Polynomial-Size Encoding of All Cuts of Small Value in Integer-Valued Symmetric Submodular Functions

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Abstract

We study connectivity functions, that is, integer-valued symmetric submodular functions on a finite ground set attaining 0 on the empty set. For a connectivity function f on an n-element set V and an integer $k \geq 0$, we show that the family of all sets $X \subseteq V$ with $f(X) = k$ admits a polynomial-size representation: it can be described by a list of at most $O(n^{4k})$ items, each consisting of a set to be included, another set to be excluded, and a partition of remaining elements, such that the union of some members of the partition and the set to be included are precisely all sets X with $f(X) = k$. We also give an algorithm that constructs this representation in time $O(n^{2k+7}\gamma + n^{2k+8} + n^{4k+2})$, where γ is the oracle time to evaluate f. This generalizes the low rank structure theorem of Bojańczyk, Pilipczuk, Przybyszewski, Sokołowski, and Stamoulis [Low rank MSO, LICS 2026] on cut-rank functions on graphs to general connectivity functions. As an application, for fixed k, we obtain a polynomial-time algorithm for finding a set A with $f(A) = k$ and a prescribed cardinality constraint on A.

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1 Introduction

A *connectivity function* on a finite set V is a function $f : 2^V \to \mathbb{Z}$ satisfying the following three conditions.

(a) (symmetric) $f(X) = f(V - X)$ for all $X \subseteq V$.
(b) (submodular) $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq V$.
(c) $f(\emptyset) = 0$.

Here are several examples of connectivity functions. See [3, 8] for more instances of symmetric submodular functions.

- Vertex cut: Given a graph G and a set X of edges, let $\nu(X)$ be the number of vertices incident with both an edge in X and an edge in $E(G) - X$.

- Edge cut: Given a graph G and a set X of vertices, let $\eta(X)$ be the number of edges having one end in X and the other end in $V(G) - X$.
- Cut-rank: Given a graph and a set X of vertices, let $\rho(X)$ be the rank of the $X \times (V(G)-X)$ 0-1 matrix over the binary field where the ij entry of the matrix is 1 if and only if the vertex i is adjacent to the vertex j. See [7].
- Matroid: Given a matroid M with the rank function r and a set X of elements, let $\lambda(X) = r(X) + r(E(M) - X) - r(E(M))$.

We prove the following theorem. This is motivated by the low rank structure theorem of Bojańczyk, Pilipczuk, Przybyszewski, Sokołowski, and Stamoulis [1, Theorem 6.1], which covers the special case of f being the cut-rank function of a graph.

▶ **Theorem 1.** *Let f be a connectivity function on an n-element set V. Then for every non-negative integer k, there exists a collection $\{(X_i, Y_i, \mathcal{P}_i)\}_{i\in I}$ of tuples of disjoint subsets X_i, Y_i of V and a partition $\mathcal{P}_i$ of $V - (X_i \cup Y_i)$ for each $i \in I$ such that $|I| \le O(n^{4k})$ and $\{X \subseteq V : f(X) = k\}$ is precisely equal to*

$$\left\{ \left(X_i \cup \bigcup_{C \in \mathcal{Q}} C \right) : \mathcal{Q} \subseteq \mathcal{P}_i \text{ for some } i \in I \right\}.$$

Furthermore, such a collection can be constructed in time $O(n^{2k+7}\gamma + n^{2k+8} + n^{4k+2})$, where γ is the time to compute $f(X)$ for any set X.

As a corollary, for each fixed k, we obtain a polynomial-time algorithm to find a set A with $f(A) = k$ and $|A \cap W| \in T$ if it exists, for an input connectivity function f on V given by an oracle and for a set $W \subseteq V$ and a set $T \subseteq \{0, 1, \dots, |W|\}$. If $T = \{\lceil |V|/2 \rceil\}$ and $W = V$, then this is the SUBMODULAR MINIMUM BISECTION problem parameterized by the function value for connectivity functions.

▶ **Corollary 2.** *Let f be a connectivity function on an n-element set V. Let W be a subset of V. In time $O(n^{2k+7}\gamma + n^{2k+8} + n^{4k+2})$, for a set $T \subseteq \{0, 1, \dots, |W|\}$ of integers, we can find a set A such that $|A \cap W| \in T$ and $f(A) = k$ or confirm that there is no such set A, where f is given by an oracle that answers $f(X)$ for any set X in time γ.*

This paper is organized as follows. Section 2 recalls definitions of interpolations of a connectivity function and discuss several properties. Section 3 defines the blocking digraphs. Section 4 discusses our key lemma, saying that the blocking digraph cannot have a structure called a skew matching. Section 5 describes all closed sets in a directed acyclic graph, when there is no large skew matching. Section 6 finally presents the proof of the main theorem. Section 7 gives the proof of the corollary.

2 Preliminaries

Let us start this section by reviewing some terms on directed graphs, also called digraphs. A vertex of a digraph is called a *sink* if it has no outgoing arcs and it is called a *source* if it has no incoming arcs. For a set X of vertices, we write $\mathrm{Out}(X)$ to denote the set of all vertices y such that there is a directed path from a vertex in X to y. Similarly, we write $\mathrm{In}(X)$ to denote the set of all vertices y such that there is a directed path from y to a vertex in X. We write $D[X]$ to denote the subgraph of D induced by X, that is the directed graph obtained by deleting all vertices outside of X and all arcs incident with a vertex out of X. A set X of vertices of a directed graph is *independent* if $D[X]$ has no arcs.

