On the Complexity of Multipacking

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Abstract

A *multipacking* in an undirected graph $G = (V, E)$ is a set $M \subseteq V$ such that for every vertex $v \in V$ and for every integer $r \geq 1$, the ball of radius r around v contains at most r vertices of M, that is, there are at most r vertices in M at a distance at most r from v in G. The Multipacking problem asks whether a graph contains a multipacking of size at least k.

For more than a decade, it remained an open question whether the Multipacking problem is NP-complete or solvable in polynomial time, although the problem is known to be polynomial-time solvable for certain graph classes (e.g., strongly chordal graphs, grids, etc). Foucaud, Gras, Perez, and Sikora [16] [*Algorithmica* 2021] made a step towards solving the open question by showing that the Multipacking problem is NP-complete for directed graphs and W[1]-hard when parameterized by the solution size.

In this paper, we prove that the Multipacking problem is NP-complete on undirected graphs, which answers the open question. Moreover, the problem is W[2]-hard on undirected graphs when parameterized by the solution size. Furthermore, we show that the problem is NP-complete and W[2]-hard (parameterized by solution size) on chordal, bipartite, and claw-free graphs, and remains NP-complete on regular and CONV graphs (intersection graphs of convex sets in the plane). Additionally, the problem is NP-complete and W[2]-hard (parameterized by the solution size) on chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs, which is a superclass of strongly chordal graphs on which the problem is polynomial-time solvable. On the positive side, we present an exact exponential-time algorithm for the Multipacking problem on general graphs that breaks the 2^n barrier, with running time $O^*(1.58^n)$, where n is the number of vertices.

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1 Introduction

Multipacking was introduced by Brewster, Mynhardt, and Teshima [4] as a natural generalization of classical packing and independence concepts in graphs (see also [1, 7, 21, 12, 24, 26]). Intuitively, multipacking can be viewed as a refined version of the independent set problem, where the feasibility of selecting vertices depends on distance constraints at all scales.

For a graph $G = (V, E)$, let $d_G(u, v)$ denote the length of a shortest path between two vertices u and v in G; we write $d(u, v)$ when there is no ambiguity. Let $N_r[v] := \{u \in V(G) : d(v, u) \leq r\}$, i.e. a ball of radius r around v. The *eccentricity* $e(w)$ of a vertex w is $\min\{r : N_r[w] = V\}$. The *radius* of the graph G is $\min\{e(w) : w \in V\}$, denoted by $\text{rad}(G)$. The *center* $C(G)$ of the graph G is the set of all vertices of minimum eccentricity, i.e., $C(G) := \{v \in V : e(v) = \text{rad}(G)\}$. Each vertex in the set $C(G)$ is called a *central vertex* of the graph G. Let $\text{diam}(G) := \max\{d(u, v) : u, v \in V(G)\}$. A diameter of G is a path of length $\text{diam}(G)$.

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A *multipacking* is a set $M \subseteq V$ in a graph $G = (V, E)$ such that $|N_r[v] \cap M| \leq r$ for each vertex $v \in V$ and for every integer $r \geq 1$. It is worth noting that in this definition, it suffices to consider values of r in the set $\{1, 2, \dots, \mathrm{diam}(G)\}$. Moreover, if c is a central vertex of the graph G, then $N_r[c] = V(G)$ for $r = \mathrm{rad}(G)$. Therefore, the size of any multipacking in G is upper-bounded by $\mathrm{rad}(G)$. Therefore, we can further restrict r (in the definition of multipacking) to the set $\{1, 2, \dots, \mathrm{rad}(G)\}$, where $\mathrm{rad}(G)$ is the radius of the graph. The *multipacking number* of G is the maximum cardinality of a multipacking of G and it is denoted by $\mathrm{mp}(G)$. A *maximum multipacking* is a multipacking M of a graph G such that $|M| = \mathrm{mp}(G)$.

MULTIPACKING problem
- **Input:** An undirected graph $G = (V, E)$, an integer $k \in \mathbb{N}$.
- **Question:** Does there exist a multipacking $M \subseteq V$ of G of size at least k?

It remained an open problem for over a decade whether the MULTIPACKING problem is NP-complete or solvable in polynomial time. This question was repeatedly addressed by numerous authors ([3, 16, 21, 26, 28, 29]), yet the problem remained as an unsolved challenge. In this paper, we answer this question by proving that the MULTIPACKING problem is NP-complete which is one of the main results of this paper.

On the algorithmic side, Beaudou, Brewster, and Foucaud [2] presented a $(2 + o(1))$-approximation algorithm for general graphs. Polynomial-time algorithms are known for trees and, more generally, for strongly chordal graphs [3], although even the case of trees requires nontrivial techniques. Approximation algorithms with ratio $(\frac{3}{2} + o(1))$ have been obtained for chordal graphs [10] and cactus graphs [11].

In 2021, Foucaud, Gras, Perez, and Sikora [16] studied the complexity of the MULTIPACKING problem on digraphs (directed graphs). They showed that the MULTIPACKING problem is NP-complete for digraphs, even for planar layered acyclic digraphs of bounded maximum degree. They further proved that the problem is W[1]-HARD (even on digraphs of maximum finite distance 3) when parameterized by the solution size.

It is well known that the MULTIPACKING problem admits a natural integer programming formulation. If M is a multipacking, we define a vector y with the entries $y_j = 1$ when $v_j \in M$ and $y_j = 0$ when $v_j \notin M$. Let $A = [a_{j,(i,k)}]$ be a matrix with entries $a_{j,(i,k)} = 1$ if $v_j \in N_k[v_i]$, and $a_{j,(i,k)} = 0$ otherwise. Let c be a vector indexed by (i, k) where $v_i \in V(G)$ and $1 \leq k \leq \mathrm{diam}(G)$, with the entries $c_{i,k} = k$.

$$\mathrm{mp}(G) = \max\{y \cdot \mathbf{1} : yA \leq c, y_j \in \{0, 1\}\}.$$

A comprehensive survey of multipacking and related problems can be found in the book [21], and a geometric variant of the problem is studied in [9].

The *maximum multipacking problem* is the dual integer program of the *optimal broadcast domination problem*. Erwin [15, 14] introduced broadcast domination in his doctoral thesis in 2001. For a graph $G = (V, E)$, a function $f : V \to \{0, 1, 2, ..., \mathrm{diam}(G)\}$ is called a *broadcast* on G. For each vertex $u \in V$, if there exists a vertex v in G (possibly, $u = v$) such that $f(v) > 0$ and $d(u, v) \leq f(v)$, then f is called a *dominating broadcast* on G. The *cost* of the broadcast f is the quantity $\sigma(f)$, which is the sum of the weights of the broadcasts over all vertices in G. So, $\sigma(f) = \sum_{v \in V} f(v)$. The minimum cost of a dominating broadcast in G (taken over all dominating broadcasts) is the *broadcast domination number* of G, denoted by $\gamma_b(G)$. An *optimal dominating broadcast* on a graph G is a dominating broadcast with a cost equal to $\gamma_b(G)$.

A standard integer linear programming formulation of broadcast domination is obtained as follows. Let c and x be the vectors indexed by (i,k) where $v_i \in V(G)$ and $1 \le k \le \operatorname{diam}(G)$, with the entries $c_{i,k} = k$ and $x_{i,k} = 1$ when $f(v_i) = k$ and $x_{i,k} = 0$ when $f(v_i) \neq k$. Let $A = [a_{j,(i,k)}]$ be a matrix with entries $a_{j,(i,k)} = 1$ if $v_j \in N_k[v_i]$, and $a_{j,(i,k)} = 0$ otherwise. Hence, the broadcast domination number can be expressed as an integer linear program:

$$\gamma_b(G) = \min\{c.x : Ax \ge \mathbf{1}, x_{i,k} \in \{0,1\}\}.$$

For general graphs, surprisingly, an optimal dominating broadcast can be found in polynomial time $O(n^6)$ [22]. The same problem can be solved in linear time for trees [3].

It is known that $\operatorname{mp}(G) \le \gamma_b(G)$ [4]. In 2019, Beaudou, Brewster, and Foucaud [2] proved that $\gamma_b(G) \le 2\operatorname{mp}(G) + 3$ and they conjectured that $\gamma_b(G) \le 2\operatorname{mp}(G)$ for every graph. In 2025, Rajendraprasad et al. [25] have shown that for general connected graphs,

$$\lim_{\operatorname{mp}(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{\operatorname{mp}(G)} \right\} = 2.$$

Stronger bounds are known for specific graph classes. In particular, for any connected chordal graph G, we have $\gamma_b(G) \le \left\lceil \frac{3}{2}\operatorname{mp}(G) \right\rceil$ [10], and for any cactus graph G, it holds that $\gamma_b(G) \le \frac{3}{2}\operatorname{mp}(G) + \frac{11}{2}$ [11].

Our Contributions. The algorithmic complexity of the Multipacking problem remained open for more than a decade ([3, 16, 26, 28, 29]). In this paper, we resolve this question by establishing hardness results for Multipacking on a wide range of graph classes, complemented by an exact exponential-time algorithm.

Our main focus is to identify graph classes for which the Multipacking problem admits a polynomial-time algorithm and to delineate a boundary between tractable and intractable cases. Whenever our reductions also imply consequences for the parameterized complexity of Multipacking on the considered graph classes, we explicitly point them out. However, we do not aim at obtaining tight parameterized complexity bounds, and we deliberately avoid more involved reductions whose sole purpose would be to strengthen such bounds.

Our main hardness result is obtained via a simple and elegant reduction from the Hitting Set problem, which is NP-complete [17] and W[2]-complete when parameterized by the solution size [8]. This reduction yields the following theorem.

▶ **Theorem 1.** *The Multipacking problem is NP-complete. Moreover, when parameterized by the solution size k, the problem is W[2]-hard. In addition, unless the Exponential Time Hypothesis (ETH)*[1] *fails, there is no algorithm for Multipacking running in time $f(k)n^{o(k)}$, where n is the number of vertices of the input graph.*

We further study the complexity of Multipacking on graph classes defined by metric properties, with a particular emphasis on $\frac{1}{2}$-hyperbolic graphs. The notion of δ-hyperbolicity was introduced by Gromov [18]. A formal definition of hyperbolic graphs is provided in the Preliminaries (Section 2). While Multipacking is solvable in polynomial time on strongly chordal graphs[2] and in linear time on trees [3], and while strongly chordal graphs are

[1] The *Exponential Time Hypothesis (ETH)* states that 3-SAT cannot be solved in time $2^{o(n)}$, where n is the number of variables of the input CNF formula.

[2] A *strongly chordal graph* is an undirected graph G which is a chordal graph and every cycle of even length (≥ 6) in G has an *odd chord*, i.e., an edge that connects two vertices that are an odd distance (> 1) apart from each other in the cycle.

$\frac{1}{2}$-hyperbolic [27], we show that this tractability does not extend to a larger class chordal $\cap$ $\frac{1}{2}$-hyperbolic. In particular, we strengthen the above result by proving that Multipacking remains NP-complete even for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs, a strict superclass of strongly chordal graphs.

Beyond metric graph classes, we establish hardness results for several classical families. Using two distinct reductions from Hitting Set, we show NP-completeness and W[2]-hardness for bipartite graphs and claw-free[3] graphs. We also prove NP-completeness for regular graphs via a reduction from the Total Dominating Set problem on cubic (3-regular) graphs [17]. Furthermore, we study Multipacking on geometric intersection graphs: we show that the problem is NP-complete for CONV graphs[4] by a reduction from Total Dominating Set on planar graphs.

