Conflict-Free Coloring Planar Graphs with 4 Colors

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Abstract

We efficiently conflict-free color every planar graph with 4 colors. An (open-neighborhood) *conflict-free coloring* assigns colors to vertices in a way that every vertex v has a neighbor w such that the color of w is distinct from the colors of the other neighbors of v (i.e., the color of w is unique in the open neighborhood of v). A previous best upper bound on the conflict-free chromatic number of planar graphs was 5, and it is known that 4 colors are sometimes necessary. Deciding whether, e.g., a planar graph admits a conflict-free coloring with 3 colors is NP-complete. Our approach uses a refined variant of the classical Gallai–Edmonds decomposition and the Four Color Theorem. In fact, our result is equivalent to the Four Color Theorem.

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1 Introduction

Conflict-free coloring was introduced in the early 2000s in geometric and combinatorial terms, motivated by applications such as frequency assignment in wireless networks [10, 24]. In this setting, one seeks a coloring such that every relevant region contains a uniquely colored element. This notion has since evolved into a rich area of research spanning combinatorics, computational geometry, and graph theory.

A natural graph-theoretic variant of the problem considers the neighborhoods of vertices as the relevant regions. An *open-neighborhood conflict-free coloring* (shortly a CFON coloring, as used e.g. in [4]) of a graph G is an assignment of colors to vertices of G such that for every vertex v having at least one neighbor, there exists an assigned color that is unique (appears exactly once) in its open neighborhood $N(v)$. Note that we always assume $v \notin N(v)$, that is, we ignore possible loops in G, and parallel edges do not matter. Several variants of this notion have been studied, including the distinction between *partial* colorings (where only a subset of vertices is colored) and *full* colorings (where each vertex of G is assigned a color), and the more restricted *proper* version in which adjacent vertices additionally must receive distinct colors. In this paper, we study full CFON colorings which are not necessarily proper.

For a historical overview, we briefly review this and other variants of conflict-free colorings on graphs in the literature. An early work of Abel et al. [1] addressed combinatorial bounds for general graphs. Later algorithmic and parameterized aspects of conflict-free colorings with respect to neighborhoods were investigated e.g. by Bodlaender, Kolay, and Pieterse [6].

Specifically for planar graphs, the line of research of combinatorial bounds for CFON colorings begins with the aforementioned Abel et al. [1], who established the first general bounds for conflict-free colorings with respect to neighborhoods. Considerable effort has

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been devoted to improving these bounds for planar graphs. Next, Bhyravarapu and Kalyanasundaram [4] showed that every planar graph admits a partial CFON coloring with at most five colors, and that every outerplanar graph admits such a coloring with at most four colors. Their work also developed general bounds for CFON colorings in terms of several structural graph parameters; these results were later extended in joint work with Mathew [5].

A breakthrough for the planar partial CFON case was obtained by Huang, Guo, and Yuan [16], who proved that every connected graph that is minor-k-colorable (that is, every minor of it is properly k-colorable) admits a partial CFON coloring with at most k colors. As a consequence, every planar graph admits a partial CFON coloring with at most four colors, and so a full CFON coloring with at most five colors by assigning a dummy fifth color to every uncolored vertex.

Note also that a lower bound of 4 colors in the worst case on planar graphs [1] is nearly trivial; take the 1-subdivision G_1 of a graph G (such as of planar $G = K_4$), then for every edge uv of G the vertices u and v must receive distinct colors in a CFON coloring of G_1, or the vertex subdividing uv does not see a unique color, and so at least $\chi(G)$ colors are needed. The question of whether the maximum CFON chromatic number in the class of planar graphs is 4 or 5 remained a central open problem in the area (e.g., [11, Conjecture 9.1]).

In parallel, several related variants of conflict-free coloring have been studied. In particular, *proper conflict-free colorings*, in which adjacent vertices must receive distinct colors, have received attention in recent years. Fabrici et al. [11] showed, among other results, that every planar graph admits a proper CFON coloring with at most eight colors, and that there exist planar graphs that do not admit such a coloring with five colors. Further refinements for planar graphs and related classes include bounds under additional structural restrictions, such as girth or forbidden cycles [7, 25]. Recent works have also investigated asymptotic and structural bounds for proper conflict-free colorings in graphs of large maximum degree [8, 18].

Beyond planar graphs, conflict-free coloring has been studied in a variety of directions, including list variants, algorithmic aspects, and complexity questions [13, 14]. In addition, variants motivated by combinatorial games and dynamic settings have recently been explored [20]. Altogether, this highlights the richness of the area and the diversity of techniques involved.

New contribution. While leaving the definitions and technical details for later, we state:

▶ **Theorem 1.1.** *Every planar graph admits a full open-neighborhood conflict-free (CFON) coloring with at most four colors, and such a coloring can be computed in polynomial time.*

Closing the gap between 4 and 5 for the maximum CFON chromatic number in the class of planar graphs has required the development of significant new methods and tools. Our approach is based on a localized analysis of the Gallai–Edmonds decomposition for maximum matchings in a graph. The Gallai–Edmonds structure [12, 9] is a classical tool in matching theory [21], and we develop a refinement that focuses on individual factor-critical components and their interaction with unmatched vertices. While the connection between CFON colorings and matchings in a graph (Observation 2.6, informally saying that the matched vertices are easily CFON colorable) has been known before [16], to the best of our knowledge, such a localized use of the Gallai–Edmonds decomposition has not been previously exploited in the context of conflict-free coloring.

The new perspective allows us to derive some structural properties of planar graphs, in particular, that there exists a maximum matching in which every unmatched vertex has degree at most five. This structural insight serves as the starting point for our analysis and enables a systematic treatment of local configurations (Figures 1–5), ultimately leading to a four-color bound for the CFON colorings of planar graphs. Our arguments also use the classical Four Color Theorem and, in turn, Theorem 1.1 implies the Four Color Theorem via the lower-bound reduction with a 1-subdivision sketched above.

Paper organization. We introduce the necessary graph coloring and matching tools in Section 2. In Section 3 we design our tools for a localized analysis of the Gallai–Edmonds decomposition. In Section 4 we prove our first cornerstone result – Theorem 4.1 about degrees of unmatched vertices, and use it to classify the local cases which have to be handled in the proof of our main result. Section 5 solves the majority of the local cases with one universal argument – Lemma 5.1, based on proper coloring of an augmented matching-contracted graph and Kempe chains. Section 6 brings an additional planarity-related tool – Lemma 6.1, which is then used in Section 7 to finish off the remaining two difficult local cases. Section 8 then summarizes the proof of the main result – Theorem 1.1, and Section 9 subsequently deals with the mostly straightforward algorithmic details.

2 Preliminaries

We work with finite and simple graphs. For a vertex v of G, we denote by $N_G(v)$ (shortly $N(v)$) its open neighborhood – the set of all vertices adjacent to v (and hence excluding v itself since we have simple graphs), and by $\deg_G(v) = |N_G(v)|$ its degree. For a set $X \subseteq V(G)$, we denote by $N_G(X) := \bigcup_{v\in X} N_G(v)$.

Colorings. A *coloring* of a graph G by/with k colors (a *k-coloring*) is an arbitrary mapping $c : V(G) \to \{1, 2, \dots, k\}$. The parameter k is often implicit and we may skip it.

A *proper coloring* of a graph G is a coloring c such that for every edge $uv \in E(G)$ we have $c(u) \neq c(v)$. (This is usually called just a “coloring”, but we want to distinguish the term from conflict-free colorings which are not required to be proper.) One of the most fundamental results of graph theory is the Four Color Theorem:

▶ **Theorem 2.1** (Appel and Haken [2, 3], Robertson, Sanders, Seymour and Thomas [22, 23]). *Every loopless planar graph can be properly colored with* 4 *colors, and this coloring can be found in quadratic time.*

We will also use the standard tool of Kempe chains. For two colors i and j in a properly colored graph G, a (i, j)-*Kempe chain* is a connected component of the subgraph induced by the vertices colored i and j.