Now let us recall the definition of an interpolation of a connectivity function, introduced in [7]. We write 3^V to denote a set of disjoint subsets $\{(X,Y) : X,Y \subseteq V,\ X \cap Y = \emptyset\}$. Let $f : 2^V \to \mathbb{Z}$ be a connectivity function. We say that $f^* : 3^V \to \mathbb{Z}$ is an *interpolation* of f if it satisfies the following four conditions.

(i) $f^*(X, V - X) = f(X)$ for all $X \subseteq V$.
(ii) (monotone) If $C \cap D = \emptyset$, $A \subseteq C$, and $B \subseteq D$, then $f^*(A,B) \leq f^*(C,D)$.
(iii) (submodular) $f^*(A,B) + f^*(C,D) \geq f^*(A \cap C, B \cup D) + f^*(A \cup C, B \cap D)$ for all $(A,B),(C,D) \in 3^V$.
(iv) $f^*(\emptyset,\emptyset) = f(\emptyset)$.

Every connectivity function admits a canonical interpolation

$$f_{\min}(A,B) = \min_{A \subseteq X \subseteq V-B} f(X)$$

[7, Proposition 4.2]. For a graph G, if we say $\rho^*_G(S,T) = \operatorname{rank} A(G)[S,T]$ where $A(G)$ is the adjacency matrix of G over the binary field, then ρ^*_G is an interpolation of the cut-rank function of the graph, see [7]. The following result implies that $f_{\min}$ is polynomial-time computable as long as f is:

▶ **Theorem 3** (Orlin [5]). *Let n be a positive integer. Let $f : 2^V \to \mathbb{Z}$ be a submodular function on an n-element set V. Let us assume that γ is the time to compute $f(X)$ for any subset X of V. Then we can find $X \subseteq V$ minimizing $f(X)$ in time $O(\gamma n^5 + n^6)$.*

Our aim is to describe subsets X of V with $f(X) = k$. The following two lemmas state that if $f(X) = k$, then there will be a pair (A,B) of small sets such that $A \subseteq X$, $B \subseteq V - X$, and $f^*(A,B) = k$.

▶ **Lemma 4.** *Let $r : 2^V \to \mathbb{Z}$ be a submodular function such that $r(X) \leq r(Y)$ for all $X \subseteq Y \subseteq V$ and $r(\emptyset) = 0$. Then there exists a set X with $r(X) = r(V)$ and $|X| \leq r(V)$. Moreover, if we can compute $r(Z)$ in time γ for any subset Z of V, then we can find such a set X in time $O(\gamma\,|V|)$.*

Proof. We proceed by induction on $|V|$. This statement is trivially true if $r(V) = 0$. Thus we may assume that $r(V) > 0$. Let $v \in V$. If $r(V - \{v\}) = r(V)$, then by the induction hypothesis applied to $V - \{v\}$, there is a subset $X \subseteq V - \{v\}$ such that $r(X) = r(V - \{v\}) = r(V)$ and $|X| \leq r(V)$, as desired. Therefore we may assume that $r(V - \{v\}) < r(V)$. By the induction hypothesis applied to $V - \{v\}$, there is a subset X of $V - \{v\}$ such that $r(X) = r(V - \{v\})$ and $|X| \leq r(V - \{v\})$.

For every $w \in V - (X \cup \{v\})$, we have $r(X) \leq r(X \cup \{w\}) \leq r(V - \{v\}) = r(X)$, and therefore $r(X \cup \{w\}) = r(X)$. Let Y be a maximal subset of V such that $X \cup \{v\} \subseteq Y$ and $r(Y) = r(X \cup \{v\})$. If $Y \neq V$, then for $w \in V - Y$, we have $r(X \cup \{w\}) + r(Y) \geq r(Y \cup \{w\}) + r(X)$ and therefore $r(Y \cup \{w\}) = r(Y)$, contradicting the assumption that Y is maximal. Therefore, $Y = V$. This implies that $r(X \cup \{v\}) = r(V)$. Note that $|X| + 1 \leq r(V - \{v\}) + 1 \leq r(V)$. Thus $X \cup \{v\}$ is a desired set. ◀

▶ **Lemma 5.** *Let f be a connectivity function on a finite set V. Let f^* be an interpolation of f. Let X be a subset of V. If $f(X) = k$, then there exist $A \subseteq X$, $B \subseteq V - X$ such that $\max(|A|,|B|) \leq k$ and $f^*(A,B) = k$. Moreover, if we can compute $f^*(S,T)$ in time γ for any pair (S,T) of disjoint subsets of V, then we can find such a pair (A,B) in time $O(\gamma\,|V|)$.*

Proof. We may assume that $k > 0$. For fixed $V - X$, the function $A' \mapsto f^*(A', V - X)$ on subsets of X is non-decreasing and submodular by the monotonicity and submodularity of f^*.

By Lemma 4, there is a subset A of X such that $f^*(A, V-X) = k$ and $|A| \leq k$. For this fixed set A, the function $B' \mapsto f^*(A, B')$ on subsets of $V-X$ is again non-decreasing and submodular. Again by Lemma 4, there is a subset B of $V-X$ such that $f^*(A,B) = k$ and $|B| \leq k$. ◀

3 Blocking digraphs

Let f be a connectivity function on a finite set V. Let f^* be an interpolation of f. For two disjoint subsets S and T of V, let us define an auxiliary digraph $D_{S,T}$ on $(V-(S\cup T))\dot\cup\{s^\circ, t^\circ\}$ as follows.

