Smallest Convex Hulls of Polygons

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Abstract

We study the problem of minimizing the area of the convex hull of k polygons with a total of n vertices in the plane, under translations and rigid motions for any fixed $k \geq 3$. For any $\varepsilon \in (0, 1)$, we give $(1 + \varepsilon)$-approximation algorithms running in $O(\varepsilon^{-1/2} \log n + \varepsilon^{1/2-k})$ time for translations, and in $O(\varepsilon^{-1/2} \log n + \varepsilon^{3/2-2k})$ time for rigid motions. We also consider minimizing the perimeter of the convex hull under translations and obtain a $(1 + \varepsilon)$-approximation algorithm running in $O(\varepsilon^{-1/2} \log n + \varepsilon^{1/2-k} \log^{k-1}(1/\varepsilon))$ time. To the best of our knowledge, these are the first results of this kind for $k \geq 3$ polygons. Furthermore, for the special case of two polygons with n_0 and n_1 vertices ($n_0 \geq n_1$), respectively, we give an $O(n_0 + n_1 \log^2(n_0 + n_1))$-time algorithm for the minimum-perimeter problem. This significantly improves upon the best-known $O((n_0 + n_1) \log^2(n_0 + n_1))$ bound by eliminating the logarithmic overhead associated with the larger input size n_0.

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1 Introduction

We study the problem of minimizing the area of the convex hull of k polygons for any fixed integer $k \geq 3$ in the plane under translations and rigid motions. The translation problem can be formulated as follows: Given k polygons $C_0, \ldots, C_{k-1}$ with a total of n vertices in the plane, determine a translation vector $\mathbf{t}_i \in \mathbb{R}^2$ for each polygon C_i such that the area of the convex hull $\mathsf{conv}(\bigcup(C_i + \mathbf{t}_i))$ is minimized. The rigid motion problem allows reorienting each input polygon. Observe that the target area remains unchanged when each polygon, not necessarily convex, is replaced by its convex hull in linear time. Thus, we assume that the input polygons are convex.

Convex hull minimization has broad applications in shape analysis and packing; it naturally arises when designing a smallest container that accommodates multiple objects under allowed motions. Motivated by such settings, we study minimizing the area (and also the perimeter) of the convex hull of multiple polygons under translations and rigid motions.

Related problems such as overlap maximization, bundling, and packing have been studied extensively under translations and rigid motions in the plane and higher dimensions (e.g., [16, 12, 15, 17, 11, 18, 8, 7]). For two convex polygons, near-linear or expected-linear-time algorithms are known (e.g., [6, 9, 13]). In higher dimensions, polynomial-time algorithms and improved bounds have also been developed (e.g., [4, 5, 2, 13]). In contrast, our smallest convex hull objective for $k \geq 3$ remains less explored due to the nonconvexity of the area function in the full translation space.

1.1 Our results

In this paper, we investigate the optimization of the convex hull area for a set of k convex polygons $C_0, \dots, C_{k-1}$ with n vertices in total in the plane, under translations and rigid motions for any fixed $k \geq 3$. We show that the area function of $\mathsf{conv}(\bigcup(C_i + \mathbf{t}_i))$ is not convex in the translation space $\mathbb{R}^{2k-2}$; it may even fail to be convex along many lines or low-dimensional flats. We characterize structural changes in the convex hull via event surfaces, bound their number by $O(n)$, and obtain an exact translation algorithm running in $O(n^\eta)$ time, where $\eta = \max\{5, 4k - 8 + \delta\}$ for any fixed $\delta > 0$.

Our core structural contribution reveals a hidden convexity in the translation space. While the area function is generally non-convex in $\mathbb{R}^{2k-2}$, we establish that it becomes strictly convex when restricted to any cleverly chosen $(k-1)$-dimensional affine subspaces. Moreover, we rigorously prove that $k-1$ is the maximum possible dimension of an affine subspace of $\mathbb{R}^{2k-2}$ on which the area function is convex. This structural breakthrough is the key that reduces our effective search space from dimension $2k-2$ to $k-1$, enabling our highly efficient $(1+\varepsilon)$-approximation algorithms, running in $O(\varepsilon^{-1/2} \log n + \varepsilon^{1/2-k})$ time for translations and $O(\varepsilon^{-1/2} \log n + \varepsilon^{3/2-2k})$ time for rigid motions for any $\varepsilon \in (0, 1)$.

To the best of our knowledge, these are the first algorithms for $k \geq 3$ polygons. Our approximation algorithms are significantly more efficient than the exact algorithm, as they identify translations in a $(k-1)$-dimensional space. Furthermore, they are simple. The translation algorithm computes an optimal translation for the k convex polygons in a $(k-1)$-dimensional space by constructing a cutting and finding the $(k-1)$-flat containing the optimal translation recursively. The algorithm for rigid motion samples orientations of each polygon and determines the approximate translations for each orientation of each polygon using the translation algorithm.

We also investigate the problem of minimizing the perimeter of the convex hull under translations. The perimeter function for a translation $\mathbf{t} \in \mathbb{T}$ is $\Psi(\mathbf{t}) = \mathsf{peri}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i + \mathbf{t}_i))$. The goal is to compute a translation $\mathbf{t}^* \in \mathbb{T}$, which we call an optimal translation, such that $\Psi(\mathbf{t}^*) \leq \Psi(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}$. Using the fact that the perimeter function $\Psi(\mathbf{t})$ is convex [6] and applying the algorithmic framework for the minimum-area variant to this problem, we obtain a $(1+\varepsilon)$-approximation algorithm running in $O(\varepsilon^{-1/2} \log n + \varepsilon^{1/2-k} \log^{k-1}(1/\varepsilon))$ time. We present an algorithm running in $O(n_0 + n_1 \log^2(n_0 + n_1))$ time for the minimum perimeter problem for two convex polygons with n_0 and n_1 vertices each $(n_0 \geq n_1)$. This significantly improves upon the best-known $O((n_0 + n_1) \log^2(n_0 + n_1))$ bound by eliminating the logarithmic overhead associated with the larger input size n_0. Details are given in Appendix A.

1.2 Sketch of our algorithms

Our approximation algorithm under translations assumes a fixed C_0 and translates C_i by a vector $\mathbf{t}_i \in \mathbb{R}^2$ for $i = 1, \dots, k-1$. Let $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{k-1}) \in \mathbb{R}^{2k-2}$ and $\Upsilon(\mathbf{t})$ be the area function of $\mathsf{conv}(C_0 \cup \bigcup(C_i + \mathbf{t}_i))$. Unlike the case for $k = 2$, $\Upsilon(\mathbf{t})$ is neither convex nor piecewise linear in the translation space of the $k-1$ polygons. Furthermore, each modification to $\mathsf{conv}(C_0 \cup \bigcup(C_i + \mathbf{t}_i))$ can be expressed as a hyperplane or quadric surface in translation space, called an *event surface*.

We identify an affine subspace of the translation space in which $\Upsilon(\mathbf{t})$ is convex and piecewise linear. This subspace is referred to as the $(\mathbf{t}_0, \mathbf{a})$-*translation space* $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$ with a fixed unit vector $\mathbf{t}_0 \in \mathbb{R}^2$ and a fixed vector $\mathbf{a} \in \mathbb{R}^{2k-2}$. We show that $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$ has the maximum dimension, $k-1$, in which $\Upsilon(\mathbf{t})$ is convex. In $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$, each event surface is a hyperplane.

We design a linear-time algorithm to compute a translation $\mathbf{t}_0^* \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ that satisfies $\Upsilon(\mathbf{t}_0^*) \leq \Upsilon(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Let $\triangle$ be an m-dimensional simplex with $m \leq k-1$ in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Let $\mathbf{t}_\triangle$ be a translation in $\triangle$ satisfying $\Upsilon(\mathbf{t}_\triangle) \leq \Upsilon(\mathbf{t})$ for all $\mathbf{t} \in \triangle$. The algorithm employs a procedure for computing an m-dimensional simplex $\triangle^* \subseteq \triangle$ such that $\mathbf{t}_\triangle \in \triangle^*$ and no event hyperplane intersects the interior of $\triangle^*$. Furthermore, the algorithm utilizes an oracle that determines the relative position of $\mathbf{t}_\triangle$ to an $(m-1)$-flat intersecting $\triangle$.

The procedure operates as follows. It constructs a $(1/2)$-cutting of $(m-1)$-dimensional flats, each containing the intersection of $\triangle$ and an event hyperplane. It obtains the cell $C_\triangle$ containing $\mathbf{t}_\triangle$ by executing the oracle with $(m-1)$-dimensional faces of the cutting. Subsequently, it triangulates $C_\triangle$, executes the oracle with $(m-1)$-dimensional faces of the triangulation, and identifies an m-dimensional simplex containing $\mathbf{t}_\triangle$. It repeats this process with the simplex and the event hyperplanes intersecting the simplex recursively until no event hyperplane intersects the interior of the resulting simplex. Finally, it computes $\triangle^*$.

The oracle works as follows. Assume that we have $\Upsilon(\mathbf{c})$ and a normalized directional derivative at each vertex $\mathbf{c}$ of $\triangle$. Given an $(m-1)$-dimensional flat F, the oracle computes an $(m-2)$-dimensional simplex $\triangle'$ that contains an optimal translation restricted to $\triangle \cap F$ in time linear in the number of event hyperplanes intersecting $\triangle$. Next, it computes a normalized directional derivative at each vertex $\mathbf{v}$ of $\triangle'$. This is done by progressively updating the derivative from a vertex $\mathbf{c}$ of $\triangle$ along the segment connecting $\mathbf{c}$ and $\mathbf{v}$, at each intersection with the event hyperplanes. This process takes time linear in the number of event hyperplanes intersecting the segment. We can determine the relative position of $\mathbf{t}_\triangle$ with respect to F in the same asymptotic time bound for computing a minimum value of $\Upsilon(\mathbf{t})$ in $\triangle \cap F$. Then, the algorithm updates the derivatives at $\mathbf{v}$ by changing their directions such that they point toward the interior of $\triangle'$. Similarly, it computes $\Upsilon(\mathbf{v})$ in time linear in the number of event hyperplanes containing $\mathbf{v}$.

For N event hyperplanes intersecting an m-dimensional simplex, let $T(N, m)$ and $T_o(N, m)$ denote the times for the procedure and the oracle, respectively. Then, $T(N, m) = O(N) + O(T_o(N, m)) + T(\lfloor N/2 \rfloor, m)$ and $T_o(N, m) = O(N) + T(N, m-1)$. We show that $T(N, m) = O(N)$ and $T_o(N, m) = O(N)$.

