The Prophet and the Voronoi Diagram

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Abstract

Consider a stream of n random points (say, from the unit square) arriving one by one, where a player must make an immediate and irreversible decision upon each point's arrival, whether to pick it. The player must pick exactly one such point, and the payoff is the area of the cell of the picked point, in the final Voronoi diagram of *all* the points. We show that there is a simple strategy so that with probability $\geq 1 - \widetilde{O}(1/\sqrt{n})$, the player's payoff is only a constant factor smaller than the optimal choice (i.e., the one made by the prophet). This competitiveness is somewhat surprising, as both the optimal payoff and this strategy's payoff are larger by a factor of $\Theta(\log n)$ than the average payoff.

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1 Introduction

The Game

Imagine that a player is given a sequence of points $P = \{p_1, \ldots, p_n\}$ one by one in the unit square $[0,1]^2$ (the player knows the value of n in advance). Whenever the player is presented with a point, they can decide to select it as their desired site. The player can choose only one such point during the game. The payoff is the area of the Voronoi cell of the point picked in the final diagram formed by *all* the points of P. The key restriction is that the player must decide immediately upon seeing a point p_i whether to pick it. A rejected point cannot be recalled later on.

A natural question is to devise a good strategy so that at the end of the game, the expected payoff for the player is (up to a constant factor) the same as that of the all-knowing (some would say all-annoying) prophet, who picks the site associated with the cell of maximum area at the end of the game.

The setting is somewhat similar to the *Secretary Problem*[1] in optimal stopping theory. Imagine interviewing a sequence of candidates for a position. One interviews them sequentially (assigning each one a score evaluating their abilities), say in random order. The interviewer must make an irreversible, immediate decision whether to hire a candidate once their interview is done. The target is to maximize the probability of hiring the best candidate. The classical strategy is to interview a $\frac{1}{e}$ fraction of the candidates, and then hire the first candidate that is better than any seen in the prefix. This strategy hires the best candidate with probability at least $1/e$. There are numerous variants of this problem, with some variants being useful for

[1] Bad name – the secretary is rarely the problem.

online auctions of ads in search results. For a nice survey of the history of the problem, see Ferguson [6]. Coming up with such guarantees usually involves proving a "prophet inequality" bounding the performance of the suggested strategy versus the "prophet".

The difference between our game and the secretary problem is that there is later interference with the quality of the choice the player has made – later points might shrink the chosen point's cell to be (relatively) small. A better metaphor might be land speculation – a piece of land bought early might be useless at the end of the game if somebody builds an obnoxious facility next to it [5]. Thus, our problem can be interpreted as modeling a land speculator's dilemma – should they invest now, or wait for a better opportunity later on?

We are not aware of any work on prophet inequalities with a model where the value of the final payoff might deteriorate after the decision is made.

The Voronoi Game

A related game, known as the Voronoi game, deals with the task of inserting points into a Voronoi diagram and trying to maximize the total area of the inserted points. For example, Cheong et al. [3] showed that in an existing Voronoi diagram of n points in the unit square, one can insert n new points, and the new cells have a total area of at least $1/2 + c$, where $c > 0$ is some absolute constant. This game is quite hard to analyze, but there are several known results, see [1].

The Voronoi Game When a Player Has Multiple Choices

(Although this is not directly related to our work, the known results are quite interesting. Hence we discuss them here briefly.) A game-theoretic variant of the problem was studied by Boppana et al. [2]. Here, each player is given a set of m possible locations, and each player can choose a location out of their available locations, under the above payoff model. They show that even in two dimensions (2D) there is no pure Nash equilibrium for three players, where each player has two possible locations (similar to Rock-Paper-Scissors, which lacks a pure equilibrium).

As a reminder, a *pure* equilibrium implies that each player has a unique choice of a point, such that in the resulting solution, no player can improve their payoff by changing their choice. The existence of a regular Nash equilibrium only guarantees the existence of a stable mixed strategy (where each player has a probability distribution over their choices).