In our paper, we study the following variant of colorings:

▶ **Definition 2.2** (Open-neighborhood conflict-free coloring). *A coloring c is an* open-neighborhood conflict-free coloring *(shortly CFON coloring) of a graph G if for every vertex $v \in V(G)$ with $N(v) \neq \emptyset$, there exists a color i such that $|c^{-1}(i) \cap N(v)| = 1$.*

Equivalently, for every vertex v with nonempty neighborhood, there is a neighbor with a color that appears exactly once among the neighbors (is unique*).*

The minimum number of colors in a CFON coloring of G is called the conflict-free chromatic number *of G and is denoted by $\chi_{\mathrm{CF}}(G)$.*

Matching-theoretic tools. We use standard notions of matching theory [21]. Given a *matching* M in a graph, that is a set of mutually non-adjacent edges, a vertex is *matched* (with respect to M) if it is incident to an edge of M, and *unmatched* otherwise. A matching is *perfect* if there is no unmatched vertex. A vertex $v \in V(G)$ is called *essential* if it is matched in every maximum matching of G, and *inessential* otherwise (i.e., it is unmatched in at least one maximum matching).

▶ **Definition 2.3** (Gallai–Edmonds decomposition [12, 9]). *For a graph G, let*

$$D(G) = \{v \in V(G) : v \textit{ is inessential}\},$$
$$A(G) = N_G(D(G)) \setminus D(G),$$
$$C(G) = V(G) \setminus (A(G) \cup D(G)).$$

So, the sets $D(G)$, $A(G)$, and $C(G)$ form a partition of $V(G)$.

▷ Claim 2.4 (e.g., [21]). The following properties hold for a Gallai–Edmonds decomposition.

a) Each connected component H of the induced subgraph $G[D(G)]$ is *factor-critical* (that is, for every vertex $v \in V(H)$, the graph $H - v$ has a perfect matching).

b) The induced subgraph $G[C(G)]$ has a perfect matching.

c) There are no edges between $C(G)$ and $D(G)$ in G.

d) Every vertex of $A(G)$ has a neighbor in $D(G)$. Moreover, in every maximum matching M of G, each vertex of $A(G)$ is incident to an edge of M whose other endpoint lies in $D(G)$, and these endpoints lie in pairwise distinct connected components of $G[D(G)]$.

e) If $G[D(G)]$ has k connected components, then for every maximum matching M of G there are $k - |A(G)|$ connected components of $G[D(G)]$ which are not adjacent by an edge of M to $A(G)$, and so there are exactly $k - |A(G)|$ unmatched vertices in M.

f) For every nonempty set $X \subseteq A(G)$, the set $N_G(X) \cap D(G)$ intersects at least $|X| + 1$ distinct connected components of $G[D(G)]$.

▶ **Theorem 2.5** (Edmonds' blossom algorithm [9]). *A matching of maximum size in a graph can be found in polynomial time.*

Perfect matching implies a conflict-free coloring via contraction. Let $F \subseteq E(G)$ be an arbitrary set of edges (however, we will usually have $F = M$ for a matching M). We define the *contracted graph* $H = H(G, F)$ of G as follows. Every connected component C of the spanning subgraph $(V(G), F)$ is contracted down into one *supervertex* of H, which we denote by $[v_1 \dots v_m]$ where $V(C) = \{v_1, \dots, v_m\}$, while suppressing loops and multiple edges so that H is simple. So, two supervertices are adjacent in H if and only if there exists an edge between their corresponding vertex sets in G. In particular, a vertex x not incident to any edge of F is kept as its own supervertex $[x]$ in H.

A coloring c of H *lifts* to a coloring c' of G if every vertex v of G which belongs to a supervertex w of H receives the color $c'(v) := c(w)$.

▶ **Observation 2.6.** *If M is a perfect matching of G, then any proper coloring of the contracted graph $H = H(G, M)$ lifts to a CFON coloring of G.*

Proof. Let c be a proper coloring of H that lifts to the coloring c' of G. Pick any vertex $v \in V(G)$, and let u be such that $uv \in M$ (since M is a perfect matching). Since $u \in N(v)$, the color $c'(u) = c'(v) = c([uv])$ appears at least once in $N(v)$. We claim it appears exactly once. Let $w \in N(v) \setminus \{u\}$ and $w' \in V(G)$ be such that $ww' \in M$. Note that $uv \neq ww'$. Since the vertices v and w are adjacent in G, their supervertices $[uv]$ and $[ww']$ are adjacent in H. Consequently, $c([uv]) \neq c([ww'])$ in a proper coloring, and hence $c'(w) = c([ww']) \neq c'(u)$. ◀

Observation 2.6 shows that when G has a perfect matching, conflict-free colorings can be obtained directly from proper colorings of the contracted graph H. Since contractions of edges preserve planarity, we can use Theorem 2.1. Consequently, the main difficulty arises from vertices that are unmatched by a maximum matching. The remainder of the paper focuses on controlling these unmatched vertices by enforcing additional color constraints in the contracted graph H and/or by locally modifying the matching.

Similar matching-based ideas have already appeared in the study of conflict-free colorings. In particular, Huang et al. [16] use maximum matchings and the Gallai–Edmonds decomposition in their analysis of partial conflict-free colorings of planar graphs. We develop this perspective further; we use a maximum matching as a global structural framework and introduce additional tools, such as alternating-path reachability and the auxiliary graphs G_x (see Section 3), which allow us to localize the analysis to individual factor-critical components.

3 Switchable vertices and the subgraph G_x

In this section, we develop the tools which will be used to analyze the structure of unmatched vertices in planar graphs. We leave most of the technical parts of the section to the full paper [15], and mark the respective statements with (*).

Let M be a fixed maximum matching of G. An *M-alternating path* is a path whose edges alternate between edges in M and edges not in M. We are now going to define several notions which depend on the choice of a matching M in our graph G, but we will for simplicity skip an explicit reference to M whenever the considered matching M is clear from the context.

▶ **Definition 3.1** (Vertex switchable from x; set $S(x)$). *Let G be a graph, M a matching and x be a vertex unmatched by M. A vertex v is* switchable from x *if there exists an M-alternating path of even length starting in x with an edge not in M and ending in v with an edge in M.*

We denote by $S(x)$ the set consisting of x together with all vertices switchable from x.

▶ **Definition 3.2** (The subgraph G_x). *Let G be a graph, M a matching and x be a vertex unmatched by M. Define G_x to be the subgraph of G induced by the vertex set*

$$V(G_x) := S(x) \cup N_G(S(x)).$$

The graph G_x is the basic local object in our analysis. It contains all vertices that may become unmatched by switching from x, together with all their neighbors, and hence G_x is in particular connected and x is the only unmatched vertex in G_x if M is maximum. In our context, G_x records the part of the graph that will be relevant for our local modifications around x. We show that the Gallai–Edmonds decomposition of each such G_x is particularly simple and that local changes of the matching within G_x do not significantly affect the neighborhoods of other unmatched vertices. We summarize our core findings as follows:

▷ Claim 3.3 (*). Let G be a graph, let M be a maximum matching of G, and let x be unmatched in M. Then the connected auxiliary graph $G_x = G\big[S(x) \cup N_G(S(x))\big]$ satisfies

$$D(G_x) = S(x), \qquad A(G_x) = N_G(S(x)) \setminus S(x), \qquad C(G_x) = \varnothing, \quad \text{and}$$
$$D(G_x) = V(G_x) \cap D(G), \qquad A(G_x) \subseteq A(G).$$

In particular, the Gallai–Edmonds decomposition of G_x consists only of D- and A-vertices, with no C-part, and x is the only vertex of G_x unmatched by the restriction of M to G_x. Consequently, if $|A(G_x)| = k$, then $G[D(G_x)]$ has $k + 1$ connected components.

Moreover, every connected component of the graph $G[D(G_x)]$ is a connected component of $G[D(G)]$, too. If $u \neq x$ is another vertex unmatched in M and D_0 is the connected component of $G[D(G_u)]$ containing u, then $V(D_0) \cap V(G_x) = \varnothing$. If P is an even M-alternating path from x to some $y \in S(x)$ and $M' = M \triangle E(P)$, then for every edge f with at least one endpoint in D_0 we have $f \in M \iff f \in M'$.

Claim 3.3 is proven in several steps. A key technical ingredient is a monotonicity property of reachability under switching: if we switch from x to a vertex $u \in S(x)$, then no new switchable vertices are created outside the original set $S(x)$. This will ensure that all subsequent local modifications remain inside G_x.