Using the linear-time algorithm in the preceding paragraphs, we devise a $(1+\varepsilon)$-approximation algorithm. We assume that C_0 has the longest diameter among all input polygons and the segment connecting a diametral pair is horizontal. The algorithm constructs $O(1/\varepsilon)$ horizontal lines and computes a translation for each $C \in \mathsf{C} \setminus \{C_0\}$ such that the topmost vertex of $C \in \mathsf{C} \setminus \{C_0\}$ is positioned on one of the horizontal lines. It enumerates $O(1/\varepsilon^{k-1})$ distinct placements for the $k-1$ topmost vertices over the horizontal lines, and for each placement, computes an optimal translation $\mathbf{t}$ of the $k-1$ polygons along their corresponding lines using the linear-time algorithm in the preceding paragraphs. Thus, in $O(n/\varepsilon^{k-1})$ time, the algorithm computes a translation $\mathbf{t}_{\mathsf{apx}}$ with $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq \Upsilon(\mathbf{t})$ for all such optimal translations $\mathbf{t}$. We show that $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Upsilon(\mathbf{t}^*)$. By approximating each input polygon into a convex polygon with $O(\varepsilon^{-1/2})$ vertices in the beginning of the algorithm, we reduce the running time to $O(\varepsilon^{-1/2} \log n + \varepsilon^{1/2-k})$.

For the smallest convex hull problem under rigid motions, we generalize the method for two polygons in [6]. We generate a set S_i of $O(1/\varepsilon)$ angles of C_i for each $i = 1, 2, \ldots, k-1$. For each distinct rotation $\mathbf{r} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$ with $\theta_i \in S_i$, we compute a translation for the rotated copies, each of C_i by θ_i for $i = 1, \ldots, k-1$, by using the algorithm in the previous paragraph. We show that there exist a rotation $\mathbf{r} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$ with $\theta_i \in S_i$ and a translation $\mathbf{t}$ obtained from the algorithm such that $\Upsilon(\mathbf{t})$ is less than $1+\varepsilon$ times the optimum area. Since there are $O(1/\varepsilon^{k-1})$ distinct rotations $\mathbf{r}$, the algorithm runs in $O(n/\varepsilon^{2k-2})$ time. By approximating each convex polygon, we reduce the running time of the algorithm to $O(\varepsilon^{-1/2} \log n + \varepsilon^{3/2-2k})$.

Our algorithm for minimizing the perimeter of the convex hull under translations works as follows. Let $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{k-1}) \in \mathbb{R}^{2k-2}$ and $\Psi(\mathbf{t})$ be the perimeter function of $\mathsf{conv}(C_0 \cup \bigcup(C_i + \mathbf{t}_i))$. Using the fact that the perimeter function $\Psi(\mathbf{t})$ is convex [6] and applying the algorithmic framework for the minimum-area variant to this problem, we design an $O(n \log^{k-1} n)$-time algorithm to compute a translation $\mathbf{t}_0^* \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ that satisfies $\Upsilon(\mathbf{t}_0^*) \leq \Upsilon(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Based on this algorithm, we devise a $(1+\varepsilon)$-approximation algorithm in the same manner as for the minimum-area variant. This leads to an algorithm with running time $O(\varepsilon^{-1/2} \log n + \varepsilon^{1/2-k} \log^{k-1}(1/\varepsilon))$ that computes a translation $\mathbf{t}_{\mathsf{apx}}$ for the polygons satisfying $\Psi(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Psi(\mathbf{t}^*)$ for any translation $\mathbf{t}^*$ minimizing the perimeter function Ψ.

For the specific case of two polygons with n and m vertices, respectively ($n \geq m$), we can apply the algorithmic framework for the minimum-area variant to the translation space directly. By showing $T_o(N, 2) = O(N + m) + T(N, 1)$ and $T(N, 1) = O(N + m \log N)$, we obtain an algorithm with running time $O(n + m \log^2(n + m))$.

The rest of the paper is organized as follows. Section 2 introduces preliminaries. Section 3 discusses the exact algorithm for computing the minimum-area convex hull for k polygons with $k \geq 3$ under translations. In Section 4, we show that $k-1$ is the maximum possible dimension of such a subspace. In Sections 5 and 6, we give $(1+\varepsilon)$-approximation algorithms for the minimum-area convex hull under translations and rigid motions, respectively. Finally, in Appendix A, we consider the problem of minimizing the perimeter of the convex hull under translation and give an exact algorithm for two polygons, and a $(1+\varepsilon)$-approximation algorithm for k polygons with $k \geq 3$.

2 Preliminaries

We work in a real-RAM model, where basic logical and arithmetic operations take $O(1)$ time. We assume that a system of $O(1)$ polynomials of fixed degree in $O(1)$ variables can be solved in $O(1)$ time. For a positive integer a, let $[a]$ denote the interval of all integers between 1 and a, inclusive. For a point $p \in \mathbb{R}^2$, we denote the x-coordinate and y-coordinate of p by $x(p)$ and $y(p)$, respectively. For two points $p, q \in \mathbb{R}^d$, we denote the line segment connecting p and q by pq, and its length by $|pq|$. We denote the lengths of its projections onto the x-axis and onto the y-axis by $x(pq)$ and $y(pq)$, respectively. The *angle* of a line ℓ is the angle swept from the x-axis in a counterclockwise direction to ℓ. So, it lies in $[0, \pi)$. The angle of a line segment s is the angle of the line containing s. For a finite set F, $|F|$ denotes its cardinality.

We treat a d-dimensional Euclidean space $\mathbb{R}^d$ as a vector space. An *affine space* in $\mathbb{R}^d$ is the Minkowski sum of a vector and a vector space. The dimension of an affine space is the dimension of its vector space. An *m-flat* is an affine m-dimensional subspace in $\mathbb{R}^d$ and an *m-simplex* is an m-dimensional polytope that is the convex hull of its $m+1$ vertices.

The *combinatorial structure* of a polytope in $\mathbb{R}^d$ is the set of faces (0-faces, 1-faces, ..., d-faces) of the polytope, ordered by inclusion, which is also called the *combinatorial type* of the polytope [20]. Two polytopes P and Q in $\mathbb{R}^d$ are combinatorially equivalent if there is a one-to-one correspondence ϕ between the set of all faces of P and the set of all faces of Q such that ϕ is inclusion-preserving, i.e. two faces F_1 and F_2 in P satisfy $F_1 \subset F_2$ if and only if $\phi(F_1) \subset \phi(F_2)$ in Q.

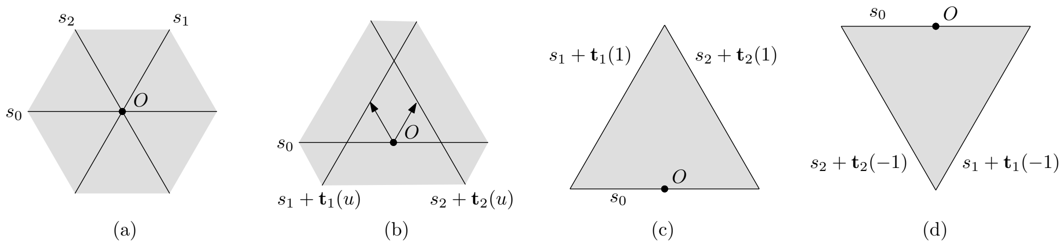

Figure 1 (a) Three segments s_0, s_1, and s_2 with midpoints at the origin O. (b) Translations of s_1 and s_2 by $\mathbf{t}_1(u) = (u/2, \sqrt{3}u/2)$ and $\mathbf{t}_2(u) = (-u/2, \sqrt{3}u/2)$, respectively, for $u \in \mathbb{R}$. Along the 1-dimensional translation, $\Upsilon(\mathbf{t}(u))$ is a concave quadratic on $[-1, 1]$, hence not convex. (c) $s_1 + \mathbf{t}_1(1)$ and $s_2 + \mathbf{t}_2(1)$. (d) $s_1 + \mathbf{t}_1(-1)$ and $s_2 + \mathbf{t}_2(-1)$.

For a compact set $S \subset \mathbb{R}^2$, let $\mathsf{conv}(S)$ denote the convex hull of S, and let $d(S)$ denote the diameter of S. Let $\mathsf{area}(S) := \mathsf{area}(\mathsf{conv}(S))$ denote the area of $\mathsf{conv}(S)$. For a fixed $\vartheta \in [0, \pi)$, the width of S in direction ϑ, denoted by $w_\vartheta(S)$, is the length of the orthogonal projection of S onto a line with angle ϑ. The width of S is $w(S) = \min_{\theta \in [0,\pi)} w_\theta(S)$.

Given a convex set $C \subset \mathbb{R}^2$, a convex set $C' \subseteq C$ is an *ε-kernel* of C for $\varepsilon \in (0, 1)$ if and only if $(1 - \varepsilon)w_\theta(C) \le w_\theta(C')$ for all $\theta \in [0, \pi)$. Agarwal et al. showed that there is a constant $c > 1$ such that, for any ε/c-kernel C' of C, the area of $C \setminus C'$ is at most $\varepsilon\mathsf{area}(C)$, implying $(1 - \varepsilon)\mathsf{area}(C) \le \mathsf{area}(C')$ [1].

▶ **Lemma 1** ([3]). *Given a convex n-gon C and $\varepsilon \in (0, 1)$, we can compute a convex polygon C' with $O(\varepsilon^{-1/2})$ vertices in $O(\varepsilon^{-1/2} \log n)$ time such that $C \subseteq C'$ and C is an ε-kernel of C'.*

We let $k \ge 3$ denote a fixed integer. Let $\mathsf{C} = \{C_0, \ldots, C_{k-1}\}$ denote the set of k convex polygons in the plane with n vertices in total such that the vertices of each polygon in C are given in a sorted array. A real-valued function f is *convex* if the line segment between any two distinct points on the graph of f lies above or on the graph of f between the two points.

▶ **Definition 2** (Event surface). *An event surface is a set of translations in the translation space that induce the same change in the combinatorial structure of the convex hull* $\mathsf{conv}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i + \mathbf{t}_i))$.

3 Minimum-area convex hull under translations

We consider the problem of minimizing the area of the convex hull of k convex polygons $C_0, \ldots, C_{k-1}$ with n vertices in total in the plane under translations for any fixed $k \ge 3$. For ease of description, we assume that no input polygon has a vertical edge. This can be achieved by a rotation if such an edge exists.

3.1 Translation space

Without loss of generality, we assume that C_0 is fixed and C_i is translated by a vector $\mathbf{t}_i \in \mathbb{R}^2$ for $i \in [k - 1]$. We represent translations of the $k - 1$ input polygons C_i by a vector $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{k-1}) \in \mathbb{R}^{2k-2}$. We call the set of all such vectors the *translation space* for C, and denote it by $\mathbb{T}$. The area function for a translation $\mathbf{t} \in \mathbb{T}$ is denoted by $\Upsilon(\mathbf{t}) = \mathsf{area}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i + \mathbf{t}_i))$. The goal is to compute a translation $\mathbf{t}^* \in \mathbb{T}$, which we call an optimal translation, such that $\Upsilon(\mathbf{t}^*) \le \Upsilon(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}$.

