is constant, but the denominator increases with g. Therefore, it follows that

$$r_1(g_{ij}) \geq \min_{-1<g\leq\beta} r_1(g) = \min_{0\leq g\leq\beta} r_1(g). \tag{21}$$

Case 2

Suppose $g_{ij} > \beta$. In this case, by Corollary 6, we have that $g_{ik} \leq \beta$ for all $k \in K_i$ and $g_{kj} \leq \beta$ for all $k \in K_j$. Thus, we may again apply Lemma 7 to both A_{ij} and A_{ji}. The only difference is that because $g_{ij} \geq \beta$, we have $\tilde{\nu}(g_{ij}^+) = \Lambda(g_{ij}^+)$ by Equation (17). So the ratio Equation (18) is lower bounded by

$$r_2(g_{ij}),\ \text{for } r_2(g) \stackrel{\text{def}}{=} \frac{2 - \Theta\big(Q(g^+)\big) + 2\sqrt{\Lambda(g^+)\left(1 - \Theta\big(Q(g^+)\big)\right)}}{2\,(1+g)}. \tag{22}$$

Since $g_{ij} \in [\beta, 1]$, we have that $r_2(g_{ij}) \geq \min_{\beta\leq g\leq 1} r_2(g)$.

Case 3

Suppose $g_{ik'} > \beta$ for some $k' \in K_i$. In this case, all other edges incident to i must have $g_{ij} \leq 1 - \beta \leq \beta$ by Corollary 6. We can then apply Lemma 8 to A_{ij} to obtain the following lower bound on Equation (18):

$$\frac{1 + f(g_{ij}, g_{ik'}) \cdot A_{ji} + \sqrt{\Theta(g_{ij}^+)}\,(f(g_{ij}, g_{ik'}) + A_{ji})}{2\,(1 + g_{ij})}. \tag{23}$$

There may or may not be some $\ell \in K_j$ with $g_{\ell j} > \beta$. As such, we split into two subcases:

Case 3a: Exactly one $g_{\ell' j} > \beta$. Then by Lemma 8, $A_{ji} \geq f(g_{ij}, g_{\ell' j})$.

Case 3b: All $g_{\ell j} \leq \beta$. Then by Lemma 7, $A_{ji} \geq \sqrt{1 - \Theta\big(Q(g_{ij}^+)\big)}$.

So far, Equation (23) for Case 3a depends on three variables g_{ij}, $g_{ik'}$ and $g_{\ell' j}$. For Case 3b, Equation (23) depends on two variables g_{ij} and $g_{ik'}$. We can relate the variables using the monogamy of entanglement claim from [13]:

▷ Claim 9 ([13]). Fix any graph $G = (V, E, w)$. Then the output $\{g_{ij}\}_{(i,j)\in E}$ in Equation (11) from the 2nd level (and higher levels) of the quantum moment-SoS hierarchy obeys $g_{ij} \leq R(g_{ik})$ and $g_{ik} \leq R(g_{ij})$ for all pairs of neighboring edges $\{(i,j),(i,k)\} \subseteq E$, where

$$R(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(\sqrt{3(1-x^2)} - x\right). \tag{24}$$

Note that R is monotonically decreasing for $x \geq \frac{1}{2}$. Together with Claim 9, we observe

$$g_{ij} \leq \min\{R(g_{ik'}), R(g_{\ell' j})\} \leq R(\beta)\,, \qquad \max\{g_{ik'}, g_{\ell' j}\} \leq R(g_{ij})\,.$$

We now assume a property of Λ to remove $g_{ik'}$ and $g_{\ell' j}$ from the minimization problem altogether.

▶ **Definition 10.** *Fix some $1/2 < \beta \leq 1$ and $\Theta \in \mathcal{C}$. Let $\mathcal{D}_{\Theta,\beta}$ be the set of functions*

$$\Lambda : [0,1] \to [0,1],$$

where f from Lemma 8 satisfies $f(x,y) \geq f^*(x)$ for all $\beta \leq y \leq R(x)$ and $-1 \leq x \leq R(\beta)$, and

$$f^*(x) \overset{\text{def}}{=} f(x, R(x)) = \sqrt{\left(1 - \Lambda(R(x)^+)\right) \cdot \left(1 - \Theta\left(Q(R(x)^+) - x^+\right)\right)}\,. \tag{25}$$

