Towards the Recognition of Oriented Interval Graphs

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Abstract

Oriented interval graphs, a recent generalization of interval graphs introduced by Gutowski et al. [GD 2022], are intersection graphs of intervals, each of which is oriented either left or right. Such a representation defines a *mixed* intersection graph: overlapping intervals with the same orientation define a (directed) *arc*; nested intervals (irrespective of the orientations of the intervals) and overlapping intervals of opposite orientations define an (undirected) *edge*. An oriented interval representation of a mixed graph G can be described combinatorially by the combination of (i) an *orientation* $\varphi\colon V(G) \to \{-1, 1\}$ of all intervals, (ii) a *clique ordering* σ, and (iii) a set $E_{\text{cont}} \subseteq E(G)$ of *containment edges*, which are represented by nested intervals. The non-trivial dependencies between these three ingredients make the recognition of oriented interval graphs a challenging problem.

In this paper, we take steps towards a general recognition algorithm by studying how orientation, clique ordering, and containment edges influence and restrict each other. We characterize the orientations that are consistent with a given set of containment edges as well as the clique orderings that are consistent with a given orientation. Based on these characterizations, we give linear-time algorithms for two constrained versions of the recognition problem where, in addition to the mixed input graph G, either the set of containment edges E_{cont} or the orientation φ is prescribed. This improves a quadratic-time algorithm of Gutowski et al. for the case that all vertices have the same orientation; an assumption that determines both the orientation and the containment edges. In particular, this also solves the recognition problem for oriented proper (or unit) interval graphs.

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1 Introduction

A *mixed graph* is a graph that may contain both (undirected) edges and (directed) *arcs*. Mixed graphs were introduced by Sotskov and Tanaev [20] and reintroduced by Hansen et al. [14] and have been studied in the context of coloring and scheduling [19, 12], (quasi-)upward planar drawings [3, 4, 8] and extensions of partial orientations [2].

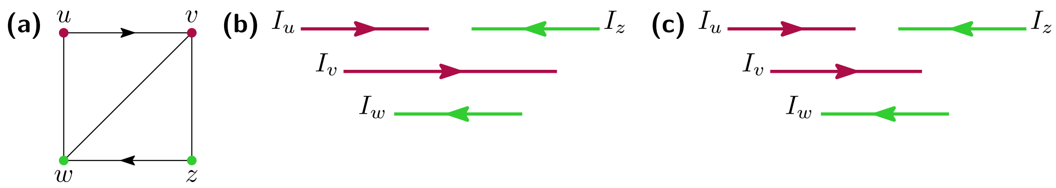

Figure 1 (a) A mixed graph G. (b,c) φ-oriented interval reps. of G, representing $\{v, w\}$ differently.

An *intersection representation* of a graph G is a mapping $\rho\colon V(G) \to \mathcal{C}$ from the vertex set $V(G)$ of G, to a class $\mathcal{C}$ of geometric objects such that $\{u, v\}$ is an edge of G if and only if $\rho(u) \cap \rho(v) \neq \emptyset$. Every class $\mathcal{C}$ of geometric objects gives rise to a class of graphs that admit an intersection representation with objects in $\mathcal{C}$. Prominent examples include interval graphs (intersection graphs of intervals on the real line), circular-arc graphs (intersection graphs of arcs on a circle), circle graphs (intersection graphs of chords of a circle), and permutation graphs (intersection graphs of line segments connecting two parallel lines).

Interval graphs form a well-understood class in structural and algorithmic graph theory, with applications spanning from scheduling problems to genome analysis [11]. Using so-called PQ-trees, one can test in linear time whether a graph G is an interval graph and, if yes, compute an interval representation of G [5]. Some problems, such as coloring or maximum clique, that are NP-hard in general can be solved in linear time for interval graphs [10, 15].

Gutowski, Mittelstädt, Rutter, Spoerhase, Wolff, and Zink [13] introduced a natural extension of interval representations where each interval is assigned an orientation, namely left or right (assuming the intervals are placed horizontally on the real line). This *oriented interval representation* yields a mixed intersection graph: overlapping intervals that have the same orientation define an arc pointing from the earlier to the later interval according to their orientation; nested intervals (regardless of their respective orientations) and overlapping intervals of opposite orientations define an edge; see Figure 1.

A natural question is whether these *oriented interval graphs* can be recognized efficiently. Even though directed and mixed graphs arise naturally in many applications, they have mostly been ignored in the study of intersection representations. Continuing the study of Gutowski et al., we show that even for very simple geometric objects, namely oriented intervals, the recognition problem for oriented intersection graphs poses significant challenges.

Classical interval graphs can be combinatorially described by a *clique ordering*, that is, a graph is an interval graph if and only if its maximal cliques can be ordered such that for each vertex v, all cliques that contain v appear consecutively [9]. For oriented interval graphs, however, the situation is more involved, as an oriented interval representation of a mixed graph G can be described combinatorially by the combination of (i) an *orientation* $\varphi\colon V(G) \to \{-1, 1\}$ of all intervals, (ii) a clique ordering σ of the underlying undirected graph, and (iii) a subset $E_{\text{cont}} \subseteq E(G)$ of the edges that are represented by *containment*, that is, nested intervals. These three elements are, however, not independent but influence and restrict each other in a non-trivial fashion. For example, if two vertices u and v have the same orientation, say to the right, but they are connected by an undirected edge, then this edge must be represented by containment; known containments may restrict the choices of the clique ordering, and clique orderings may decide certain containments and overlaps, which influences the orientations.

For a mixed graph G, the recognition problem boils down to finding a triplet $(\varphi, \sigma, E_{\text{cont}})$ whose elements are consistent with each other in the sense that together they describe an oriented interval representation of G. So far, the recognition problem has only been solved in the case where all intervals are oriented to the right, which, in addition to the orientation φ, uniquely determines the containment edges E_{cont}. Gutowkski et al. [13] showed that, in this setting, a clique ordering that is consistent with the orientation (and the containment edges) can be computed in time quadratic in the number of graph vertices.

Our Contribution. In this paper, we take steps towards the general recognition problem for oriented interval graphs by studying the interconnection between σ, φ and E_{cont}. We call a triplet $(\varphi, \sigma, E_{\text{cont}})$ *consistent* if there is a corresponding oriented interval representation, and we call any pair of two such elements *consistent* if there is a third element that results in a consistent triplet. In Section 3, we give a combinatorial description of all orientations φ such that, for a given set E_{cont} of containment edges, the pair $(\varphi, E_{\text{cont}})$ is consistent. This yields an algorithm for computing a clique ordering σ such that, for a given consistent pair $(\varphi, E_{\text{cont}})$, the triplet $(\varphi, \sigma, E_{\text{cont}})$ is consistent. We prove the following.

▶ **Theorem 1.** *Given a mixed graph G and $E_{\text{cont}} \subseteq E(G)$, there is a linear-time algorithm that decides whether G admits an oriented interval representation where precisely the edges in E_{cont} are represented by containment.*

A *proper* interval graph is an interval graph that admits a representation where no interval properly contains another interval. In other words, a proper interval graph is an interval graph with $E_{\text{cont}} = \emptyset$. A *unit* interval graph is an interval graph that admits a representation where all intervals have the same length. The classes of proper interval graphs and the class of unit interval graphs are the same [18]. Using the algorithm behind Theorem 1, we can recognize oriented proper (or unit) interval graphs.