These results are summarized in the following theorem.

▶ **Theorem 2.** *The* Multipacking *problem is NP-complete for each of the following graph classes (See Fig. 1):*

- *chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs,*
- *bipartite graphs,*
- *claw-free graphs,*
- *regular graphs, and*
- *CONV (convex intersection) graphs.*

Moreover, when parameterized by the solution size, the problem is W[2]-hard for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs, bipartite graphs, and claw-free graphs.

Finally, complementing our hardness results, we design an exact exponential-time algorithm for Multipacking on general graphs.

▶ **Theorem 3.** *There exists an exact algorithm for computing a maximum multipacking in an n-vertex graph with running time $O^*(1.58^n)$.*

Due to space constraints in the conference version, some lemma proofs have been moved to the appendix. The proofs of lemmas marked with $(*)$ are deferred to the appendix.

Organisation. In Section 2, we recall some definitions and notations. In Section 3, we prove that the Multipacking problem is NP-complete and also the problem is W[2]-hard when parameterized by the solution size. In Section 4, we discuss the hardness of the Multipacking problem on some subclasses. In Section 5, we present an exact exponential algorithm for the problem on general graphs. We conclude in Section 6 by presenting several important questions and future research directions.

2 Preliminaries

A complete bipartite graph $K_{1,3}$ is called a *claw*, and the vertex that is incident with all the edges in a claw is called the *claw-center*. A *Claw-free graph* is a graph that does not contain a claw as an induced subgraph.

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[3] *Claw-free graph* is a graph that does not contain a claw ($K_{1,3}$) as an induced subgraph.
[4] A graph G is a *CONV graph* or *Convex Intersection Graph* if and only if there exists a family of convex sets on a plane such that the graph has a vertex for each convex set and an edge for each intersecting pair of convex sets.

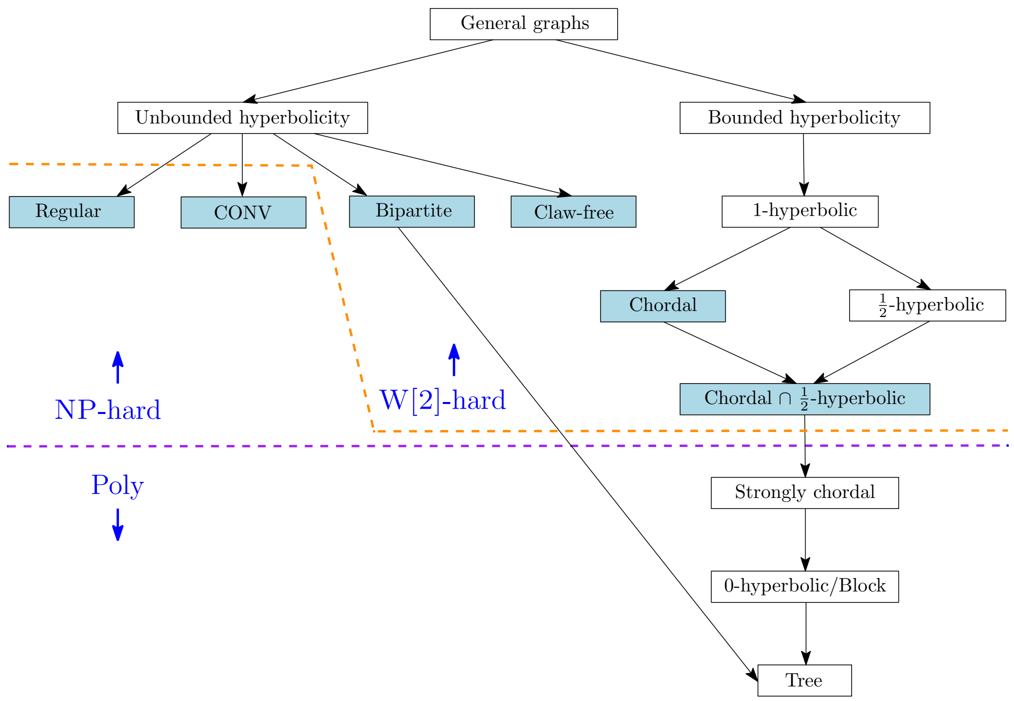

Figure 1 Inclusion diagram for graph classes mentioned in this paper (and related ones). If a class A has a downward path to class B, then B is a subclass of A. The Multipacking problem is NP-hard for the graph classes above the dashed purple line and polynomial-time solvable for the graph classes below it. Moreover, the problem is W[2]-hard for the graph classes above the dashed darkorange curve. In this paper, we discuss the hardness of the Multipacking problem for the graph classes in the colored (lightblue) block.

Let G be a connected graph. For any four vertices $u, v, x, y \in V(G)$, we define $\delta(u, v, x, y)$ as half of the difference between the two larger of the three sums $d(u, v) + d(x, y)$, $d(u, y) + d(v, x)$, $d(u, x) + d(v, y)$. The *hyperbolicity* of a graph G, denoted by $\delta(G)$, is the value of $\max_{u,v,x,y \in V(G)} \delta(u, v, x, y)$. For every $\delta \geq \delta(G)$, we say that G is δ-*hyperbolic*. A graph class $\mathcal{G}$ is said to be *hyperbolic* (or bounded hyperbolicity class) if there exists a constant δ such that every graph $G \in \mathcal{G}$ is δ-hyperbolic.

The complement of a graph G, denoted by G^c, is the graph with vertex set $V(G^c) = V(G)$, where two distinct vertices are adjacent in G^c if and only if they are non-adjacent in G. Two graphs G and H are said to be *isomorphic* if there exists a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$ for all $u, v \in V(G)$.

3 Hardness of the Multipacking problem (Proof of Theorem 1)

In this section, we present a reduction from the Hitting Set problem to prove our first hardness result for the Multipacking problem.

▶ **Lemma 4.** *The Multipacking problem is NP-complete. Moreover, the problem is W[2]-hard when parameterized by the solution size.*

Proof. We reduce the well-known NP-complete (also W[2]-complete when parametrized by solution size) problem Hitting Set [8, 17] to the Multipacking problem.

HITTING SET problem
- **Input:** A finite set U, a collection $\mathcal{F}$ of subsets of U, and an integer $k \in \mathbb{N}$.
- **Question:** Does there exist a *hitting set* $H \subseteq U$ of size at most k; that is, a set of at most k elements from U such that each set in $\mathcal{F}$ contains at least one element from H?

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$ where $S_i \subseteq U$ for each i. We construct a graph G in the following way: (i) Include each element of $\mathcal{F}$ in the vertex set of G and $\mathcal{F}$ forms a clique in G. (ii) Include $k-2$ length path $u_i = u_i^1 u_i^2 \ldots u_i^{k-1}$ in G, for each element $u_i \in U$. (iii) Join u_i^1 and S_j by an edge iff $u_i \notin S_j$. Fig. 2 gives an illustration.

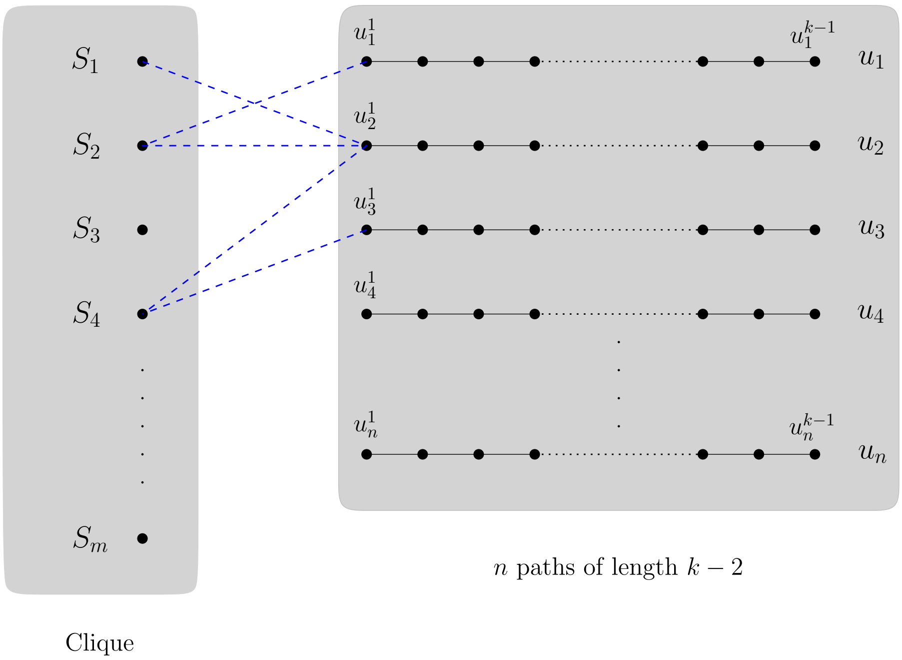

Figure 2 An illustration of the construction of the graph G used in the proof of Lemma 4, demonstrating the hardness of the MULTIPACKING problem.

Claim 4.1. $\mathcal{F}$ has a hitting set of size at most k iff G has a multipacking of size at least k, for $k \geq 2$.

Proof. (If part) Suppose $\mathcal{F}$ has a hitting set $H = \{u_1, u_2, \ldots, u_k\}$. We want to show that $M = \{u_1^{k-1}, u_2^{k-1}, \ldots, u_k^{k-1}\}$ is a multipacking in G. If $r < k-1$, then $|N_r[v] \cap M| \leq 1 \leq r$, $\forall v \in V(G)$, since $d(u_i^{k-1}, u_j^{k-1}) \geq 2k-2$, $\forall i \neq j$. If $r > k-1$, then $|N_r[v] \cap M| \leq |M| = k \leq r$, $\forall v \in V(G)$. Suppose $r = k-1$. If M is not a multipacking, there exists some $v \in V(G)$ such that $|N_{k-1}[v] \cap M| = k$. Therefore, $M \subseteq N_{k-1}[v]$. Suppose $v = u_p^q$ for some p and q. In this case, $d(u_p^q, u_i^{k-1}) \geq k$, $\forall i \neq p$. Therefore, $v \notin \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq k-1\}$. This implies that $v \in \{S_1, S_2, \ldots, S_m\}$. Let $v = S_t$ for some t. If there exists $i \in \{1, 2, \ldots, k\}$ such that S_t is not adjacent to u_i^1, then $d(S_t, u_i^{k-1}) \geq k$. Therefore, $M \subseteq N_{k-1}[S_t]$ implies that S_t is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\}$. Then $u_1, u_2, \ldots, u_k \notin S_t$. Therefore, H is not a hitting set of $\mathcal{F}$. This is a contradiction. Hence, M is a multipacking of size k.

(Only-if part). Suppose G has a multipacking M of size k. Let $H = \{u_i : u_i^j \in M, 1 \leq i \leq n, 1 \leq j \leq k-1\}$. Then $|H| \leq |M| = k$. Without loss of generality, let $H = \{u_1, u_2, \ldots, u_{k'}\}$ where $k' \leq k$. Now we show that H is a hitting set of $\mathcal{F}$. Suppose H is not a hitting set.