(a) For $x \in V-(S\cup T)$, if $f^*(S, T\cup\{x\}) > f^*(S,T)$, then $(s^\circ, x) \in A(D_{S,T})$.
(b) For $x \in V-(S\cup T)$, if $f^*(S\cup\{x\}, T) > f^*(S,T)$, then $(x, t^\circ) \in A(D_{S,T})$.
(c) For distinct $x, y \in V-(S\cup T)$, if $f^*(S\cup\{x\}, T\cup\{y\}) > f^*(S,T)$, then $(x,y) \in A(D_{S,T})$.

Let us call this digraph the *blocking digraph* for (S,T) with respect to f^*. This is a digraph appearing in the proof for properties on *blocking sequences* introduced by Geelen [4, Chapter 5]. A set X of vertices of a digraph D is a *closed set* if D has no arcs leaving X.

▶ **Lemma 6.** *Let f be a connectivity function on a finite set V and f^* be an interpolation of f. Let A, B, S, T be pairwise disjoint subsets of V. Let D be the blocking digraph for (S,T) with respect to f^*. Then $f^*(S\cup A, T\cup B) = f^*(S,T)$ if and only if D has no arcs from $A\cup\{s^\circ\}$ to $B\cup\{t^\circ\}$.*

Proof. Let $k = f^*(S,T)$.

The forward direction is easy. Suppose that D has an arc from $A\cup\{s^\circ\}$ to $B\cup\{t^\circ\}$. Then there is a subset A' of A and a subset B' of B such that $|A'| \leq 1$, $|B'| \leq 1$, and $f^*(S\cup A', T\cup B') > k$. By the monotonicity of f^*, $f^*(S\cup A, T\cup B) \geq f^*(S\cup A', T\cup B') > k$, contradicting the assumption.

The backward direction can be found in the proof of [6, Proposition 3.1] or any proofs for applying blocking sequences. We include the proof for completeness. We proceed by induction on $|A|+|B|$. If $|A|+|B| = 0$, then $A = B = \emptyset$ and the statement is trivial.

If $|A| \geq 2$, let $v \in A$. By the induction hypothesis, we have $f^*(S\cup(A-\{v\}), T\cup B) = k$, $f^*(S\cup\{v\}, T\cup B) = k$, and $f^*(S, T\cup B) = k$. By submodularity,

$$f^*(S\cup(A-\{v\}), T\cup B) + f^*(S\cup\{v\}, T\cup B) \geq f^*(S\cup A, T\cup B) + f^*(S, T\cup B).$$

Hence $f^*(S\cup A, T\cup B) \leq k$. Since f^* is monotone, we have $f^*(S\cup A, T\cup B) \geq f^*(S,T) = k$. Therefore $f^*(S\cup A, T\cup B) = k$. This completes this case.

If $|B| \geq 2$, then by the same argument, we deduce that $f^*(S\cup A, T\cup B) = k$.

Now $|A| \leq 1$ and $|B| \leq 1$. If $A = \emptyset$ and $B = \{w\}$, then there is no arc from s° to w, and therefore $f^*(S, T\cup\{w\}) = k$. If $A = \{v\}$ and $B = \emptyset$, then there is no arc from v to t°, and therefore $f^*(S\cup\{v\}, T) = k$. If $A = \{v\}$ and $B = \{w\}$, then there is no arc from v to w, and so $f^*(S\cup\{v\}, T\cup\{w\}) = f^*(S,T)$. Hence $f^*(S\cup A, T\cup B) = k$ in all remaining cases. ◀

Now our aim is to describe all subsets X of $V-(S\cup T)$ with $f(S\cup X) = k$ for every pair (S,T) of disjoint subsets with $\max(|S|, |T|) \leq f^*(S,T) = k$. Let D be the blocking digraph for (S,T) with respect to f^*. Let X be a closed set of vertices of D with $s^\circ \in X$ and $t^\circ \notin X$. Then for every strongly connected component C of D, either $V(C) \subseteq X$ or $V(C)\cap X = \emptyset$. Thus X cannot split a strongly connected component and therefore we can reduce the problem to a directed acyclic graph obtained by contracting each strongly

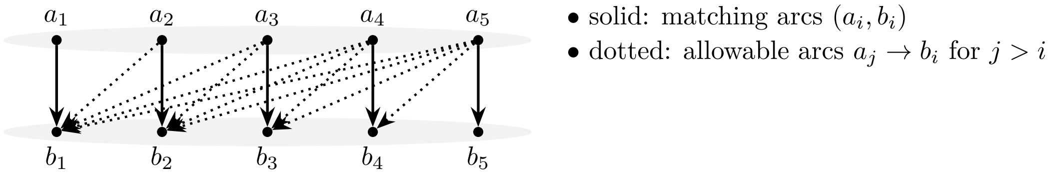

Figure 1 A skew matching of size 5. The solid arcs form the matching (a_i, b_i), while dotted arcs indicate other allowable arcs. No arcs $a_i \to b_j$ with $i < j$ and no arcs $b_i \to a_j$ are present. Arcs between a_i and a_j or between b_i and b_j are allowed.

connected component. Furthermore, for the purposes of the upcoming arguments (and the resulting algorithm), we can remove from D any vertex v that either can be reached from s° by a directed path or can reach t° by a directed path: in the former case, this vertex has to be in X, and in the latter, it cannot be in X. Then what is left for us is to characterize all sets X of vertices of a *directed acyclic graph* having no arcs leaving X, called a *closed set*.

4 Forbidden skew matching

We say that an interpolation f^* is *symmetric* if $f^*(X,Y) = f^*(Y,X)$ for all disjoint subsets X, Y. Since $f_{\min}$ is symmetric, every connectivity function admits a symmetric interpolation.