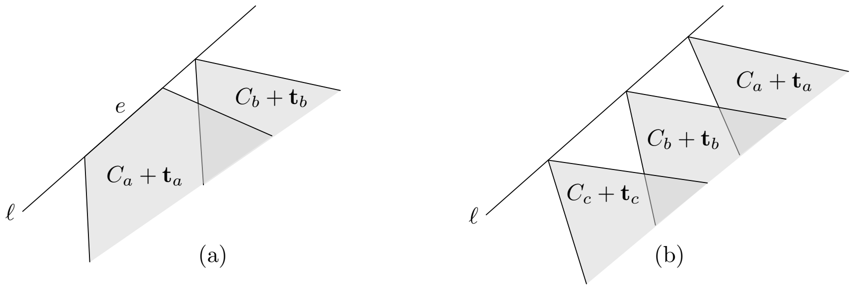

Figure 2 (a) Degeneracy Type 1. (b) Degeneracy Type 2.

For $k = 2$, it is known that $\Upsilon(\mathbf{t})$ is convex and piecewise linear [6]. Using this property, there is a linear-time algorithm to find one minimizing $\Upsilon(\mathbf{t})$ among all translations $\mathbf{t}$ [13]. However, $\Upsilon(\mathbf{t})$ is not necessarily convex for $k \geq 3$. Let s_i be a segment with length 2 and angle $i\pi/3$ such that the midpoint of s_i is at the origin for $i = 0, 1, 2$ as shown in Figure 1. For $u \in \mathbb{R}$, let $\mathbf{t}_1(u) = (u/2, \sqrt{3}u/2)$ and $\mathbf{t}_2(u) = (-u/2, \sqrt{3}u/2)$, and let $\mathbf{t}(u) = (\mathbf{t}_1(u), \mathbf{t}_2(u))$. Then $\Upsilon(\mathbf{t}(u)) = 3\sqrt{3}/2 - \sqrt{3}u^2/2$ for $u \in [-1, 1]$. Observe that $\Upsilon(\mathbf{t}(u))$ is neither convex nor piecewise linear. In Section 4, we show that $k - 1$ is the maximum possible dimension of affine subspaces of $\mathbb{T}$ in which $\Upsilon(\mathbf{t})$ is convex.

3.2 Event surfaces and their arrangement

While $\mathbf{t}$ varies continuously over $\mathbb{T}$, there are translations at which the combinatorial structure of $\mathsf{conv}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i + \mathbf{t}_i))$ changes. A change in the combinatorial structure occurs when a vertex of a polygon emerges from the interior to the boundary of the convex hull, or submerges from the boundary into the interior. In either case, such a change corresponds to the following degeneracy types:

1. $C_b + \mathbf{t}_b$ touches the supporting line ℓ of an edge of $C_a + \mathbf{t}_a$ for $a \neq b$ and all input polygons are placed on the same side of ℓ. See Figure 2(a).
2. $C_a + \mathbf{t}_a, C_b + \mathbf{t}_b$, and $C_c + \mathbf{t}_c$ touch a line ℓ simultaneously and all input polygons are placed on the same side of ℓ. See Figure 2(b).

We define event surfaces that represent translation vectors corresponding to Degeneracy Type 1. Fix an edge e of C_a. Consider the set T of ordered pairs $(\mathbf{t}_a, \mathbf{t}_b)$ of translation vectors for (C_a, C_b) with $a \neq b$ such that $C_b + \mathbf{t}_b$ touches the line containing $e + \mathbf{t}_a$. Let T' be the set of $(k-2)$-tuples of translation vectors for polygons in $\mathsf{C} \setminus \{C_a, C_b\}$ that place the polygons on the side of the supporting line of $e + \mathbf{t}_a$ where $C_a + \mathbf{t}_a$ and $C_b + \mathbf{t}_b$ lie. There are two cases.

(a) If $0 \in \{a, b\}$, T can be expressed as a line in $\mathbb{R}^2$, and T' can be expressed as a Cartesian product of $k - 2$ half-planes. Thus, an event surface is the Cartesian product of a line in $\mathbb{R}^2$ and $k - 2$ half-planes in $\mathbb{R}^2$.

(b) If $0 \notin \{a, b\}$, the set T can be expressed as a hyperplane in $\mathbb{R}^4$ and T' can be expressed as a Cartesian product of $k-3$ half-planes in $\mathbb{R}^2$. Thus, an event surface is the Cartesian product of a hyperplane in $\mathbb{R}^4$ and $k - 3$ half-planes in $\mathbb{R}^2$.

There is a hyperplane in $\mathbb{T}$ containing the set of translation vectors for each case. We call such a hyperplane an *event surface* of type 1.

Now we define event surfaces that represent translation vectors corresponding to Degeneracy Type 2. Consider the set of translation vectors $\mathbf{t} \in \mathbb{T}$ that correspond to Degeneracy Type 2 with three polygons $C_a, C_b, C_c \in \mathsf{C}$. Let T be the set of triples $(\mathbf{t}_a, \mathbf{t}_b, \mathbf{t}_c)$ of translation vectors for C_a, C_b, and C_c such that $C_a + \mathbf{t}_a, C_b + \mathbf{t}_b$, and $C_c + \mathbf{t}_c$ touch a line ℓ

simultaneously from the same side of ℓ. Let T' be the set of $(k-3)$-tuples of translation vectors for the $k-3$ polygons in $\mathsf{C} \setminus \{C_a, C_b, C_c\}$ that place the polygons in the side of ℓ where $C_a + \mathbf{t}_a, C_b + \mathbf{t}_b, C_c + \mathbf{t}_c$ lie. There are two cases:

(a) If $0 \in \{a, b, c\}$, T can be expressed as part of a quadric surface in $\mathbb{R}^4$, and T' can be expressed as a Cartesian product of $k-3$ half-planes. Thus, an event surface is a Cartesian product of a quadric surface in $\mathbb{R}^4$ and $k-3$ half-planes.

(b) If $0 \notin \{a, b, c\}$, T can be expressed as part of a quadric surface in $\mathbb{R}^6$, and T' can be expressed as a Cartesian product of $k-4$ half-planes. Thus, an event surface is a Cartesian product of a quadric surface in $\mathbb{R}^6$ and $k-4$ half-planes.

There is a quadric surface in $\mathbb{T}$ containing the set of translation vectors for each case. We call such a quadric surface an *event surface* of type 2.

Each event surface represents a set of translation vectors corresponding to the degeneracy type. We can check whether a translation belongs to the set in $O(1)$ time. We analyze the number of event surfaces of type 1. For a fixed edge e of $C_a + \mathbf{t}_a$, $C_b + \mathbf{t}_b$ touches the supporting line of e from the same side at the same vertex (or edge) of $C_b + \mathbf{t}_b$. Thus, it is bounded by the total number of edges of the polygons, which is $O(n)$. For event surfaces of type 2, since every polygon $C \in \mathsf{C}$ is convex, each vertex v of C has a unique interval of orientations ϑ in $[0, 2\pi)$ such that C is tangent to the line of angle $\vartheta \bmod \pi$ containing v. For any two distinct vertices of C, their intervals are interior-disjoint. Thus, the intervals of polygon vertices in C partition $[0, 2\pi)$ into at most n intervals of orientations. For each direction, each convex polygon has exactly one supporting vertex or edge; as direction varies, the identity changes only $O(n)$ times over all vertices, hence the number of event surfaces of type 2 is $O(n)$.

Since $\mathbb{T}$ has dimension $d = 2k-2$, applying the vertical decomposition for $O(n)$ semi-algebraic event surfaces in $\mathbb{R}^d$ yields a time complexity of $O(n^{2d-4+\delta}) = O(n^{4k-8+\delta})$ for any fixed $\delta > 0$ [14]. Evaluating $\Upsilon(\mathbf{t})$ for any fixed $\mathbf{t}$ in a cell takes $O(n)$ time, which is $O(n^{2k-1})$ for all $O(n^{2k-2})$ cells. Thus, an optimal translation for Υ can be computed in $O(n^\eta)$ time, where $\eta = \max\{4k-8+\delta, 2k-1\} = \max\{4k-8+\delta, 5\}$ $(2k-1 \geq 5$ for $k \geq 3)$.

▶ **Theorem 3.** *Given k convex polygons with n vertices in total for any fixed integer $k \geq 3$ in the plane, we can compute an optimal translation for the polygons that minimizes the area function Υ in $O(n^\eta)$ time, where $\eta = \max\{5, 4k-8+\delta\}$ for any fixed $\delta > 0$.*

4 Dimensions in which $\Upsilon(\mathbf{t})$ is convex

In Section 3 we gave an example (Figure 1) with three line segments for which $\Upsilon(\mathbf{t})$ is neither convex nor piecewise linear. In this section, we show that $k-1$ is the maximum possible dimension of an affine subspace of $\mathbb{T}$ in which $\Upsilon(\mathbf{t})$ is convex.

▶ **Lemma 4.** *For any fixed integer $k \geq 3$ and for any set of k polygons in $\mathbb{R}^2$, no k-dimensional affine subspace of $\mathbb{T}$ allows $\Upsilon(\mathbf{t})$ to be globally convex. Thus, $k-1$ is the maximum possible dimension of affine subspaces of $\mathbb{T}$ in which $\Upsilon(\mathbf{t})$ is convex.*

Proof. It suffices to show that $\Upsilon(\mathbf{t})$ is not convex for any affine subspace $\tilde{\mathbb{T}}$ with dimension k. Since $\tilde{\mathbb{T}}$ has dimension k, there are two polygons, say $C_1, C_2 \in \mathsf{C}$, such that C_1 has two degrees of freedom in $\tilde{\mathbb{T}}$ and C_2 has at least one degree of freedom in $\tilde{\mathbb{T}}$. Assume C_2 has one degree of freedom along the direction $(1, 0)$ in $\tilde{\mathbb{T}}$. Let Q be the convex hull of $k-2$ polygons in $\mathsf{C} \setminus \{C_1, C_2\}$.