We begin with this monotonicity statement and then use it to show that x is the unique vertex of G_x unmatched by the restricted matching $M \cap E(G_x)$. From there we determine the Gallai–Edmonds decomposition of G_x itself, compare it with the global Gallai–Edmonds decomposition of G and derive the desired conclusions.

4 Degree bound and a classification of configurations

The main structural message is the following new degree bound in planar graphs: from any unmatched vertex, one can switch to a low-degree unmatched vertex.

▶ **Theorem 4.1** (Low-degree switchable vertex in planar graphs). *Let G be a planar graph and M a maximum matching of G. If x is unmatched in M, then there exists a vertex v switchable from x, $v \in S(x)$, with $\deg_G(v) \leq 5$ (possibly $v = x$).*

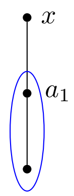
The proof of Theorem 4.1 uses the Gallai–Edmonds decomposition and standard planarity arguments, and we include it in the Appendix for the sake of completeness.

Using also Claim 3.3, we now easily derive:

▶ **Corollary 4.2** (*). *Every planar graph G admits a maximum matching M such that either M is perfect, or every vertex unmatched by M has degree at most 5.*

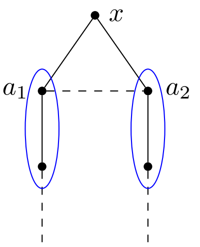
The classification of configurations. By Observation 2.6, the only obstruction to turning a proper coloring of the contracted graph $H(G, M)$ of a matching M in G into a conflict-free coloring of G comes from vertices that are unmatched by M. Moreover, by Corollary 4.2, we may assume that every vertex unmatched by M has degree at most 5 in G.

We therefore consider a maximum matching M in G and an unmatched vertex x of degree at most 5. Since M is a maximum matching, each vertex of the neighborhood $N(x)$ is matched, and so either matched to another vertex of $N(x)$, or matched to a vertex outside of $N(x)$. We classify all possible configurations at x, up to *isomorphism*, with respect to *the degree of* x and the way *its neighbors are matched* (inside $N(x)$ or outwards), in Figures 1–5.

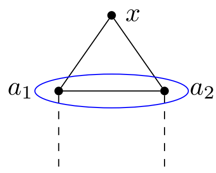


Case 1A: Edges matched outwards.

Figure 1 Degree one vertex; note that other edges could be incident to depicted vertices except x.

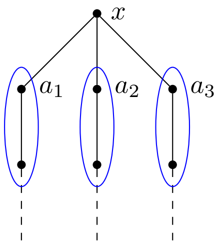


Case 2A: Edges matched outwards.

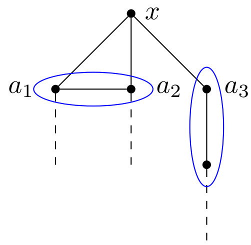


Case 2B: Matched neighbor pair.

Figure 2 Degree two vertices; again, other edges could exist in the picture except those incident to x.

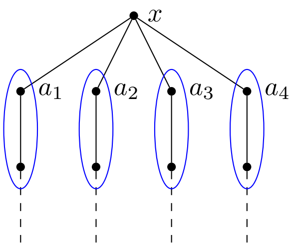


Case 3A: All edges matched outwards.

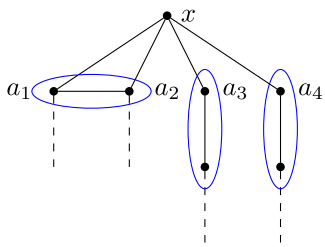


Case 3B: Matched one neighbor pair.

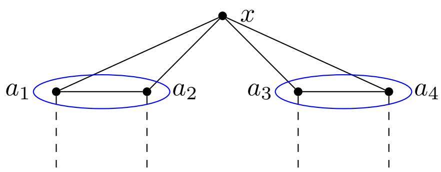
Figure 3 Degree three vertices; again, other edges could exist in the picture except those incident to x, in particular edges between neighbors of x that are not matched together.



Case 4A: All edges matched outwards.



Case 4B: Matched one neighbor pair.

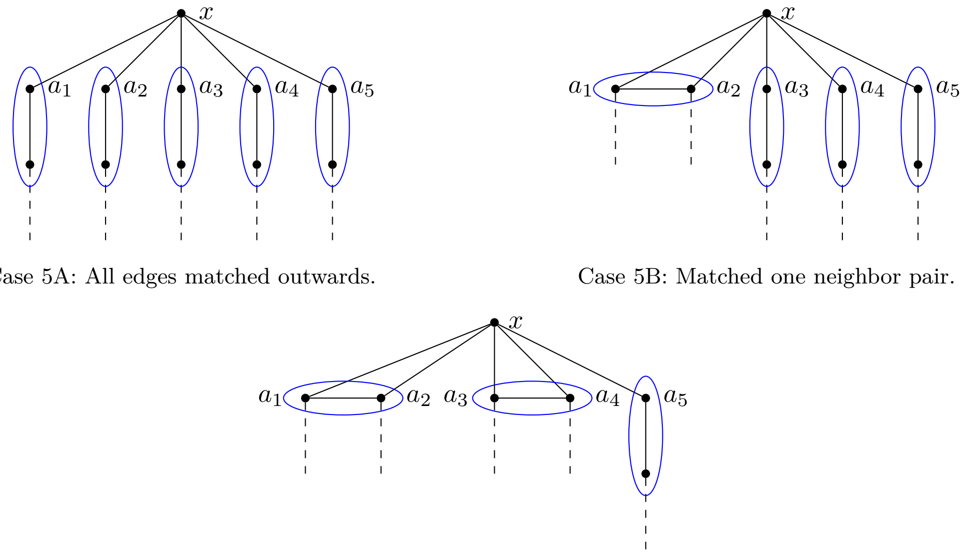


Case 4C: Matched two neighbor pairs.

Figure 4 Degree four vertices; other edges could exist in the picture except those incident to x, and the planar cyclic ordering of the edges at x is not necessarily the same as depicted here.

As depicted, there are 11 such configurations without isolated vertices, and we further divide them into three groups based on how they will be approached in the coming proofs:

- *Contractible cases:* Case 1A, Case 2A, Case 3A, Case 3B, Case 4B, Case 5A, Case 5C.
- *Contractible with Kempe chaining, or Kempe-contractible cases:* Case 4A, Case 5B.
- *Difficult cases (marked in italic in the pictures):* Case 2B, Case 4C.



Case 5A: All edges matched outwards.

Case 5B: Matched one neighbor pair.

Case 5C: Matched two neighbor pairs.

Figure 5 Degree five vertices; other edges could exist in the picture except those incident to x, and the planar cyclic ordering of the edges at x is not necessarily the same as depicted here.

5 Resolving the contractible cases

The task of this section is to provide the full proof of an "easier part" of Theorem 1.1 which simply builds on the ideas of Observation 2.6 and on Corollary 4.2. However, since this part does not resolve all of the configurations enumerated in Section 4 (the two difficult cases remain to be resolved later), the formulation of this partial result is quite technical in order to transparently allow for future local modifications concerning the difficult cases.

For any planar graph H with a fixed planar embedding and a vertex $x \in V(H)$, we define the following operation. The *wheel augmentation at x* adds to H a new edge between every non-adjacent pair of consecutive neighbors of x in this embedding of H. Informally, this construction "makes a wheel (graph)" around x. If, for some independent set of vertices $X \subseteq V(H)$, H' is the graph obtained from H by a sequence of wheel augmentations at every vertex $x \in X$ (the order of these augmentations does not matter since X is independent), then H' is called the *wheel augmented graph* of H at X. Independent wheel augmentations trivially preserve planarity of H.

Let M be a maximum matching of G and $H = H(G, F)$ be the contracted graph of G for some $F \subseteq E(G)$. Note that, unlike in Observation 2.6, the edge set F which we contract in G while constructing our coloring, will not always be equal to the matching M – in specific cases we may choose F to be "slightly different" from M, but the difference will occur only in the closed neighborhood of vertices unmatched by M. While the proof of Lemma 5.1 itself only needs to contract the matching edges for the purpose of constructing our coloring, proofs of the difficult cases (see Section 7) will require to contract, e.g., three vertices on a path of length 2 into one supervertex. In the first reading of the statements and proofs below, it is safe to assume "$F = M$" and consider the fine details regarding F later on.