We say that a set of arcs (a_1, b_1), (a_2, b_2), …, (a_ℓ, b_ℓ) in a directed graph D is a *skew matching* if $a_1, a_2, \ldots, a_\ell, b_1, b_2, \ldots, b_\ell$ are pairwise disjoint vertices of D such that

(i) $(a_i, b_i) \in E(D)$ for all $i \in [\ell]$,
(ii) $(a_i, b_j) \notin E(D)$ for all $1 \le i < j \le \ell$,
(iii) $(b_i, a_j) \notin E(D)$ for all $i, j \in [\ell]$.

See Figure 1. We say ℓ is the size of the skew matching. Our next lemma shows that if f^* is a symmetric interpolation of a connectivity function f and D is the blocking digraph for (S,T) with respect to f^*, then $D - (\mathrm{Out}(s^\circ) \cup \mathrm{In}(t^\circ))$ does not admit a skew matching of size $2f^*(S,T) + 1$.

▶ **Lemma 7.** *Let f^* be a symmetric interpolation of a connectivity function f on a finite set V. Let S, T be disjoint subsets of V. If there exist distinct elements $a_1, a_2, \ldots, a_\ell, b_1, b_2, \ldots, b_\ell$ in $V - (S \cup T)$ such that*

(i) *$f^*(S \cup \{a_i\}, T \cup \{b_i\}) > f^*(S,T)$ for all $i \in \{1, 2, \ldots, \ell\}$,*
(ii) *$f^*(S \cup \{a_i\}, T \cup \{b_j\}) = f^*(S,T)$ for all $1 \le i < j \le \ell$,*
(iii) *$f^*(S \cup \{b_i\}, T \cup \{a_j\}) = f^*(S,T)$ for all $i, j \in \{1, 2, \ldots, \ell\}$, and*
(iv) *$f^*(S \cup \{a_i\}, T) = f^*(S, T \cup \{b_i\}) = f^*(S,T)$ for all $i \in \{1, 2, \ldots, \ell\}$,*

then $\ell \le 2f^*(S,T)$.

Proof. Let $k = f^*(S,T)$. For $i \in \{1, 2, \ldots, \ell\}$, let $A_i = \{a_1, \ldots, a_i\}$ and $B_i = \{b_1, \ldots, b_i\}$. By the assumption, $(S \cup T) \cap (A_\ell \cup B_\ell) = \emptyset$. First we claim that $f^*(S \cup A_i, T \cup B_i) \ge k + i$ for all $i \in \{1, 2, \ldots, \ell\}$. This is obviously true if $i = 1$. If $i > 1$, then by submodularity and monotonicity, we have

$$
\begin{aligned}
&f^*(S \cup A_i, T \cup B_i) + f^*(S \cup A_{i-1}, T \cup \{b_i\}) \\
&\ge f^*(S \cup A_{i-1}, T \cup B_i) + f^*(S \cup A_i, T \cup \{b_i\}) \\
&\ge f^*(S \cup A_{i-1}, T \cup B_{i-1}) + f^*(S \cup \{a_i\}, T \cup \{b_i\})
\end{aligned}
$$

and by the induction hypothesis and (i),

$$\geq (k+i-1)+(k+1).$$

Note that by (ii), (iv), and Lemma 6, $f^*(S\cup A_{i-1}, T\cup\{b_i\})=k$. Therefore we deduce that $f^*(S\cup A_i, T\cup B_i)\geq k+i$.

By the claim, we deduce that $f^*(S\cup A_\ell, T\cup B_\ell)\geq k+\ell$. Since f^* is submodular and $f^*(\emptyset,\emptyset)=f(\emptyset)=0$, we have

$$\begin{aligned} f^*(S\cup A_\ell, B_\ell) &\geq f^*(S\cup A_\ell, T\cup B_\ell)+f^*(S\cup A_\ell,\emptyset)-f^*(S\cup A_\ell, T)\\ &\geq f^*(S\cup A_\ell, T\cup B_\ell)-f^*(S\cup A_\ell, T),\\ f^*(A_\ell, B_\ell) &\geq f^*(S\cup A_\ell, B_\ell)+f^*(\emptyset, B_\ell)-f^*(S,B_\ell)\geq f^*(S\cup A_\ell, B_\ell)-f^*(S,B_\ell). \end{aligned}$$

By Lemma 6 and (iv), $f^*(S\cup A_\ell, T)=k$ and $f^*(S,B_\ell)\leq f^*(S,T\cup B_\ell)=k$. Therefore we deduce that

$$f^*(A_\ell, B_\ell)\geq k+\ell-2k=\ell-k.$$

Now, $f^*(B_\ell, A_\ell)\leq f^*(S\cup B_\ell, T\cup A_\ell)=k$ by Lemma 6 and (iii). Since f^* is symmetric, this is equal to $f^*(A_\ell, B_\ell)\geq \ell-k$. Thus, we deduce that $\ell\leq 2k$. ◀

5 Covering all closed sets

For a set S of vertices of a directed graph, let us write $N_D^-(S)$ to denote the set of all in-neighbors of vertices in S outside of S and write $N_D^+(S)$ to denote the set of all out-neighbors of vertices in S outside of S. We omit the subscript D to write $N^-(S)$ and $N^+(S)$, when the underlying digraph D is clear from the context.

▶ **Lemma 8.** *Let D be a directed graph and let S_1, S_2 be disjoint sets of vertices of D such that no two vertices of $S_1\cup S_2$ are joined by a directed path of length at most 3. Let M be a minimal subset of $S_1\cup S_2$ such that*[1]

$$\begin{aligned} N^+(M\cap S_1) &= N^+(S_1),\\ N^-(M\cap S_1) &= N^-(S_1),\\ N^+(M\cap S_2)-N^+(S_1) &= N^+(S_2)-N^+(S_1),\\ N^-(M\cap S_2)-N^-(S_1) &= N^-(S_2)-N^-(S_1). \end{aligned} \tag{1}$$

Then D has a skew matching of size $|M|$.