For a compact set C, let $B(C)$ be a disk centered at a point in C and containing C. Without loss of generality, assume that $B(C_1), B(C_2)$, and $B(\mathsf{Q})$ have the same radius $\delta > 0$, $B(\mathsf{Q})$ and $B(C_1)$ are centered at the origin, and $B(C_2)$ is centered at (p_x, p_y) for a fixed

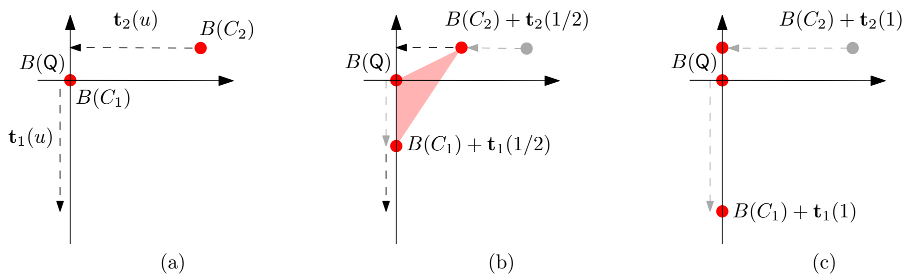

Figure 3 (a) $B(C_1)$ and $B(C_2)$ are translated by $\mathbf{t}_1(u)$ and $\mathbf{t}_2(u)$, respectively, while the others are stationary. (b) $B(C_1)+\mathbf{t}_1(1/2)$ and $B(C_2)+\mathbf{t}_2(1/2)$. (c) $B(C_1)+\mathbf{t}_1(1)$ and $B(C_2)+\mathbf{t}_2(1)$.

$p_x \geq \max\{64\delta, p_y\}$. See Figure 3. Let $\mathbf{t}(u) = (\mathbf{t}_1(u), \mathbf{t}_2(u))$ be the translation vector with $\mathbf{t}_1(u) = (0, -p_x u)$ for C_1 and $\mathbf{t}_2(u) = (-p_x u, 0)$ for C_2. Since C_1 has two degrees of freedom and C_2 has one degree of freedom with direction $(1,0)$, $\mathbf{t}(u)$ is in $\tilde{\mathbb{T}}$.

$$\begin{aligned}
\Upsilon(\mathbf{t}(0)) &\leq 2\delta\sqrt{p_x^2+p_y^2}+\delta^2\pi \leq 2\sqrt{2}\delta p_x+\delta^2\pi\\
\Upsilon(\mathbf{t}(1)) &\leq 2\delta(p_x+p_y)+\delta^2\pi \leq 4\delta p_x+\delta^2\pi\\
\Upsilon(\mathbf{t}(1/2)) &\geq p_x^2/8.
\end{aligned}$$

Since $\Upsilon(\mathbf{t}(1/2)) > \max\{\Upsilon(\mathbf{t}(0)), \Upsilon(\mathbf{t}(1))\}$ for $p_x \geq \max\{64\delta, p_y\}$, $\Upsilon(\mathbf{t})$ is not convex for $\mathbf{t} \in \tilde{\mathbb{T}}$. A similar argument is shown in [6]. ◀

If we consider $\mathbf{t}(u)$ for $u < 0$ or $u > 1$ in Lemma 4, we have the following corollary.

▶ **Corollary 5.** *For any fixed integer $k \geq 3$, $k-1$ is the maximum possible dimension of affine subspaces of $\mathbb{T}$ in which $\Upsilon(\mathbf{t})$ is unimodal.*

5 Efficient $(1+\varepsilon)$-approximation algorithms

For k convex polygons $C_0, \dots, C_{k-1}$ with n vertices in total in the plane for any fixed $k \geq 3$, we give an efficient $(1+\varepsilon)$-approximation algorithm for $\varepsilon \in (0,1)$ that returns a translation $\mathbf{t}_{\mathsf{apx}} \in \mathbb{T}$ satisfying $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Upsilon(\mathbf{t}^*)$ in $O(\varepsilon^{-1/2}\log n + \varepsilon^{1/2-k})$ time.

5.1 $(\mathbf{t}_0, \mathbf{a})$-translation space

As shown in Section 3.1, $\Upsilon(\mathbf{t})$ is not necessarily convex for $k \geq 3$ in $\mathbb{T}$. However, we show that it is convex in certain affine subspaces of dimension $k-1$ of $\mathbb{T}$. In Section 4, we proved that $k-1$ is the maximum possible dimension of such a subspace.

For a fixed unit vector $\mathbf{t}_0 \in \mathbb{R}^2$ and a fixed vector $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_{k-1}) \in \mathbb{T}$ with $\mathbf{a}_i \in \mathbb{R}^2$ for $i \in [k-1]$, we define the affine subspace $\{(\mathbf{a}_1+x_1\mathbf{t}_0, \dots, \mathbf{a}_{k-1}+x_{k-1}\mathbf{t}_0) \mid x_i \in \mathbb{R} \text{ for } i \in [k-1]\}$ with dimension $k-1$ of $\mathbb{T}$ as the $(\mathbf{t}_0, \mathbf{a})$-*translation space*, denoted by $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Let $\mathbf{x} = (x_1, x_2, \dots, x_{k-1})$ with $x_i \in \mathbb{R}$ for $i \in [k-1]$, and define $\mathbf{t}(\mathbf{x}) = (\mathbf{a}_1+x_1\mathbf{t}_0, \dots, \mathbf{a}_{k-1}+x_{k-1}\mathbf{t}_0)$. We show that the function $f : \mathbb{R}^{k-1} \mapsto \mathbb{R}$ defined by $f(\mathbf{x}) = \Upsilon(\mathbf{t}(\mathbf{x}))$ is convex and piecewise linear. We abuse the notation and define $\mathsf{conv}(\mathbf{x}) = \mathsf{conv}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i + x_i\mathbf{t}_0))$.

▶ **Lemma 6.** *$\Upsilon(\mathbf{t})$ is convex and piecewise linear in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$.*

Proof. Let $\mathbf{t}_0 = (1,0)$ and D be a subspace of $\mathbb{T}$ with the same combinatorial structure of $\mathsf{conv}(\mathbf{x})$ for any $\mathbf{x} \in \mathsf{D}$. We show that $f(\mathbf{x})$ is linear for $\mathbf{x} \in \mathsf{D}$. Each vertex of $\mathsf{conv}(\mathbf{x})$ is a vertex of a polygon in C. For an edge e of $\mathsf{conv}(\mathbf{x})$, let $\alpha(e)$ and $\beta(e)$ be indices such

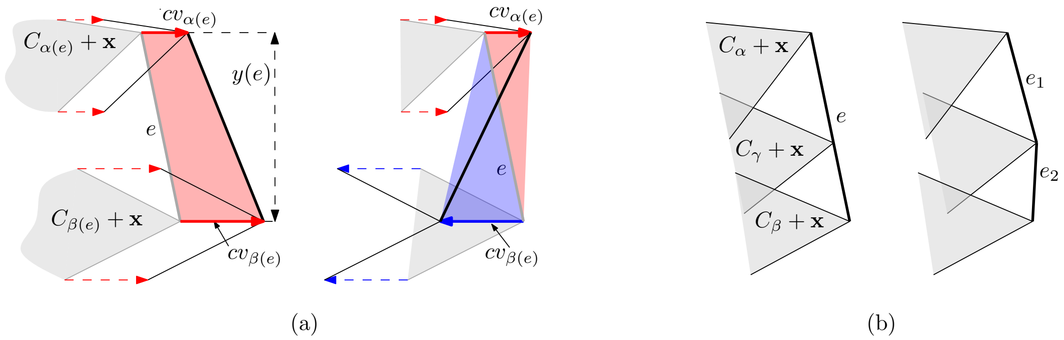

Figure 4 (a) The area swept by a right edge e is $cv_{\alpha(e)}y(e)/2+cv_{\beta(e)}y(e)/2$. Left: $v_{\alpha(e)}, v_{\beta(e)}>0$. Right: $v_{\alpha(e)}>0$, $v_{\beta(e)}<0$. (b) Left: an edge event for a right edge e of $\mathsf{conv}(\mathbf{x})$. Right: two right edges e_1 and e_2 of $\mathsf{conv}(\mathbf{x}+\delta\mathbf{v})$ for small $\delta>0$.

that e connects a vertex of $C_{\alpha(e)}\in\mathsf{C}$ and a vertex of $C_{\beta(e)}\in\mathsf{C}$, possibly with $\alpha(e)=\beta(e)$. If e is on the left chain of $\mathsf{conv}(\mathbf{x})$, it is a *left* edge; otherwise, it is a *right* edge. Let $\mathbf{v}=(v_1,v_2,\ldots,v_{k-1})$ be a fixed unit vector in $\mathbb{R}^{k-1}$. Let $v_0=0$ to represent that C_0 is fixed and its translation vector is zero. Let c be a positive constant such that $\mathsf{conv}(\mathbf{x}+c\delta\mathbf{v})$ has the same combinatorial structure for every $\delta\in[0,1]$. The area change between $\mathsf{conv}(\mathbf{x})$ and $\mathsf{conv}(\mathbf{x}+c\mathbf{v})$ incurred by an edge e of $\mathsf{conv}(\mathbf{x})$ is $cv_{\alpha(e)}y(e)/2+cv_{\beta(e)}y(e)/2$. See Figure 4(a). Thus,

$$f(\mathbf{x}+c\mathbf{v})-f(\mathbf{x})=c\sum_{e\in\mathsf{R}}\Big(\frac{v_{\alpha(e)}}{2}+\frac{v_{\beta(e)}}{2}\Big)y(e)-c\sum_{e\in\mathsf{L}}\Big(\frac{v_{\alpha(e)}}{2}+\frac{v_{\beta(e)}}{2}\Big)y(e),$$

where R is the set of right edges of $\mathsf{conv}(\mathbf{x})$, and L is the set of left edges of $\mathsf{conv}(\mathbf{x})$. Thus, f is linear along the translation by $c\mathbf{v}$, and $f(\mathbf{x})$ is linear for $\mathbf{x}$ in D. Therefore, $f(\mathbf{x})=\Upsilon(\mathbf{t}(\mathbf{x}))$ is piecewise linear for $\mathbf{x}$ in $\mathbb{T}(\mathbf{t}_0,\mathbf{a})$.