▶ **Corollary 2.** *Given a mixed graph G, we can decide in linear time whether G admits an oriented proper (or unit) interval representation.*

In Section 4, we give a combinatorial description of all clique orderings σ such that, for a given orientation φ, the pair (φ, σ) is consistent. This allows us to design an algorithm for computing a set E_{cont} of containment edges such that, for a given consistent pair (φ, σ), the triplet $(\varphi, \sigma, E_{\text{cont}})$ is consistent. Generalizing and speeding up the algorithm of Gutowski et al. [13], we prove the following.

▶ **Theorem 3.** *Given a mixed graph G with an orientation φ, there is a linear-time algorithm that decides whether G admits an oriented interval representation with orientation φ.*

Statements whose proofs are in the full version [1] are marked with a clickable ($\star$).

2 Preliminaries

Interval Graphs and Interval Representations. We consider sets of intervals on the real line in non-degenerate general position, i.e., all their endpoints are distinct. Two intervals I and I' *overlap* if their endpoints alternate, whereas I *contains* I' if both endpoints of I' lie between the two endpoints of I.

Given a set $\mathcal{I}$ of intervals, its *intersection graph* is the graph that has $\mathcal{I}$ as its vertex set and two different vertices I and I' are adjacent if and only if they have a non-empty intersection. An undirected graph G is called an *interval graph* if G is isomorphic to an intersection graph of a set $\mathcal{I}$ of intervals. In this case, we call $\mathcal{I}$ an *interval representation* of G, and, for each vertex u of G, we let the interval in $\mathcal{I}$ that corresponds to u be denoted by I_u. If $I_u \cap I_v \neq \emptyset$, then we say that they *realize* the corresponding edge of G. If I_u contains I_v, then we say that the edge $\{u, v\}$ is a *containment edge*.

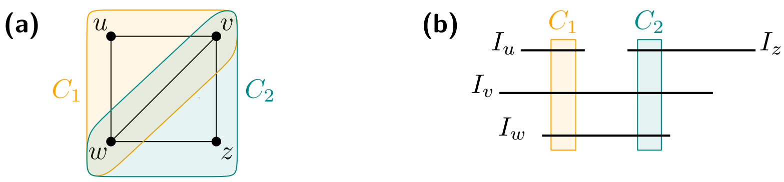

Figure 2 (a) A graph G with $\mathcal{C}(G) = \{C_1, C_2\}$. (b) An interval rep. $\mathcal{I}$ of G and its clique order.

For $W \subseteq V(G)$, we let $G[W]$ denote the subgraph of G induced by the vertices of W. Its induced representation is denoted by $\mathcal{I}[W]$. For a graph G, let $\mathcal{C}(G)$ be the set of maximal cliques of G. It is well-known that a graph G is an interval graph if and only if $\mathcal{C}(G)$ admits a linear ordering in which, for every vertex u of G, the set $\{C \in \mathcal{C}(G) \mid u \in C\}$ is consecutive [9]. Such an ordering is called a *clique order*; see Figure 2b.

Oriented Interval Representations. A graph containing directed arcs and undirected edges is a *mixed graph*. Given a mixed graph G, let $\vec{E}(G)$ denote the set of arcs of G, and let $E(G)$ denote the set of edges of G. For a mixed graph G, we let $U(G)$ denote the *underlying undirected graph* that contains an edge $\{u, v\}$ whenever G contains an edge or an arc with endpoints u and v. An *orientation* of G is a mapping $\varphi\colon V(G) \to \{-1, 1\}$, which, in the context of an interval representation $\mathcal{I} = \{I_v \mid v \in V(G)\}$ of G, we interpret as an orientation of the respective intervals, where an interval $I_v \in \mathcal{I}$ is *right-oriented* if $\varphi(v) = +1$ and it is *left-oriented* if $\varphi(v) = -1$. An *interval representation* of a mixed graph G is an interval representation of its underlying undirected graph $U(G)$. By contrast, an *oriented* interval representation takes the directions of the arcs of G into account as follows; see Figure 3.

▶ **Definition 4.** *Let G be a mixed graph. An* oriented interval representation *of G is a pair $(\mathcal{I}, \varphi)$ consisting of an interval representation $\mathcal{I}$ of G and an orientation φ of G such that for each pair of intervals $I_u = [l_u, r_u]$ and $I_v = [l_v, r_v]$ from $\mathcal{I}$ with $l_u < l_v$:*
- *$(u, v) \in \vec{E}(G)$ if and only if I_u and I_v overlap, and $\varphi(I_u) = \varphi(I_v) = +1$.*
- *$(v, u) \in \vec{E}(G)$ if and only if I_u and I_v overlap, and $\varphi(I_u) = \varphi(I_v) = -1$.*
- *$\{u, v\} \in E(G)$ if and only if*
  - *I_u and I_v overlap and $\varphi(I_u) \neq \varphi(I_v)$ or*
  - *I_u contains I_v, regardless of the orientations of the intervals.*

We also call (I, φ) φ-oriented.

A mixed graph G is an *oriented interval graph*[1] if it admits an oriented interval representation. We call an interval representation $\mathcal{I}$ of G *φ-orientable* if $(\mathcal{I}, \varphi)$ is an oriented interval representation of G. Figure 2b depicts an interval representation that is not φ-orientable for the orientation of the graph from Figure 1a. A *clique* of G is a clique in its underlying undirected graph $U(G)$, and we write $\mathcal{C}(G)$ for $\mathcal{C}(U(G))$.

Matching Representations and Permutation Graphs. A *matching representation* of an undirected graph G is a pair $(<_L, <_R)$ of linear orders of the vertex set $V(G)$ such that two vertices $u, v \in V(G)$ are ordered differently in $<_L$ and $<_R$ if and only if they are adjacent in G. We depict such a representation as a diagram with the vertices in $<_L$ and $<_R$ placed on two parallel horizontal lines in their respective order and connecting the two representatives of a vertex v on $<_L$ and $<_R$ by a segment; see Figure 4c. A graph that allows a matching

[1] Gutowski et al. [13] called such graphs *bidirectional* interval graphs.