Proof. Let us partition M into four sets M_1, M_2, M_3, M_4 as follows.

$$\begin{aligned} M_1 &= \{x\in M : N^+(M\cap S_1-\{x\})\neq N^+(S_1)\},\\ M_2 &= \{x\in M-M_1 : N^+(M\cap S_2-\{x\})-N^+(S_1)\neq N^+(S_2)-N^+(S_1)\},\\ M_3 &= \{x\in M-(M_1\cup M_2) : N^-(M\cap S_2-\{x\})-N^-(S_1)\neq N^-(S_2)-N^-(S_1)\},\\ M_4 &= \{x\in M-(M_1\cup M_2\cup M_3) : N^-(M\cap S_1-\{x\})\neq N^-(S_1)\}. \end{aligned}$$

Since M is minimal, we have $M=M_1\cup M_2\cup M_3\cup M_4$. The four sets are pairwise disjoint by construction, and some of them may be empty.

[1] Note that such a minimal M exists since $M=S_1\cup S_2$ satisfies (1).

Let $a_1, a_2, \ldots, a_p$ be the vertices of M_1. For each $i \leq p$, let b_i be a vertex in $N^+(S_1) - N^+(M \cap S_1 - \{a_i\})$.

Let $a_{p+1}, a_{p+2}, \ldots, a_q$ be the vertices of M_2. For each $p < i \leq q$, let b_i be a vertex in $N^+(S_2) - N^+(M \cap S_2 - \{a_i\}) - N^+(S_1)$.

Let $b_{q+1}, b_{q+2}, \ldots, b_r$ be the vertices of M_3. For each $q < i \leq r$, let a_i be a vertex in $N^-(S_2) - N^-(M \cap S_2 - \{b_i\}) - N^-(S_1)$.

Let $b_{r+1}, b_{r+2}, \ldots, b_s$ be the vertices of M_4. For each $r < i \leq s$, let a_i be a vertex in $N^-(S_1) - N^-(M \cap S_1 - \{b_i\})$.

We claim that (a_1, b_1), (a_2, b_2), …, (a_s, b_s) is a skew matching of D. Clearly, $(a_i, b_i) \in E(D)$ for all $i \leq s$.

If $i < j \leq q$, then $(a_i, b_j) \notin E(D)$ by the choice of b_j. If $q < i < j$, then $(a_i, b_j) \notin E(D)$ by the choice of a_i. If $i \leq q < j$, then $(a_i, b_j) \notin E(D)$ because otherwise D has a directed path of length 1 from $a_i \in M$ to $b_j \in M$. Therefore we deduce that $(a_i, b_j) \notin E(D)$ for all $i < j$.

Finally, if $i, j \leq q$, then $(b_i, a_j) \notin E(D)$ because otherwise D has a directed path $a_i b_i a_j$ of length 2 from $a_i \in M$ to $a_j \in M$. If $i, j > q$, then $(b_i, a_j) \notin E(D)$ because otherwise D has a directed path $b_i a_j b_j$ of length 2 from $b_i \in M$ to $b_j \in M$. If $i \leq q < j$, then $(b_i, a_j) \notin E(D)$ because otherwise D has a directed path $a_i b_i a_j b_j$ of length 3 from $a_i \in M$ to $b_j \in M$. If $j \leq q < i$, then $(b_i, a_j) \notin E(D)$ because D has no directed path of length 1 from $b_i \in M$ to $a_j \in M$. ◀

▶ **Lemma 9.** *Let D be a directed acyclic graph with no skew matching of size $\ell + 1$. Let X be a closed set. Let S_1 be the set of all sources of $D[X]$. Let S_2 be the set of all sinks of $D - (X \cup \mathrm{In}(S_1))$. Let M be a minimal subset of $S_1 \cup S_2$ satisfying (1). Then $|M| \leq \ell$ and for $M_1 = M \cap S_1$, $M_2 = M \cap S_2$, and every subset Y of $(S_1 \cup S_2) - M$, the set $Z = X \triangle Y$ satisfies the following.*

(a) $M_1 \cup M_2 \cup (V(D) - (\mathrm{Out}(M_1) \cup \mathrm{In}(M_1) \cup \mathrm{In}(M_2)))$ *is an independent set.*

(b) $\mathrm{Out}(M_1) \subseteq Z$.

(c) $Z \cap \mathrm{In}(M_1) = M_1$.

(d) $Z \cap \mathrm{In}(M_2) = \emptyset$.

Proof. We first check that Lemma 8 applies to S_1 and S_2. Since S_1 is a set of sources in $D[X]$ and X is closed, D has no directed path between distinct vertices of S_1. Similarly, as S_2 is a set of sinks of $D - X$ and X is closed, D has no directed path joining distinct vertices of S_2. Since X is closed, D has no directed path from S_1 to S_2. As $S_2 \cap \mathrm{In}(S_1) = \emptyset$, D has no directed path from S_2 to S_1. Therefore no two vertices of $S_1 \cup S_2$ are joined by a directed path in D. By Lemma 8, $|M| \leq \ell$. To prove (a), let

$$U := \mathrm{Out}(M_1) \cup \mathrm{In}(M_1) \cup \mathrm{In}(M_2).$$

We first show that

$$V(D) - U \subseteq S_1 \cup S_2. \tag{2}$$

Let $v \in V(D) - (S_1 \cup S_2)$. Then $v \in \mathrm{Out}(S_1) - S_1$, $v \in \mathrm{In}(S_1) - S_1$, or $v \in \mathrm{In}(S_2) - S_2$. By (1), $\mathrm{Out}(S_1) - S_1 \subseteq \mathrm{Out}(M_1)$, $\mathrm{In}(S_1) - S_1 \subseteq \mathrm{In}(M_1)$, and $\mathrm{In}(S_2) - S_2 \subseteq \mathrm{In}(M_2) \cup \mathrm{In}(M_1)$. Thus, we deduce that $v \in U$. This proves (2).