If $\Lambda \in \mathcal{D}_{\Theta,\beta}$, then in Case 3 of Lemma 8, we can lower-bound both $f(g_{ij}, g_{ik'})$ and $f(g_{ij}, g_{\ell' j})$ with $f(g_{ij}, R(g_{ij}))$, since $g_{ik'}, g_{\ell' j} \leq R(g_{ij})$. Thus, if $\Lambda \in \mathcal{D}_{\Theta,\beta}$, we can without loss of generality minimize Equation (23) over $-1 \leq g_{ij} \leq R(\beta)$ with fixed $g_{ik'} = R(g_{ij})$ and fixed $g_{j\ell'} = R(g_{ij})$. Thus, in Case 3a, Equation (23) is lower bounded by

$$r_3(g_{ij}),\ \text{for } r_3(g) \overset{\text{def}}{=} \frac{1 + f^*(g)^2 \;+\; 2\sqrt{\Theta(g^+)}f^*(g)}{2(1+g)}, \tag{26}$$

and $r_3(g_{ij}) \geq \min_{-1<g\leq R(\beta)} r_3(g)$. In Case 3b, we find that Equation (23) is lower bounded by

$$\begin{aligned} &\frac{1 + f^*(g_{ij})\sqrt{1 - \Theta(Q(g_{ij}^+))} \;+\; \sqrt{\Theta(g_{ij}^+)}\left(f^*(g_{ij}) + \sqrt{1 - \Theta(Q(g_{ij}^+))}\right)}{2(1+g_{ij})} \\ &\geq \min\{r_1(g_{ij}), r_3(g_{ij})\} \geq \min\left\{\min_{0\leq g\leq R(\beta)} r_1(g),\ \min_{-1<g\leq R(\beta)} r_3(g)\right\}. \end{aligned} \tag{27}$$

The first inequality in Equation (27) is due to the fact that

$$f^*(g), \sqrt{1 - \Theta(Q(g^+))} \geq \min\left\{f^*(g), \sqrt{1 - \Theta(Q(g^+))}\right\} \quad \forall g \in \mathbb{R}.$$

The second inequality in Equation (27) is due to Equation (21). Note also that $R(\beta) \leq \beta$, so that
$\min_{0\leq g\leq R(\beta)} r_1(g) \geq \min_{0\leq g\leq \beta} r_1(g)$.

We lower bound the approximation ratio by combining all cases (Equations (20), (22), and (23)):

▷ Claim 11. Suppose ν has the form Equation (17) for some $\beta > \frac{1}{2}$, function $\Theta \in \mathcal{C}$, and function $\Lambda \in \mathcal{D}_{\Theta,\beta}$. Then Algorithm 1 with this choice of ν has approximation ratio at least

$$\alpha \;\geq\; \min\left\{\min_{0\leq g\leq \beta} r_1(g)\,,\ \min_{\beta\leq g\leq 1} r_2(g)\,,\ \min_{-1<g\leq R(\beta)} r_3(g)\right\} \tag{28}$$

for functions r_1, r_2 and r_3 as defined in Equations (20), (22), and (26) respectively.

It remains to choose β, $\Theta \in \mathcal{C}$, and $\Lambda \in \mathcal{D}_{\Theta,\beta}$ that obtain a large value of Equation (28). We provide such a choice in the following theorem.

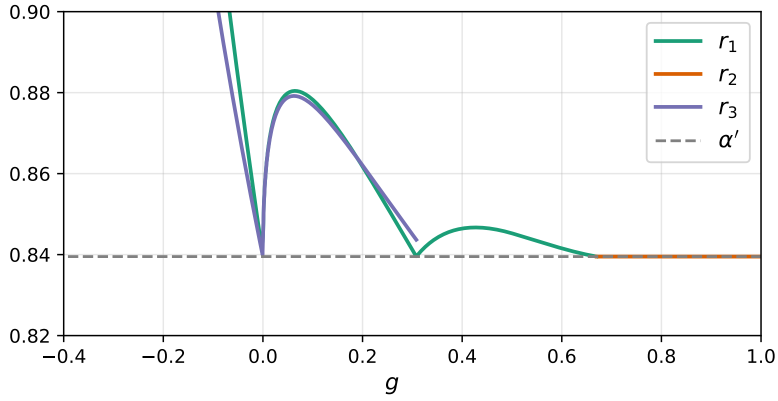

Figure 1 Values of the functions r_1, r_2 and r_3 as defined in Equations (20), (22), and (26) respectively.