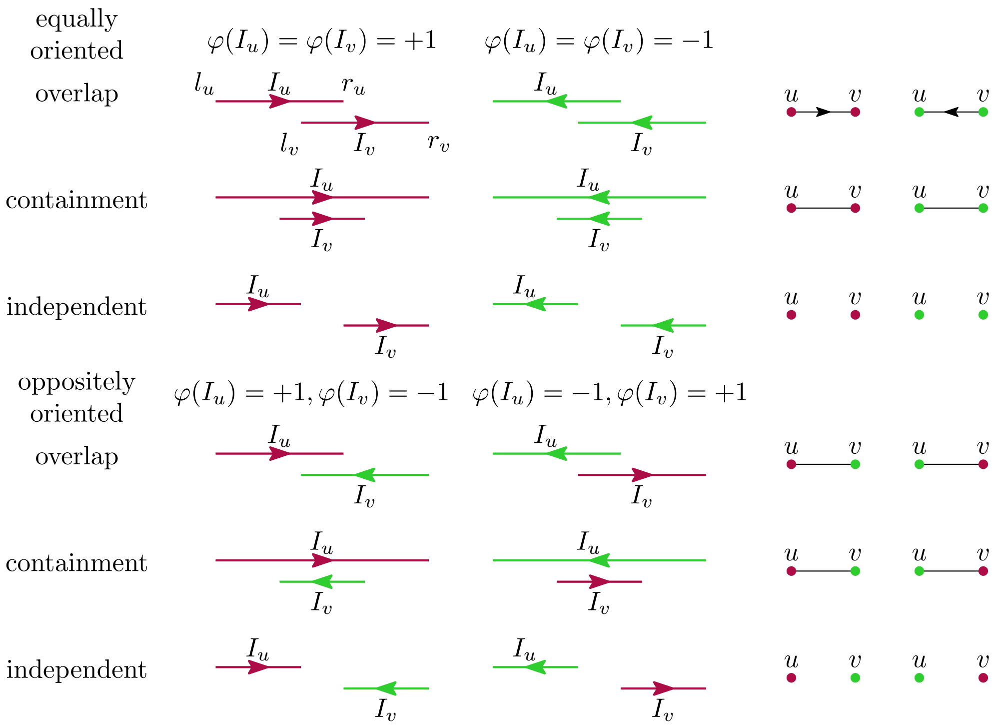

Figure 3 Possible positions of oriented intervals when $l_u < l_v$ and the edges they represent. In our illustrations, color ruby encodes right (positive) orientation, while lime means left (negative).

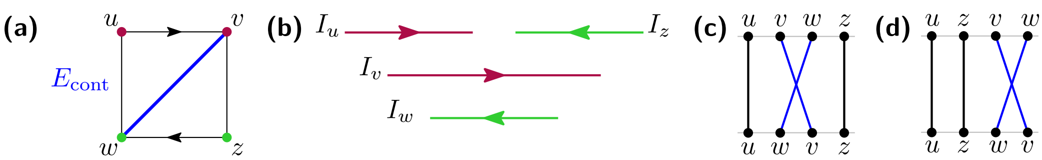

Figure 4 (a) A mixed graph G with set $E_{\text{cont}} = \big\{\{v, w\}\big\}$ of containment edges; (b) an oriented interval representation of G; (c) diagram of a matching representation for $H = (V(G), E_{\text{cont}})$ obtained from the interval representation; (d) another matching representation for H that corresponds to no interval representation.

representation is called a *permutation graph*. This is the case if and only if both G and its complement $\overline{G}$ admit a *transitive orientation*, i.e., an orientation of edges, where the presence of arcs (u, v) and (v, w) yields also the arc (u, w). In particular, such transitive orientations of G and $\overline{G}$ can be obtained from the matching representation by the intersection (as relations) of $<_R$ with $>_L$ and $<_R$ with $<_L$, respectively; in the diagram, the orientation of $\overline{G}$ corresponds to the left-to-right ordering of non-intersecting segments [7]; see Figure 4c.

Moreover, by choosing $<_L$ and $<_R$ as the orders of left and right endpoints, respectively, every interval representation yields a matching representation of a permutation graph G whose edges represent nested intervals. A transitive orientation of G therefore naturally orders the corresponding intervals by inclusion.

3 Prescribed Containment Edges

In this section we present an algorithm for testing whether a mixed graph G with given containment edges E_{cont} is an oriented interval graph. This contains the case of oriented proper interval representations as a special case, where $E_{\text{cont}} = \emptyset$.

Our approach works in two steps. First, we show that we can compute in linear time an orientation φ of G such that the pair $(\varphi, E_{\text{cont}})$ is consistent if and only if G admits an oriented interval representation with containment edges E_{cont}. We call such a φ *feasible*. For a feasible φ, we then check the existence of a representation.

3.1 Computing a Feasible Orientation

A key observation is that if $G - E_{\text{cont}}$ is connected, its feasible orientation φ (if any) is unique up to reversal as choosing the orientation of a vertex u determines the orientations of all its neighbors in $G - E_{\text{cont}}$; namely, they have the same orientation if the two vertices are connected by an arc and they have opposite orientations if they are connected by an (undirected) edge. Observe that such an orientation φ can be easily computed in linear time. If $G - E_{\text{cont}}$ is not connected, this reasoning can be applied to each individual connected component of $G - E_{\text{cont}}$. In the rest of this section, we show that these orientations, which are unique up to reversal, can be combined arbitrarily.

For an orientation φ of G and $W \subseteq V(G)$ define the *reversal orientation* $\text{rev}_W(\varphi)$ by

$$v \mapsto \begin{cases} -\varphi(v) & v \in W \\ \varphi(v) & v \notin W \end{cases}$$

as the orientation where precisely the vertices in W have their orientation reversed. Observe that for two disjoint sets $U, W \subseteq V(G)$ we have $\text{rev}_U \circ \text{rev}_W = \text{rev}_{U \cup W}$, while for two sets in inclusion $U \subseteq W$ we have $\text{rev}_U \circ \text{rev}_W = \text{rev}_{W \setminus U}$.

We are interested in conditions under which reversing the orientations of a subset W preserves the feasibility of orientations. Let $\mathcal{I}$ be a φ-orientable interval representation of G. We say that an interval I (not necessarily in $\mathcal{I}$) is *overlap-free with respect to $\mathcal{I}$* if there exists no interval $J \in \mathcal{I}$ that overlaps I. We define $\text{cont}_{\mathcal{I}}(I) \coloneqq \{v \in V(G) \mid \mathcal{I}(v) \subseteq I\}$ as the set of vertices of G whose intervals are contained in I. We let $\text{mirr}_I(\mathcal{I})$ denote the interval representation that we obtain from $\mathcal{I}$ by mirroring the subrepresentation $\mathcal{I}[W]$ at the midpoint of I where $W = \text{cont}_{\mathcal{I}}(I)$. We show that this mirroring of the subrepresentation $\mathcal{I}[W]$ is consistent with reversing the orientations of the vertices in W, that is, $\text{mirr}_I(\mathcal{I})$ is a $\text{rev}_W(\varphi)$-orientable interval representation of G.

▶ **Lemma 5.** **(⋆)** *Let G be a mixed graph and let $\mathcal{I}$ be a φ-orientable interval representation of G. Let I be an overlap-free interval with respect to $\mathcal{I}$ and let $W = \mathrm{cont}_{\mathcal{I}}(I)$. Then $\mathrm{mirr}_I(\mathcal{I})$ is a $\mathrm{rev}_W(\varphi)$-orientable interval representation of G.*

We use this to show that reversing the orientations of any subset W of vertices such that all edges between W and $V \setminus W$ are containment edges, preserves the feasibility of an orientation. We say that an interval I (not necessarily in $\mathcal{I}$) *spans* the intervals of $\mathcal{I}[W]$ if every interval of $\mathcal{I}[W]$ is properly contained in I.