Boppana et al. show that deciding whether there exists a *pure* Nash equilibrium is NP-hard for $m = 4$ in one dimension (here, each player is assigned the Voronoi interval counterclockwise from their chosen location on the circle). Strangely, if the payoff is the length of the Voronoi cell (on the circle) for a player, then a pure equilibrium always exists. To address this, Boppana et al. study the game where the locations of the points provided for each player are chosen randomly, analyzing the number of pure Nash equilibria (all on a circle).

Stochastic Geometry

There is a vast amount of work on the geometric properties of Voronoi diagrams and similar geometric structures when the input points are picked randomly [8, 9, 7, 4]. We need several such facts for our analysis, and we provide, for completeness, self-contained proofs of them. We emphasize, however, that none of these results seems to be new. Our proofs of these facts are elementary and might be of independent interest. In particular, we bound the size of the largest cell in a random Voronoi diagram (Lemma 6), and prove that many cells are almost as large (Lemma 9). The latter is a surprisingly easy implication of known bounds on balls-into-bins.

Paper Organization

In Section 2, we cover some preliminaries, including some background on martingales, and balls-into-bins. In Section 3, we prove the bounds we need on the distribution of the area of the largest cells in a random Voronoi diagram. In Section 4, we deploy these tools to present and analyze the player's strategy that guarantees $O(1)$-competitiveness with probability close to one. Some concluding remarks are given in Section 5.

2 Preliminaries

All logs in the paper are natural logs.

Torus Topology

In the following, consider the unit cube $\mathcal{H}_d = [0,1]^d$ under the torus topology. In one dimension, the distance between two points $p_1, q_1 \in [0,1]$ is $d_T(p_1, q_1) = \min(|p_1 - q_1|, 1 - |p_1 - q_1|)$. More generally, the ***torus distance*** between two points $p, q \in \mathcal{H}_d$ is

$$d_T(p,q) = \sqrt{\sum_{i=1}^{d} d_T(p_i, q_i)^2}.$$

▶ Remark 1 (Voronoi diagrams and torus topology). In the following, we deal with sets of points $P \subseteq [0,1]^2$, and their associated Voronoi diagram restricted to $[0,1]^2$, where one uses the torus distance as defined above (i.e., torus topology) to compute the diagram. To avoid annoying repetition, we emphasize that all Voronoi diagrams in the following are computed under these settings.

2.1 A Bit on Balls-into-Bins

Imagine throwing n balls into $m = \alpha n$ bins, for $\alpha = 4/\log n$, uniformly at random. For simplicity of exposition, assume that $n = t^4$, for some integer t. The probability that a specific bin has k balls at the end of this process is

$$\tau_k = \binom{n}{k} p^k (1-p)^{n-k},$$

where $p = 1/m = 1/\alpha n$.

Let $X_k = f_k(b_1, \ldots, b_n)$ be the number of bins containing exactly k balls, where $b_1, \ldots, b_n$ are the locations of the n balls. Observe that $\mathbb{E}[X_k] = m\tau_k$. A standard way to think about such processes is as an *exposure martingale*. That is, let $Z_i = \mathbb{E}[f_k \mid b_1, \ldots, b_i]$ (i.e., we expose the locations of the balls sequentially), for $i = 0, \ldots, n$. Clearly, $Z_n = X_k$, while $Z_0 = \mathbb{E}[X_k]$. It is not hard to verify that $f_k(\cdots)$ is 2-Lipschitz – moving a single ball from one bin to another changes the status of at most two bins. That implies that $|Z_i - Z_{i-1}| \leq 2$. The sequence $Z_0, \ldots, Z_n$ is a martingale.

▶ **Theorem 2** (Azuma's Inequality). *Let $Z_0, \ldots, Z_n$ be a martingale with $Z_0 = \mu$, and $|Z_{i+1} - Z_i| \leq c_i$, for $i = 0, \ldots, n-1$. Then, for any $\lambda > 0$, we have $\mathbb{P}[|Z_n - \mu| > \lambda\sqrt{n}] < 2\exp\Bigl(-\lambda^2 n / (2\sum_i c_i^2)\Bigr)$.*

2.1.1 Proving Concentration of Empty and Singleton Bins

The following are pretty standard estimates – we include the detailed proofs for the sake of completeness.