Now, we show $f(\mathbf{x})$ is convex by showing that $\nabla_{\mathbf{v}}f(\mathbf{x})$ increases along a line ℓ with direction $\mathbf{v}$. Assume that $\mathbf{x}$ lies on a line ℓ with direction $\mathbf{v}$. Letting $c\to 0$, we obtain

$$\nabla_{\mathbf{v}}f(\mathbf{x})=\sum_{e\in R}\Big(\frac{v_{\alpha(e)}}{2}+\frac{v_{\beta(e)}}{2}\Big)y(e)-\sum_{e\in L}\Big(\frac{v_{\alpha(e)}}{2}+\frac{v_{\beta(e)}}{2}\Big)y(e). \tag{1}$$

Recall that the combinatorial structure of $\mathsf{conv}(\mathbf{x})$ changes when a vertex either sticks out from the convex hull or sinks into the convex hull. This is equivalent to an edge of $\mathsf{conv}(\mathbf{x})$ appearing or disappearing. Consider the case where a right edge e disappears from $\mathsf{conv}(\mathbf{x})$. This occurs when a polygon $C_\gamma\in\mathsf{C}$ touches e during the translation by $\mathbf{x}$. Then e is replaced by two edges e_1 and e_2. See Figure 4(b). Let $\alpha=\alpha(e)=\alpha(e_1)$, $\beta=\beta(e)=\beta(e_2)$, and $\gamma=\beta(e_1)=\alpha(e_2)$, possibly $\alpha=\beta$. Observe that $v_\gamma>(v_\alpha y(e_2)+v_\beta y(e_1))/(y(e_1)+y(e_2))$. For a small real $\delta>0$,

$$\begin{aligned}\nabla_{\mathbf{v}}f(\mathbf{x})-\nabla_{\mathbf{v}}f(\mathbf{x}-\delta\mathbf{v}) &= \Big(\frac{v_\alpha}{2}+\frac{v_\gamma}{2}\Big)y(e_1)+\Big(\frac{v_\gamma}{2}+\frac{v_\beta}{2}\Big)y(e_2)-\Big(\frac{v_\alpha}{2}+\frac{v_\beta}{2}\Big)y(e)\\ &= \frac{v_\gamma}{2}y(e)-\frac{v_\alpha}{2}y(e_2)-\frac{v_\beta}{2}y(e_1) \quad \because y(e)=y(e_1)+y(e_2)\\ &> \frac{v_\alpha y(e_2)+v_\beta y(e_1)}{2}-\frac{v_\alpha}{2}y(e_2)-\frac{v_\beta}{2}y(e_1)=0.\end{aligned}$$

Thus, $\nabla_{\mathbf{v}}f(\mathbf{x})$ increases. Analogously, we can show that $\nabla_{\mathbf{v}}f(\mathbf{x})$ increases for other cases. Thus, $f(\mathbf{x})=\Upsilon(\mathbf{t}(\mathbf{x}))$ is convex for $\mathbf{x}$ in $\mathbb{T}(\mathbf{t}_0,\mathbf{a})$. ◀

By Lemma 4 and Lemma 6, $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$ is an affine subspace with the maximum possible dimension $k-1$ in which $\Upsilon(\mathbf{t})$ is convex.

Consider the event surfaces in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Every event surface of type 1 is a hyperplane in $\mathbb{T}$, so it is also a hyperplane in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. For an event surface of type 2, let p_1, p_2, and p_3 be collinear points in $\mathbb{R}^2$. Then $p_1 + a\mathbf{t}_0$, $p_2 + a\mathbf{t}_0$, and $p_3 + a\mathbf{t}_0$ are collinear for any $a \in \mathbb{R}$, and there is a constant $c \in \mathbb{R}$ such that p_1, $p_2 + b\mathbf{t}_0$, and $p_3 + bc\mathbf{t}_0$ are collinear for any $b \in \mathbb{R}$. Thus, every event surface of type 2 is also a hyperplane in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. We call each event surface in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$ an *event hyperplane*. There are $O(n)$ event hyperplanes in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Each event hyperplane represents a set of translation vectors corresponding to its degeneracy type. Let E be the set of all event hyperplanes in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$.

Suppose that $\mathbf{x}$ moves continuously in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. We maintain the combinatorial structure of $\mathsf{conv}(\mathbf{x})$ and $\nabla_{\mathbf{v}} f(\mathbf{x})$, and compute $\nabla_{\mathbf{v}'} f(\mathbf{x})$ for a unit vector $\mathbf{v}' \neq \mathbf{v}$ in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$ efficiently.

▶ **Lemma 7.** *Given* $\mathsf{conv}(\mathbf{x})$*, we can update* $\nabla_{\mathbf{v}} f(\mathbf{x})$ *in* $O(1)$ *time when* $\mathbf{x}$ *hits an event hyperplane. Given* $\nabla_{\mathbf{v}} f(\mathbf{x})$ *and* $\mathsf{conv}(\mathbf{x})$ *for fixed* $\mathbf{v}$ *and* $\mathbf{x}$*, we can compute* $\nabla_{\mathbf{v}'} f(\mathbf{x})$ *for a unit vector* $\mathbf{v}' \neq \mathbf{v}$ *in time linear in the number of event hyperplanes containing* $\mathbf{x}$*.*

Proof. Let $\mathbf{t}_0 = (1, 0)$. Suppose $\mathbf{x}$ hits an event hyperplane H. If $\mathbf{x}$ hits two or more event hyperplanes simultaneously, we handle them one by one. We check for changes in the combinatorial structure of $\mathsf{conv}(\mathbf{x})$ by H in $O(1)$ time. If it changes, at most two edges appear and at most two edges disappear in $\mathsf{conv}(\mathbf{x})$. Since $\nabla_{\mathbf{v}} f(\mathbf{x})$ is determined by the component of $\mathbf{v}$ and $y(e)$ of the edges e involved in the change by Lemma 6, we can update it in $O(1)$ time by adding and subtracting at most two lengths, one for each edge appearing and disappearing. Note that $y(e)$ of an edge e of $\mathsf{conv}(\mathbf{x})$ is constant for any $\mathbf{v}$ under the same combinatorial structure of $\mathsf{conv}(\mathbf{x})$ because $\mathbf{t}_0 = (1, 0)$. Using the notations in Lemma 6, Equation 1 can be expressed as

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \sum_{i=1}^{k-1} \Big(v_i \sum_{e \in \mathsf{R}_i^\alpha} \frac{y(e)}{2}\Big) + \sum_{i=1}^{k-1} \Big(v_i \sum_{e \in \mathsf{R}_i^\beta} \frac{y(e)}{2}\Big) - \sum_{i=1}^{k-1} \Big(v_i \sum_{e \in \mathsf{L}_i^\alpha} \frac{y(e)}{2}\Big) - \sum_{i=1}^{k-1} \Big(v_i \sum_{e \in \mathsf{L}_i^\beta} \frac{y(e)}{2}\Big),$$

where $\mathsf{R}_i^\alpha = \{e \in \mathsf{R} \mid i = \alpha(e)\}$, $\mathsf{R}_i^\beta = \{e \in \mathsf{R} \mid i = \beta(e)\}$, $\mathsf{L}_i^\alpha = \{e \in \mathsf{L} \mid i = \alpha(e)\}$, $\mathsf{L}_i^\beta = \{e \in \mathsf{L} \mid i = \beta(e)\}$ for the set R of right edges and the set L of left edges of $\mathsf{conv}(\mathbf{x})$. We can compute $\nabla_{\mathbf{v}'} f(\mathbf{x})$ from $\nabla_{\mathbf{v}} f(\mathbf{x})$ in $O(1)$ time for any $\mathbf{x}$ in the interior of a cell in the arrangement of E in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$.

Assume $\mathbf{x}$ lies on h event hyperplanes, where $h \geq 1$. Given $\mathbf{v}'$, consider the intersections of the plane spanned by $\mathbf{v}$ and $\mathbf{v}'$, with event hyperplanes containing $\mathbf{x}$. We check for changes in the combinatorial structure of $\mathsf{conv}(\mathbf{x})$ by each such intersection. This can be done in $O(1)$ time as each event hyperplane represents a set of translation vectors corresponding to its degeneracy type. Since at most two edges appear and at most two edges disappear in $\mathsf{conv}(\mathbf{x})$ by such an intersection, we add and subtract at most $2h$ values, one for each edge appearing and disappearing in $\mathsf{conv}(\mathbf{x})$. Then, we compute $\nabla_{\mathbf{v}'} f(\mathbf{x})$ from $\nabla_{\mathbf{v}} f(\mathbf{x})$ by updating $\mathbf{v}'$. This can be done in $O(h)$ time. ◀

We can compute $f(\mathbf{x}')$ from $f(\mathbf{x})$ for $\mathbf{x}' \neq \mathbf{x}$ efficiently.

▶ **Lemma 8.** *For any two points* $\mathbf{x}$ *and* $\mathbf{x}'$ *in* $\mathbb{R}^{k-1}$*, once we have* $\mathsf{conv}(\mathbf{x})$*,* $\nabla_{\mathbf{x}'-\mathbf{x}} f(\mathbf{x})$*,* $f(\mathbf{x})$*, and the event hyperplanes intersecting the segment connecting* $\mathbf{x}$ *and* $\mathbf{x}'$*, we can compute* $f(\mathbf{x}')$ *in time linear in the number of event hyperplanes intersecting the segment.*

Proof. Let $\mathbf{z} = \mathbf{x}' - \mathbf{x}$, z be the line segment connecting $\mathbf{x}'$ and $\mathbf{x}$, and $|z|$ denote the length of z. Let h denote the number of event hyperplanes intersecting z. If $h = 0$, we can compute $f(\mathbf{x}')$ in $O(1)$ time from $\nabla_{\mathbf{z}} f(\mathbf{x})$ and $f(\mathbf{x})$.

For $h \geq 1$, let $D(a) = f(\mathbf{x} + a\mathbf{z}/|z|)$ and let $D'(a) = \nabla_{\mathbf{z}} f(\mathbf{x} + a\mathbf{z}/|z|)$ for $a \in [0, |z|]$. By assumption, $D(0) = f(\mathbf{x})$ and $D'(0)$ are given. We compute the set V of at most h values for a such that $\mathbf{x} + a\mathbf{z}/|z|$ lies on $z \cap H$ and is contained in the set of translations stored in H for $H \in \mathsf{E}$. This can be done in $O(h)$ time. Each element in V corresponds to at most two values to add to $D(0)$ and at most two values to subtract from $D(0)$ as in Lemma 7. Since $D(|z|) = D(0) + \int_0^{|z|} D'(s)\,ds$, we can compute $D(|z|)$ from $D(0)$ by adding $|z|D'(0)$, and by adding and subtracting at most $2|\mathsf{V}|$ values in $O(|\mathsf{V}|)$ time. Therefore, we can compute $f(\mathbf{x}') = D(|z|)$ in $O(h)$ time. ◀

5.2 Linear-time algorithm in $(\mathbf{t}_0, \mathbf{a})$-translation space

We present a linear-time algorithm that computes a translation $\mathbf{t}_0^* \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ such that $f(\mathbf{t}_0^*) \leq f(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Recall that the dimension of $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$ is $k - 1$. Given n hyperplanes in $\mathbb{R}^d$, a $(1/r)$-*cutting* is a collection of d-simplices with disjoint interiors, together covering $\mathbb{R}^d$, such that the interior of each simplex intersects at most n/r hyperplanes.