▶ **Lemma 6.** **(⋆)** *Let G be a mixed graph, and let $W \subseteq V$ be a vertex set such that all edges between the vertices of W and $V \setminus W$ are containment edges. Then G has a φ-orientable interval representation if and only if G has a $\mathrm{rev}_W(\varphi)$-orientable interval representation.*

We show now that a feasible orientation (if one exists) can be computed in linear time.

▶ **Lemma 7.** *There is a linear-time algorithm that given G and E_{cont} computes a feasible orientation φ of G if it admits one.*

Proof. Let G be a mixed graph and let K be a connected component of $G - E_{\mathrm{cont}}$. Observe that since K is connected by non-containment edges, choosing the orientation for a single vertex $v \in V(K)$ uniquely determines the orientations of all other vertices in K. That is, if it exists, the consistent orientation φ_K of K is unique up to reversal. We can compute φ_K in time linear in the size of K. If K has no consistent orientation, G is a no-instance and we return an arbitrary orientation of G. Otherwise, we return the orientation φ of G defined by $\varphi(v) = \varphi_K(v)$, where K is the connected component of $G - E_{\mathrm{cont}}$ that contains v.

It remains to show that φ is feasible. Clearly, if G admits a φ-orientable interval representation, then G is an oriented interval graph. Conversely, assume G is an oriented interval graph and admits an oriented interval representation $(\mathcal{I}, \varphi')$ for some orientation φ'. We choose $(\mathcal{I}, \varphi')$ such that the number of vertices for which it coincides with φ is maximized. If $\varphi = \varphi'$, then we are done. Otherwise, there is a vertex v with $\varphi(v) \neq \varphi'(v)$. Let K be the connected component of $G - E_{\mathrm{cont}}$ that contains v. Since the orientation for each connected component K of $G - E_{\mathrm{cont}}$ is unique up to reversal, we have $\varphi(v) = -\varphi'(v)$ for all $v \in V(K)$. By Lemma 6 we have that since G has a φ'-orientable interval representation, it has a $\mathrm{rev}_K(\varphi')$-orientable interval representation. Then $\mathrm{rev}_K(\varphi')$ coincides with φ for more vertices than φ'. This is a contradiction to the choice of φ'. ◀

3.2 Constructing an Interval Representation for a Given Orientation

Based on Lemma 7 we may assume that, in addition to the mixed graph G and the containment edges E_{cont}, we are also given a feasible orientation φ. That is, the pair $(\varphi, E_{\mathrm{cont}})$ is consistent if G admits an oriented interval representation with containment edges E_{cont}.

Our algorithm is based on the fact that the graph H with $V(H) = V(G)$ and $E(H) = E_{\mathrm{cont}}$ is a permutation graph that admits a special type of matching representation if and only if the given mixed interval graph G admits an oriented interval representation.

Let u and v be two vertices of G with $(u, v) \in \vec{E}(G)$. Then any φ-orientable interval representation orders u and v in the sense that for $I_u = [l_u, r_u]$ and $I_v = [l_v, r_v]$, the order of l_u and l_v is the same as the order of r_u and r_v. If $\varphi(u) = \varphi(v) = 1$, then the direction of (u, v) implies that $l_u < l_v$ and $r_u < r_v$ in any φ-orientable interval representation. Similarly, $\varphi(u) = \varphi(v) = -1$ implies that $l_v < l_u$ and $r_v < r_u$ in any φ-orientable interval representation. We capture these conditions in the *placement order* $\prec_\varphi$ defined as $u \prec_\varphi v$ if and only if $((u, v) \in \vec{E}(G) \wedge \varphi(u) = \varphi(v) = 1) \vee ((v, u) \in \vec{E}(G) \wedge \varphi(u) = \varphi(v) = -1)$.

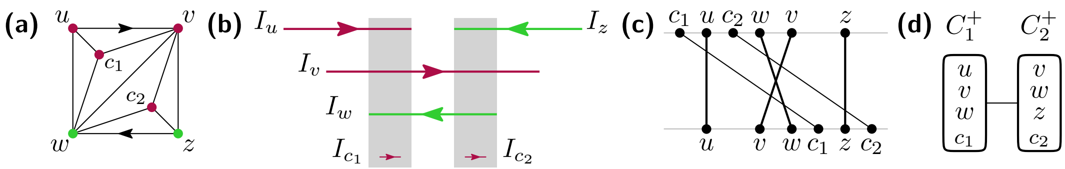

Figure 5 (a) The augmentation G^+ of the graph G from Figure 4a, (b) an interval representation $\mathcal{I}^+$ of G^+, (c) diagram of a matching representation of H^+, and (d) the clique order of $\mathcal{I}^+$.

By definition, $\prec_\varphi$ is antisymmetric and acyclic on $V(G)$. Note that $\prec_\varphi$ is not necessarily transitive, we have chosen this symbol and name as it well captures the relative position of two vertices connected by an arc. The following lemma establishes the connection between matching representations and containments in interval representations.

▶ **Lemma 8.** **(⋆)** *Let G be a mixed graph and let $(\mathcal{I}, \varphi)$ be its oriented interval representation with containment edges E_{cont}. Then $H = (V(G), E_{\text{cont}})$ is a permutation graph and its complement $\overline{H}$ admits a transitive orientation $\prec_{\overline{H}}$ that extends $\prec_\varphi$.*

We note that Dushnik and Miller [6] established the connection between containment interval graphs and posets of dimension 2. Lemma 8 gives necessary conditions for when G is a φ-orientable interval graph, which are, however, not sufficient, as we may lose information about the non-containment adjacencies of G; an example is given in [1].

However, this information is crucial for deciding if G admits an oriented interval representation. In order to encode this information in H we introduce the following construction.

For a mixed graph G, the *augmented graph G^+* is obtained by adding a clique point vertex c_i for each $C_i \in \mathcal{C}(G)$ to G, such that c_i is adjacent to a vertex $v \in G$ if and only if $v \in C_i$. We consider the edges incident to clique point vertices as elements of the containment edges E^+_{cont} of G^+. The orientation of the clique point vertices is thus irrelevant and we extend φ to an orientation φ^+ of G^+ by setting $\varphi^+(c_i) := 1$ for every c_i.

The *augmented interval representation $\mathcal{I}^+$* is obtained from $\mathcal{I}$ by adding I_{c_i} strictly inside $\bigcap \mathcal{I}[C_i]$ for every c_i. Sets $C_1^+ = C_1 \cup \{c_1\}, \ldots, C_k^+ = C_k \cup \{c_k\}$ are the *augmented maximal cliques*; see Figure 5. In the augmented representation, the augmented maximal cliques are ordered the same way as the original ones and with a slight abuse of notation, we also denote this ordering by σ.

▶ **Lemma 9.** **(⋆)** *A graph G has a φ-oriented interval representation if and only if G^+ has a φ^+-oriented interval representation such that no clique point interval properly contains another interval.*

Assume G is an oriented interval graph with an oriented interval representation $(\mathcal{I}, \varphi)$. By Lemma 9 we obtain that G^+ admits an oriented interval representation $(\mathcal{I}^+, \varphi^+)$, where every clique point interval is inclusion-wise minimal. By a slight abuse of notation we use the symbol $\prec_\varphi$ also for the placement order on G^+, because clique point vertices do not participate in any of its comparisons, and hence both placement orders on G and G^+ are identical as sets.