In the next definition we aim to capture changes of colorings of H which "leave intact" (up to a permutation of the colors) vertices of H coming by contractions from chosen components of the graph $G[D(G)]$ defined with respect to M. For any two colorings c and $\bar{c}$ of H and a subset U of vertices of G unmatched by M, we say that $\bar{c}$ is a *U-protecting recoloring* of c if the following holds for every vertex $x \in U$ and the connected component D_x of $G[D(G)]$ containing x: if $V_x \subseteq V(H)$ are the vertices into which some vertex of D_x is contracted via F, then the coloring $\bar{c}$ differs from c on V_x only by a permutation of the colors.

The next lemma, Lemma 5.1, informally claims that we "can resolve" – in the sense of obtaining a CFON 4-coloring (analogously to Observation 2.6), all matched vertices of G and also all unmatched vertices of G, except those whose local configuration is one of the two difficult cases (Case 2B, Case 4C) or which are in a very specific local situation. Importantly, the cumbersome formulation of Lemma 5.1 gives us the power to resolve these resolvable unmatched vertices *after* we (in yet undetermined way) resolve the difficult cases in the coming sections, and without conflicts or discrepancies between the cases. The formulation also ensures that nothing else than the cases described in Lemma 5.1 c) remains "unresolved".

▶ **Lemma 5.1.** *Let G be a planar graph without isolated vertices. There exists a maximum matching M of G and a subset $U \subseteq V(G) \setminus V(M)$ of unmatched vertices such that the following hold. Let $W = V(G) \setminus (U \cup V(M))$ be the (independent) set of unmatched vertices complementary to U, and G_W denote (with respect to M) the union of the subgraphs G_x for $x \in W$.*

a) *Every vertex of $V(G) \setminus V(M)$ is of degree at most 5 in G (cf. Corollary 4.2).*

b) *For any set of edges $F \subseteq E(G)$ such that $F \cap E(G_W) = M \cap E(G_W)$, let $H = H(G, F)$ be the corresponding contracted graph and H' be the wheel augmented graph of H at W. Then every proper 4-coloring c_1 of H' admits a U-protecting recoloring to a proper 4-coloring c of H' such that, denoting by c' the 4-coloring of G into which the coloring c lifts (cf. Observation 2.6), every vertex $x \in W$ has a neighbor u in G whose color $c'(u)$ is unique in the open neighborhood $N(x)$.*

c) *For every vertex $x \in U$ and every $z \in S(x)$ (including $z = x$) such that the degree of z in G is at most 5, let M' be any matching obtained from M in one of the following ways:*
- *$M' := M \triangle E(P)$ for an even M-alternating x–z path in G (switching from x to z),*
- *$M' := M \triangle E(Q)$ for $x = z$ and an M-alternating cycle Q which is contained in the same connected component of $G[D(G)]$ as x (switching a cycle "nearby x").*

Then, with respect to M';
- *the configuration at z is one of the two difficult cases (Case 2B, Case 4C), or*
- *the configuration at z is one of the two Kempe-contractible cases (Case 4A, Case 5B) and there exist 4 pairwise disjoint paths between the neighbors of z and the set $A(G)$.*

Proof. We fix a planar embedding of G. For a maximum matching M' of G, we say that an unmatched vertex $v \in V(G) \setminus V(M')$ is *bad* if the configuration at v with respect to M' is either one of the two difficult cases, or one of the two Kempe-contractible cases with existing 4 pairwise disjoint paths between $N(v)$ and $A(G)$. We pick M as a maximum matching of G such that all unmatched vertices are of degree at most 5 and that the set U of bad vertices unmatched by M is minimal over all such M. So, in particular, condition a) holds true.

Verification of condition c) is straightforward; if there were $x \in U$ and $z \in S(x)$ or $z = x$ of degree at most 5, and a switched matching M' as in c), such that z is not bad with respect to M', then we would consider M' in place of M. By Claim 3.3, this switch to M' does not change the local matching configuration at any of the unmatched vertices other than z, and so it contradicts our minimality assumption on U. It thus remains to verify condition b) for all $x \in W$ (the non-bad unmatched vertices).

Let $c_1 : V(H') \to \{1, 2, 3, 4\}$ be an arbitrary proper coloring of the graph H' which is a wheel augmented graph of planar $H = H(G, F)$, and so H' is planar and Theorem 2.1 applies. Pick an unmatched vertex $x \in W$ such that the configuration at x is one of the contractible cases in the matching M. By the assumptions, F coincides with M in the neighborhood of x. Observe in Figures 1–5 that in each of these cases at least one of the adjacent matching edges $f \in M$ (and so $f \in F$ as well) has exactly one of its ends adjacent to x in G (matched outwards in the picture), and the degree of $[x]$ is at most 3 in contracted H except in Case 5A where it is 5.

Let $[f]$ denote the supervertex of f in the contracted graph H. If $\deg_H(x) \leq 3$, then $[f]$ is adjacent to every other neighbor of x in H', and so $[f]$ has a unique color among the neighbors of x in any proper coloring c_1 of H'. And since exactly one end of f is a neighbor of x in G, the lifted coloring c' of G assigns this color $c_1([f])$ to a unique vertex in the neighborhood of x in G, too. In Case 5A, the neighbors of x in H' span a 5-cycle, and thus they receive 3 distinct colors in any proper 4-coloring c_1 of H' and one of these 3 colors is unique. Since all neighbors of x are matched outwards in Case 5A, this color is also unique for x in the lifted coloring c' of G. We have verified condition b) for such x and any $c = c_1$.

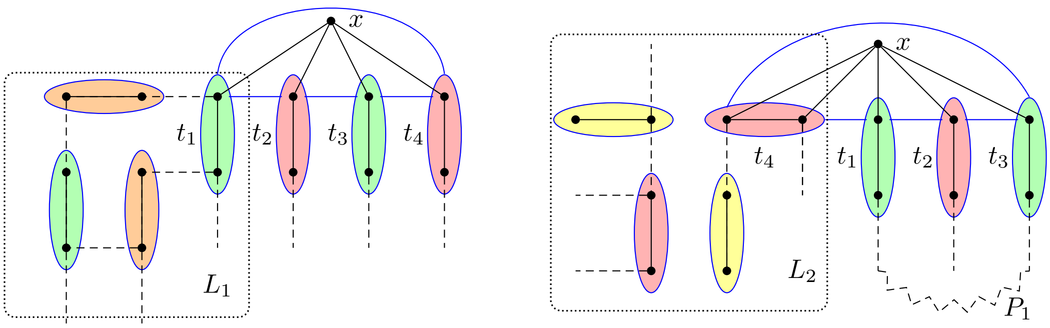
Now, pick an unmatched vertex $x \in W$ such that the configuration at x is one of the two Kempe-contractible cases. Our aim is to show that, since x is not bad ($x \notin U$), condition b) holds true for x. Again, we will use that F coincides with the matching M in the neighborhood of x. So, in each of Case 4A and Case 5B the degree of $[x]$ in H is 4, and whenever a 4-coloring c_1 assigns 3 distinct colors to the neighbors of $[x]$ in H', we get a unique color among them, which yields a unique color among the neighbors of x in G in the lifted coloring c' for any $c = c_1$.

We may hence further assume that c_1 gives only 2 colors to the 4 neighbors of $[x]$ in H' in a proper way. More specifically, and without loss of generality, let t_1, t_2, t_3, t_4 be the neighbors of x in H' in this cyclic order, and $c_1(t_1) = c_1(t_3) = 1$ and $c_1(t_2) = c_1(t_4) = 2$. Let D_0 be the connected component of $G[D(G)]$ containing x, and let D_0' result from D_0 by contracting the edges of $M \cap E(D_0)$. Furthermore, note that $M \cap E(D_0) = F \cap E(D_0)$ (so $D_0' \subseteq H'$), all vertices of D_0 except x are matched by $M \cap E(D_0)$, and no edge of $F \setminus M$ is incident to D_0 by the assumption $F \cap E(G_W) = M \cap E(G_W)$.