▶ **Theorem 12** (formal restatement of Theorem 1). *The value of the minimization problem Equation (28) is at least* $\alpha \geq \alpha' \stackrel{\text{def}}{=} 0.839511$ *with the following choice of parameters:*

- ν *is the function*

$$\nu(x) \stackrel{\text{def}}{=} \arcsin \sqrt{\tilde{\nu}(x)}\,, \qquad \tilde{\nu}(x) \stackrel{\text{def}}{=} \begin{cases} \Theta(x^+), & \text{if } x \leq \beta\,, \\ \Lambda(x^+), & \text{if } x > \beta\,, \end{cases}$$

- $\beta = 0.67,\ \gamma = 0.049,$
- Θ *is the piecewise linear function defined by the points*

$$\left\{(0,0)\,, (Q(\beta), \gamma)\,, \left(\beta, \frac{(\gamma/2 + \alpha'(1+\beta) - 1)^2}{1-\gamma} \approx 0.1913\right), (1, 2(1-\alpha') \approx 0.3210)\right\}. \tag{29}$$

- Λ *is the function*

$$\Lambda(x) = \frac{\left(\frac{1}{2}\Theta\left(Q(x)\right) + \alpha'(1+x) - 1\right)^2}{1 - \Theta\left(Q(x)\right)}. \tag{30}$$

As a consequence, Algorithm 1 with this choice of ν *has approximation ratio* $\alpha \geq 0.839511$.

We provide some intuition for the choice of parameters. Λ is chosen such that $r_2(g)$ is exactly α' in the entire domain. Θ is chosen such that $r_1(g)$ is exactly α' at points 0 and β. We then choose β and γ such that $\Lambda \in \mathcal{D}$. We plot functions r_1 through r_3 in Figure 1. We also provide an interactive online plot here. We defer the proof of Theorem 12 to the appendix of the full version.

5 Limits on approximability

We now provide some upper bounds on the approximation that can be achieved by the methods outlined in our work.

5.1 Limitations from analysis

Our first limitation comes from our assumptions in the analysis of Algorithm 1. Specifically, we assume that θ_{ij} depends *only* on g_{ij}, and that we take the *worst-case* ratio over edges. With mild assumptions, *any* such analysis obtains an α-approximation where $\alpha < 0.839512$. (Recall that Theorem 12 shows the existence of $\alpha \geq 0.839511$.)

▶ **Lemma 13.** *Using a worst-case edge analysis, Algorithm 1 with any choice of ν where $\nu(0) = 0$ is at most a 0.839512-approximation for the EPR problem.*

Proof. Consider the four-node path graph P_4. It is known that the SDP Equation (10) returns $(g_{12}, g_{23}, g_{34}) = (\sqrt{3}/2, 0, \sqrt{3}/2)$ [19, Section 4.2.2]. We analyze the approximation ratio of Algorithm 1 on this graph using Lemma 2 (which is tight on triangle-free graphs) and a worst-case edge analysis. Hence, the approximation ratio α satisfies

$$\begin{aligned}\alpha &\leq \min\left\{\frac{\langle\psi_G|\, h_{23}\,|\psi_G\rangle}{1+g_{23}}, \frac{\langle\psi_G|\, h_{12}\,|\psi_G\rangle}{1+g_{12}}\right\}\\ &\leq \min\left\{\frac{1+\cos^2\theta_s + 2\cos\theta_s\sin\theta_m}{2}, \frac{1+\cos\theta_m + (1+\cos\theta_m)\sin\theta_s}{2+\sqrt{3}}\right\},\end{aligned}$$

where $\theta_s = \nu(\sqrt{3}/2)$ and $\theta_m = \nu(0)$. If we assume $\theta_m = \nu(0) = 0$, then

$$\alpha \leq \max_{-1\leq x\leq 1} \min\left\{\frac{2-x^2}{2}, \frac{2+2x}{2+\sqrt{3}}\right\} = \max_{0\leq x\leq 1} \min\left\{\frac{2-x^2}{2}, \frac{2+2x}{2+\sqrt{3}}\right\},$$

where $x = \sin\theta_s$. Since $\frac{2-x^2}{2}$ is a concave parabola and $\frac{2+2x}{2+\sqrt{3}}$ is a positively sloped line which intersect exactly once in the interval $[0, 1]$ we deduce that α is given by the common value at the point of intersection. Solving the corresponding quadratic equation

$$\frac{2-x^2}{2} = \frac{2+2x}{2+\sqrt{3}},\ 0\leq x\leq 1 \implies x = \frac{-2+\sqrt{10+4\sqrt{3}}}{2+\sqrt{3}},\ \alpha = \frac{2\big(\sqrt{3}+\sqrt{10+4\sqrt{3}}\big)}{(2+\sqrt{3})^2} \simeq 0.8395111.$$ ◀

We do not view $\nu(0) = 0$ as restrictive. In fact, if $\nu(\epsilon) \nrightarrow 0$ as $\epsilon \to 0^+$, then Algorithm 1 only achieves an 0.5-approximation on the complete bipartite graph $K_{a,a}$ for large values of a [19].