Let H^+ be the *containment graph* of G^+, i.e., $V(H^+) = V(G^+)$ and $E(H^+) = E^+_{\text{cont}}$. By Lemma 8, H^+ is a permutation graph whose complement $\overline{H^+}$ has a transitive orientation $\prec^+_{\overline{H}}$ that extends $\prec_\varphi$. By orienting H^+ by inclusion of the corresponding intervals, i.e., choosing $u \prec^+_H v \iff I_u \subset I_v$, we obtain a transitive orientation $\prec^+_H$ of H^+ such that $c_i \prec^+_H v$ for all $v \in C_i$. We show that for H^+, these necessary conditions are also sufficient.

▶ **Theorem 10.** **($\star$)** *Let φ be an orientation of a mixed interval graph G. Let $H^+ = (V^+, E^+_{\text{cont}})$ be the containment graph of G^+. Then G admits a φ-oriented interval representation if and only if*

(i) *H^+ admits a transitive orientation $\prec^+_H$ such that each c_i is a minimal element and*

(ii) *$\overline{H^+}$ admits a transitive orientation $\prec^+_{\overline{H}}$ that extends $\prec_\varphi$.*

The following theorem is the main result of this section.

▶ **Theorem 1.** *Given a mixed graph G and $E_{\text{cont}} \subseteq E(G)$, there is a linear-time algorithm that decides whether G admits an oriented interval representation where precisely the edges in E_{cont} are represented by containment.*

Proof. By Lemma 7 we compute a feasible orientation φ of G in linear time. Thus, G is an oriented interval graph if and only if it has a φ-oriented interval representation. Recall that by Theorem 10, G is a φ-orientable interval graph, if and only if (i) $H^+ = (V^+, E^+_{\text{cont}})$ admits a transitive orientation $\prec^+_H$ such that each c_i is a minimal element and (ii) $\overline{H^+}$ admits a transitive orientation $\prec^+_{\overline{H}}$ that extends $\prec_\varphi$. The problem of extending a partial transitive orientation of a graph with n vertices and m edges can be solved in $O(n+m)$ time [17]. In our setting, the difficulty with linear time lies in checking the existence of $\prec^+_{\overline{H}}$ as the complement graph $\overline{H^+}$ may be too dense. To solve this issue, we reduce checking the existence of $\prec^+_H$ and $\prec^+_{\overline{H}}$ to checking the existence of a matching representation of H^+ that satisfies certain constraints. We then solve the latter problem by using lowest common ancestors and modular decompositions; see [1] for the details. ◀

We note that the same characterization and algorithms apply to the case where, in addition, we prescribe a clique ordering $\sigma = C_1 < C_2 < \cdots < C_k$ of the required interval representation, as it suffices to add $(c_1, c_2), (c_2, c_3), \ldots, (c_{k-1}, c_k)$ to $\prec_\varphi$ in Theorem 10. Since the additional constraints are of the same type, the linear-time algorithm can handle them without modification.

▶ **Corollary 11.** *Let G be a mixed graph with containment edges $E_{\text{cont}} \subseteq E$, let φ be an orientation of G, and let σ be a clique order of G. There is a linear-time algorithm that decides whether G admits a φ-oriented interval representation with clique order σ where precisely the edges in E_{cont} are represented by interval inclusions.*

4 Prescribed Orientations

In this section, we assume that φ is given as part of the input. We outline[2] an algorithm that recognizes φ-orientable interval graphs using clique orderings. The proof works in two steps. First, we show that we can compute a clique ordering σ of G such that G has a φ-oriented interval representation $\mathcal{I}$ if and only if the pair (φ, σ) is consistent. Then, we construct a φ-orientable interval representation of G with clique ordering σ if and only if G is a φ-oriented interval graph.

A *PQ-tree* T over a finite universe U is a rooted ordered tree whose leaves correspond to the elements of U and whose internal nodes are of two types called *P-* and *Q-nodes*. Such a PQ-tree represents all permutations of its leaves that can be obtained by arbitrarily reordering the children of P-nodes and reversing the order of children of any set of Q-nodes. The PQ-tree of an interval graph G is the PQ-tree that has as leaves the maximal cliques

[2] The full version of this section can be found in [1].

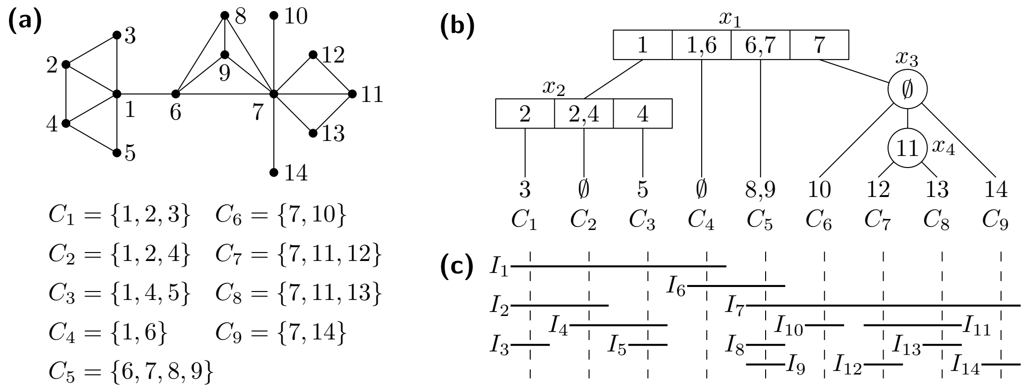

Figure 6 (a) An interval graph G and its maximal cliques; (b) an MPQ-tree T of G (P-nodes are round, Q-nodes are boxes); (c) the interval representation corresp. to the ordering of the leaves in T.

of G and represents the clique orders of G [5]. We call the edges of the PQ-tree *links* and for a Q-node x with children $y_1, \ldots, y_k$ (in this order), we call a set of consecutive links $S_{i,j} = \{xy_i, \ldots, xy_j\}$ a *segment* of x.

To describe the clique orderings of G we use modified PQ-trees, also called MPQ-trees [16]; see Figure 6 for an example. An *MPQ-tree* of an interval graph G consists of the PQ-tree of G with additional annotations that associate the vertices of G (and not just its maximal cliques) with certain parts of the PQ-tree. Namely, we assign every vertex v in G to the node x in T that is the lowest common ancestor of all cliques containing v. For a node x of the MPQ-tree, we let W_x denote the set of vertices assigned to x. In Figure 6b, for example, we have $W_{x_1} = \{1,6,7\}$ and $W_{x_4} = \{11\}$.