▶ **Lemma 9** ([10])**.** *Given n hyperplanes in $\mathbb{R}^d$ and any $r \in (1, n)$, we can compute a $(1/r)$-cutting of size $O(r^d)$ in $O(nr^{d-1})$ time. Also, we can compute the hyperplanes intersecting the interior of each d-simplex of the cutting in the same time bound.*

For a compact set $C \subset \mathbb{R}^d$, we use $\mathrm{int}(C)$ to denote the interior of C. Let $\triangle$ be an m-simplex with $m \leq k - 1$ such that N event hyperplanes intersect $\mathrm{int}(\triangle)$. Let $\mathbf{t}_\triangle$ be a translation in $\triangle$ satisfying $f(\mathbf{t}_\triangle) \leq f(\mathbf{t})$ for all translations $\mathbf{t} \in \triangle$. Let $T(N, m)$ be the time for computing an m-simplex $\triangle^* \subseteq \triangle$ such that $\mathbf{t}_\triangle \in \triangle^*$ and no event hyperplane intersects $\mathrm{int}(\triangle^*)$. Let $T_o(N, m)$ be the oracle time for computing the relative position of $\mathbf{t}_\triangle$ to an $(m-1)$-flat Γ in $\triangle$.

▶ **Lemma 10.** *$T(N, m) = O(N) + O(T_o(N, m)) + T(\lfloor N/2 \rfloor, m)$.*

Proof. Let $\triangle$ be an m-simplex, and let $\mathbf{t}_\triangle$ be a translation in $\triangle$ satisfying $f(\mathbf{t}_\triangle) \leq f(\mathbf{t})$ for all translations $\mathbf{t} \in \triangle$. For a set $\mathsf{E}_\triangle$ of N event hyperplanes intersecting $\mathrm{int}(\triangle)$, let $\mathsf{F}_\triangle$ be the set consisting of $(m-1)$-flats $\triangle \cap H$ for each $H \in \mathsf{E}_\triangle$, with $|\mathsf{F}_\triangle| = N$.

We find an m-simplex $\triangle' \subseteq \triangle$ such that at most $\lfloor N/2 \rfloor$ event hyperplanes intersect $\mathrm{int}(\triangle')$ as follows. First, we compute a $(1/2)$-cutting of $\mathsf{F}_\triangle$ in $\triangle$ with size $O(1)$ using Lemma 9. For each $(m-1)$-face g of the cutting, we determine the relative position of $\mathbf{t}_\triangle$ to the $(m-1)$-flat γ with $g \subset \gamma$. Then we find the cell P containing $\mathbf{t}_\triangle$ in the arrangement of the $(m-1)$-flats γ with $g \subset \gamma$ for each $(m-1)$-face g. Let E_P be the set of event hyperplanes in $\mathsf{E}_\triangle$ that intersect $\mathrm{int}(P)$. Then $|\mathsf{E}_P| \leq \lfloor N/2 \rfloor$ by Lemma 9. We triangulate P, and then compute an m-simplex $\triangle'$ in the triangulation containing $\mathbf{t}_\triangle$ by running the oracle for each $(m-1)$-face of the triangulation. Also, we compute the set F_P consisting of $(m-1)$-flats $\triangle' \cap H'$ for each $H' \in \mathsf{E}_P$, with $|\mathsf{F}_P| \leq \lfloor N/2 \rfloor$. We repeat the above on $\triangle'$ and F_P recursively until we find an m-simplex $\triangle^*$ such that $\mathbf{t}_\triangle \in \triangle^*$ and no event hyperplane in $\mathsf{E}_\triangle$ intersects $\mathrm{int}(\triangle^*)$.

We analyze the time complexity of $T(N, m)$. It takes $O(N)$ time in total, except the oracle time and the time for the recursion on an m-simplex $\triangle'$ and a set F_P of $\lfloor N/2 \rfloor$ $(m-1)$-flats. Since both the triangulation and the cutting have size $O(1)$, the oracle takes $O(T_o(N, m))$ time in total. Thus, we have $T(N, m) = O(N) + O(T_o(N, m)) + T(\lfloor N/2 \rfloor, m)$ ◀

By Lemma 6, we can prove the following lemma in a way similar to Lemma 3.5 in [13] by extending the argument from the two-dimensional case to $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$.

▶ **Lemma 11.** *Let R be a convex region in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$ containing $\mathbf{t}_0^*$. For a hyperplane H intersecting $\mathrm{int}(R)$, we can determine the relative position of $\mathbf{t}_0^*$ in R with respect to H in the same asymptotic time for computing $\min_{\mathbf{t}\in H'\cap R}\Upsilon(\mathbf{t})$ for any fixed hyperplane H'.*

We analyze the oracle time $O(T_o(N, m))$.

▶ **Lemma 12.** *Assume that $\nabla_{\mathbf{w}'-\mathbf{w}} f(\mathbf{w})$, $f(\mathbf{w})$, and $\mathsf{conv}(\mathbf{w})$ are given for every pair $(\mathbf{w}, \mathbf{w}')$ of vertices of an m-simplex. For every $m \in [k-1]$ and $N \in [|\mathsf{E}|]$, $T_o(N, m) = O(N) + T(N, m-1)$.*

Proof. Let Q be the intersection of an $(m-1)$-flat and an m-simplex $\triangle$. Then Q is an $(m-1)$-dimensional convex polytope with $O(m) = O(1)$ vertices, each on an edge of $\triangle$. We compute $\nabla_{\mathbf{v}'-\mathbf{v}} f(\mathbf{v})$ for a vertex $\mathbf{v}'$ of Q as follows. Since $\nabla_{\mathbf{w}'-\mathbf{w}} f(\mathbf{w})$ are given for every pair $(\mathbf{w}, \mathbf{w}')$ of vertices of $\triangle$, we have $\nabla_{\mathbf{v}-\mathbf{w}} f(\mathbf{w})$. We compute $O(N)$ intersection points of event hyperplanes and segment $\mathbf{wv}$. Then we check them one-by-one if the combinatorial structure of $\mathsf{conv}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i + e_i\mathbf{t}_0))$ changes for each intersection point $\mathbf{e} = (e_1, e_2, \ldots, e_{k-1})$. If it changes, we add and subtract at most two values from $\nabla_{\mathbf{v}-\mathbf{w}} f(\mathbf{w})$ as in Lemma 7. Thus, we can compute $\nabla_{\mathbf{v}-\mathbf{w}} f(\mathbf{v})$ in $O(N)$ time, and $\nabla_{\mathbf{v}'-\mathbf{v}} f(\mathbf{v})$ from $\nabla_{\mathbf{v}-\mathbf{w}} f(\mathbf{v})$ in $O(N)$ time. Also, we can compute $\nabla_{\mathbf{v}'-\mathbf{v}} f(\mathbf{v})$ for every pair $(\mathbf{v}, \mathbf{v}')$ of vertices of Q in $O(N)$ time, and $f(\mathbf{v})$ in $O(N)$ time for all vertices $\mathbf{v}$ of Q.

If Q is an $(m-1)$-simplex, $T_o(N, m) = O(N) + T(N, m-1)$ by Lemma 11. If Q is not an $(m-1)$-simplex, we triangulate Q, and then compute an $(m-1)$-simplex in the triangulation containing a translation minimizing $f(\mathbf{t})$ for all translations $\mathbf{t} \in Q$ by running the oracle for each $(m-2)$-face of the triangulation. Then $T_o(N, m) = O(N) + O(T_o(N, m-1)) + T(N, m-1)$ by Lemma 11. Since $T(N, m-1) = O(N) + O(T_o(N, m-1)) + T(\lfloor N/2 \rfloor, m-1)$ by Lemma 10, $T_o(N, m) = O(N) + T(N, m-1)$. ◀

By Lemmas 10 and 12, we have $T(N, m) = O(N) + O(T(N, m-1)) + T(\lfloor N/2 \rfloor, m)$.

▶ **Lemma 13.** *We can compute $\mathbf{t}_0^* \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ with $f(\mathbf{t}_0^*) \le f(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ in $O(n)$ time.*

Proof. For $T(N, 1)$, we compute event hyperplanes intersecting a 1-simplex with endpoints $\mathbf{v}_1$ and $\mathbf{v}_2$. Assume that $\nabla_{\mathbf{v}_i-\mathbf{v}_j} f(\mathbf{v}_i)$, $f(\mathbf{v}_i)$, and the combinatorial structure of $\mathsf{conv}(\mathbf{v}_i)$ are given for $i, j \in \{1, 2\}$ with $i \ne j$. We apply the recursive decimation algorithm by Ahn et al. (Lemma 15 in [3]). Let $\mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1$, z be the line segment connecting $\mathbf{v}_1$ and $\mathbf{v}_2$, and $|z|$ be its length. Let $B(a) = f(\mathbf{v}_1 + a\mathbf{v}_{21}/|z|)$ for $a \in [0, |z|]$. We compute the set V of at most N values a such that the point $\mathbf{v}_1 + a\mathbf{v}_{21}/|z|$ lies on $z \cap H$ and is contained in the set of translations stored in H for $H \in \mathsf{E}$. This can be done in $O(N)$ time. Let $B^+(a) = \nabla_{\mathbf{v}_{21}} f(\mathbf{v}_1 + a\mathbf{v}_{21}/|z|)$ and $B^-(a) = \nabla_{\mathbf{v}_1-\mathbf{v}_2} f(\mathbf{v}_1 + a\mathbf{v}_{21}/|z|)$. If $B^+(0) \ge 0$ or $B^-(|z|) \ge 0$, we are done. So assume $B^+(0) < 0$ and $B^-(|z|) < 0$. Our goal is to find a value $a^* \in (0, |z|)$ such that $B^+(a^*) \ge 0$ and $B^-(a^*) \ge 0$.

At each iteration, we maintain an open interval (a_0, a_1), $B^+(a_0)$, and $B^-(a_1)$. Initially, $(a_0, a_1) = (0, |z|)$. Then, a_0 corresponds to $\mathbf{v}_1$, and a_1 corresponds to $\mathbf{v}_2$. In a recursive step, we compute the median a_2 of V in $O(|\mathsf{V}|)$ time. Each $a' \in \mathsf{V}$ with $a_0 < a' \le a_2$ corresponds to an event hyperplane that defines at most two values to add and subtract from $B^+(a_0)$. Thus, we compute $B^+(a_2)$ from $B^+(a_0)$ in $O(|\mathsf{V}|)$ time by adding and subtracting at most $2|\mathsf{V}|$ values from $B^+(a_0)$. Similarly, we compute $B^-(a_2)$ from $B^-(a_1)$ in $O(|\mathsf{V}|)$ time.