5.2 Limitations from ansatz

The ansatz we use in this work (and used in the initial algorithm of [11]) has a natural upper bound of 0.873. This was suggested by [20]; here we give a formal proof.

▶ **Lemma 14.** *Algorithm 1 achieves at most a $\frac{3+\sqrt{5}}{6} \approx 0.8727$-approximation on the EPR problem, even if each angle θ_{ij} could depend on the entire SDP output $\{g_{ij}\}_{(i,j)\in E}$.*

Proof. Consider the unweighted cycle C_4 defined by

$$V = \{1, 2, 3, 4\}, \quad E = \{(1, 2), (2, 3), (3, 4), (4, 1)\}.$$

Let $\{a, b, c, d\} = \{\theta_{12}, \theta_{23}, \theta_{34}, \theta_{41}\}$ denote the four angles in Algorithm 1. Using Lemma 2, we have

$$\begin{aligned}2\,\langle\chi|H(C_4)|\chi\rangle = 1 &+ \cos d\cos b + (\cos d + \cos b)\sin a + 1 + \cos a\cos c + (\cos a + \cos c)\sin b\\ &+1 + \cos b\cos d + (\cos b + \cos d)\sin c + 1 + \cos c\cos a + (\cos c + \cos a)\sin d,\end{aligned}$$

where each set of three terms corresponds to a single edge. Since C_4 has no triangles, note that the equation in Lemma 2 is an equality. To upper bound $\langle\chi|H(C_4)|\chi\rangle$, we rewrite each term of the form $\sin u\cos v$ as

$$\sin u\,\cos v = \left(\frac{1}{\sqrt{x}}\sin u\right)\left(\sqrt{x}\cos v\right),$$

for some arbitrary scalar $x > 0$. Then, defining the vectors

$$\mathbf{a} = (\cos d, \cos a, \cos b, \cos c) \,||\, \frac{1}{\sqrt{x}} (\sin a, \sin b, \sin c, \sin d) \,||\, \sqrt{x}\, (\cos b, \cos c, \cos d, \cos a)\,,$$
$$\mathbf{b} = (\cos b, \cos c, \cos d, \cos a) \,||\, \sqrt{x}\, (\cos d, \cos a, \cos b, \cos c) \,||\, \frac{1}{\sqrt{x}} (\sin a, \sin b, \sin c, \sin d)\,,$$

where $||$ denotes concatenation allows us to express

$$\begin{aligned} 2\,\langle \chi | H(C_4) | \chi \rangle &= 4 + \mathbf{a}\cdot\mathbf{b} \le 4 + \|\mathbf{a}\|_2 \, \|\mathbf{b}\|_2 = 4 + \|\mathbf{a}\|_2^2 \\ &= 4 + (1+x)\left(\cos^2 a + \cos^2 b + \cos^2 c + \cos^2 d\right) + \frac{1}{x}\left(\sin^2 a + \sin^2 b + \sin^2 c + \sin^2 d\right), \end{aligned}$$

where in the first line we apply the Cauchy–Schwarz inequality. Choosing x such that $1+x = \frac{1}{x}$ yields $x = (\sqrt{5}-1)/2$. Using $\cos^2\theta + \sin^2\theta = 1$ then yields

$$\langle \chi | H(C_4) | \chi \rangle \le 3 + \sqrt{5}\,. \tag{31}$$

In fact, by picking $a = b = c = d = \tan^{-1} x$ one obtains that $\langle \chi | H(C_4) | \chi \rangle = 3 + \sqrt{5}$. We know that $\lambda_{\max}\{H(C_4)\} = 6$ because C_4 is complete and bipartite [14, 19]. So this algorithm is an α-approximation of *at most* $\alpha \le \frac{3+\sqrt{5}}{6}$ with a tight upper bound. ◀

Several works for the EPR problem [11, 2, 3, 20] use this ansatz in Equation (2). Lemma 14 demonstrates that a new approach is required to boost $\alpha > 0.873$.