Next we show

$$S_1 \cup S_2 \subseteq M_1 \cup M_2 \cup (V(D) - U). \tag{3}$$

It is enough to prove that U contains no vertex in $(S_1 \cup S_2) - (M_1 \cup M_2)$. Suppose that $v \in (S_1 \cup S_2) \cap U$. Then $v \in \mathrm{Out}(M_1) \cup \mathrm{In}(M_1) \cup \mathrm{In}(M_2)$. If $v \in \mathrm{Out}(M_1) \cup \mathrm{In}(M_1)$, then $v \in X \cup \mathrm{In}(S_1)$ and therefore $v \notin S_2$, which implies that $v \in S_1$. So, we conclude that $v \in M_1$ in this case. If $v \in \mathrm{In}(M_2) - (\mathrm{Out}(M_1) \cup \mathrm{In}(M_1))$, then $v \notin S_1$, because otherwise D has a directed path from $v \in S_1 \subseteq X$ to some vertex in M_2 and $M_2 \cap X = \emptyset$, contradicting the fact that X is closed. So $v \in S_2$. Since $v \in \mathrm{In}(M_2)$, we conclude that $v \in M_2$. This proves (3).

By (2) and (3),

$$S_1 \cup S_2 = M_1 \cup M_2 \cup (V(D) - (\mathrm{Out}(M_1) \cup \mathrm{In}(M_1) \cup \mathrm{In}(M_2))).$$

Since $S_1 \cup S_2$ is independent, (a) follows.

As X is a closed set and $M_1 \subseteq X$, we deduce that $\mathrm{Out}(M_1) \subseteq X$. Since $Y \cap \mathrm{Out}(M_1) = \emptyset$, we obtain (b).

Since M_1 is a set of sources of $D[X]$, $\mathrm{In}(M_1) \cap X = M_1$. Since $Y \cap \mathrm{In}(M_1) = \emptyset$, we deduce (c).

Note that $X \cap \mathrm{In}(M_2) = \emptyset$, because otherwise X contains a vertex in $M_2 \subseteq S_2$, as X is closed. Then (d) follows because $Y \cap \mathrm{In}(M_2) = \emptyset$. This completes the proof. ◀

▶ **Lemma 10.** *Let D be a directed acyclic graph and let M_1, M_2 be disjoint sets of vertices and let Z be a set of vertices. If*
(a) $M_1 \cup M_2 \cup (V(D) - (\mathrm{Out}(M_1) \cup \mathrm{In}(M_1) \cup \mathrm{In}(M_2)))$ *is an independent set,*
(b) $\mathrm{Out}(M_1) \subseteq Z$*,*
(c) $Z \cap \mathrm{In}(M_1) = M_1$*, and*
(d) $Z \cap \mathrm{In}(M_2) = \emptyset$*,*
then Z is a closed set.

Proof. Let $x \in Z$ and y be an out-neighbor of x. We want to show that $y \in Z$. If $x \in \mathrm{Out}(M_1)$, then trivially $y \in \mathrm{Out}(M_1) \subseteq Z$. So we may assume that $x \notin \mathrm{Out}(M_1)$. In particular, $x \notin M_1$ because $M_1 \subseteq \mathrm{Out}(M_1)$. By (c), $x \notin \mathrm{In}(M_1)$ and therefore x is in $V(D) - (\mathrm{In}(M_1) \cup \mathrm{Out}(M_1))$. By (d), $x \notin \mathrm{In}(M_2)$. By (a), $y \in \mathrm{Out}(M_1) \cup \mathrm{In}(M_1) \cup \mathrm{In}(M_2) - (M_1 \cup M_2)$. However, since $x \notin \mathrm{In}(M_2)$, we know that $y \notin \mathrm{In}(M_2)$. And as $x \notin \mathrm{In}(M_1)$, we deduce that $y \notin \mathrm{In}(M_1)$. Thus, $y \in \mathrm{Out}(M_1)$, implying that $y \in Z$ by (b). ◀

▶ **Lemma 11.** *Let D be an n-vertex directed acyclic graph with no skew matching of size $\ell + 1$. Then there is a list of at most $O(n^\ell)$ pairs (X_i, Y_i) of disjoint subsets X_i, Y_i of $V(D)$ such that*
(i) *for every subset U of $V(D) - (X_i \cup Y_i)$, $X_i \cup U$ is a closed set,*
(ii) *for every closed set K, there is an index i such that $K \cap (X_i \cup Y_i) = X_i$.*
Moreover, such a list can be found in time $O(n^\ell(n + |E(D)|))$.