If $B^+(a_2) \ge 0$ and $B^-(a_2) \ge 0$, $B(a_2) = B(a^*)$ and we are done. If $B^+(a_2) < 0$ and $B^-(a_2) > 0$, we update V by removing those values in $[a_0, a_2]$ from V and recurse on (a_2, a_1). If $B^+(a_2) > 0$ and $B^-(a_2) < 0$, we update V by removing those values in $[a_2, a_1]$ from V

and recurse on (a_0, a_2). This can be done in $O(|\mathsf{V}|)$ time. Since the size of V halves in each iteration, we can compute a^* in $O(N)$ time. By Lemma 8, we can compute $B(a^*)$ in $O(N)$ time.

We have $T(N,1) = O(N)$, and $T(N,m) = O(N) + T(\lfloor N/2 \rfloor, m) = O(N)$. So, we can compute a $(k-1)$-dimensional simplex $\triangle$ such that $\mathbf{t}_0^* \in \triangle$ and no event hyperplane in E intersects $\mathrm{int}(\triangle)$ in $T(|\mathsf{E}|, k-1) = O(n)$ time. We can evaluate the linear function $f(\mathbf{x})$ in $\triangle$ in $O(n)$ time. Thus, we can compute $\mathbf{t}_0^* \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ satisfying $f(\mathbf{t}_0^*) \leq f(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ in $O(n)$ time. ◀

5.3 Approximation algorithm

We give an approximation algorithm for the minimum-area convex hull problem. For a fixed $\vartheta \in [0, \pi)$, let $W_\vartheta = \max_{C \in \mathsf{C}} w_\vartheta(C)$. Let Q^* be an optimal convex hull of k polygons in C.

▶ **Lemma 14.** *$w_\vartheta(\mathsf{Q}^*) \leq 2W_\vartheta$ for every $\vartheta \in [0, \pi)$.*

Proof. Assume that every polygon in C is already placed in Q^*. Suppose $w_\vartheta(\mathsf{Q}^*) > 2W_\vartheta$ for some $\vartheta \in [0, \pi)$. Let v and w be two vertices of Q^* such that $w_\vartheta(vw) = w_\vartheta(\mathsf{Q}^*)$. Let C_v and C_w denote the sets of polygons in C that have v and w as vertices, respectively. Let $\mathsf{Q}_v = \mathsf{conv}(\mathsf{C}_v)$ and $\mathsf{Q}_w = \mathsf{conv}(\mathsf{C}_w)$. Since $w_\vartheta(C) \leq W_\vartheta$ for every $C \in \mathsf{C}_v$, $w_\vartheta(\mathsf{Q}_v) \leq W_\vartheta$. Similarly, $w_\vartheta(\mathsf{Q}_w) \leq W_\vartheta$.

Since $v \in \mathsf{Q}_v$ and $w \in \mathsf{Q}_w$, and $w_\vartheta(vw) > 2W_\vartheta$, Q_v and Q_w are disjoint. Thus, there is a translate Q' of either Q_v or Q_w such that Q' touches the other and $\mathsf{Q}' \subset \mathsf{conv}(\mathsf{Q}_v \cup \mathsf{Q}_w) \subset \mathsf{Q}^*$. Without loss of generality, assume Q' is a translate of Q_v. Let $\mathbf{e}$ denote the translation vector satisfying $\mathsf{Q}' = \mathsf{Q}_v + \mathbf{e}$. Since the internal angle at v of Q^* and Q_v is less than π, $v \notin \mathsf{Q}'$ and $v \notin \mathsf{conv}(\mathsf{Q}' \cup \mathsf{Q}_w)$. For $\mathsf{C}'_v = \{C + \mathbf{e} \mid C \in \mathsf{C}_v\}$, $\mathsf{conv}((\mathsf{C} \setminus \mathsf{C}_v) \cup \mathsf{C}'_v) \subset \mathsf{Q}^*$ and $v \notin \mathsf{conv}((\mathsf{C} \setminus \mathsf{C}_v) \cup \mathsf{C}'_v)$. Thus, $\mathsf{area}((\mathsf{C} \setminus \mathsf{C}_v) \cup \mathsf{C}'_v) < \mathsf{area}(\mathsf{Q}^*)$, which contradicts the optimality of Q^*. ◀

We now state an approximation algorithm.

▶ **Theorem 15.** *Given k convex polygons with n vertices in total for any fixed integer $k \geq 3$ in the plane and $\varepsilon \in (0,1)$, we can compute a translation $\mathbf{t}_{\mathsf{apx}}$ for the polygons in $O(n/\varepsilon^{k-1})$ time satisfying $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Upsilon(\mathbf{t}^*)$ for any translation $\mathbf{t}^*$ minimizing the area function Υ.*

Proof. Assume C_0 has diameter $D = \max_{C \in \mathsf{C}} d(C)$ with a diametral pair (v, w) such that vw is horizontal, and the topmost vertex of C_0 is at the origin. Let H be the set of $O(1/\varepsilon)$ horizontal lines $y = b$ for $b = -W_{\pi/2} + j\varepsilon W_{\pi/2}/4$ for each $j = 0, 1, \ldots, \lfloor 8/\varepsilon \rfloor$.

We compute a translation $\mathbf{t}_{\mathsf{apx}} \in \mathbb{T}$ for the $k-1$ polygons in $\mathsf{C} \setminus \{C_0\}$ under the constraint that each polygon must be placed with its topmost vertex lying on one line in H. Observe that there are $O(1/\varepsilon^{k-1})$ distinct placements for the $k-1$ polygons over $O(1/\varepsilon)$ lines in H that satisfy the constraint. We enumerate every such placement and compute an optimal translation $\mathbf{t}$ of the polygons along their corresponding lines using the algorithm in Lemma 13. We compute in $O(n/\varepsilon^{k-1})$ time a translation $\mathbf{t}_{\mathsf{apx}}$ with $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Upsilon(\mathbf{t}^*)$ for any translation $\mathbf{t}^*$ minimizing Υ.

Let $\mathbf{t}^* = (\mathbf{t}_1^*, \mathbf{t}_2^*, \ldots, \mathbf{t}_{k-1}^*)$ be an optimal translation for Υ. By the construction of H and Lemma 14, there exist a line $\ell \in \mathsf{H}$ and a translation $\bar{\mathbf{t}}_i \in \mathbb{R}^2$ with $p_i + \bar{\mathbf{t}}_i \in \ell$ satisfying $x(\mathbf{t}_i^*) = x(\bar{\mathbf{t}}_i)$ and $|y(\mathbf{t}_i^*) - y(\bar{\mathbf{t}}_i)| \leq \varepsilon W_{\pi/2}/8$ for each $i \in [k-1]$. Observe that $(\bar{\mathbf{t}}_1, \bar{\mathbf{t}}_2, \ldots, \bar{\mathbf{t}}_{k-1})$ is in $\mathbb{T}(\mathbf{t}_0, \mathbf{t}^*)$, where $\mathbf{t}_0 = (0,1)$. By Lemmas 6 and 14, $w_0(\mathsf{Q}^*) \leq 2W_0 \leq 2D$. Since $DW_{\pi/2}/2 \leq \Upsilon(\mathbf{t}^*)$, $\Upsilon(\mathbf{t}_{\mathsf{apx}}) - \Upsilon(\mathbf{t}^*) \leq 2w_0(\mathsf{Q}^*) \times \frac{\varepsilon W_{\pi/2}}{8} \leq \frac{\varepsilon D W_{\pi/2}}{2} \leq \varepsilon \Upsilon(\mathbf{t}^*)$. ◀

We improve the time bound in Theorem 15 further by approximating each polygon in C into a convex polygon with $O(\varepsilon^{-1/2})$ vertices by Lemma 1 in the beginning of the algorithm.

▶ **Theorem 16.** *Given k convex polygons with n vertices in total for any fixed integer $k \geq 3$ in the plane and $\varepsilon \in (0,1)$, we can compute a translation $\mathbf{t}_{\mathsf{apx}}$ for the polygons in $O(\varepsilon^{-1/2}\log n + \varepsilon^{1/2-k})$ time satisfying $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Upsilon(\mathbf{t}^*)$ for any translation $\mathbf{t}^*$ minimizing the area function Υ.*

Proof. Using Lemma 1, we compute a convex polygon $\tilde{C}_i$ with $O(\varepsilon^{-1/2})$ vertices for each $C_i \in \mathsf{C}$ such that $C_i \subseteq \tilde{C}_i$ and C_i is an $\varepsilon/(3c)$-kernel of $\tilde{C}_i$ for a positive constant c in $O(\varepsilon^{-1/2}\log n)$ time in total. Let $\varepsilon_3 = \varepsilon/3$. Let $\tilde{\Upsilon}(\mathbf{t}) = \mathsf{area}(\tilde{C}_0 \cup \bigcup_{i=1}^{k-1}(\tilde{C}_i + \mathbf{t}_i))$ for vectors $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{k-1}) \in \mathbb{T}$. Then $(1-\varepsilon_3)\tilde{\Upsilon}(\mathbf{t}) \leq \Upsilon(\mathbf{t})$ for every $\mathbf{t} \in \mathbb{T}$. Let $\tilde{\mathbf{t}}_{\mathsf{opt}}$ be one minimizing $\tilde{\Upsilon}(\mathbf{t})$ among all $\mathbf{t} \in \mathbb{T}$. By Theorem 15, we can compute a translation $\mathbf{t}_{\mathsf{apx}} \in \mathbb{T}$ in $O(\varepsilon^{1/2-k})$ time satisfying $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq \tilde{\Upsilon}(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon_3)\tilde{\Upsilon}(\tilde{\mathbf{t}}_{\mathsf{opt}}) \leq (1+\varepsilon_3)\tilde{\Upsilon}(\mathbf{t}^*) \leq \frac{1+\varepsilon_3}{1-\varepsilon_3}\Upsilon(\mathbf{t}^*) < (1+\varepsilon)\Upsilon(\mathbf{t}^*)$. ◀

6 Minimum-area convex hull under rigid motions

We consider the problem of minimizing the area of the convex hull of k convex polygons $C_0, \ldots, C_{k-1}$ with n vertices in total in the plane under rigid motions for any fixed $k \geq 3$. We assume C_0 is fixed. For a compact convex set C in the plane, we denote by $C(\theta)$ the rotated copy of C by θ around the midpoint of a diametral pair of C. For a rotation vector $\mathbf{r} \in [0, 2\pi)^{k-1}$ with $\mathbf{r} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$, we define the area function $\Phi(\mathbf{r})$ as the minimum area of the convex hull of C_0 and $C_i(\theta_i)$'s for all $i \in [k-1]$ over all translations $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{k-1}) \in \mathbb{T}$, that is, $\Phi(\mathbf{r}) = \min_{\mathbf{t}\in\mathbb{T}} \mathsf{area}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i(\theta_i) + \mathbf{t}_i))$.