References

1 Anurag Anshu, David Gosset, and Karen Morenz. Beyond product state approximations for a quantum analogue of Max Cut. *LIPIcs, Volume 158, TQC 2020*, 158:7:1–7:15, 2020. `doi:10.4230/LIPIcs.TQC.2020.7`.

2 Anuj Apte, Eunou Lee, Kunal Marwaha, Ojas Parekh, and James Sud. Improved Algorithms for Quantum MaxCut via Partially Entangled Matchings, April 2025. `doi:10.48550/arXiv.2504.15276`.

3 Anuj Apte, Ojas Parekh, and James Sud. Conjectured Bounds for 2-Local Hamiltonians via Token Graphs, June 2025. `doi:10.48550/arXiv.2506.03441`.

4 Sergey Bravyi, Arvid J Bessen, and Barbara M Terhal. Merlin-arthur games and stoquastic complexity. *arXiv preprint quant-ph/0611021*, 2006.

5 Sabine Burgdorf, Igor Klep, and Janez Povh. *Optimization of polynomials in non-commuting variables*. SpringerBriefs in Mathematics. Springer, 2016.

6 Sevag Gharibian and Ojas Parekh. Almost optimal classical approximation algorithms for a quantum generalization of Max-Cut. *LIPIcs, Volume 145, APPROX/RANDOM 2019*, 145:31:1–31:17, 2019. `doi:10.4230/LIPIcs.APPROX-RANDOM.2019.31`.

7 Michel X. Goemans and David P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the ACM*, 42(6):1115–1145, November 1995. `doi:10.1145/227683.227684`.

8 Sander Gribling, Lennart Sinjorgo, and Renata Sotirov. Improved approximation ratios for the Quantum Max-Cut problem on general, triangle-free and bipartite graphs, April 2025. `doi:10.48550/arXiv.2504.11120`.

9 Zackary Jorquera, Alexandra Kolla, Steven Kordonowy, Juspreet Singh Sandhu, and Stuart Wayland. Monogamy of Entanglement Bounds and Improved Approximation Algorithms for Qudit Hamiltonians, November 2024. `doi:10.48550/arXiv.2410.15544`.

10 Nathan Ju and Ansh Nagda. Improved approximation algorithms for the EPR Hamiltonian, April 2025. `doi:10.48550/arXiv.2504.10712`.

11 Robbie King. An Improved Approximation Algorithm for Quantum Max-Cut. *Quantum*, 7:1180, November 2023. `doi:10.22331/q-2023-11-09-1180`.

12 A. Kitaev, A. Shen, and M. Vyalyi. *Classical and Quantum Computation*, volume 47 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, May 2002. `doi:10.1090/gsm/047`.

13 Eunou Lee and Ojas Parekh. An improved Quantum Max Cut approximation via matching, February 2024. `arXiv:2401.03616`.

14 Elliott Lieb and Daniel Mattis. Ordering Energy Levels of Interacting Spin Systems. *Journal of Mathematical Physics*, 3(4):749–751, July 1962. `doi:10.1063/1.1724276`.

15 Kunal Marwaha and James Sud. Quantum maxcut reference, 2025. URL: `https://marwahaha.github.io/quantum-maxcut-reference/`.

16 Miguel Navascues, Stefano Pironio, and Antonio Acin. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(7):073013, July 2008. `doi:10.1088/1367-2630/10/7/073013`.

17 Ojas Parekh and Kevin Thompson. Application of the Level-2 Quantum Lasserre Hierarchy in Quantum Approximation Algorithms. *LIPIcs, Volume 198, ICALP 2021*, 198:102:1–102:20, 2021. `doi:10.4230/LIPIcs.ICALP.2021.102`.

18 Ojas Parekh and Kevin Thompson. An Optimal Product-State Approximation for 2-Local Quantum Hamiltonians with Positive Terms, June 2022. `doi:10.48550/arXiv.2206.08342`.

19 Jun Takahashi, Chaithanya Rayudu, Cunlu Zhou, Robbie King, Kevin Thompson, and Ojas Parekh. An SU(2)-symmetric Semidefinite Programming Hierarchy for Quantum Max Cut, August 2023. `doi:10.48550/arXiv.2307.15688`.

20 Wenxuan Tao and Fen Zuo. A Refined Algorithm For the EPR model, June 2025. `doi:10.48550/arXiv.2506.08547`.