Then there exists t such that $S_t \cap H = \emptyset$. Therefore, S_t is adjacent to each vertex of the set $\{u_1^1, u_2^1, \dots, u_{k'}^1\}$. Then $M \subseteq N_{k-1}[S_t]$. This implies that $|N_{k-1}[S_t] \cap M| = k$. This is a contradiction. Therefore, H is a hitting set of size at most k. ◁

Since the above reduction is a polynomial-time, the Multipacking problem is NP-complete. Moreover, the reduction is also an FPT reduction. Hence, the Multipacking problem is W[2]-hard when parameterized by the solution size. ◀

▶ **Corollary 5.** *Multipacking problem is NP-complete for chordal graphs. Moreover, Multipacking problem is W[2]-hard for chordal graphs when parameterized by the solution size.*

Proof. We prove that the graph G constructed in the proof of Lemma 4 is chordal. Note that if G has a cycle $C = c_0c_1c_2 \dots c_{t-1}c_0$ of length at least 4, then there exists an index i such that $c_i, c_{i+2 \pmod t} \in \mathcal{F}$ because no two vertices of the set $\{u_1^1, u_2^1, \dots, u_n^1\}$ are adjacent. Since $\mathcal{F}$ forms a clique, so c_i and $c_{i+2 \pmod t}$ are endpoints of a chord. Therefore, the graph G is chordal. ◀

It is known that, unless ETH fails, there is no $f(k)(m+n)^{o(k)}$-time algorithm for the Hitting Set problem, where k is the solution size, $|U| = n$ and $|\mathcal{F}| = m$ [8].

▶ **Lemma 6.** *Unless ETH fails, there is no $f(k)n^{o(k)}$-time algorithm for the Multipacking problem, where k is the solution size and n is the number of vertices of the graph.*

Proof. Consider the construction of G in the proof of Lemma 4. Let $|V(G)| = n_1$. From the construction, we have $n_1 = m + n(k-1) = O(m+n^2)$. Therefore, the construction of G takes $O(m+n^2)$-time. Suppose that there exists an $f(k)n_1^{o(k)}$-time algorithm for the Multipacking problem. Therefore, we can solve the Hitting Set problem in $f(k)n_1^{o(k)} + O(m+n^2) = f(k)(m+n^2)^{o(k)} = f(k)(m+n)^{o(k)}$ time. This is a contradiction, since there is no $f(k)(m+n)^{o(k)}$-time algorithm for the Hitting Set problem, assuming ETH. ◀

4 Hardness results on several subclasses (Proof of Theorem 2)

In this section, we present hardness results on some subclasses.

4.1 Chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs

In Corollary 5, we have already shown that, for chordal graphs, the Multipacking problem is NP-complete and the problem is W[2]-hard when parameterized by the solution size. Here, we strengthen this result. It is known that the Multipacking problem is solvable in cubic time for strongly chordal graphs [3]. We prove hardness for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs, a superclass of strongly chordal graphs [27]. To show this, we need the following result.

▶ **Theorem 7** ([5]). *If G is a chordal graph, then its hyperbolicity $\delta(G) \leq 1$. Moreover, $\delta(G) = 1$ if and only if G contains one of the graphs in Fig. 3 as an isometric subgraph[5]. Equivalently, a chordal graph G is $\frac{1}{2}$-hyperbolic (i.e., $\delta(G) \leq \frac{1}{2}$) if and only if G does not contain any of the graphs in Fig. 3 as an isometric subgraph.*

[5] A subgraph H of a graph G is *isometric* if for any $u, v \in V(H)$, it holds that $d_H(u,v) = d_G(u,v)$.

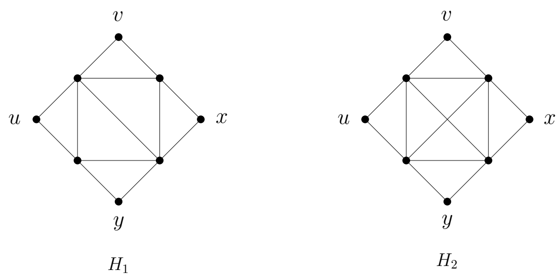

Figure 3 Forbidden isometric subgraphs for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs.

▶ **Lemma 8.** *MULTIPACKING problem is NP-complete for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs. Moreover, MULTIPACKING problem is W[2]-hard for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs when parameterized by the solution size.*

Proof. We reduce the HITTING SET problem to the MULTIPACKING problem to show that the MULTIPACKING problem is NP-complete for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$ where $S_i \subseteq U$ for each i. We construct a graph G in the following way: (i) Include each element of $\mathcal{F}$ in the vertex set of G. (ii) Include $k-2$ length path $u_i = u_i^1 u_i^2 \ldots u_i^{k-1}$ in G, for each element $u_i \in U$. (iii) Join u_i^1 and S_j by an edge iff $u_i \notin S_j$. (iv) Include the vertex set $Y = \{y_{i,j} : 1 \leq i < j \leq n\}$ in G. (v) For each i, j where $1 \leq i < j \leq n$, $y_{i,j}$ is adjacent to u_i^1 and u_j^1. (vi) $\mathcal{F} \cup Y$ forms a clique in G. Fig. 4 gives an illustration.

Claim 8.1. G is a chordal graph.

Proof. If G has a cycle $C = c_0 c_1 c_2 \ldots c_{t-1} c_0$ of length at least 4, then there exists an index i such that $c_i, c_{i+2 \pmod t} \in \mathcal{F} \cup Y$ because no two vertices of the set $\{u_1^1, u_2^1, \ldots, u_n^1\}$ are adjacent. Since $\mathcal{F} \cup Y$ forms a clique, c_i and $c_{i+2 \pmod t}$ are endpoints of a chord. Therefore, the graph G is chordal. ◁

Claim 8.2. G is a $\frac{1}{2}$-hyperbolic graph.

Proof. Let G' be the induced subgraph of G on the vertex set $\mathcal{F} \cup Y \cup \{u_1^1, u_2^1, \ldots, u_n^1\}$. We want to show that $\text{diam}(G') \leq 2$. Note that the distance between any two vertices of the set $\mathcal{F} \cup Y$ is 1, since $\mathcal{F} \cup Y$ forms a clique in G. Furthermore, the distance between any two vertices of the set $\{u_1^1, u_2^1, \ldots, u_n^1\}$ is 2, since they are non-adjacent and have a common neighbor in the set Y. Suppose $v \in \mathcal{F} \cup Y$ and $u_i^1 \in \{u_1^1, u_2^1, \ldots, u_n^1\}$ where v and u_i^1 are non-adjacent. In that case, $u_i y_{1,i} v$ is a 2-length path that joins v and u_i^1 since $\mathcal{F} \cup Y$ forms a clique in G. Hence, $\text{diam}(G') \leq 2$. By Claim 8.1 we have that G is chordal. From Theorem 7, we can say that the hyperbolicity of G is at most 1. Suppose G has hyperbolicity exactly 1. In that case, from Theorem 7, we can say that G contains at least one of the graphs H_1 or H_2 (Fig. 3) as isometric subgraphs. Note that every vertex of H_1 and H_2 is a vertex of a cycle. Therefore, H_1 or H_2 does not have a vertex from the set $\{u_i^j : 1 \leq i \leq n, 2 \leq j \leq k-1\}$. This implies that H_1 or H_2 are isometric subgraphs of G'. But both H_1 and H_2 have diameter 3, while $\text{diam}(G') \leq 2$. This is a contradiction. Hence, G is a $\frac{1}{2}$-hyperbolic graph. ◁

Claim 8.3. $\mathcal{F}$ has a hitting set of size at most k iff G has a multipacking of size at least k, for $k \geq 3$.

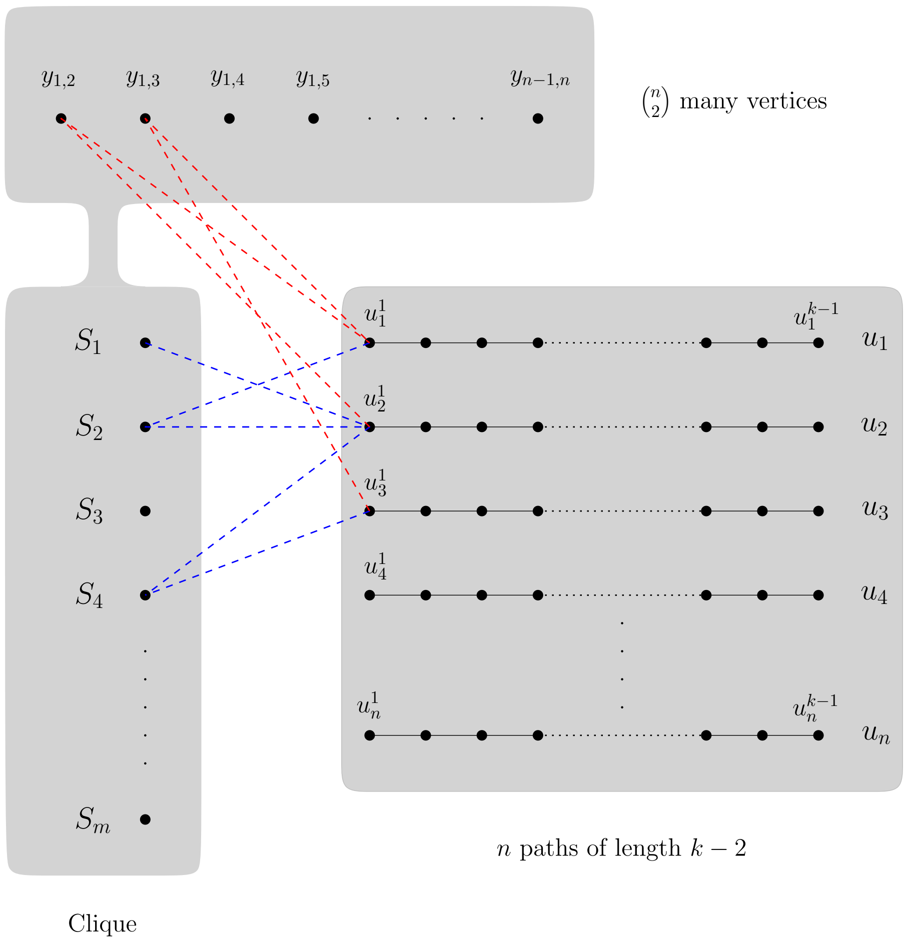

Figure 4 An illustration of the construction of the graph G used in the proof of Lemma 8, demonstrating the hardness of the MULTIPACKING problem for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs.

Proof. (If part) Suppose $\mathcal{F}$ has a hitting set $H = \{u_1, u_2, \ldots, u_k\}$. We want to show that $M = \{u_1^{k-1}, u_2^{k-1}, \ldots, u_k^{k-1}\}$ is a multipacking in G. If $r < k-1$, then $|N_r[v] \cap M| \leq 1 \leq r$, $\forall v \in V(G)$, since $d(u_i^{k-1}, u_j^{k-1}) \geq 2k-2$, $\forall i \neq j$. If $r > k-1$, then $|N_r[v] \cap M| \leq |M| = k \leq r$, $\forall v \in V(G)$. Suppose $r = k-1$. If M is not a multipacking, there exists some $v \in V(G)$ such that $|N_{k-1}[v] \cap M| = k$. Therefore, $M \subseteq N_{k-1}[v]$. Suppose $v = u_p^q$ for some p and q. In this case, $d(u_p^q, u_i^{k-1}) \geq k$, $\forall i \neq p$. Therefore, $v \notin \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq k-1\}$. This implies that $v \in \mathcal{F} \cup Y$. Suppose $v \in Y$. Let $v = y_{p,q}$ for some p and q. Then $d(y_{p,q}, u_i^{k-1}) \geq k$, $\forall i \in \{1, 2, \ldots, n\} \setminus \{p, q\}$. Therefore, $v \notin Y$. This implies that $v \in \mathcal{F}$. Let $v = S_t$ for some t. If there exist $i \in \{1, 2, \ldots, k\}$ such that S_t is not adjacent to u_i^1, then $d(S_t, u_i^{k-1}) \geq k$. Therefore, $M \subseteq N_{k-1}[S_t]$ implies that S_t is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\}$. Then $u_1, u_2, \ldots, u_k \notin S_t$. Therefore, H is not a hitting set of $\mathcal{F}$. This is a contradiction. Hence, M is a multipacking of size k.