▶ **Lemma 3.** *Consider throwing $n \geq 2$ balls into $m = \alpha n$ bins, for $\alpha = \beta/\log n$, where $\beta \geq 4$ is a constant and $n \geq c_0$, where c_0 is some* sufficiently *large constant. Let X_0 be the number of empty bins. We have $X_0 = \Theta(n^{1-1/\beta}/\log n)$ with probability $\geq 1 - n^{-c}$, where $c \geq 8$ is an arbitrarily large constant.*

Proof. For $k = 0$, using $1 - x \leq e^{-x}$, we have

$$\mathbb{E}[X_0] = \alpha n\left(1 - \frac{1}{\alpha n}\right)^n \leq \alpha n \exp\left(-\frac{1}{\alpha n}\right)^n = \alpha n \exp\left(-\frac{1}{\alpha}\right)$$
$$= \alpha n \exp\left(-\frac{\log n}{\beta}\right) = \beta\frac{n^{1-1/\beta}}{\log n}.$$

Using $1 - x \geq \exp(-x - x^2)$, we have

$$\mathbb{E}[X_0] \geq \alpha n \exp\left(-\frac{1}{\alpha n} - \frac{1}{\alpha^2 n^2}\right)^n = \alpha n \exp\left(-\frac{1}{\alpha}\right)\exp\left(-\frac{1}{\alpha^2 n}\right) \geq \frac{\beta n^{1-1/\beta}}{2\log n},$$

for n sufficiently large. The natural exposure martingale for X_0, implies by Azuma's inequality, that $\mathbb{P}[|X_0 - \mathbb{E}[X_0]| \geq \lambda\sqrt{n}] \leq 2\exp(-\lambda^2/8)$. The claim now follows, by taking (say) $\lambda = n^{1/8}$. ◀

▶ Remark 4. Somewhat confusingly, in the above lemma, we throw n balls into significantly fewer bins (i.e., fewer by a logarithmic factor). It is easy to verify that there are no empty bins, with high probability, if the number of bins is (say) $\beta\frac{n}{\log n}$ for $\beta = 1/10$. The lemma states that if $\beta \geq 4$, then the number of empty bins is polynomially large with high probability. So everything here is about the exact value of β, as there is a phase transition phenomenon for the number of empty bins in the interval $\beta \in [1/10, 4]$.

▶ **Lemma 5.** *For the number X_1 of bins containing exactly one ball, we have $X_1 \geq n^{1-1/\beta}/(4\log n)$ with high probability (i.e., $X_1 = \Theta(n^{1-1/\beta}/\log n)$).*

Proof. Arguing as above, as $\alpha = \beta/\log n$, we have

$$\mathbb{E}[X_1] = \alpha n \cdot n\frac{1}{\alpha n}\left(1 - \frac{1}{\alpha n}\right)^{n-1} \leq n\exp\left(-\frac{n-1}{\alpha n}\right) \leq 2n\exp\left(-\frac{1}{\alpha}\right)$$
$$= 2n\exp\left(-\frac{\log n}{\beta}\right) = 2n^{1-1/\beta}.$$

Similarly, we have $\mathbb{E}[X_1] = n\left(1 - \frac{1}{\alpha n}\right)^{n-1} \geq n\exp\left(-\frac{1}{\alpha} - \frac{1}{\alpha^2 n}\right) \geq \frac{n^{1-1/\beta}}{2}$. Applying Azuma's inequality to the exposure martingale for X_1 yields the result. ◀

3 The Distribution of Area of Cells in a Voronoi Diagram

3.1 The Largest Voronoi Cell Is Not That Large

▶ **Lemma 6.** *Consider throwing $n \geq 2$ points $P = \{p_1, \ldots, p_n\}$, uniformly and independently, into $\mathcal{H} = [0,1]^2$, and let $\mathcal{V} = \mathcal{V}(P)$ be the Voronoi diagram of the points of P. Let A be the area of the largest Voronoi cell in $\mathcal{V}$. Then, for any constant $c > 1$, we have $\mathbb{P}[A \geq 4c\frac{\log n}{n}] \leq \frac{1}{n^{2c-4}}$.*

Proof. Let p be a point in P, and assume it has a cell C in $\mathcal{V}$. Let u be the furthest point from p in C, with $r = d_T(p, u)$. Observe that $\text{area}(C) \leq \pi r^2$, implying that $r = d_T(p, u) \geq \sqrt{\text{area}(C)/\pi}$.