The next arguments are illustrated in Figure 6.

Let L_1 denote the $(1, 3)$-Kempe chain in $H' - x$ containing t_1. If $L_1 \subseteq V(D_0')$ and $t_3 \notin L_1$, then we exchange the colors 1 and 3 within L_1; more precisely, we define a new coloring c_2 of H' by setting $c_2(v) = c_1(v)$ for $v \in V(H' - x) \setminus L_1$, $c_2(x) = 4$ and $c_2(v) = 4 - c_1(v)$ for $v \in L_1$. Then c_2 is again proper on H' and c_2 assigns 3 distinct colors to the neighbors of x, and hence condition b) holds true for x in c_2, as argued above. Importantly, the new coloring c_2 is a U-protecting recoloring of our (so far arbitrary) c_1 by Claim 3.3, since $V(D_0) \cap V(G_u) = \varnothing$ for the other unmatched vertices $u \neq x$. In particular, the validity of b) is preserved (from $c = c_1$ to $c = c_2$) for the other unmatched vertices.

We can perform a similar U-protecting Kempe-chain exchange in the case that L_1 contains a path $P_1' \subseteq H' - [x]$ from t_1 to t_3 which lifts into a t_1–t_3 path P_1 in $G - x$ such that $|V(P_1) \cap A(G)| \leq 1$. In this case we let L_2 be the union of all $(2, 4)$-Kempe chains in $H' - [x]$ which are drawn in the face of the cycle $P_1' + x$ which contains t_4. We now exchange the colors 2 and 4 within L_2; we define a new coloring by $c_2(v) = c_1(v)$ for $v \in V(H' - [x]) \setminus L_2$, $c_2(x) = 3$ and $c_2(v) = 6 - c_1(v)$ for $v \in L_2$, which is proper on H'. Again, we arrive at the case of 3 colors in the neighborhood of x solved above, thus verifying b) for x. Furthermore, since $|V(P_1) \cap A(G)| \leq 1$, no other component of $G[D(G)]$ is "split" by the cycle $P_1 + x$ drawn within G, and since the vertices of P_1' do not participate in our color exchange, the



(a) The $(1,3)$-Kempe chain L_1 containing t_1 in $H' - x$ does not include t_3 and does not leave the component D'_0. Switching colors green and orange within L_1 yields a unique color in the neighborhood of x.

(b) The $\{1,3\}$-colored path P_1 forms with x a cycle separating the $(2,4)$-Kempe chains of $H' - x$ into two groups, and in one of them (say, L_2 including the contracted vertex t_4) we switch colors red and yellow. This again yields a unique color (red of t_2) in the neighborhood of x in G.

Figure 6 Illustrating the proof of Lemma 5.1: Kempe-chain exchanges in Case 4A and Case 5B – both cases are covered by the same collection of arguments since the local situation looks "the same" in the contracted graph $H(G,M)$. The wheel augmentation at x in H' is depicted by blue edges between the vertices t_1, t_2, t_3, t_4 which result by contractions of the depicted matching edges of G. Colors 1 and 2 of the neighbors t_1, t_2, t_3, t_4 of x in H' are in order green and red in the picture.

other components of $G[D(G)]$ are only possibly affected by a permutation of colors. Hence, c_2 indeed is a U-protecting recoloring of c_1 by definition and the validity of b) is not affected for the other unmatched vertices.

Since there are no edges between $V(D_0)$ and $C(G)$ by Claim 2.4, the only remaining case is that the lift of L_1 to G contains a path Q_1 from a neighbor a_1 of x (where a_1 contracts into t_1 in H') to the set $A(G)$. We may further assume that Q_1 is contained in D_0 except its end in $A(G)$. By the previous arguments and symmetry, analogous paths Q_2, Q_3, Q_4 exist from other neighbors of x which contract to t_2, t_3, t_4 in this order. Now $Q_1 \cup Q_3$ is disjoint from $Q_2 \cup Q_4$ by their definition via Kempe chains. If, say, Q_1 intersected Q_3, then there would be a t_1–t_3 path $P_1 \subseteq Q_1 \cap Q_3$ as considered in the previous paragraph. Consequently, Q_1, Q_2, Q_3, Q_4 are pairwise disjoint and x is thus bad, contradicting our assumption $x \notin U$. ◀

6 Global planarity constraints

With Lemma 5.1 at hand, for the rest of the paper, we only need to focus on resolving the remaining difficult local matching configurations (cf. Lemma 5.1 c)) – Case 2B and Case 4C, and on the subcase of having 4 disjoint paths between $N(x)$ and $A(G)$.

In their analysis, planarity of the graph G will play an important role captured via the following lemma whose proof, and related comments, are left for the full paper [15].

▶ **Lemma 6.1 (*).** *Let G_1 be a simple graph with a vertex partition $V(G_1) = A \cup B$ where $|A| = k$. Assume that the vertices of B can be partitioned into exactly $k+1$ disjoint subsets, each of which forms, as a subgraph of G_1, a* gadget *with some vertices of A (not necessarily distinct between the gadgets) that is of one of the following types:*

(I) *a single vertex $b \in B$ of degree 4, adjacent to four vertices of A;*

(II) *a copy of $K_{2,3}$ with bipartition (X, Y), where $X \subseteq B$, $|X| = 2$, and $Y \subseteq A$, $|Y| = 3$;*

(III) *a triangle T on three vertices of B, together with two vertices $a_1, a_2 \in A$ such that each of a_1, a_2 is adjacent to all three vertices of T.*

Then G_1 is not planar.

We will use Lemma 6.1 as follows. Note that the subcase of having 4 disjoint paths between $N(x)$ and $A(G)$ in the Kempe-contractible cases of Lemma 5.1 c) readily yields a minor G_1 of G containing one of the gadget types of Lemma 6.1 – type **(I)**. In this specific application, the set A is the Gallai–Edmonds set $A(G_x)$ of a particular unmatched vertex x, while the vertex $b \in B$ is the contraction of the component of $G[D(G_x)]$ containing z.

Analogously, for any unmatched vertex x and each of the difficult configurations of Lemma 5.1 c) around x, we will either be able to locally find a possibly another matching and a coloring of the contracted graph satisfying also the condition of Lemma 5.1 b), or we will show that each of the factor-critical components of $G[D(G_x)]$ yields (as a minor on the B-side) one of the three gadgets of Lemma 6.1. Since, by Claim 3.3, there are $|A(G_x)| + 1 = k + 1$ such components, and hence gadgets, Lemma 6.1 would then contradict planarity of G_x.

7 Difficult cases: 2B and 4C

Having finished the preparatory work in Lemma 5.1 and Lemma 6.1, we turn to a local in-depth analysis of each of the difficult cases; Case 2B and Case 4C. The rather lengthy and technical details are left entirely for the full paper [15], and here we only briefly illustrate the proof techniques used there on the (in our view more interesting) Case 4C.

Case 4C sketched (*). Consider a maximum matching M and a vertex $x \in U$ as in Lemma 5.1, such that x is Case 4C. Referring to Figure 4, we assume that x has neighbors a_1, a_2, a_3, a_4 in this cyclic order, such that $a_1a_2 \in M$ and $a_3a_4 \in M$ (observe that $a_1a_3, a_2a_4 \in M$ would violate planarity). The rough ideas of our solution are as follows.