(Only-if part). Suppose G has a multipacking M of size k. Let $H = \{u_i : u_i^j \in M, 1 \leq i \leq n, 1 \leq j \leq k-1\}$. Then $|H| \leq |M| = k$. Without loss of generality, let $H = \{u_1, u_2, \ldots, u_{k'}\}$ where $k' \leq k$. Now we show that H is a hitting set of $\mathcal{F}$. Suppose H is not a hitting set. Then there exists t such that $S_t \cap H = \emptyset$. Therefore, S_t is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_{k'}^1\}$. Then $M \subseteq N_{k-1}[S_t]$. This implies that $|N_{k-1}[S_t] \cap M| = k$. This is a contradiction. Therefore, H is a hitting set of size at most k. ◁

Since the above reduction is a polynomial-time, the Multipacking problem is NP-complete for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs. Moreover, the reduction is also an FPT reduction. Hence, the Multipacking problem is W[2]-hard for chordal $\cap$ $\frac{1}{2}$-hyperbolic graphs when parameterized by the solution size. ◀

4.2 Bipartite graphs

Here, we discuss the hardness of the Multipacking problem for bipartite graphs.

▶ **Lemma 9** (∗). *Multipacking problem is NP-complete for bipartite graphs. Moreover, Multipacking problem is W[2]-hard for bipartite graphs when parameterized by the solution size.*

4.3 Claw-free graphs

Next, we discuss the hardness of the Multipacking problem for claw-free graphs.

▶ **Lemma 10** (∗). *Multipacking problem is NP-complete for claw-free graphs. Moreover, Multipacking problem is W[2]-hard for claw-free graphs when parameterized by the solution size.*

4.4 Regular graphs

Next, we prove NP-completeness for regular graphs. We need the following result to prove our theorem.

From the Erdős-Gallai Theorem [13] and the Havel-Hakimi criterion [19, 20], we can say the following.

▶ **Lemma 11** ([13, 19, 20]). *Let n and d be nonnegative integers. If $d < n$ and nd is even, then there exists a simple d-regular graph on n vertices. Moreover, such a graph can be constructed in polynomial time.*

We use Lemma 11 to prove the NP-completeness of the Multipacking problem on regular graphs.

▶ **Lemma 12.** *Multipacking problem is NP-complete for regular graphs.*

Proof. It is known that the Total Dominating Set problem is NP-complete for cubic (3-regular) graphs [17]. We reduce the Total Dominating Set problem on cubic graphs to the Multipacking problem on regular graphs.

Total Dominating Set problem
- **Input:** An undirected graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.
- **Question:** Does there exist a *total dominating set* $S \subseteq V$ of size at most k; that is, a set of at most k vertices such that every vertex in V has at least one neighbor in S?

Let (G, k) be an instance of the Total Dominating Set problem, where G is a 3-regular graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ where $n \geq 6$. Therefore, n is even. Let $d = n - 4$. So, d is also even. Now we construct a graph G' in the following way (Fig. 5 gives an illustration).

(i) For every vertex $v_a \in V(G)$, we construct a graph H_a in G' as follows.
- Add the vertex sets $S_a^i = \{u_a^{i,j} : 1 \leq j \leq d\}$ for all $2 \leq i \leq k - 2$ in H_a. Each vertex of S_a^i is adjacent to each vertex of S_a^{i+1} for each i.
- Add a vertex u_a^1 in H_a where u_a^1 is adjacent to each vertex of S_a^2. Moreover, S_a^2 forms a clique.

- Add the vertex set $T_a = \{u_a^{k-1,j} : 1 \le j \le d^2\}$ in H_a where the induced subgraph $H_a[T_a]$ is a $(2d-1)$-regular graph. We can construct such a graph in polynomial-time by Lemma 11 since d is an even number and $2d-1 < d^2$ (since $n \ge 6$).
- Let $U_1, U_2, \ldots, U_d$ be a partition of T_a into d disjoint sets, each of size d, that is, $T_a = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_d$ and $|U_j| = d$ for each $1 \le j \le d$. Each vertex in U_j is adjacent to $u_a^{k-2,j}$ for all $1 \le j \le d$.

(ii) If v_i and v_j are different and not adjacent in G, we make u_i^1 and u_j^1 adjacent in G' for every $1 \le i, j \le n, i \ne j$.

Note that G' is a $2d$-regular graph.

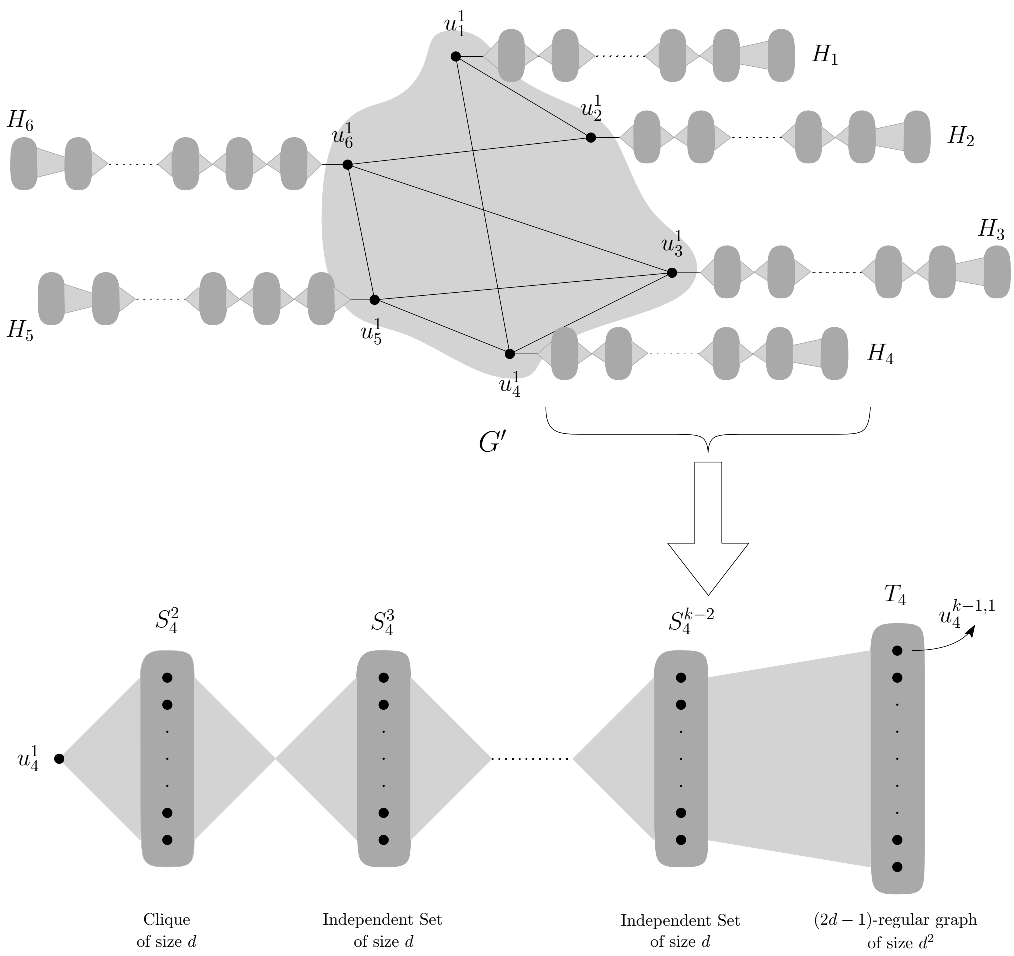

Figure 5 An illustration of the construction of the graph G' used in the proof of Lemma 12, demonstrating the hardness of the MULTIPACKING problem for regular graphs. The induced subgraph $G'[u_1^1, u_2^1, \ldots, u_6^1]$ is isomorphic to G^c.

Claim 12.1. *G has a total dominating set of size at most k iff G' has a multipacking of size at least k, for $k \ge 2$.*

Proof. (If part) Suppose G has a total dominating set $D = \{v_1, v_2, \ldots, v_k\}$. We want to show that $M = \{u_1^{k-1,1}, u_2^{k-1,1}, \ldots, u_k^{k-1,1}\}$ is a multipacking in G'. If $r < k-1$, then $|N_r[v] \cap M| \le 1 \le r$, $\forall v \in V(G')$, since $d(u_i^{k-1,1}, u_j^{k-1,1}) \ge 2k-3$, $\forall i \ne j$. If $r > k-1$, then

$|N_r[v] \cap M| \leq |M| = k \leq r$, $\forall v \in V(G')$. Suppose $r = k-1$. If M is not a multipacking, there exists some vertex $v \in V(G')$ such that $|N_{k-1}[v] \cap M| = k$. Therefore, $M \subseteq N_{k-1}[v]$. Suppose $v \in V(H_p) \setminus \{u_p^1\}$ for some p. In this case, $d(v, u_i^{k-1,1}) \geq k$, $\forall i \neq p$. Therefore, $v \notin V(H_p) \setminus \{u_p^1\}$ for any p. This implies that $v \in \{u_i^1 : 1 \leq i \leq n\}$. Let $v = u_t^1$ for some t. If there exist $i \in \{1, 2, \ldots, k\}$ such that neither $u_t^1 = u_i^1$ nor u_t^1 is adjacent to u_i^1, then $d(u_t^1, u_i^{k-1,1}) \geq k$. Therefore, $M \subseteq N_{k-1}[u_t^1]$ implies that either u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\}$ when $u_t^1 \notin \{u_1^1, u_2^1, \ldots, u_k^1\}$ or u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\} \setminus \{u_t^1\}$ when $u_t^1 \in \{u_1^1, u_2^1, \ldots, u_k^1\}$. Therefore, in the graph G, either v_t is not adjacent to any vertex of D when $v_t \notin D$ or v_t is not adjacent to any vertex of $D \setminus \{v_t\}$ when $v_t \in D$. In both cases, D is not a total dominating set of G. This is a contradiction. Hence, M is a multipacking of size k in G'.