Let G be a set of n^4 points formed by a uniform grid in $\mathcal{H} = [0, 1]^2$, such that the distance of any point in $\mathcal{H}$ to the nearest point in G is at most $\sqrt{2}/n^2$. Observe that if there is a disk of radius r, for $r > 1/n$, in $\mathcal{H}$ containing no points of P, then there is such a disk of radius $r - \sqrt{2}/n^2 \geq R = r(1 - 1/8)$ centered at a point of G.

If the area of C is at least $A = 4c\frac{\log n}{n}$, then $r \geq \sqrt{A/\pi} \geq \sqrt{c\frac{\log n}{n}}$, and $R \geq \frac{7}{8}\sqrt{c\frac{\log n}{n}}$. By the union bound, the probability that there exists a disk of radius R centered at some point of G that is empty, is at most

$$n^4 \bigl(1 - \pi R^2\bigr)^n \leq n^4 \exp\bigl(-\pi R^2 n\bigr) \leq n^4 \exp\left(-\pi \frac{49 c \log n}{64 n} \cdot n\right) \leq n^{4-2c}. \qquad \blacktriangleleft$$

3.2 There Are Many Large Voronoi Cells

3.2.1 A Voronoi Cell of a Point vs. the Boundary of a Square

In the following, let C be a square in the plane. For a point $p \in C$, let D_p be the locus of points closer to p than any point on the boundary of C. Consider the function $f(p) = \text{area}(D_p)/\text{area}(C)$.

▶ **Lemma 7.** *For a random point $p \in C$, we have $\mathbb{P}[f(p) \geq 1/15] \geq 1/15$.*

Proof. Let ℓ be the side length of C, and let C' be the square of length $\delta = \ell/(1 + 2\sqrt{2})$ centered at the center of C, and assume that both C and C' are axis-aligned. We require that the minimum distance of any point in C' to ∂C is at least $\sqrt{2}\delta$. Namely, we have that $\ell = 2\sqrt{2}\delta + \delta$, see Figure 1 – this is how the value of δ was chosen.

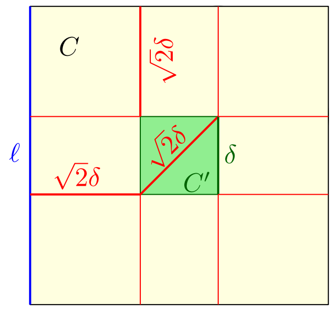

Figure 1 A point inside C' contains it in its Voronoi cell.

Thus, the probability that a random point p falls into C' is $1/(1 + 2\sqrt{2})^2 \geq 1/15$. If p falls inside C' then $C' \subseteq D_p$ (i.e., all the points of C' are closer to p than any point of ∂C), implying the claim. ◀

▶ Remark 8. A significantly more tedious argument shows that $\mathbb{P}[f(p) \geq 1/12] \geq 1/12$. Numerical simulations suggest that the correct statement of this type is $\mathbb{P}[f(p) \geq 1/6] \geq 1/6$. A simple and elegant proof of this would be quite interesting, although for our purposes, any constant suffices.

3.2.2 Many Large Cells

▶ **Lemma 9.** *Let P be a set of n points picked uniformly and independently at random from $[0,1]^2$, where n is sufficiently large. Then, there is a constant γ, such that in the Voronoi diagram of P in $[0,1]^2$, there are at least $\sqrt{n}/30$ cells of area at least $\gamma(\log n)/n$. Furthermore, these cells are relatively fat – there is a radius $R = \Theta(\sqrt{(\log n)/n})$, such that the disk of radius R centered at the site of each of these large cells contains no other point of P.*

Proof. Let t be the largest integer such that

$$4\frac{n}{\log n} \leq t^2 \leq 8\frac{n}{\log n}.$$

We partition $[0,1]^2$ into $m = t^2$ cells by using the uniform grid $t \times t$. Observe that $m = \beta n/\log n$, for some $4 \leq \beta \leq 8$. Thus, by Lemma 5, we have that there are at least $X_1 \geq n^{1-1/\beta}/(4\log n) \geq \sqrt{n}$ cells of this grid that contain exactly one point, and this happens with high probability.