Let x be a Q-node with children $y_1, \ldots, y_k$ and let v be a vertex that is assigned to x. It follows from the construction of the PQ-tree for interval graphs that if the subtree rooted at the child y_i contains a leaf that corresponds to a maximal clique of G that contains v, then all leaves in this subtree correspond to maximal cliques containing v [16]. Consider the set of links $\{x, y_i\}$ where all leaves of the subtree rooted at y_i are maximal cliques containing v. Since G is an interval graph, the leaves containing v are consecutive in T. Hence, the links in this set are consecutive and form a segment of x. We call this segment the *segment of v* and denote it by S_v. In Figure 6b, for example, we have $S_1 = \{x_1x_2, x_1C_4\}$ and $S_2 = \{x_2C_1, x_2C_2\}$. For $1 \le i \le k$, we let $L_i \subseteq W_x$ denote the set of vertices that are assigned to x and whose segment contains $\{x, y_i\}$. In Figure 6b, for example, we have $L_1 = \{1\}$, $L_2 = \{1,6\}$, $L_3 = \{6,7\}$ and $L_4 = \{7\}$ for x_1. The maximal clique represented by a leaf x of T is the set $W_x \cup W_P \cup W_Q$, where W_P denotes the set containing all vertices of G that are assigned to P-nodes on the path p from x to the root, and W_Q denotes the set containing the vertices in L_i for every edge qy_i in p where y_i is a child of a Q-node q.

We say that a vertex v of G is *above* a node x if v is assigned to a node on the path from x to the root in T, excluding x. Similarly, we say that v is *below* x if v is assigned to a node on a path from x to some leaf in the subtree T_x of T rooted at x, excluding x. The set of all vertices of G that are above node x is referred to as A_x^T and the set of all vertices below x as B_x^T. In Figure 6b, for example, we have $A_{x_2}^T = \{1,6,7\}$ and $B_{x_2}^T = \{3,5\}$.

Since T is an ordered tree, the leaves of T are ordered. We say that an interval representation $\mathcal{I}$ of G *agrees* with T if the order of the maximal cliques of G in $\mathcal{I}$ coincides with the order of the maximal cliques described by the leaves of T. We let π_x^T denote the order of the children of x in T. A *rotation* of an MPQ-tree T is an MPQ-tree T' obtained from T

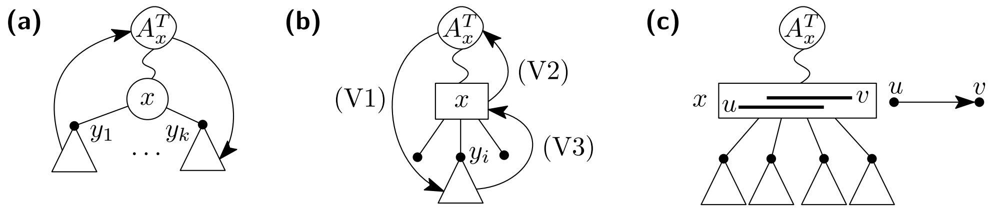

Figure 7 (a) Constrained P-node with constraining children y_1 and y_k; (b) examples for the three types of vertical arcs constraining a Q-node x; (c) a horizontal arc constraining a Q-node x.

by arbitrarily permuting the order of the children of P-nodes and reversing the order of the children of any subset of Q-nodes in T. We say that an MPQ-tree T is *φ-orientable* if it agrees with some φ-orientable interval representation of G. Let a be an arc in G with endpoints u and v. If u and v are assigned to the same node of T, then we call a *horizontal*. Otherwise, a is *vertical*, and u and v are assigned to distinct nodes x_u and x_v of T, respectively. In this case, if x_u is an ancestor of x_v in T, then u is called the *upper* and v the *lower endpoint* of a .

We show that certain vertical and horizontal arcs impose restrictions on the clique order, and thus on the order of the children of some nodes in T. Let x be a node of T with children $y_1, \dots, y_k$ and let $B_i = B_{y_i}^T \cup W_{y_i}$ with $i \in \{1, \dots, k\}$. We say that y_i is a *constraining child* of x if x is a P-node and there exists a vertical arc a in G with the lower endpoint in B_i and the upper endpoint in $W_x \cup A_x^T$. We say that x is *constrained* if one of the following holds:

- x is a P-node and x has a constraining child y_i or
- x is a Q-node and there is an arc a with endpoints u and v such that
  - **(V1)** a is vertical, $u \in B_x^T$, and $v \in A_x^T$ or
  - **(V2)** a is vertical, $u \in W_x$, $S_u \neq S_{1,k}$, and $v \in A_x^T$ or
  - **(V3)** a is vertical, $u \in B_x^T$, and $v \in W_x$ or
  - **(H)** a is horizontal, $u, v \in W_x$, and $S_u \neq S_v$.

Figure 7 illustrates constrained nodes. We say that the position of a child y_i of x is *fixed*, if y_i has the same position in π_x^T for every φ-oriented MPQ-tree T of G. We say that π_x^T is fixed if it is the same for every φ-orientable rotation T of the MPQ-tree of G. Next, we show that the pair (φ, σ) is consistent, that is, G admits a φ-orientable interval representation with clique order σ, if and only if the rotation T of the MPQ-tree of G corresponding to σ satisfies the following constraints.

(a) If x is a constrained P-node, then the position of each constraining child y_i in π_x^T is fixed and must be either the first or the last child of x.

(b) If x is a constrained Q-node, then π_x^T is fixed, i.e., for every φ-orientable rotation T' of T, we have $\pi_x^{T'} = \pi_x^T$.

We say that a clique order σ of G is *φ-consistent* if it is the clique order defined by a rotation T of the MPQ-tree of G that satisfies properties (a) and (b). We give a proof sketch for the following lemma, which is the first part of the main result of this section.

▶ **Lemma 12.** **($\star$)** *Let G be a mixed graph, let φ be an orientation of G and let σ be a φ-consistent clique order of G. Then G admits a φ-orientable interval representation if and only if it admits one where the clique order is σ.*

Sketch of proof. Recall that a clique order σ of G corresponds to the leaf order of an MPQ-tree T of G. Assume G admits a φ-orientable interval representation $\mathcal{I}$ and let σ be a clique order of G. We show that G admits a φ-orientable interval representation with clique order σ if and only if σ is φ-consistent. We give an example for the necessity; the full proof is given in [1]. Let x be a P-node with constraining child y_i and assume there exists a vertical arc (u, v) in G such that $u \in B_i$ and $v \in W_x \cup A_x^T$. Observe that y_i must be the first child of x if $\varphi(u) = \varphi(v) = 1$: If, for the sake of contradiction, some y_j, $i \neq j$ is the first child of x, then there exists a vertex $w \in B_j$ that is not adjacent to u. Consider the intervals $I_u = [l_u, r_u]$, $I_v = [l_v, r_v]$, and $I_w = [l_w, r_w]$ in $\mathcal{I}$. We have $l_u < l_v$, as (u, v) is an arc in G and $\varphi(u) = \varphi(v) = 1$, and $r_w < l_u$, as T puts y_j before y_i; see Figure 8. Therefore, $r_w < l_v$ and thus I_v and I_w are disjoint. This is a contradiction to the properties of the MPQ-tree, by which every vertex in $W_x \cup A_x^T$ must be adjacent to every vertex in B_x^T. We show similar statements for Q-nodes in the full version [1].