(Only-if part). Suppose G' has a multipacking M of size k. Let $D = \{v_i : V(H_i) \cap M \neq \emptyset, 1 \leq i \leq n\}$. Then $|D| \leq |M| = k$. Without loss of generality, let $D = \{v_1, v_2, \ldots, v_{k'}\}$ where $k' \leq k$. Now we show that D is a total dominating set of G. Suppose D is not a total dominating set. Then there exists t such that either $v_t \notin D$ and no vertex in D is adjacent to v_t or $v_t \in D$ and no vertex in $D \setminus \{v_t\}$ is adjacent to v_t. Therefore, in G', either u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\}$ when $u_t^1 \notin \{u_1^1, u_2^1, \ldots, u_k^1\}$ or u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\} \setminus \{u_t^1\}$ when $u_t^1 \in \{u_1^1, u_2^1, \ldots, u_k^1\}$. Then $M \subseteq N_{k-1}[u_t^1]$. This implies that $|N_{k-1}[u_t^1] \cap M| = k$. This is a contradiction. Therefore, D is a total dominating set of size at most k in G. ◁

Hence, the Multipacking problem is NP-complete for regular graphs. ◀

4.5 Convex Intersection graphs

Here, we discuss the hardness of the Multipacking problem for a geometric intersection graph class: CONV graphs. A graph G is a *CONV graph* if and only if there exists a family of convex sets on a plane such that the graph has a vertex for each convex set and an edge for each intersecting pair of convex sets. We show that the Multipacking problem is NP-complete for CONV graphs. To prove this we will need the following result.

▶ **Theorem 13** ([23]). *The complement of a planar graph (also called co-planar graph) is a CONV graph.*

We use Theorem 13 to prove the NP-completeness of the Multipacking problem on CONV graphs.

▶ **Lemma 14** ($*$). *Multipacking problem is NP-complete for CONV graphs.*

5 Exact Exponential Algorithm (Proof of Theorem 3)

In this section, we present an exact exponential-time algorithm for computing a maximum multipacking in a graph. Using a naive approach, it achieves a running time of $O^*(1.58^n)$, where n is the number of vertices in the graph.

Let $\mathcal{M}(G)$ denote the set of all multipackings of a graph G. Suppose G is a connected graph and T is a spanning tree of G. Then $d_T(u,v) \geq d_G(u,v)$ for any $u, v \in V(G)$. Therefore, any multipacking of G is a multipacking of T. Hence $\mathcal{M}(G) \subseteq \mathcal{M}(T)$.

In our algorithm, we compute a family of sets of size $O(1.58^n)$ that contains all elements of $\mathcal{M}(T)$. A maximum multipacking is then obtained by searching within this family.

To describe the algorithm, we first define some notions related to rooted trees. A *rooted tree* is a tree in which one vertex is designated as the *root*, giving the tree a hierarchical structure. In a rooted tree, every vertex except the root has a unique *parent*, which is the neighboring vertex on the unique path from that vertex to the root. A vertex that lies directly below another in this hierarchy is called its *child*. If a vertex u is the parent of v, and v is the parent of w, then u is the *grandparent* of w. The maximum distance (or number of edges) from the root to any vertex in a rooted tree T is called the *height* of T, which is denoted by $\mathrm{height}(T)$.

Before describing the algorithm, we introduce some notation. For two families of sets $\mathcal{S}_1$ and $\mathcal{S}_2$, we define $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{M_1 \cup M_2 : M_1 \in \mathcal{S}_1, M_2 \in \mathcal{S}_2\}$. For a vertex u and a family of sets $\mathcal{S}$, we define $u \oplus \mathcal{S} := \{\{u\}\} \oplus \mathcal{S}$. If H is a subgraph of a graph G, we write $G \setminus H := G[V(G) \setminus V(H)]$, that is, the subgraph of G induced by the vertex set $V(G) \setminus V(H)$. We say that two rooted trees T_u and T_v, rooted at u and v respectively, are *isomorphic*, and we denote this by $T_u \cong T_v$, if there exists a bijection $f : V(T_u) \to V(T_v)$ such that $xy \in E(T_u)$ if and only if $f(x)f(y) \in E(T_v)$, and moreover $f(u) = v$.

First, we present a simple exact exponential-time algorithm for computing a maximum multipacking that runs in time $O^*(1.62^n)$. Subsequently, we describe an improved exponential-time algorithm with a better running time.

▶ **Proposition 15.** *There is an exact exponential-time algorithm for computing a maximum multipacking in a graph with n vertices, which has running time $O^*(1.62^n)$.*

Proof. Let G be a connected graph with n vertices and T be a BFS (Breadth-First Search) tree of G rooted at v. Note that every multipacking in G is a multipacking in T. For $n = 1$, computing a maximum multipacking is trivial. Assume $n \geq 2$. Let w be a farthest vertex from v and y be the parent of w. Let T_y be the subtree of T rooted at y. If a multipacking M_1 of T contains w, then no other vertices of T_y belong to M_1; otherwise two vertices of M_1 would belong to $N_1[y]$. Therefore, $M_1 \in w \oplus \mathcal{M}(T \setminus T_y)$. If a multipacking M_2 of T does not contain w, then $M_2 \in \mathcal{M}(T \setminus w)$ where $T \setminus w$ is the subtree of T formed by deleting w from T. Therefore, $\mathcal{M}(T) \subseteq (w \oplus \mathcal{M}(T \setminus T_y)) \cup \mathcal{M}(T \setminus w)$. This implies the recursive relation $|\mathcal{M}(T)| \leq |\mathcal{M}(T \setminus T_y)| + |\mathcal{M}(T \setminus w)|$, since $|w \oplus \mathcal{M}(T \setminus T_y)| = |\mathcal{M}(T \setminus T_y)|$. Let $f(n)$ be the maximum number of multipackings in a tree on n vertices. So, the recursive relation yields that $f(n) \leq f(n-2) + f(n-1)$, which is the Fibonacci recurrence. Solving this recurrence yields $f(n) = O(1.62^n)$. Therefore, we can compute a family of subsets of $V(G)$ having size $O(1.62^n)$ that contains all the multipackings of G. By searching within this family, we can find a maximum multipacking of G in $O^*(1.62^n)$ time. ◀

Now we refine the above branching idea to obtain a better running time. Let $H_1(k)$ denote the rooted tree with vertex set $\{x, u_1, u_2, \ldots, u_k\}$, rooted at x, where x is adjacent to each u_i for $1 \leq i \leq k$; that is, $H_1(k)$ is a complete bipartite graph with partite sets $\{x\}$ and $\{u_1, u_2, \ldots, u_k\}$. Let $H_2(k_1, k_2)$ denote the rooted tree rooted at y and vertex set $\{y, a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_1}, c_1, c_2, \ldots, c_{k_2}\}$, whose edge set is $\{a_ib_i : 1 \leq i \leq k_1\} \cup \{a_iy : 1 \leq i \leq k_1\} \cup \{c_iy : 1 \leq i \leq k_2\}$ (see Fig. 6).

▶ **Lemma 16.**

1) *If $|V(H_1(k))| = m$, then $|\mathcal{M}(H_1(k))| = m + 1$, and the family $\mathcal{M}(H_1(k))$ can be computed in time $O(|\mathcal{M}(H_1(k))|)$.*

2) *If $|V(H_2(k_1, k_2))| = m$ and $m \geq 4$, then $|\mathcal{M}(H_2(k_1, k_2))| \leq \frac{3}{8}m^2 - \frac{1}{2}m + \frac{17}{8}$, and the family $\mathcal{M}(H_2(k_1, k_2))$ can be computed in time $O(|\mathcal{M}(H_2(k_1, k_2))|)$.*

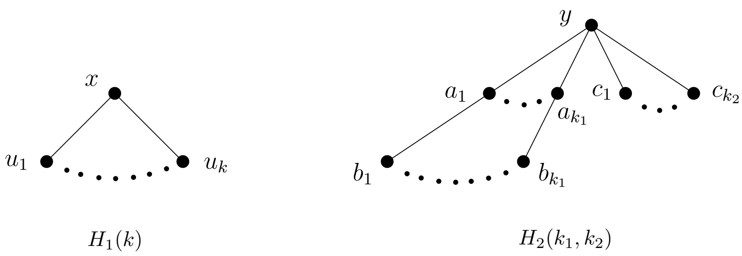

Figure 6 Rooted trees $H_1(k)$ and $H_2(k_1, k_2)$.

Proof.

1) The radius of $H_1(k)$ is at most 1, which implies $0 \leq \mathrm{mp}(H_1(k)) \leq 1$. Therefore, $\mathcal{M}(H_1(k)) = \{\{u\} : u \in V(H_1(k))\} \cup \{\emptyset\}$. Hence $|\mathcal{M}(H_1(k))| = m + 1$, and the family $\mathcal{M}(H_1(k))$ can be computed in time $O(|\mathcal{M}(H_1(k))|)$.

2) Let $A = \{a_1, a_2, \dots, a_{k_1}\}$, $B = \{b_1, b_2, \dots, b_{k_1}\}$, and $C = \{c_1, c_2, \dots, c_{k_2}\}$. Therefore, $V(H_2(k_1, k_2)) = \{y\} \cup A \cup B \cup C$. Since $H_2(k_1, k_2)$ has radius at most 2, it has multipackings of size at most 2. Let M_0, M_1, M_2 be the sets of multipackings of size $0, 1, 2$ respectively. Therefore, $\mathcal{M}(H_2(k_1, k_2)) = M_0 \cup M_1 \cup M_2$. Clearly, $M_0 = \{\emptyset\}$ and $M_1 = \{\{u\} : u \in V(H_2(k_1, k_2))\}$. Therefore, $|M_0| = 1$ and $|M_1| = m$. Note that $M_2 = \{\{b, c\} : b \in B, c \in C\} \cup \{\{b_i, b_j\} : b_i, b_j \in B, i \neq j\} \cup \{\{a_i, b_j\} : a_i \in A, b_j \in B, i \neq j\}$. Therefore, $|M_2| = k_1k_2 + \binom{k_1}{2} + k_1(k_1 - 1)$. Since $m = 2k_1 + k_2 + 1$,

$$\begin{aligned}
|M_2| &= k_1k_2 + \binom{k_1}{2} + k_1(k_1 - 1) \\
&= k_1(m - 2k_1 - 1) + \binom{k_1}{2} + k_1(k_1 - 1) \\
&= \frac{2m - 5}{2}\, k_1 - \frac{1}{2}\, k_1^2.
\end{aligned}$$

Let $f(k_1) = \frac{2m-5}{2}\, k_1 - \frac{1}{2}\, k_1^2$. Since $m = 2k_1 + k_2 + 1$, we have $0 \leq k_1 \leq \frac{m-1}{2}$. Note that $\frac{df(k_1)}{dk_1} = \frac{2m-5}{2} - k_1 \geq 0$, for $m \geq 4$ and $0 \leq k_1 \leq \frac{m-1}{2}$. Hence $f(k_1)$ is a non-decreasing function when $0 \leq k_1 \leq \frac{m-1}{2}$ and $m \geq 4$. Therefore, $f(k_1) \leq f(\frac{m-1}{2}) = \frac{2m-5}{2} \cdot \frac{m-1}{2} - \frac{1}{2}\,(\frac{m-1}{2})^2$. Hence

$$\begin{aligned}
|\mathcal{M}(H_2(k_1, k_2))| &= |M_0| + |M_1| + |M_2| \\
&= 1 + m + f(k_1) \\
&\leq 1 + m + \frac{2m - 5}{2} \cdot \frac{m - 1}{2} - \frac{1}{2}\left(\frac{m - 1}{2}\right)^2 \\
&= \frac{3}{8}m^2 - \frac{1}{2}m + \frac{17}{8}.
\end{aligned}$$

From the constructions of the sets M_0, M_1 and M_2, the family $\mathcal{M}(H_2(k_1, k_2))$ can be computed in time $O(|\mathcal{M}(H_2(k_1, k_2))|)$. ◀

▶ **Corollary 17.** *If $|H_2(k_1, k_2)| = m$ and $m \geq 4$, then $|\mathcal{M}(H_2(k_1, k_2))| \leq 1.58^m$ and the family $\mathcal{M}(H_2(k_1, k_2))$ can be computed in time $O(1.58^m)$.*

Proof. From Lemma 16, we have $|\mathcal{M}(H_2(k_1, k_2))| \leq \frac{3}{8}m^2 - \frac{1}{2}m + \frac{17}{8}$. The maximum value of $(\frac{3}{8}m^2 - \frac{1}{2}m + \frac{17}{8})^{1/m}$ for $m \geq 4$ is 1.5732... attained at $m = 4$. Hence, the corollary holds. ◀

Algorithm 1 $\mathcal{M}_v(T)$: A set that contains all multipackings of a tree T with root v.