Consider such a cell C with a single point p in it (a "singleton" cell). By Lemma 7, with probability $\geq \psi = 1/15$, the point p is closer to at least a ψ-fraction of this cell than to the boundary of the cell. Thus, the Voronoi cell of p in the Voronoi diagram of P, with probability at least ψ, is at least of area $\xi = \psi/t^2 \geq \frac{\log n}{15 \cdot 8n} = \frac{\log n}{120n}$. Namely, the cell of this point is quite large with probability at least ψ. The probability that all such singleton cells fail to generate a large Voronoi cell is at most $(1-\psi)^{X_1} = O\big(\exp(-\sqrt{n}/15)\big)$, which is absurdly small. Moreover, let Y be the number of singleton cells in the grid that succeed in generating a cell of size $\geq \xi$. We have that $X_1 \geq \sqrt{n}$, and thus $\mu = \mathbb{E}[Y] \geq \psi X_1 \geq \sqrt{n}/15$. A standard application of Chernoff's inequality now implies that

$$\mathbb{P}\big[Y \leq \sqrt{n}/30\big] \leq \mathbb{P}[Y \leq \mu/2] \leq \exp(-\mu/8) \leq \exp(-\sqrt{n}/128),$$

which is still absurdly small.

The claim about the radius is a byproduct of the above analysis – each successful singleton cell has a point close to the cell's center, and a disk of radius proportional to the sidelength of the cell is fully contained in the Voronoi cell of p, see Figure 1. ◀

4 The Player's Strategy

For any i, let $P_i = \{p_1, \ldots, p_i\}$ denote the prefix of the first i points.

In the following, we assume n is sufficiently large, and the player is exposed to the random points of $P = \{p_1, \ldots, p_n\}$ one by one. As a reminder, the points are picked uniformly at random from the unit square $[0,1]^2$. The player's strategy is to ignore the first (say) $n - f$ points, where

$$f = c\sqrt{n}\log n, \tag{1}$$

and c is a sufficiently large constant to be determined shortly. The player then picks the first point p_i, in the remaining suffix of the permutation, such that the disk D_i of radius $R = \Theta(\sqrt{(\log n)/n})$ (the exact value is specified by the analysis of Lemma 9) centered at p_i, contains no earlier point in the permutation. The player made their decision, and now has to hope for the best.

▶ **Theorem 10.** *Let P be a sequence of n points picked uniformly at random from $\mathcal{H} = [0,1]^2$. The sequence is provided in an online fashion, one point at a time, and the player must irrevocably pick one of the points in the sequence when it is provided (without knowing the* *future points). Then, the above strategy results in the player having a Voronoi cell of area $\Omega(\Delta)$, where $\Delta = {}^{\log n}/_{n}$ and the largest cell in the Voronoi diagram of P (under the torus topology) has area at most $O(\Delta)$. That is, the player's strategy is $O(1)$-competitive with the optimal strategy that has full knowledge of P. This guarantee holds with probability $\geq 1 - O({}^{\log^2 n}/_{\sqrt{n}})$.*

Proof. By Lemma 9, there are at least $U \geq c'\sqrt{n}$ sites in P, containing a disk of radius $R = \Theta(\sqrt{(\log n)/n})$ centered at the site, that does not contain any other point of P, where c' is some absolute constant. This implies that in expectation, by Eq. (1), there are at least

$$N = \frac{U}{n} f \geq \frac{c'\sqrt{n}}{n} \cdot c\sqrt{n}\log n \geq c'' \log n,$$

such sites in the suffix $p_{n-f+1}, \ldots, p_n$, and one can make the leading constant c'' arbitrarily large, by increasing the constant c in the strategy used to define f. The probability that none of these U large sites fall in the suffix $P \setminus P_{n-f}$ is clearly polynomially small. For those of little faith, this probability is

$$\eta = \frac{n-U}{n} \cdot \frac{n-U-1}{n-1} \cdots \frac{n-U-f+1}{n-f+1} \leq \left(\frac{n-U}{n}\right)^f$$
$$\leq \exp\left(-\frac{Uf}{n}\right) = \exp(-N) = \frac{1}{n^{c''}},$$

as one imagines picking the last f points first, avoiding the U "forbidden" large sites. As we can make this bad probability arbitrarily small, this means that with high probability, a large cell would be encountered in the last f sites.