Let D_0 be the connected component of $G[D(G_x)]$ containing x. For start we show that if any two consecutive vertices of a_1, a_2, a_3, a_4 are connected by a path $P \subseteq D_0$ (which is not a single edge and which avoids x and a_1, a_2, a_3, a_4 except at the ends), then we can resolve this case locally, or we contradict planarity of G or the conclusions of Lemma 5.1:

- Since $P \subseteq D_0$, every internal vertex of P is reachable from x by some M-alternating path inside D_0 via one of a_1, a_2, a_3, a_4, and so using an edge a_1a_2 or a_3a_4. With help of an easy case analysis, one can use this fact to construct an M-alternating path $P' \subseteq D_0$ between two consecutive vertices of a_1, a_2, a_3, a_4 (possibly not the same two as the ends of P).
- If the two ends of P' are a_1, a_2 up to symmetry, see Figure 7, then we can switch M by the M-alternating cycle $P' + a_1a_2$, obtaining contractible Case 4B. This is in contrary to Lemma 5.1 c) (in other words, we would directly resolve Case 4B instead).
- If, up to symmetry, the two ends of P' are a_2, a_3 and $a_1a_4 \in E(G)$, then the same solution is obtained by switching the M-alternating cycle $P' + a_1a_2 + a_1a_4 + a_3a_4$.
- If the two ends of P' are a_2, a_3, see Figure 8, then we choose a set $F \subseteq E(G)$ from switched $M \triangle E(P')$ by adding the two edges xa_1 and xa_4, and define H' as the contracted graph $H(G, F)$ with added edge a_2a_3 (which preserves planarity since a_2,a_3 are consecutive around x). As a result, any proper 4-coloring of H' assigns the supervertex $[a_1xa_4]$ one color, say 1, and two other colors, say 2 and 3, to the supervertices of a_2 and a_3. Then, in the lifted coloring of G, x sees a unique color 2 in a_2, and both a_1 and a_4 see a unique color 1 in x (since $a_1a_4 \notin E(G)$). The rest follows from Lemma 5.1.

Otherwise, we examine possible vertex degrees in D_0. Since each vertex of $V(D_0) \subseteq S(x)$ (which in this case include a_1, a_2, a_3, a_4) is switchable from x, Lemma 5.1 implies that every vertex of D_0 has degree 2, 4 or at least 6, or it is in Case 5B with 4 disjoint paths to $A(G)$. We then apply an inductive edge-density argument within D_0 which shows that there is a path $P'' \subseteq D_0$ between some two of a_1, a_2, a_3, a_4, or it eventually yields a gadget of Lemma 6.1,

With respect to the starting argument, we may hence assume that P'' is an M-alternating path and the ends of P'' are not consecutive neighbors of x. See Figure 9. Moreover, by a straightforward planarity argument, the vertices a_1 and a_3 can be joined by an edge not crossing the given drawing of G (informally, drawn on either side along P''), unless there exist, for $j = 2, 4$, a path Q_j from a_j to $V(P'') \setminus \{a_1, a_3\}$. Since there is no path between consecutive neighbors of x in D_0, both paths Q_2, Q_4 intersect $A(G)$. Consequently, after wheel augmentation at x in G, the cycle $P'' + x$ together with Q_2, Q_4 and the augmenting edges a_1a_4, a_2a_3 yield a gadget of type **(III)**.

Otherwise, we can (analogously as above) choose a set F from switched $M \triangle E(P'')$ and the two edges xa_2 and xa_4, and define H' as the contracted graph $H(G, F)$ with added edge a_1a_3 which now preserves planarity. As a result, again, any proper 4-coloring of H' assigns the supervertex $[a_2xa_4]$ one color, say 1, and two other colors, say 2 and 3, to the supervertices of a_1 and a_3. So, in the lifted coloring of G, x sees a unique color 2 in a_1, and both a_2 and a_4 see a unique color 1 in x. The rest again follows from Lemma 5.1.

8 Completing the main proof

We now assemble all previous ingredients into a proof of the main result. Again, we refer to the full paper [15] for additional details.

▶ **Theorem 8.1** (Combinatorial side of Theorem 1.1). *Every planar graph G admits an open-neighborhood conflict-free coloring with at most 4 colors.*

Sketch of Proof. For start, we remove all isolated vertices of G (since Definition 2.2 does not impose restriction on them). Let $F_1 \subseteq E(G)$ be an edge set defining the contracted graph $H_1 = H(G, F_1)$, let $M_1 \subseteq F_1$ be a matching (not necessarily maximum) such that no edge of M_1 shares a vertex with an edge of $F_1 \setminus M_1$, and $U_1 \subseteq V(G) \setminus V(M_1)$. Assume that c_1 is a proper 4-coloring of H_1 which lifts into a 4-coloring c'_1 of G, satisfying that every vertex of $V(G) \setminus (U_1 \cup V(M_1))$ has a neighbor in G of a unique color in c'_1.

First observe that if $U_1 = \emptyset$, then c'_1 is a CFON 4-coloring of G. Indeed, if $v \in V(M_1)$ and w is the M_1-neighbor of v, then the color $c'_1(w) = c'_1(v)$ is unique in the neighborhood of v in G since c_1 is proper in H_1. If, otherwise, $v \in V(G) \setminus V(M_1)$, then (since $U_1 = \emptyset$) v has a neighbor in G with a unique color in c'_1 by our assumption.

In order to find such M_1 and $U_1 = \emptyset$ in given G, we proceed as follows. We start with fixing a maximum matching M of the graph G and a subset U of unmatched vertices as in Lemma 5.1, which satisfy the previous initial conditions when setting $M_1 = F_1 = M$ and $U_1 = U$ by Lemma 5.1 b). Let $W = V(G) \setminus (U \cup V(M))$ and G_W denote (with respect to M) the union of the subgraphs G_x for $x \in W$. Among all choices of a matching M_1 obtained from M by a sequence of switchings avoiding $E(G_W)$, a set $F_1 \subseteq E(G)$, $F_1 \supseteq M_1$ such that no edge of M_1 shares a vertex with an edge of $F_1 \setminus M_1$, and $U_1 \subseteq U$, we assume one such that: U_1 is inclusion-minimal for which there is a proper 4-coloring c_1 of $H_1 = H(G, F_1)$, obtained from an arbitrary proper 4-coloring of H_1 by a U-protecting recoloring as in Lemma 5.1 b), such that c_1 satisfies the initial assumptions stated above. Note that, in particular, $M_1 \cap E(G_W) = M \cap E(G_W) = F_1 \cap E(G_W)$.

As sketched in Section 7 and detailed in the full paper [15], for any vertex $x \in U_1 \neq \emptyset$, one of the following three possibilities must hold.

- In every connected component D_0 of $G[D(G_x)]$ we obtain, by a possible independent wheel augmentation and suitable contractions restricted to D_0 (which both preserve planarity), one of the gadgets of Lemma 6.1. Since there are $|A(G_x)| + 1$ such gadgets by Claim 3.3, Lemma 6.1 then contradicts planarity of G.
- Within one of the connected components of $G[D(G_x)]$ we can switch the matching M to obtain one of the contractible or Kempe-contractible cases. This contradicts the conclusions of Lemma 5.1 c) or yields the gadget **(I)** as covered by the previous point.
- There is $z \in S(x)$ such that we switch the matching M_1 to M_2 along an M_1-alternating path from x to z (this does not interfere with already resolved/fixed vertices of U, in other words it does not enlarge U_1), and locally modify the contract set F_1 to F_2 within the component of $G[D(G_x)]$ containing z. The local configuration at z is Case 2B or Case 4C, and as shown in the detailed analysis of these two cases, in each case M_2 and $H_2 = H(G, F_2)$ can be chosen to satisfy the initial assumptions of this argument for $U_2 = U_1 \setminus \{x\}$. This contradicts minimality of U_1.

Hence, indeed, $U_1 = \emptyset$ and the proof is finished. ◀

9 Conflict-free 4-coloring algorithmically

The proof of Theorem 8.1, as presented above, is itself almost entirely efficiently algorithmic. There are two points in this procedure that deserve comment in advance.

First, we start in Lemma 5.1 with a specific maximum matching of the input graph G, namely the one of Corollary 4.2, and this is not simply provided by the algorithm of Theorem 2.5. Instead, we will use a two-step self-reduction approach.

Second, our proof relies on excluding the possible gadgets of Lemma 6.1 before trying to color the neighborhood of an unmatched vertex. In the algorithm, we instead blindly try the modifications and recolorings anticipated by our proof at the unmatched vertices, and either find an admissible resolution, or mark this case as yielding a gadget (which we do not need to know explicitly). This paradigm shift leads to necessity to re-iterate some of the proof steps in the algorithm, such as the step of proper 4-coloring of a contracted graph.

In view of the previous remarks, we assert:

▶ **Theorem 9.1** (Algorithmic side of Theorem 1.1). *An open-neighborhood conflict-free 4-coloring of an input planar graph G can be found in polynomial time.*

Proof. The algorithm works, on a high level, in three phases: (i) initialization of a matching, (ii) resolution of all unmatched vertices, and (iii) coloring and postprocessing. As noted above, in some cases the algorithm can return from phase (iii) to phase (ii) with a new unmatched vertex.