Input: A tree T with root v.
Output: A set that contains all multipackings of T.
if height$(T) \geq 2$ **then**
  Compute the set of farthest vertices from v in T, denote it by S_v.
  For each $w \in S_v$, let w_1 and w_2 denote the parent and the grandparent of w in T, respectively.
  if there exists $w \in S_v$ such that $T_{w_1} \cong H_1(k)$ for some $k \geq 2$ (See Fig. 6) **then**
    return $(w \oplus \mathcal{M}_v(T \setminus T_{w_1})) \cup \mathcal{M}_v(T \setminus w)$
  end if
  if there exists $w \in S_v$ such that $T_{w_2} \cong H_2(1,0)$ (See Fig. 6) **then**
    return $(w \oplus \mathcal{M}_v(T \setminus T_{w_2})) \cup \mathcal{M}_v(T \setminus w)$
  end if
  return $\mathcal{M}_v(T \setminus T_{w_2}) \oplus \mathcal{M}(T_{w_2})$
end if
if height$(T) \leq 1$ **then**
  return $\mathcal{M}(T)$
end if

▶ **Lemma 18** (∗). *Let T be a rooted tree on n vertices with root v. The Algorithm 1 computes a family of sets $\mathcal{M}_v(T)$ of size $O(1.58^n)$ that contains all multipackings of T, and it has running time $O^*(1.58^n)$.*

▶ **Theorem 3.** *There exists an exact algorithm for computing a maximum multipacking in an n-vertex graph with running time $O^*(1.58^n)$.*

Proof. Let G be a connected graph and T be its spanning tree. Let v be a vertex of G. Consider T as a rooted tree with root v. Since $d_T(u,v) \geq d_G(u,v)$ for any $u,v \in V(G)$, $\mathcal{M}(G) \subseteq \mathcal{M}(T)$. From Lemma 18, $\mathcal{M}_v(T)$ can be computed in time $O^*(1.58^n)$ and $|\mathcal{M}_v(T)| = O(1.58^n)$. Now $\mathcal{M}(G) \subseteq \mathcal{M}_v(T)$, since $\mathcal{M}(T) \subseteq \mathcal{M}_v(T)$. For any set $M \in \mathcal{M}_v(T)$, we can check whether M is a multipacking of G in $O(n^2)$ time. Therefore, we can search a maximum multipacking of G in the family of sets $\mathcal{M}_v(T)$ in time $O^*(1.58^n)$. If G is not a connected graph, then a maximum multipacking can be computed independently for each connected component using the same approach. ◀

The exact exponential-time algorithm computes a maximum multipacking in an n-vertex graph in $O^*(1.58^n)$ time using a naive approach. We emphasize that this algorithm is not based on, and is not similar to, the known polynomial-time algorithm for trees [3]; here the spanning tree is used only to generate a superset of the set of possible solutions, since every multipacking of G is also a multipacking of any spanning tree of G. The algorithm can be refined by exploiting structural properties of the rooted spanning tree. In particular, the current version considers only subtrees of height at most 2; extending it to handle subtrees of greater height so that the possible multipackings within them can be computed using their structure could further improve the running time for finding a maximum multipacking of the whole graph. Note that in our algorithm the root of the spanning tree is fixed at each step. Another possible improvement is to update the root of the spanning tree at each step.

▶ **Lemma 19.** *If P_n is a path on n vertices, then the number of multipackings of P_n is $O(1.47^n)$.*

Proof. Let $P_n = \{v_1, v_2, \ldots, v_n\}$ and P_i be the induced subgraph of P_n with vertex set $\{v_1, v_2, \ldots, v_i\}$, where $i \leq n$. Suppose M is a multipacking of P_n. If $v_n \in M$, then $v_{n-1}, v_{n-2} \notin M$ since $|N_1[v_{n-1}] \cap M| \leq 1$. From this observation, we can say that the number of multipackings of P_n that include v_n is $|\mathcal{M}(P_{n-3})|$ and the number of multipackings of P_n that do not include v_n is $|\mathcal{M}(P_{n-1})|$. Therefore, we have the recursive relation $|\mathcal{M}(P_n)| = |\mathcal{M}(P_{n-1})| + |\mathcal{M}(P_{n-3})|$ with initial conditions $|\mathcal{M}(P_1)| = 2$, $|\mathcal{M}(P_2)| = 3$, and $|\mathcal{M}(P_3)| = 4$. Solving this recurrence yields $|\mathcal{M}(P_n)| \leq 2 \cdot (1.46557...)^{n-1}$. ◀

Lemma 19 yields the following.

▶ **Theorem 20.** *Let G be an n-vertex graph, and suppose that a Hamiltonian path[6] P of G is given as part of the input. Then a maximum multipacking of G can be computed in time $O^*(1.47^n)$.*

▶ **Proposition 21.** *If P_n is a path on n vertices, then the number of maximal multipackings of P_n is $O(1.33^n)$.*

Proof. Let $P_n = \{v_1, v_2, \ldots, v_n\}$ and P_i be the induced subgraph of P_n with vertex set $\{v_1, v_2, \ldots, v_i\}$, where $i \leq n$. Let $\mathcal{M}'(P_i)$ denote the set of all maximal multipackings in P_i. Suppose $M \in \mathcal{M}'(P_n)$. Then only one among v_n, v_{n-1}, v_{n-2} is in M, since $|N_1[v_{n-1}] \cap M| \leq 1$. The number of maximal multipackings of P_n that include v_n is $|\mathcal{M}'(P_{n-3})|$, since these multipackings cannot contain v_{n-1} or v_{n-2}. The number of maximal multipackings of P_n that include v_{n-1} is $|\mathcal{M}'(P_{n-4})|$, since these multipackings cannot contain v_n, v_{n-2} or v_{n-3}. Similarly, the number of maximal multipackings of P_n that include v_{n-2} is $|\mathcal{M}'(P_{n-5})|$, since these multipackings cannot contain v_n, v_{n-1}, v_{n-3} or v_{n-4}. Therefore, we have the recursive relation $|\mathcal{M}'(P_n)| = |\mathcal{M}'(P_{n-3})| + |\mathcal{M}'(P_{n-4})| + |\mathcal{M}'(P_{n-5})|$ with initial conditions $|\mathcal{M}'(P_1)| = 1$, $|\mathcal{M}'(P_2)| = 2$, and $|\mathcal{M}'(P_3)| = 3$. Solving this recurrence yields $|\mathcal{M}'(P_n)| \leq (1.3247...)^n$. ◀

6 Conclusion

We conclude with some remarks and open problems concerning the Multipacking problem.

It is known that planar graphs are SEG graphs[7] [6]. In 1998, Kratochvíl and Kuběna [23] posed the question of whether the complement of any planar graph (i.e., a co-planar graph) is also a SEG graph. As far as we know, this is still an open question. A positive answer to this question would imply that the Multipacking problem is NP-complete for SEG graphs, since in that case the induced subgraph $G'[u_1^1, u_2^1, \ldots, u_n^1]$ in the proof of Lemma 14 would be a SEG graph, which in turn would imply that G' itself is a SEG graph. However, the complexity of the problem on planar graphs still remains unresolved.

Our results in this paper naturally lead to several open questions that merit further investigation:

1. What is the complexity status of the problem on planar graphs? Does an FPT algorithm for planar graphs exist when parameterized by solution size?
2. Does there exist an FPT algorithm for general graphs when parameterized by treewidth?

[6] A path P in a graph G is called a *Hamiltonian path* if it contains every vertex of G exactly once.
[7] A graph G is a *SEG graph* (or *Segment Graph*) if and only if there exists a family of straight-line segments in a plane such that the graph has a vertex for each straight-line segment and an edge for each intersecting pair of straight-line segments.

3. Can one design a subexponential-time algorithm (that is, an algorithm running in time $2^{o(n)}$) for general graphs?
4. While a $(2 + o(1))$-factor approximation algorithm is known [2], can the approximation factor be improved? Moreover, does the problem admit a PTAS?

Addressing these questions would provide a more complete understanding of the computational landscape of the Multipacking problem and could lead to important algorithmic developments.

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A Appendix

A.1 Proof of Lemma 9

Proof. We reduce the Hitting Set problem to the Multipacking problem to show that the Multipacking problem is NP-complete for bipartite graphs.

Let $U = \{u_1, u_2, \dots, u_n\}$ and $\mathcal{F} = \{S_1, S_2, \dots, S_m\}$ where $S_i \subseteq U$ for each i. We construct a graph G in the following way: (i) Include each element of $\mathcal{F}$ in the vertex set of G. (ii) Include a vertex C in G where all the vertices of $\mathcal{F}$ are adjacent to the vertex C. (iii) Include $k-2$ length path $u_i = u_i^1 u_i^2 \dots u_i^{k-1}$ in G, for each element $u_i \in U$. (iv) Join u_i^1 and S_j by an edge iff $u_i \notin S_j$. Fig. 7 gives an illustration.

Note that G is a bipartite graph where the partite sets are $B_1 = \mathcal{F} \cup \{u_i^j : 1 \le i \le n \text{ and } j \text{ is even}\}$ and $B_2 = \{C\} \cup \{u_i^j : 1 \le i \le n \text{ and } j \text{ is odd}\}$.

Claim 9.1. $\mathcal{F}$ has a hitting set of size at most k iff G has a multipacking of size at least k, for $k \ge 2$.