All the player can do is to check that the disk D_i is empty of points of P_{i-1}. By the above, the player would encounter such a disk in the last f sites, with high probability. Sadly, if somewhat unlikely, one of the final $(n - i \leq f)$ remaining sites might still hit D_i. This is considered to be a failure, and the probability for that, using the union bound, is

$$f \cdot \mathrm{area}(D_i) = O\left(\frac{\log n}{n}\sqrt{n}\log n\right) = O\left(\frac{\log^2 n}{\sqrt{n}}\right).$$

If the disk D_i the player chose is not hit, then at least a quarter of its area would belong to the final Voronoi cell of p_i. Indeed, this cell is at least the Voronoi cell of p_i versus all the boundary points of D_i. ◀

▶ Remark 11. One can reduce the probability of failure by observing that at least two sites must hit D_i for the area the player gets to become catastrophically smaller. Indeed, a single site hitting D_i can "steal" at most half the area of the Voronoi cell assigned to p_i before it got hit.

5 Conclusions

We proposed a new geometric variant of the secretary problem where the payoff is a function computed at the end of the game – specifically, the area of the Voronoi cell the player picked. Unlike in the secretary problem, the payoff is not guaranteed at the time of selection and is subject to deterioration due to interference by later elements appearing in the sequence. We are unaware of previous work on such variants, and we think they are both interesting and natural.

Our result should extend to higher dimensions $d > 2$ in a straightforward fashion, as we used surprisingly few geometric properties of Voronoi diagrams. Our analysis relied on showing that there are many sites that are far from other sites and thus have large cells, enabling the player to wait for almost the end of the sequence before making their choice. There are many natural problems for further research:

(I) Different interference models.

(II) What if the sites are predetermined, but their order is random.

(III) What if the player can make k choices (and keep all of them, or only the best one)?

References

1 H.-K. Ahn, S.-W. Cheng, O. Cheong, M. Golin, and R. van Oostrum. Competitive facility location: The Voronoi game. *Theo. Comp. Sci.*, 310:457–467, January 2004. `doi:10.1016/j.tcs.2003.09.004`.

2 Meena Boppana, Rani Hod, Michael Mitzenmacher, and Tom Morgan. Voronoi choice games. In Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, editors, *Proc. 42nd Int. Colloq. Automata Lang. Prog.* (ICALP), volume 55 of *LIPIcs*, pages 23:1–23:13. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2016. `doi:10.4230/LIPIcs.ICALP.2016.23`.

3 O. Cheong, S. Har-Peled, N. Linial, and J. Matoušek. The one-round Voronoi game. *Discrete Comput. Geom.*, 31(1):125–138, 2004. `doi:10.1007/S00454-003-2951-4`.

4 Sung Nok Chiu, Dietrich Stoyan, Wilfrid S Kendall, and Joseph Mecke. *Stochastic Geometry and Its Applications.* John Wiley & Sons, Chichester, UK, 3 edition, 2013. `doi:10.1002/9781118658222`.

5 Richard L. Church and Zvi Drezner. Review of obnoxious facilities location problems. *Comput. & Oper. Res.*, 138:105468, February 2022. `doi:10.1016/j.cor.2021.105468`.

6 Thomas S. Ferguson. Who solved the secretary problem? *Stat. Sci.*, 4(3):282–289, 1989. `doi:10.1214/ss/1177012493`.

7 A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations: Concepts and applications of Voronoi diagrams.* Probability and Statistics. John Wiley & Sons, Inc., New York, NY, USA, 2nd edition edition, 2000. URL: `http://okabe.t.u-tokyo.ac.jp/okabelab/{Voronoi}/`.

8 L. Santalo. *Introduction to Integral Geometry.* Paris, Hermann, 1953.

9 Luis A Santaló. *Integral Geometry and Geometric Probability.* Cambridge University Press, Cambridge, 2 edition, 2004. Classic Reprint. `doi:10.1017/CBO9780511617331`.