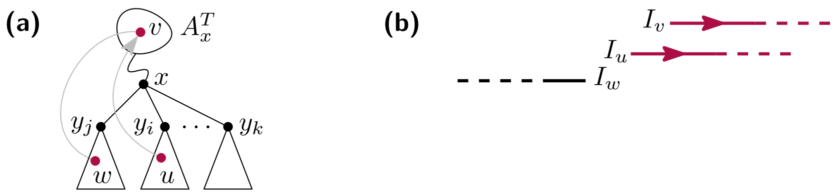

Figure 8 (a) MPQ-tree where y_i is not the last child of x; (b) the implied interval representation.

For the converse, let T be an MPQ-tree of G that satisfies the constraints. Let T' be the rotation of the MPQ-tree of G that agrees with $\mathcal{I}$. We choose $\mathcal{I}$ such that T' is as similar to T as possible, meaning that the number of nodes for which the order of their children is the same in both trees is maximized. If $T = T'$ we are done. Otherwise, let x be the lowest node in T for which $\pi_x^T \neq \pi_x^{T'}$. Let T'' be the MPQ-tree we obtain from T' by changing $\pi_x^{T'}$ to π_x^T. Observe that for each node x the set of nodes assigned to x as well as the sets A_x^T and B_x^T are the same for all three trees T, T', T''. We claim that T'' agrees with a φ-oriented interval representation $\mathcal{I}'$ of G, which contradicts the choice of T', and hence $T = T'$ follows.

The full proof considers two cases, namely, x is a P-node and x is a Q-node. In both cases, we use the fact that there exists a *normalized* interval representation $\mathcal{I}'$ in which the following properties hold for all nodes x in T:

- intervals that contain each other in $\mathcal{I}$ also contain each other in $\mathcal{I}'$ and
- if x is a P-node, then for every non-constraining child y_i of x, every edge between B_i and $W_x \cup A_x^T$ is represented by containment in $\mathcal{I}'$ or
- if x is an unconstrained Q-node, then every edge between (i) B_x^T and A_x^T, (ii) B_x^T and $u \in W_x$ with $S_u = S_{1,k}$ and (iii) $u \in W_x \cup A_x^T$ and $v \in W_x$ with $S_v \neq S_{1,k}$ is represented by containment in $\mathcal{I}'$.

The existence of a normalized representation is shown in the full version [1]. For a P-node x with children $y_1, \ldots, y_k$, we then show the following. Property (a) gives us that π_x^T and $\pi_x^{T''}$ agree on the positions of the constraining children. In other words, π_x^T and $\pi_x^{T''}$ differ only with respect to the order of the non-constraining children of x. Observe that $\pi_x^{T''}$ can be obtained from π_x^T by repeatedly switching the order of two consecutive non-constraining children of x. In order to show that there exists a φ-oriented interval representation that agrees with T'', we show the following claim in the full version [1].

$\triangleright$ Claim. ***($\star$)*** Let $T^\star$ be the rotation of T' we get by switching the order of two consecutive non-constraining children y_i, y_{i+1} of x in T'. There exists a φ-orientable interval representation $\mathcal{I}'$ that agrees with $T^\star$.

The idea is that since x is not constrained, each interval of $\mathcal{I}'[B_i]$ and $\mathcal{I}'[B_{i+1}]$ is properly contained in every interval of $\mathcal{I}'[W_x \cup A_x^{T'}]$. We can therefore apply Lemma 5 twice – first with $W = \mathrm{cont}_{\mathcal{I}'}(I)$ where I is the smallest interval that spans $\mathcal{I}'[B_i] \cup \mathcal{I}'[B_{i+1}]$ and then with

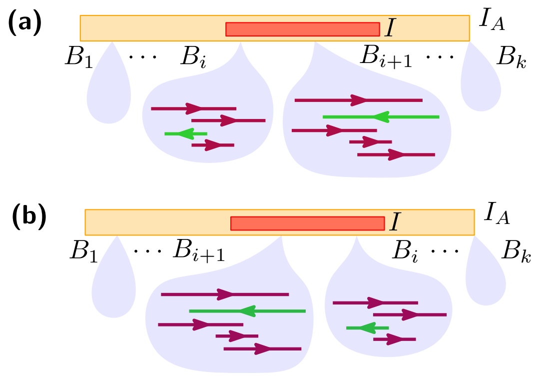

Figure 9 (a) Before and (b) after the swap of two consecutive unconstrained children using Lemma 5 in a normalized representation.

$W_i = \mathrm{cont}_{\mathcal{I}''}(I_i)$ and $W_{i+1} = \mathrm{cont}_{\mathcal{I}''}(I_{i+1})$ where I_i and I_{i+1} are the smallest $\mathcal{I}'' = \mathrm{mirr}_I(\mathcal{I}')$ – to obtain a φ-orientable interval representation that agrees with $T^\star$. That is, the order of $\mathcal{I}'[B_i]$ and $\mathcal{I}'[B_{i+1}]$ coincides with the order of y_i and y_{i+1} in $T^\star$; see Figure 9.

By repeatedly switching consecutive non-constraining children of x, we can therefore construct a φ-orientable interval representation that agrees with T''. This contradicts our choice of T'.

For a Q-node x, observe that since π_x^T and $\pi_x^{T'}$ differ, it follows from Property (b) that x is not constrained, since otherwise π_x^T would be fixed and therefore the same in T and T'. Let $U \subseteq W_x$ be the set of vertices u of G that are assigned to x and for which $S_u \neq S_{1,k}$. To show that T agrees with a φ-orientable interval representation, first show that (i) $G[U]$ is φ-orientable if and only if it is $\mathrm{rev}_U(\varphi)$-orientable and that (ii) there are only (undirected) edges between $B_x^T \cup U$ and $A_x^T \cup (W_x \setminus U)$. We then apply Lemma 6 to obtain a rev_φ-orientable interval representation $\mathcal{I}''$. Altogether, we show that there exists a φ-orientable interval representation of G that agrees with T'', a contradiction to our choice of T'. ◀

We remark that the MPQ-tree T and the constrained children of each node can be computed in polynomial time. Therefore, in polynomial time we can decide whether a mixed graph admits a φ-consistent clique order and compute one in the positive case. In fact, by the following lemma, which we prove in the full version, this can be achieved in linear time.

▶ **Lemma 13.** **($\star$)** *Let G be a mixed graph and let φ be an orientation of G. There is a linear-time algorithm that computes a clique order of G that is φ-consistent if G admits a φ-consistent clique order.*

This result shows that, given an orientation φ of G, we can construct an MPQ-tree that represents every clique order σ of G such that the pair (φ, σ) is consistent if and only if G is an oriented interval graph. It remains to construct a representation from (φ, σ).

Recall the definition of the augmented graph G^+ from Section 3.2, and let $C = \{c_1, \dots, c_k\}$ be the set of clique point vertices. Let $V^L = \{v \in V(G) \mid \varphi(v) = -1\}$, let $G^L = G^+[V^L \cup C]$, and let $\sigma^L = C_1^L, \dots, C_k^L$, where, for $i \in \{1, \dots, k\}$, $C_i^L = C_i^+ \cap V(G^L)$; see Figure 10c. Define V^R, G^R, $C_1^R, \dots, C_k^R$, and σ^R analogously; see Figure 10e. Let φ^L and φ^R denote the orientations of G^L and G^R that assign to each vertex the orientation -1 and $+1$, respectively. We show the following statement.