Proof. (If part) Suppose $\mathcal{F}$ has a hitting set $H = \{u_1, u_2, \dots, u_k\}$. We want to show that $M = \{u_1^{k-1}, u_2^{k-1}, \dots, u_k^{k-1}\}$ is a multipacking in G. If $r < k-1$, then $|N_r[v] \cap M| \le 1 \le r$, $\forall v \in V(G)$, since $d(u_i^{k-1}, u_j^{k-1}) \ge 2k-2$, $\forall i \neq j$. If $r > k-1$, then $|N_r[v] \cap M| \le |M| = k \le r$, $\forall v \in V(G)$. Suppose $r = k-1$. If M is not a multipacking, there exists some $v \in V(G)$ such that $|N_{k-1}[v] \cap M| = k$. Therefore, $M \subseteq N_{k-1}[v]$. Suppose $v = u_p^q$ for some p and q. In this case, $d(u_p^q, u_i^{k-1}) \ge k$, $\forall i \neq p$. Therefore, $v \notin \{u_i^j : 1 \le i \le n, 1 \le j \le k-1\}$. Moreover, $v \neq C$, since $d(C, u_i^{k-1}) \ge k$, $\forall 1 \le i \le n$. This implies that $v \in \{S_1, S_2, \dots, S_m\}$. Let $v = S_t$ for some t. If there exist $i \in \{1, 2, \dots, k\}$ such that S_t is not adjacent to u_i^1, then

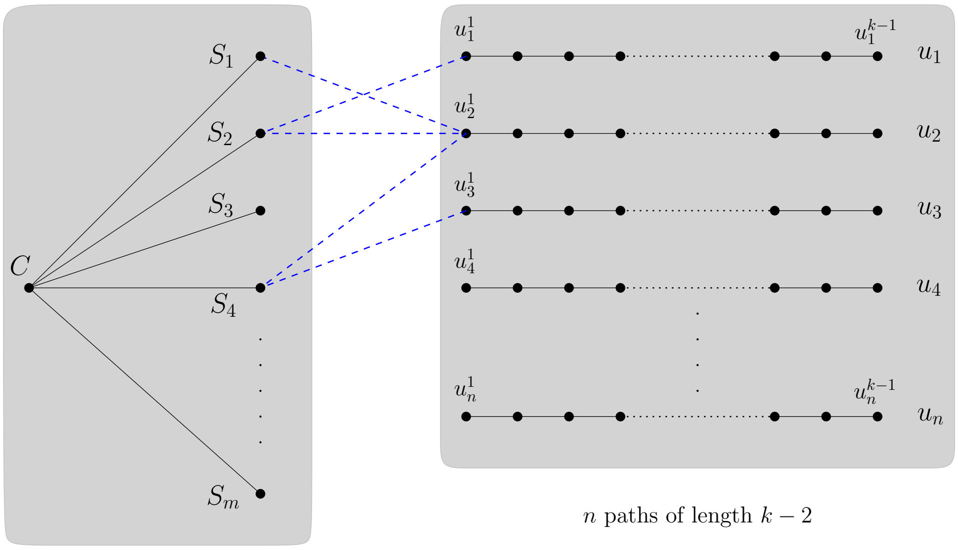

Figure 7 An illustration of the construction of the graph G used in the proof of Lemma 9, demonstrating the hardness of the MULTIPACKING problem for bipartite graphs.

$d(S_t, u_i^{k-1}) \geq k$. Therefore, $M \subseteq N_{k-1}[S_t]$ implies that S_t is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\}$. Then $u_1, u_2, \ldots, u_k \notin S_t$. Therefore, H is not a hitting set of $\mathcal{F}$. This is a contradiction. Hence, M is a multipacking of size k.

(Only-if part). Suppose G has a multipacking M of size k. Let $H = \{u_i : u_i^j \in M, 1 \leq i \leq n, 1 \leq j \leq k-1\}$. Then $|H| \leq |M| = k$. Without loss of generality, let $H = \{u_1, u_2, \ldots, u_{k'}\}$ where $k' \leq k$. Now we show that H is a hitting set of $\mathcal{F}$. Suppose H is not a hitting set. Then there exists t such that $S_t \cap H = \emptyset$. Therefore, S_t is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_{k'}^1\}$. Then $M \subseteq N_{k-1}[S_t]$. This implies that $|N_{k-1}[S_t] \cap M| = k$. This is a contradiction. Therefore, H is a hitting set of size at most k. ◁

Since the above reduction is a polynomial-time, the MULTIPACKING problem is NP-complete for bipartite graphs. Moreover, the reduction is also an FPT reduction. Hence, the MULTIPACKING problem is W[2]-hard for bipartite graphs when parameterized by the solution size. ◀

A.2 Proof of Lemma 10

Proof. We reduce the HITTING SET problem to the MULTIPACKING problem to show that the MULTIPACKING problem is NP-complete for claw-free graphs.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$ where $S_i \subseteq U$ for each i. We construct a graph G in the following way: (i) Include each element of $\mathcal{F}$ in the vertex set of G and $\mathcal{F}$ forms a clique in G. (ii) Include $k-3$ length path $u_i = u_i^1 u_i^2 \ldots u_i^{k-2}$ in G, for each element $u_i \in U$. (iii) Include the set of vertices $W = \{w_{j,i} : u_i \notin S_j, 1 \leq i \leq n, 1 \leq j \leq m\}$ in G. Join S_j and u_i^1 by a 2-length path $S_j w_{j,i} u_i^1$. (iv) Join $w_{j,i}$ and $w_{q,p}$ by an edge iff either $j = q$ or $i = p$. Fig. 8 gives an illustration.

Claim 10.1. G is a claw-free graph.

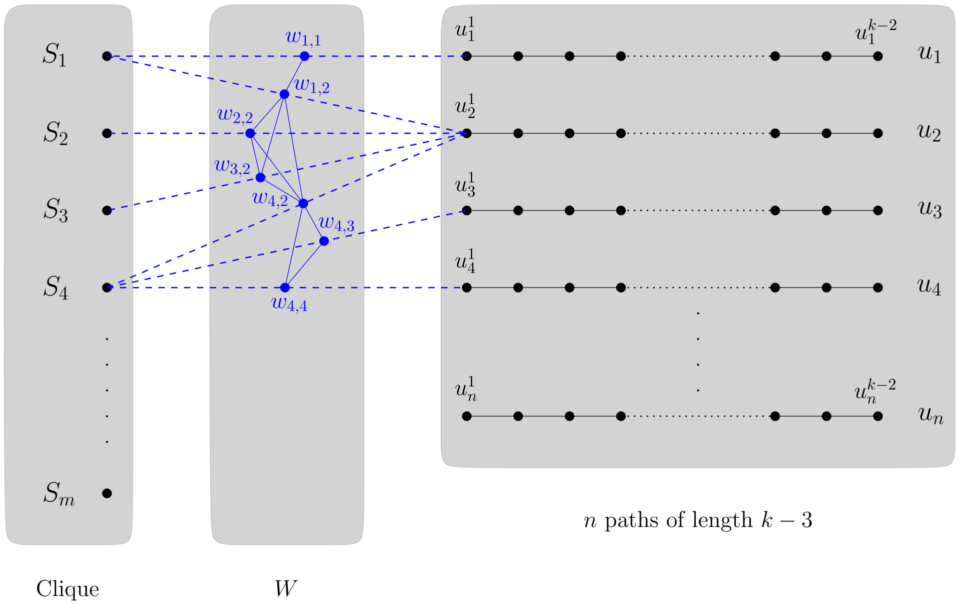

Figure 8 An illustration of the construction of the graph G used in the proof of Lemma 10, demonstrating the hardness of the MULTIPACKING problem for claw-free graphs.

Proof. Suppose G contains an induced claw C whose vertex set is $V(C) = \{c, x, y, z\}$ and the claw-center is c. Note that $c \notin \{u_i^j : 1 \leq i \leq n, 2 \leq j \leq k-2\}$, since each element of this set has degree at most 2.

Suppose $c \in \{u_i^1 : 1 \leq i \leq n\}$. Let $c = u_t^1$ for some t. Therefore, $\{x, y, z\} \subseteq N(c) = \{u_t^2\} \cup \{w_{j,t} : u_t \notin S_j, 1 \leq j \leq m\}$. Then at least 2 vertices of the set $\{x, y, z\}$ belong to the set $\{w_{j,t} : u_t \notin S_j, 1 \leq j \leq m\}$. Note that $\{w_{j,t} : u_t \notin S_j, 1 \leq j \leq m\}$ forms a clique in G. This implies that at least 2 vertices of the set $\{x, y, z\}$ are adjacent. This is a contradiction, since C is an induced claw in G. Therefore, $c \notin \{u_i^1 : 1 \leq i \leq n\}$.

Suppose $c \in \mathcal{F}$. Let $c = S_t$ for some t. Therefore, $\{x, y, z\} \subseteq N(c) = \{S_j : j \neq t, 1 \leq j \leq m\} \cup \{w_{t,i} : u_i \notin S_t, 1 \leq i \leq n\}$. Note that both sets $\{S_j : j \neq t, 1 \leq j \leq m\}$ and $\{w_{t,i} : u_i \notin S_t, 1 \leq i \leq n\}$ form cliques in G. This implies that at least 2 vertices of the set $\{x, y, z\}$ are adjacent. This is a contradiction, since C is an induced claw in G. Therefore, $c \notin \mathcal{F}$.

Suppose $c \in W$. Let $c = w_{q,p}$ for some q and p. Let $A_q = \{S_q\} \cup \{w_{q,i} : u_i \notin S_q, i \neq p, 1 \leq i \leq n\}$ and $B_p = \{u_p^1\} \cup \{w_{j,p} : u_p \notin S_j, j \neq q, 1 \leq j \leq m\}$. Note that both sets A_q and B_p form cliques in G and $\{x, y, z\} \subseteq N(c) = A_q \cup B_p$. This implies that at least 2 vertices of the set $\{x, y, z\}$ are adjacent. This is a contradiction, since C is an induced claw in G. Therefore, $c \notin W$.

Therefore, G is a claw-free graph. $\lhd$

Claim 10.2. $\mathcal{F}$ has a hitting set of size at most k iff G has a multipacking of size at least k, for $k \geq 2$.

Proof. **(If part)** Suppose $\mathcal{F}$ has a hitting set $H = \{u_1, u_2, \ldots, u_k\}$. We want to show that $M = \{u_1^{k-2}, u_2^{k-2}, \ldots, u_k^{k-2}\}$ is a multipacking in G. If $r < k-1$, then $|N_r[v] \cap M| \leq 1 \leq r$, $\forall v \in V(G)$, since $d(u_i^{k-2}, u_j^{k-2}) \geq 2k-3$, $\forall i \neq j$. If $r > k-1$, then $|N_r[v] \cap M| \leq |M| = k \leq r$, $\forall v \in V(G)$. Suppose $r = k-1$. If M is not a multipacking, there exists some $v \in V(G)$ such that $|N_{k-1}[v] \cap M| = k$. Therefore, $M \subseteq N_{k-1}[v]$.

Case 1: $v \in \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq k-2\}$**.** Let $v = u_p^q$ for some p and q. In this case, $d(u_p^q, u_i^{k-2}) \geq k$, $\forall i \neq p$. Therefore, $v \notin \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq k-2\}$.

Case 2: $v \in W$**.** Let $v = w_{p,q}$ for some p and q. Therefore, $u_q \notin S_p$. Since $M \subseteq N_{k-1}[w_{p,q}]$, we have $d(w_{p,q}, u_i^{k-2}) \leq k-1$, $\forall 1 \leq i \leq k$. This implies that $d(w_{p,q}, u_i^1) \leq 2$, $\forall 1 \leq i \leq k$. Note that $d(w_{p,q}, u_i^1) \neq 1$, $\forall i \neq q, 1 \leq i \leq k$. Therefore, $d(w_{p,q}, u_i^1) = 2$, $\forall i \neq q, 1 \leq i \leq k$. Suppose $t \neq q$ and $1 \leq t \leq k$. Then $d(w_{p,q}, u_t^1) = 2$. Therefore, $w_{p,q}$ is adjacent to $w_{h,t}$, for some h. This is only possible when $h = p$, since $t \neq q$. Therefore, u_t^1 and S_p are joined by a 2-length path $u_t^1 - w_{p,t} - S_p$. This implies that $u_t \notin S_p$. Hence, $u_1, u_2, \ldots, u_k \notin S_p$. Therefore, H is not a hitting set of $\mathcal{F}$. This is a contradiction. Therefore, $v \notin W$.