Proof. Enumerate all pairs (M_1^i, M_2^i) of disjoint sets of vertices such that

$$\left|M_1^i \cup M_2^i\right| \leq \ell, \qquad \mathrm{Out}(M_1^i) \cap M_2^i = \emptyset,$$

and

$$M_1^i \cup M_2^i \cup (V(D) - (\mathrm{Out}(M_1^i) \cup \mathrm{In}(M_1^i) \cup \mathrm{In}(M_2^i)))$$

is an independent set. There are at most $O(n^\ell)$ such pairs. For each pair, set

$$X_i = \mathrm{Out}(M_1^i), \qquad Y_i = (\mathrm{In}(M_1^i) \cup \mathrm{In}(M_2^i)) - M_1^i.$$

Since D is acyclic, $\mathrm{Out}(M_1^i) \cap \mathrm{In}(M_1^i) = M_1^i$. Also, $\mathrm{Out}(M_1^i) \cap \mathrm{In}(M_2^i) = \emptyset$; otherwise some vertex of M_2^i would be reachable from M_1^i, contradicting $\mathrm{Out}(M_1^i) \cap M_2^i = \emptyset$. Hence $X_i \cap Y_i = \emptyset$.

We prove (i). Let $U \subseteq V(D) - (X_i \cup Y_i)$ and put $Z = X_i \cup U$. We verify the four assumptions of Lemma 10. The independence assumption is part of the enumeration. Clearly $\mathrm{Out}(M_1^i) = X_i \subseteq Z$. Moreover, $Z \cap \mathrm{In}(M_1^i) = M_1^i$ and $Z \cap \mathrm{In}(M_2^i) = \emptyset$, because $X_i \cap \mathrm{In}(M_1^i) = M_1^i$, $X_i \cap \mathrm{In}(M_2^i) = \emptyset$, and U is disjoint from Y_i. Thus Lemma 10 implies that $Z = X_i \cup U$ is closed.

We prove (ii). Let K be a closed set. Apply Lemma 9 to $X = K$ and $Y = \emptyset$. It gives sets $M_1 \subseteq K$ and $M_2 \cap K = \emptyset$ with $|M_1 \cup M_2| \leq \ell$ satisfying the independence condition above, and it gives $\mathrm{Out}(M_1) \subseteq K$. Therefore $\mathrm{Out}(M_1) \cap M_2 = \emptyset$, so this pair appears in our enumeration. For the corresponding (X_i, Y_i), Lemma 9 also gives $K \cap \mathrm{In}(M_1) = M_1$ and $K \cap \mathrm{In}(M_2) = \emptyset$. Since $M_1 \subseteq X_i = \mathrm{Out}(M_1)$, we obtain $K \cap (X_i \cup Y_i) = X_i$.

The list can be constructed within the stated time bound by enumerating the $O(n^\ell)$ candidate pairs and, for each pair, computing the relevant reachability sets and checking independence in $O(n + |E(D)|)$ time. ◀

▶ **Proposition 12.** *Let D be an n-vertex directed graph with no skew matching of size $\ell + 1$. Then there is a list of at most $O(n^\ell)$ triples $(X_i, Y_i, \mathcal{P}_i)$ of disjoint subsets X_i, Y_i of $V(D)$ and a partition $\mathcal{P}_i$ of $V(D) - (X_i \cup Y_i)$ such that*

- **(i)** *for any subset $\mathcal{Q}$ of $\mathcal{P}_i$, $X_i \cup \bigcup_{C \in \mathcal{Q}} C$ is a closed set,*
- **(ii)** *for every closed set K, there is i such that $K \cap (X_i \cup Y_i) = X_i$ and $C \cap K \in \{\emptyset, C\}$ for all $C \in \mathcal{P}_i$.*

Moreover, this list can be found in time $O(n^{\ell+2})$.

Proof. Let D' be the directed acyclic graph obtained from D by contracting each strongly connected component into a single vertex. We claim that D' has no skew matching of size $\ell + 1$. Suppose for a contradiction that D' has a skew matching $(A_1, B_1), \ldots, (A_{\ell+1}, B_{\ell+1})$, where each A_i, B_i is a vertex of D'. By the definition of a contraction, each vertex of D' corresponds to a strongly connected component of D, and an arc $A_i \to B_i$ in D' means there is an arc from some vertex of the component A_i to some vertex of the component B_i in D. Choose such vertices $a_i \in A_i$ and $b_i \in B_i$ with $a_i \to b_i$ in D. Since the skew matching in D' uses pairwise distinct vertices, all components A_i, B_i are distinct and thus the chosen vertices are pairwise distinct as well. Moreover, for $i < j$, the non-edge condition $(A_i, B_j) \notin E(D')$ implies that there is no arc from any vertex of A_i to any vertex of B_j in D, and the non-edge condition $(B_i, A_j) \notin E(D')$ implies that there is no arc from any vertex of B_i to any vertex of A_j in D. Hence $(a_1, b_1), \ldots, (a_{\ell+1}, b_{\ell+1})$ is a skew matching in D, contradicting the assumption. Now, we deduce the conclusion by applying Lemma 11. If (X_i', Y_i') is a member of the list obtained from Lemma 11 for D', then we construct $(X_i, Y_i, \mathcal{P}_i)$ by expanding each vertex of D' into a set of vertices in the strongly connected components and taking $\mathcal{P}_i$ be the partition of $V(D) - (X_i \cup Y_i)$ into strongly connected components. ◀

6 Final proof

Proof of Theorem 1. Let $f^*(S,T) = \min_{S \subseteq X \subseteq V-T} f(X)$. By Theorem 3, f^* can be computed in time $O(n^5\gamma + n^6)$. For each $X \subseteq V$ with $f(X) = k$, there is a pair (S, T) of disjoint subsets such that

$$f^*(S,T) = f(X) = k, \;\; S \subseteq X \subseteq V - T, \text{ and } \max(|S|, |T|) \leq k$$

by Lemma 5. Let D be the blocking digraph for (S,T) with respect to f^*. Let $D' = D - (\mathrm{Out}(s^\circ) \cup \mathrm{In}(t^\circ))$. Then Y is a closed set containing s° and not containing t° if and only if $Y = \mathrm{Out}(s^\circ) \cup Y'$ for a closed set Y' of D'.