(i) In the initial phase, we first compute a matching M as in Corollary 4.2. This is done by two calls to the algorithm of Theorem 2.5; first one computes any maximum matching and the number t of unmatched vertices of G. Then, denoting by G^+ the graph obtained from G by adding t new vertices adjacent to all vertices of degree ≤ 5 in G, the second call computes a perfect matching M^+ of G^+ and sets M to be the restriction of M^+ to G. Subsequent computation of the Gallai–Edmonds decomposition of G with respect to M and of the components of $G[D(G)]$ can be done by local search in linear time.

(ii) In this phase we process, in a greedy order, all currently unmatched vertices x (with respect to M) that are not yet marked as unresolvable. In each case of x, we determine the local configuration at x and at the vertices switchable from x by a local search within G_x. Then:

- If the local configuration at $x = z$ is one of the contractible cases, or x can be switched to z being in a contractible case, then we annotate it as needing a wheel augmentation at z and mark the vertex z as resolved.
- Otherwise, if x is one of the two Kempe-contractible cases, then we annotate this case with a wheel augmentation at x and local modification steps anticipated by the proof of Lemma 5.1, and mark x as potentially resolved.
- If x is one of the two difficult cases (note that x already cannot be switched to a contractible case), and the specific local modification steps anticipated by the proofs of these cases can be carried out there, then we annotate the case with these modification steps and again mark x as potentially resolved. If not, we can conclude that the component of $G[D(G)]$ containing x yields a gadget of Lemma 6.1, mark x as unresolvable and continue the main cycle with switching to another unmatched vertex of degree at most 5.

(iii) In the last phase, we first complete the previously annotated wheel augmentations and local modifications, and make the contracted graph H. We find a proper 4-coloring of H by the algorithm of Theorem 2.1, apply previously annotated recolorings at the vertices marked as potentially resolved, and lift the resulting coloring back to G. If we succeed, we stop the algorithm with this lifted coloring as the output.

If the lifted coloring is not conflict-free at some "potentially resolved" vertex x, then we again conclude that the component of $G[D(G)]$ containing x yields a gadget of Lemma 6.1 and mark x as unresolvable. We switch the matching from x to another vertex $y \in S(x)$ of degree at most 5 such that y is not yet marked as unresolvable, and we return to phase (ii) with y.

Since G is planar, not every connected component of $G[D(G_x)]$ may yield a gadget of Lemma 6.1, and so we must eventually succeed in a resolved case above.

The runtime of our algorithm is dominated by three parts; computing a maximum matching in a planar graph which is $\mathcal{O}(n\sqrt{n})$ by [19]; the overall time for processing and postprocessing components of all inessential vertices of G, upper-bounded by $\mathcal{O}(n^2)$; and the time needed to compute a 4-coloring of a planar graph which is $\mathcal{O}(n \log n)$ by [17], multiplied by up to $\mathcal{O}(n)$ iterations. These sum to overall $\mathcal{O}(n^2 \log n)$ runtime.

A complete pseudocode can be, due to space restriction, found in the Appendix. ◀

10 Concluding remarks and questions

We have shown that planar graphs are CFON 4-colorable by lengthy technical case analysis, resolving an open question, and have given an efficient algorithm alongside. The first natural question is how to simplify our analysis and, especially, streamline the flow of arguments.

Secondly it would be nice to improve the algorithm runtime a bit; perhaps by searching for the gadget obstructions first, which would eliminate the need for repeated coloring phase.

Another direction is to explore generalizations of our result. More broadly, our method relies heavily on the interaction between planarity and the structure of factor-critical components. It is therefore natural to ask how far this approach extends beyond planar graphs. In particular, can Corollary 4.2 be generalized to wider interesting graph classes?

From the complexity point of view, one can ask how difficult it is to CFON color a planar graph with 2 or 3 colors. In [1] the authors proved that *partial* CFON coloring of planar graphs with 1, 2 or 3 colors is NP-hard, but it seems complicated to translate their result to the *full* coloring case. In the Appendix, we give a simple proof that CFON 3-coloring of planar graphs is NP-hard.

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A Additions to Section 4 (Degree bound and a classification of configurations)

▶ **Theorem 4.1** (Low-degree switchable vertex in planar graphs). *Let G be a planar graph and M a maximum matching of G. If x is unmatched in M, then there exists a vertex v switchable from x, $v \in S(x)$, with $\deg_G(v) \leq 5$ (possibly $v = x$).*

Proof. We focus on the planar induced subgraph $G_x = G\big[S(x) \cup N(S(x))\big]$ of G within the proof. By Claim 3.3, the matching M leaves exactly one vertex unmatched in G_x, namely x. In the Gallai–Edmonds decomposition of G_x we have $V(G_x) = A \cup D$ where $D = D(G_x) = S(x)$ and $A = A(G_x)$. Let $D_1, \dots, D_k$ be the connected components of $G_x[D]$.

Suppose for a contradiction that every vertex in D has degree at least 6 in G. Since G_x contains all neighbors of $S(x) = D$, we have $\deg_{G_x}(v) = \deg_G(v) \geq 6$ for all $v \in D$.

Pick a component D_i and let $n_i = |D_i|$. Then, denoting by $e(D_i)$ the number of edges inside D_i and $e(D_i, A)$ the number of edges between D_i and A, our assumption yields

$$2e(D_i) + e(D_i, A) \;=\; \sum_{v \in D_i} \deg_{G_x}(v) \;\geq\; 6|D_i| = 6n_i. \tag{1}$$

Since D_i is planar, we have $e(D_i) \leq 3n_i - 3$, and substituting into (1) gives $e(D_i, A) \geq 6n_i - 2(3n_i - 3) = 6$. Summing over all k components of $G_x[D]$,

$$e(D, A) \;\geq\; 6k. \tag{2}$$

Again by planarity of G_x, $|E(G_x)| < 3|V(G_x)|$. Thus

$$\sum_{v \in V(G_x)} \deg_{G_x}(v) = 2|E(G_x)| < 6|V(G_x)|.$$

Focusing now on the degrees in A and using $\deg_{G_x}(v) \geq 6$ for $v \in D$, we continue

$$\sum_{a\in A} \deg_{G_x}(a) \;=\; \sum_{v\in V(G_x)} \deg_{G_x}(v) - \sum_{v\in D} \deg_{G_x}(v) \;<\; 6|V(G_x)| - 6|D| \;=\; 6|A|.$$

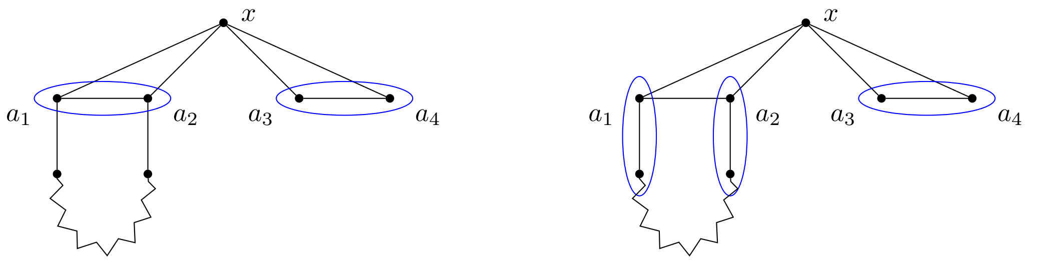
Since every edge between D and A contributes to the degree of some vertex of A,

$$e(D, A) \;\leq\; \sum_{a\in A} \deg_{G_x}(a) \;<\; 6|A|. \tag{3}$$

Finally, by Claim 2.4 e) we have $|A| < k$ (actually, $|A| = k - 1$ in this case). Substituting into (3), we get $e(D, A) < 6k$. Combining (2) and (3) we hence arrive at a contradiction $6k \leq e(D, A) < 6k$. Therefore, some vertex in $D = S(x)$ has degree at most 5 in G. ◀

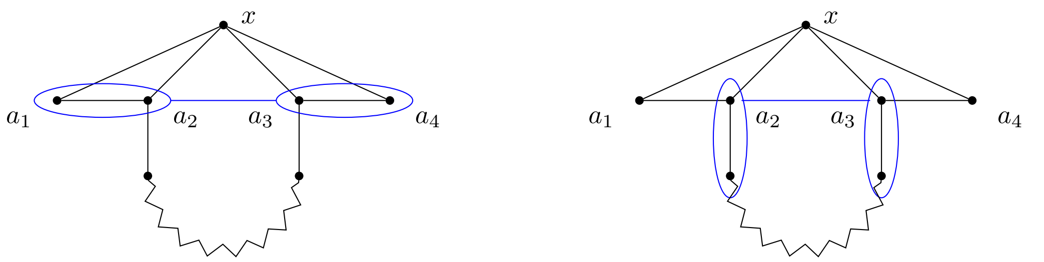
B Additions to Section 7 (Difficult cases: 2B and 4C)

We provide additional illustrating pictures for Section 7 in Figures 7–9 which did not fit into the main body due to the page limit.