Case 3: $v \in \mathcal{F}$**.** Let $v = S_t$ for some t. If there exist $i \in \{1, 2, \ldots, k\}$ such that $u_i \in S_t$, then $d(S_t, u_i^1) \geq 3$. Therefore, $d(S_t, u_i^{k-2}) \geq k$. This implies that $u_i^{k-2} \notin N_{k-1}[S_t]$. Therefore, $M \nsubseteq N_{k-1}[S_t]$. Hence, $M \subseteq N_{k-1}[S_t]$ implies that $u_1, u_2, \ldots, u_k \notin S_t$. Then H is not a hitting set of $\mathcal{F}$. This is a contradiction. Therefore, $v \notin \mathcal{F}$.

Hence, M is a multipacking of size k.

(Only-if part). Suppose G has a multipacking M of size k. Let $H = \{u_i : u_i^j \in M, 1 \leq i \leq n, 1 \leq j \leq k-1\}$. Then $|H| \leq |M| = k$. Without loss of generality, let $H = \{u_1, u_2, \ldots, u_{k'}\}$ where $k' \leq k$. We want to show that H is a hitting set of $\mathcal{F}$. Suppose H is not a hitting set. Then there exists t such that $S_t \cap H = \emptyset$. Therefore, S_t and u_i^1 are joined by a 2-length path $S_t - w_{t,i} - u_i^1$, for each $i \in \{1, 2, \ldots, k'\}$. So, $M \subseteq N_{k-1}[S_t]$. This implies that $|N_{k-1}[S_t] \cap M| = k$. This is a contradiction. Therefore, H is a hitting set of size at most k. ◁

Since the above reduction is a polynomial-time, the MULTIPACKING problem is NP-complete for claw-free graphs. Moreover, the reduction is also an FPT reduction. Hence, the MULTIPACKING problem is W[2]-hard for claw-free graphs when parameterized by the solution size. ◀

A.3 Proof of Lemma 14

Proof. It is known that the TOTAL DOMINATING SET problem is NP-complete on planar graphs [17]. We reduce the TOTAL DOMINATING SET problem on planar graphs to the MULTIPACKING problem on CONV graphs to show that the MULTIPACKING problem is NP-complete on CONV graphs.

Let (G, k) be an instance of the TOTAL DOMINATING SET problem, where G is a planar graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. We construct a graph G' in the following way (Fig. 9 gives an illustration).

(i) For every vertex $v_i \in V(G)$, we introduce a path $u_i = u_i^1 u_i^2 \ldots u_i^{k-1}$ of length $k-2$ in G'.

(ii) If v_i and v_j are different and not adjacent in G, we make u_i^1 and u_j^1 adjacent in G' for every $1 \leq i, j \leq n, i \neq j$.

Claim 14.1. G' is a CONV graph.

Proof. Note that the induced subgraph $G'[u_1^1, u_2^1, \ldots, u_n^1]$ is isomorphic to G^c. Since G is a planar graph, $G'[u_1^1, u_2^1, \ldots, u_n^1]$ is a CONV graph by Lemma 13. Hence, G' is a CONV graph. ◁

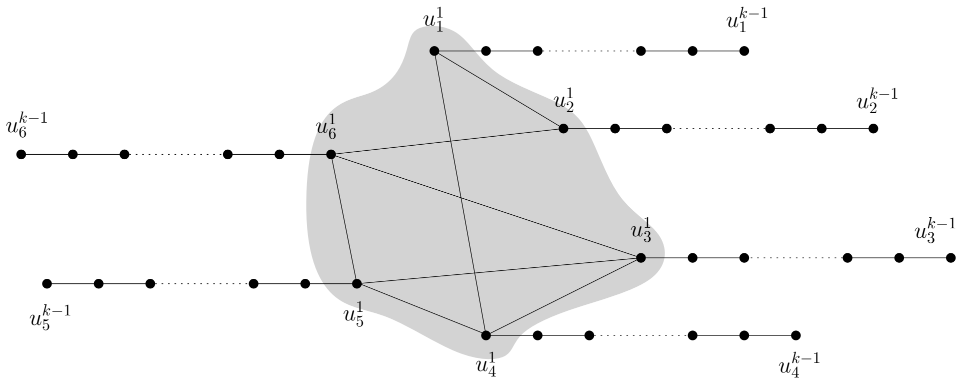

Figure 9 An illustration of the construction of the graph G' used in the proof of Lemma 14, demonstrating the hardness of the MULTIPACKING problem for CONV graphs. The induced subgraph $G'[u_1^1, u_2^1, \ldots, u_6^1]$ (the colored region) is isomorphic to G^c.

Claim 14.2. G has a total dominating set of size at most k iff G' has a multipacking of size at least k, for $k \geq 2$.

Proof. (If part) Suppose G has a total dominating set $D = \{v_1, v_2, \ldots, v_k\}$. We want to show that $M = \{u_1^{k-1}, u_2^{k-1}, \ldots, u_k^{k-1}\}$ is a multipacking in G'. If $r < k-1$, then $|N_r[v] \cap M| \leq 1 \leq r$, $\forall v \in V(G')$, since $d(u_i^{k-1}, u_j^{k-1}) \geq 2k-3$, $\forall i \neq j$. If $r > k-1$, then $|N_r[v] \cap M| \leq |M| = k \leq r$, $\forall v \in V(G')$. Suppose $r = k-1$. If M is not a multipacking, there exists some $v \in V(G')$ such that $|N_{k-1}[v] \cap M| = k$. Therefore, $M \subseteq N_{k-1}[v]$. Suppose $v = u_p^q$ for some p and q where $1 \leq p \leq n, 2 \leq q \leq k-1$. In this case, $d(u_p^q, u_i^{k-1}) \geq k$, $\forall i \neq p$. Therefore, $v \notin \{u_i^j : 1 \leq i \leq n, 2 \leq j \leq k-1\}$. This implies that $v \in \{u_i^1 : 1 \leq i \leq n\}$. Let $v = u_t^1$ for some t. If there exist $i \in \{1, 2, \ldots, k\}$ such that neither $u_t^1 = u_i^1$ nor u_t^1 is adjacent to u_i^1, then $d(u_t^1, u_i^{k-1}) \geq k$. Therefore, $M \subseteq N_{k-1}[u_t^1]$ implies that either u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\}$ when $u_t^1 \notin \{u_1^1, u_2^1, \ldots, u_k^1\}$ or u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\} \setminus \{u_t^1\}$ when $u_t^1 \in \{u_1^1, u_2^1, \ldots, u_k^1\}$. Therefore, in the graph G, either v_t is not adjacent to any vertex of D when $v_t \notin D$ or v_t is not adjacent to any vertex of $D \setminus \{v_t\}$ when $v_t \in D$. In both cases, D is not a total dominating set of G. This is a contradiction. Hence, M is a multipacking of size k in G'.

(Only-if part). Suppose G' has a multipacking M of size k. Let $D = \{v_i : u_i^j \in M, 1 \leq i \leq n, 1 \leq j \leq k-1\}$. Then $|D| \leq |M| = k$. Without loss of generality, let $D = \{v_1, v_2, \ldots, v_{k'}\}$ where $k' \leq k$. Now we show that D is a total dominating set of G. Suppose D is not a total dominating set. Then there exists t such that either $v_t \notin D$ and no vertex in D is adjacent to v_t or $v_t \in D$ and no vertex in $D \setminus \{v_t\}$ is adjacent to v_t. Therefore, in G', either u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\}$ when $u_t^1 \notin \{u_1^1, u_2^1, \ldots, u_k^1\}$ or u_t^1 is adjacent to each vertex of the set $\{u_1^1, u_2^1, \ldots, u_k^1\} \setminus \{u_t^1\}$ when $u_t^1 \in \{u_1^1, u_2^1, \ldots, u_k^1\}$. Then $M \subseteq N_{k-1}[u_t^1]$. This implies that $|N_{k-1}[u_t^1] \cap M| = k$. This is a contradiction. Therefore, D is a total dominating set of size at most k in G. ◁

Hence, the MULTIPACKING problem is NP-complete for CONV graphs. ◀

A.4 Proof of Lemma 18

Proof. In Algorithm 1, S_v denotes the set of vertices farthest from v in T, and for each $w \in S_v$, w_1 and w_2 denote the parent and the grandparent of w in T, respectively. Let $f(n)$ be the maximum number of multipackings in a tree on n vertices. We first consider the case height$(T) \geq 2$.

Suppose there exists $w \in S_v$ such that $T_{w_1} \cong H_1(k)$ for some $k \geq 2$. If any $M \in \mathcal{M}(T)$ contains w, then $M \in (w \oplus \mathcal{M}_v(T \setminus T_{w_1}))$. If M does not contain w, then $M \in \mathcal{M}_v(T \setminus w)$. Therefore, $\mathcal{M}(T) \subseteq (w \oplus \mathcal{M}_v(T \setminus T_{w_1})) \cup \mathcal{M}_v(T \setminus w)$. So, the recursive relation yields that $f(n) \leq f(n-3) + f(n-1)$, since $|V(T \setminus T_{w_1})| \leq n-3$ and $|V(T \setminus w)| = n-1$. Solving this recurrence yields $f(n) = O(1.47^n)$.

Suppose there exists $w \in S_v$ such that $T_{w_2} \cong H_2(1,0)$. If any $M \in \mathcal{M}(T)$ contains w, then $M \in (w \oplus \mathcal{M}_v(T \setminus T_{w_2}))$. If M does not contain w, then $M \in \mathcal{M}_v(T \setminus w)$. Therefore, $\mathcal{M}(T) \subseteq (w \oplus \mathcal{M}_v(T \setminus T_{w_2})) \cup \mathcal{M}_v(T \setminus w)$. So, the recursive relation yields that $f(n) \leq f(n-3) + f(n-1)$, since $|V(T \setminus T_{w_2})| \leq n-3$ and $|V(T \setminus w)| = n-1$. Solving this recurrence yields $f(n) = O(1.47^n)$.

Suppose there does not exist any $w \in S_v$ such that $T_{w_1} \cong H_1(k)$ for any $k \geq 2$ or $T_{w_2} \cong H_2(1,0)$. Then $T_{w_2} \cong H_2(k_1,k_2)$ for some $(k_1,k_2) \in \mathbb{N} \times (\mathbb{N} \cup \{0\}) \setminus \{(1,0)\}$. In this case, $\mathcal{M}(T) \subseteq \mathcal{M}_v(T \setminus T_{w_2}) \oplus \mathcal{M}(T_{w_2})$. Let $|V(T_{w_2})| = m$. Note that $m \geq 4$. Therefore, $|\mathcal{M}(T_{w_2})| \leq 1.58^m$ and the family $\mathcal{M}(T_{w_2})$ can be computed in time $O(1.58^m)$, by Corollary 17. Thus, we obtain the recurrence $f(n) \leq f(n-m) \cdot 1.58^m$. The solution of this recursion is $f(n) = O(1.58^n)$.

If $\text{height}(T) \leq 1$, then $T \cong H_1(k)$ for some k. In this case, $\mathcal{M}(T)$ can be computed in $O(|V(T)|)$ time, by Lemma 16.

Hence, the Algorithm 1 computes a set $\mathcal{M}_v(T)$ of size $O(1.58^n)$ that contains all multipackings of T, and it has running time $O^*(1.58^n)$. ◀