By Lemma 7, D' has no skew matching of size $2k+1$. By Proposition 12, we obtain a list of at most $O(n^{2k})$ triples $(X_i, Y_i, \mathcal{P}_i)$ of disjoint subsets X_i, Y_i of $V(D')$ and a partition $\mathcal{P}_i$ of $V(D') - (X_i \cup Y_i)$ such that

(i) for any subset $\mathcal{Q}$ of $\mathcal{P}_i$, $X_i \cup \bigcup_{C \in \mathcal{Q}} C$ is a closed set of D',

(ii) for every closed set K of D', there is i such that $K \cap (X_i \cup Y_i) = X_i$ and $K \cap C \in \{\emptyset, C\}$ for all $C \in \mathcal{P}_i$.

Now we construct our list by adding all vertices of S and $\mathrm{Out}(s^\circ) - \{s^\circ\}$ into X_i and all vertices of T and $\mathrm{In}(t^\circ) - \{t^\circ\}$ into Y_i. There are at most $O(n^{2k})$ pairs (S,T) of disjoint subsets $S, T \subseteq V$ of size at most k, from which it directly follows that $|I| \le O(n^{4k})$.

In order to construct all the required tuples algorithmically, we iterate all $O(n^{2k})$ pairs (S,T). For each, we construct the blocking digraph D from the definition within $O(n^2)$ evaluations of f^* and then directly apply Proposition 12. The time complexity bound follows. ◀

7 Application to submodular minimum bisection

The SUBMODULAR MINIMUM BISECTION problem is for an input connectivity function f on an n-element set V with an integer k to find a partition of V into two sets A and B such that $|A| = \lfloor n/2 \rfloor$, $|B| = \lceil n/2 \rceil$, and $f(A) \le k$.

More generally, let us try to find a set A with $|A| \in T$ for some fixed set T, while having $f(A) \le k$. Furthermore, we can replace $|A|$ with $|A \cap W|$ for a fixed set W. We can prove a stronger statement by replacing $f(A) \le k$ with $f(A) = k$, because we can run the algorithm $k+1$ times. We remark that Pilipczuk and Sokołowski observed previously that their low-rank structure theorem [1, Theorem 6.1] implies such an algorithm for the specific case of cut-rank function of graphs, i.e., for the case where $f = \rho$.

Proof of Corollary 2. By Theorem 1, in time $O(n^{2k+7}\gamma + n^{2k+8} + n^{4k+2})$ we can construct the list $\{(X_i, Y_i, \mathcal{P}_i)\}_{i \in I}$ with $|I| \le O(n^{4k})$. It is enough to show that for each i, we can decide whether there exists a subset $\mathcal{Q}$ of $\mathcal{P}_i$ such that

$$\left| \left(X_i \cup \bigcup_{U \in \mathcal{Q}} U \right) \cap W \right| \in T.$$

Let $A_1, A_2, \ldots, A_\ell$ be the elements of $\mathcal{P}_i$. Let $s_0 = |X_i \cap W|$ and $s_j = |A_j \cap W|$ for all $j \in [\ell]$.

We present an algorithm based on dynamic programming. For each $j \in \{0, 1, \ldots, \ell\}$ and each integer $t \in \{0, 1, \ldots, |W|\}$, we maintain a value $I_j(t) \in 2^{[j]} \cup \{\perp\}$, where $I_j(t) \ne \perp$ means that

$$s_0 + \sum_{p \in I_j(t)} s_p = t.$$

Initialize $I_0(s_0) = \emptyset$ and $I_0(t) = \perp$ for all $t \ne s_0$. For each $j \in [\ell]$ and each $t \in \{0, 1, \ldots, |W|\}$, update as follows:

- If $I_{j-1}(t) \ne \perp$, set $I_j(t) = I_{j-1}(t)$.
- Otherwise, if $t \ge s_j$ and $I_{j-1}(t - s_j) \ne \perp$, set $I_j(t) = I_{j-1}(t - s_j) \cup \{j\}$.
- Otherwise set $I_j(t) = \perp$.

Therefore, for this fixed i, a feasible set exists if and only if $I_\ell(t) \neq \perp$ for some $t \in T$. In that case, let $\mathcal{Q} = \{A_p : p \in I_\ell(t)\}$ and output

$$A := X_i \cup \bigcup_{U \in \mathcal{Q}} U.$$

By construction, $|A \cap W| = t \in T$, and by Theorem 1, $f(A) = k$.

For fixed i, the dynamic programming has $O(\ell\,|W|) \leq O(n^2)$ states and transitions. Hence it runs in $O(n^2)$ time. Repeating this for all i adds $O(n^2 \cdot |I|) = O(n^{4k+2})$ time. Therefore, the total running time is $O(n^{2k+7}\gamma + n^{2k+8} + n^{4k+2})$. ◀

Recall that for a graph G and a subset X of $V(G)$, $\eta(X)$ is the number of edges between X and $V(G)-X$. As discussed in Section 1, η is a connectivity function on $V(G)$. The minimum bisection problem for η was previously studied by Cygan, Lokshtanov, Pilipczuk, Pilipczuk, and Saurabh [2], who showed that it is fixed-parameter tractable.

We would like to ask whether the minimum bisection problem for other connectivity functions are fixed-parameter tractable. More generally is it true that the submodular minimum bisection is fixed-parameter tractable for all connectivity functions?

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