Before switching along the zigzag path. After switching along this path.

Figure 7 An M-alternating path in Case 4C between two consecutive matched neighbors a_1, a_2.

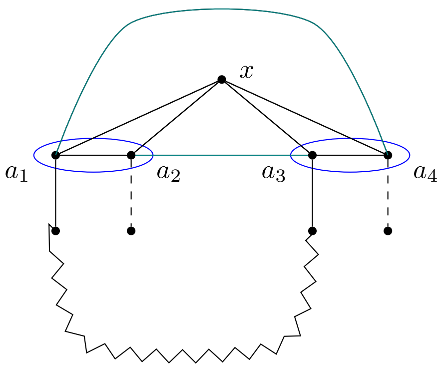


Before switching along the zigzag path. After switching along this path.

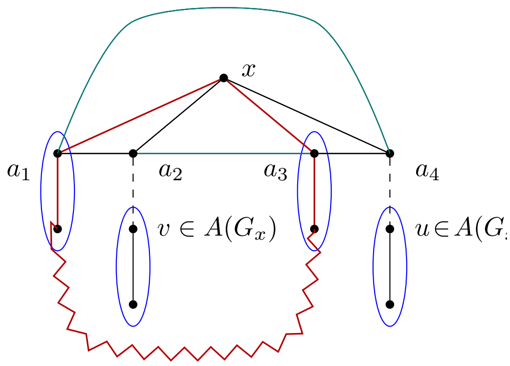
Figure 8 An M-alternating path P' in Case 4C between two consecutive non-matched neighbors a_2, a_3. We switch the matching by P' and contract all three vertices a_1, x, a_4 into one before coloring (we have $a_1x, a_4x \in F$).

C Additions to Section 9 (Conflict-free 4-coloring algorithmically)

In the algorithm pseudocode below we distinguish the current matching M from the contraction set F, which defines the contracted graph $H(G, F)$. Initially $F = M$. During the algorithm, M may be changed by switching along alternating paths, while F is updated accordingly, except for the local modifications explicitly prescribed by the difficult cases.



Before switching along the zigzag path.



After switching and showing gadget **(III)**.

Figure 9 An M-alternating path between two non-consecutive neighbors a_1, a_3 in Case 4C, and the switched situation in which we attempt to create an edge between a_1 and a_3 (along the zigzag path) or find a path connecting a_2 or a_4 to either a_1 or a_3 entirely within the factor-critical component, or we obtain gadget **(III)** with $A = A(G_x)$ and the gadget triangle drawn in red.

We shall also use the term *local resolution* for Cases 2B and 4C. More precisely, a local resolution is a modification supported inside the relevant factor-critical component of $G[D(G)]$, and it is of one of the following two types. Either we switch the matching locally so that the resulting unmatched vertex falls into one of the already contractible configurations, or we modify the contraction set F locally by adding or deleting the prescribed contraction edges. In both cases the resulting contracted graph remains planar, previously fixed components are not affected, and the configuration is either resolved immediately or recorded for the final protected recoloring step.

D Additions to Section 10 (Concluding remarks and questions)

▶ **Proposition D.1.** *The following problem is* NP*-complete: Given a planar graph G, is there a CFON coloring of G with 3 colors?*

Proof. Given a planar graph H, we construct H_1 by subdividing every edge of H with a new vertex, and H_2 by adding a new leaf to every vertex of $V(H)$ in H_1. Then H is properly 3-colorable, if and only if H_2 is CFON 3-colorable, which we prove now.

$\Rightarrow$: We take a proper 3-coloring of H and assign the same colors to the vertices of $V(H)$ in H_2. Then we give color 1 to each vertex of $V(H_1) \setminus V(H)$ in H_2, and color 2 to every vertex of $V(H_2) \setminus V(H_1)$ in H_2. This is clearly a CFON coloring.

$\Leftarrow$: Each vertex $x \in V(H_1) \setminus V(H)$ in H_2 can see a unique color in a coloring of H_2 if and only if the two neighbors of x in H_2 have distinct colors. Hence, if and only if the edge of H which is subdivided by x in H_1 receives two distinct colors. Therefore, a CFON coloring of H_2 restricts to a proper coloring of H.

The conclusion follows since proper 3-colorability of planar graphs is NP-complete. ◀

Algorithm 1 Constructing a CFON **4**-coloring of a planar graph.

Require: A planar graph G without isolated vertices
Ensure: An open-neighborhood conflict-free coloring of G with at most 4 colors
1: Compute a planar embedding of G
2: Compute the size of a maximum matching of G and the matching defect t
3: Compute a maximum matching M of G conforming to Corollary 4.2, using self-reduction based on the parameter t
4: Compute the Gallai–Edmonds decomposition of G with respect to M
5: Let $F := M$. This set will define the final contracted graph.
6: Initialize $\mathcal{U}$ as the set of unmatched vertices still under consideration.
7: Initialize $\mathcal{P} := \emptyset$ as the set of already processed unmatched representatives.
8: Initialize $\mathcal{R} := \emptyset$ as the set of vertices scheduled for the final recoloring step.
9: Initialize the set of annotations describing local changes of F.
10: **while** $\mathcal{U} \neq \emptyset$ **do**
11: Choose an unmatched vertex x from $\mathcal{U}$
12: **if** x is in or can be switched into a contractible configuration **then**
13: Switch M along the M-alternating path.
14: Update F accordingly
15: Remove x from $\mathcal{U}$
16: **else if** x is represented by Kempe-contractible case **then**
17: Add x to $\mathcal{R}$ and remove x from $\mathcal{U}$
18: **else if** x is represented by Case 2B or Case 4C and local resolution can be applied **then**
19: Make appropriate local resolutions and annotate any contractions needed not implied by matching (in Case 4C) or any contractions that should not be performed (in Case 2B), update F accordingly
20: Remove x from $\mathcal{U}$
21: **else** (x is represented by Case 2B or Case 4C and no local resolution can be applied)
22: Perform switch to another non processed vertex v of degree less than 6
23: Update F accordingly
24: Remove x from $\mathcal{U}$ and add v to $\mathcal{U}$
25: **end if**
26: Add x to $\mathcal{P}$ (mark it as processed).
27: **end while**
28: Contract the components of $(V(G), F)$, respecting all annotations, and let $H := H(G, F)$.
29: Make a wheel in H around each vertex representing a contractible configuration
30: Compute a proper 4-coloring of H
31: Lift this coloring of H to the original graph
32: **for** each vertex $x \in \mathcal{R}$ **do**
33: **if** the lifted coloring is already conflict-free on G_x **then**
34: Continue
35: **end if**
36: Perform the Kempe-chain recoloring on H as described in the proof of Lemma 5.1
37: **if** the recoloring does not succeed **then**
38: Conclude that the factor-critical component containing x yields a forbidden gadget minor (Lemma 5.1)
39: Discard the current contracted graph H
40: Perform switch to another non processed vertex v of degree less than 6 in some other factor-critical component of G_x
41: Update F accordingly
42: add v to $\mathcal{U}$ and remove x from $\mathcal{R}$
43: Go back to line 10
44: **end if**
45: **end for**
46: **return** the resulting 4-CFON coloring of G