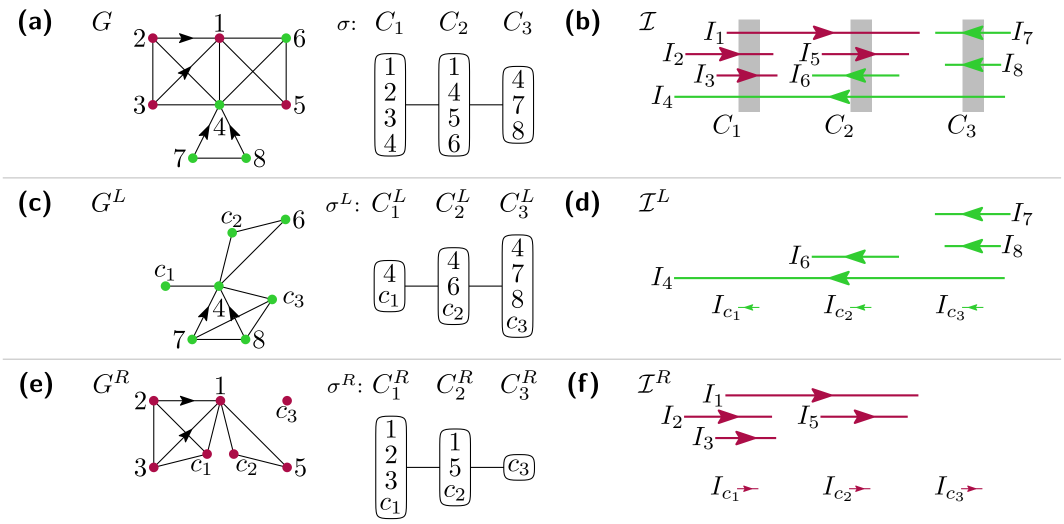

Figure 10 Example illustrating the proof of Lemma 14.

▶ **Lemma 14.** *A mixed graph G admits a φ-orientable interval representation with clique order σ if and only if*

- *the graph G^L admits a φ^L-orientable interval representation with clique order σ^L and*
- *the graph G^R admits a φ^R-orientable interval representation with clique order σ^R.*

Given representations of G^L and G^R, a φ-orientable interval representation of G can be computed from representations of G^L and G^R in linear time.

Proof. "⇒": We take the φ-orientable interval representation $\mathcal{I}$ of G with clique order σ and split it into two representations, a representation $\mathcal{I}^L$ for the left-oriented intervals and a representation $\mathcal{I}^R$ for the right-oriented intervals. To both of these, we add, for each clique points in C, a very short interval that intersects all intervals that form the corresponding clique. The first resulting representation is φ^L-oriented and has clique order σ^L, the second one is φ^R-oriented and has clique order σ^R, as desired.

"⇐": Let $\mathcal{I}^L$ be the φ^L-orientable interval representation of $G^L - C$, and let $\mathcal{I}^R$ be the φ^R-orientable interval representation $\mathcal{I}^R$ of $G^R - C$. We join $\mathcal{I}^L$ and $\mathcal{I}^R$ as follows. Since σ^L and σ^R both have a clique for each clique point vertex, we can simply set, for each $i \in \{1, \dots, k\}$, $C_i = (C_i^L \cup C_i^R) \setminus \{c_i\}$ and $\sigma = C_1, \dots, C_k$. It remains to determine, for each $i \in \{1, \dots, k\}$, the order of the interval endpoints within C_i; see Figure 11. To this end, we first place the left endpoints of the intervals in $V(G)$ that start in C_i and extend to the right; then we place the right endpoints of the intervals in $V(G)$ that start in C_i and extend to the left. We order the left endpoints as follows: we start with the intervals in V^L in the order in which they appear in $\mathcal{I}^L$; then we add the intervals in V^R in the order in which they appear in $\mathcal{I}^R$. We order the right endpoints accordingly. This fixes a set $\mathcal{I}$ of intervals that contains an interval for each vertex of G; it can be computed in linear time by merging $\mathcal{I}^L$ and $\mathcal{I}^R$.

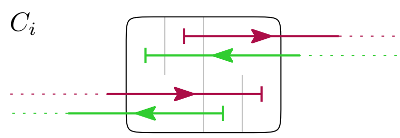

Figure 11 How to place the endpoints of left- and right-oriented intervals within a clique.

Note that two intervals in $\mathcal{I}$ intersect if and only if they are both contained in a maximal clique C_i. This ensures that all undirected edges of G are realized in $\mathcal{I}$. Every arc (u, v) of G is realized in $\mathcal{I}$ because either both u and v lie in $V^L \setminus C$ and (u, v) is realized in $\mathcal{I}^L$, or they both lie in $V^R \setminus C$ and (u, v) is realized in $\mathcal{I}^R$. Since the order of the endpoints of intervals in $V^L \setminus C$ (in $V^R \setminus C$) is the same in $\mathcal{I}^L$ (in $\mathcal{I}^R$) and in $\mathcal{I}$, the arc (u, v) is realized in $\mathcal{I}$. ◀

We are now ready to prove the main result of this section.

▶ **Theorem 3.** *Given a mixed graph G with an orientation φ, there is a linear-time algorithm that decides whether G admits an oriented interval representation with orientation φ.*

Proof. We use Lemma 13 to compute, in linear time, a clique order σ such that G admits a φ-orientable interval representation if and only if G admits such a representation with clique order σ. Then we split (G, σ) into (G^L, σ^L) and (G^R, σ^R) in linear time. Since all vertices of G^L have the same orientation, the undirected edges of G^L are precisely the containment edges, and we can use Corollary 11 to check in linear time whether G^L admits a φ^L-orientable interval representation with clique order σ^L. Then we do the corresponding check for G^R. If both checks are successful, we use Lemma 14 to combine the resulting interval representations of G^L and G^R in linear time to a φ-orientable interval representation of G. ◀

5 Concluding Remarks

In this work, we have initiated a systematic study of the interconnections between clique orders, orientations, and containment edges of oriented interval representations. Given a mixed graph G, we have characterized the orientations φ of G that, given a set $E_{\text{cont}} \subseteq E(G)$, form a consistent pair $(\varphi, E_{\text{cont}})$ as well as the clique orders σ of G that, given an orientation φ of G, form a consistent pair (φ, σ). We also presented linear-time algorithms for two constrained versions of the recognition problem, where, in addition to G, either the set E_{cont} or the orientation φ is prescribed. The problem of recognizing oriented interval graphs given a clique order σ of G seems to be a natural next step for future work. The structural insights that we have gained in this paper could be a good starting point for tackling this problem, as well as the general recognition problem, both of which we leave open here. We note, though, that even if a polynomial-time algorithm exists for the latter, it is not clear that it would also solve the constrained recognition problems that we have studied in this paper.

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