We present a $(1+\varepsilon)$-approximation algorithm by generalizing the technique in [6]. We generate a set S_i of $O(1/\varepsilon)$ angles of C_i for each $i \in [k-1]$. For each distinct rotation $\mathbf{r} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$ with $\theta_i \in S_i$, we compute a translation $\mathbf{t}_{\mathsf{apx}}$ with $\Upsilon(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Upsilon(\mathbf{t}^*)$ for C_0 and $C_i(\theta_i)$'s for all $i \in [k-1]$ by using the algorithm in Theorem 15.

Let $\mathbf{r}^* = (\theta_1^*, \theta_2^*, \ldots, \theta_{k-1}^*)$ denote a rotation satisfying $\Phi(\mathbf{r}^*) \leq \Phi(\mathbf{r})$ for all rotations $\mathbf{r} \in [0, 2\pi)^{k-1}$. Let C_i^* be a translate of $C_i(\theta_i^*)$ for each $i \in [k-1]$ such that $\Phi(\mathbf{r}^*) = \mathsf{area}(C_0 \cup \bigcup_{i=1}^{k-1} C_i^*)$. Let $\mathsf{Q}^* = \mathsf{conv}(C_0 \cup \bigcup_{i=1}^{k-1} C_i^*)$. Let $D = \max_{C\in\mathsf{C}} d(C)$ and $W = \max_{C\in\mathsf{C}} w(C)$. Without loss of generality, we assume that C_0 has diameter D.

▶ **Lemma 17.** $\Phi(\mathbf{r}^*) \geq DW/2$

Proof. By Lemma 6 in [6], $\mathsf{area}(C_0 \cup C_i^*) \geq DW/2$ for C_i^* with $w(C_i^*) = W$. ◀

We now give a bound on the angle between the line through a diametral pair of C_0 and the line through a diametral pair of C_i^*.

▶ **Lemma 18.** *For the smaller angle ϕ_i between the line through a diametral pair of C_0 and the line through a diametral pair of C_i^*, $\sin\phi_i \leq 2W/d(C_i)$.*

Proof. Let (p,q) be a diametral pair of C_0 and let (p_i,q_i) be a diametral pair of C_i^*. Let ϕ_i be the smaller angle between the line through p,q and the line through p_i,q_i. Since both pq and p_iq_i are contained in Q^*, and their convex hull has area at least $(|pq||p_iq_i|/2)\sin\phi_i = (Dd(C_i)/2)\sin\phi_i$, $\Phi(\mathbf{r}^*) \geq (Dd(C_i)/2)\sin\phi_i$.

Let R_i be a rectangle circumscribed to C_i with side lengths $d(C_i)$ and $w(C_i)$ for each $i \in [k-1]$, and let R be a rectangle circumscribed to C_0 with side lengths D and W. Observe that there is a rotation $\mathbf{r} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$ such that the rotated copy of R_i by θ_i is contained in R under translations. Thus, $\Phi(\mathbf{r}^*) \leq \Phi(\mathbf{r}) \leq \mathsf{area}(R) = DW$. Then $(Dd(C_i)/2)\sin\phi_i \leq \Phi(\mathbf{r}^*) \leq DW$. ◀

Proof for Lemma 19 is given in Appendix B.

▶ **Lemma 19.** *$\Phi(\mathbf{r}^*+\boldsymbol{\phi}) \leq (1+\varepsilon)\Phi(\mathbf{r}^*)$ for $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_{k-1})$ with $\phi_i \leq \min\left\{\frac{\varepsilon W}{16d(C_i)}, \frac{\pi}{6}\right\}$ for each $i \in [k-1]$ and $\varepsilon > 0$.*

We now state an approximation algorithm.

▶ **Theorem 20.** *Given k convex polygons with n vertices in total for a fixed integer $k \geq 3$ in the plane and $\varepsilon \in (0,1)$, we can compute a rotation $\mathbf{r}_{\mathsf{apx}}$ for the polygons in $O(n/\varepsilon^{2k-2})$ time satisfying $\Phi(\mathbf{r}_{\mathsf{apx}}) \leq (1+\varepsilon)\Phi(\mathbf{r}^*)$ for any rotation $\mathbf{r}^*$ minimizing the area function Φ.*

Proof. Let $\varepsilon_3 = \varepsilon/3$ and $N_i = W/d(C_i)$ for each $i \in [k-1]$. We can compute D, W, and N_i for all $i \in [k-1]$ in linear time. For each $i \in [k-1]$, we sample angles at intervals of $\varepsilon_3 N_i/16$, but omit angles θ where the diameters of C_0 and $C_i(\theta)$ make angle ϕ with $\sin\phi > 2N_i$. Let S_i be the set of sampled angles for each $i \in [k-1]$ with $|S| = O(1/\varepsilon_3)$.

For each distinct rotation $\mathbf{r} = (\theta_1, \theta_2, \ldots, \theta_{k-1})$ with $\theta_i \in S_i$ for $i \in [k-1]$, we apply the algorithm in Theorem 15 on $C_i(\theta_i)$'s for every $i \in [k-1]$ and compute a translation that yields a $(1+\varepsilon_3)$-approximation to the optimal translation for Υ in $O(n/\varepsilon^{k-1})$ time. Then we give a rotation vector $\mathbf{r}_{\mathsf{apx}}$ achieving the minimum area among all distinct rotations. Since there are $O(1/\varepsilon_3^{k-1})$ distinct rotations, it takes $O(n/\varepsilon^{2k-2})$ time in total. By Theorem 15 and Lemmas 18 and 19, $\Phi(\mathbf{r}_{\mathsf{apx}}) \leq (1+\varepsilon/3)^2\Phi(\mathbf{r}^*) \leq (1+2\varepsilon/3+\varepsilon^2/9)\Phi(\mathbf{r}^*) \leq (1+\varepsilon)\Phi(\mathbf{r}^*)$. ◀

By approximating the input polygons as in Theorem 16, we have the following theorem.

▶ **Theorem 21.** *Given k convex polygons with n vertices in total for a fixed integer $k \geq 3$ in the plane and $\varepsilon \in (0,1)$, we can compute a rotation $\mathbf{r}_{\mathsf{apx}}$ for the polygons in $O(\varepsilon^{-1/2}\log n + \varepsilon^{3/2-2k})$ time satisfying $\Phi(\mathbf{r}_{\mathsf{apx}}) \leq (1+\varepsilon)\Phi(\mathbf{r}^*)$ for any rotation $\mathbf{r}^*$ minimizing the area function Φ.*

Note that the algorithms in Theorems 20 and 21 compute a rigid motion for k convex polygons, including the case for $k = 2$, that achieves $(1+\varepsilon)$-approximation to the rigid motion that attains the minimum area convex hull.

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A Minimum-perimeter convex hull under translations

We consider the problem of minimizing the perimeter of the convex hull of k convex polygons $C_0, \ldots, C_{k-1}$ with n vertices in total in the plane under translations for any fixed $k \geq 3$. We adopt the notations introduced in Section 3 and Section 5. For a compact set $S \subset \mathbb{R}^2$, let $\mathsf{peri}(S) := \mathsf{peri}(\mathsf{conv}(S))$ denote the perimeter of $\mathsf{conv}(S)$. We apply the algorithmic framework in Lemma 13 to this problem. The perimeter function for a translation $\mathbf{t} \in \mathbb{T}$ is denoted by

$$\Psi(\mathbf{t}) = \mathsf{peri}(C_0 \cup \bigcup_{i=1}^{k-1}(C_i + \mathbf{t}_i)).$$

The goal is to compute a translation $\mathbf{t}^* \in \mathbb{T}$, which we call an optimal translation, such that $\Psi(\mathbf{t}^*) \leq \Psi(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}$. We use the following property.

▶ **Lemma 22** ([6])**.** *The perimeter function $\Psi(\mathbf{t})$ is convex.*

Since the same degeneracy types described in Section 3.2 arise in this setting, $\Psi(\mathbf{t})$ has the same event surfaces as those defined for the area function.

A.1 Two convex polygons

Let n_0 and n_1 be the number of vertices of C_0 and C_1, respectively. We assume that $n_0 \geq n_1$. We present an algorithm running in $O(n_0 + n_1 \log^2(n_0 + n_1))$ time for the problem. This improves the previous $O((n_0+n_1)\log^2(n_0+n_1))$ bound by reducing the log-factor dependence on the larger input size n_0. The running time becomes linear for $n_1 \leq n_0/\log^2(n_0 + n_1)$.

Since Degeneracy Type 2 requires three polygons to touch a line simultaneously, it does not occur in this setting where only $k = 2$ polygons are considered. Thus, event surfaces can be expressed as lines in $\mathbb{T}$, which we call *event lines*. So, we can directly apply the algorithm in Lemma 10 to $\mathbb{T}$. Then it remains to bound $T_o(N, 2)$ in terms of $T(N, 1)$.

Consider the edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$. Then the edges can be classified into three types: edges of C_0 (type 0), edges of $C_1 + \mathbf{t}_1$ (type 1), and edges connecting a vertex of C_0 to a vertex of $C_1 + \mathbf{t}_1$ (type 2). Observe that the total length of the edges of type 0 and type 1 remains constant as long as the combinatorial structure of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$ remains unchanged. So, the local minimum of $\Psi(\mathbf{t}_1)$ is determined by the total length of edges of type 2. Let $\Psi_2(\mathbf{t}_1)$ be the sum of the length of type 2 edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$. Observe that the gradient of Ψ_2 is equivalent to the gradient of Ψ; that is, $\nabla\Psi_2 = \nabla\Psi$.

For each vertex v of $C_1 + \mathbf{t}_1$, there are at most two type 2 edges incident to v in $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$. Thus, the total number of type 2 edges in $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$ is $O(n_1)$, implying that $\Psi_2(\mathbf{t}_1)$ can be evaluated in $O(n_1)$ time. The following lemma shows how to efficiently update type 2 edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}'_1))$ from type 2 edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$ for $\mathbf{t}_1 \neq \mathbf{t}'_1$.

▶ **Lemma 23.** *For any two points $\mathbf{t}_1$ and $\mathbf{t}'_1$ in $\mathbb{T}$, given the type 2 edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$ and the event lines intersecting the segment connecting $\mathbf{t}_1$ and $\mathbf{t}'_1$, we can compute the type 2 edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}'))$ in time linear in the number of event lines intersecting the segment.*

Proof. Let E_2 be the set of type 2 edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}_1))$ and let T be the set of intersections of the event lines and the segment connecting $\mathbf{t}_1$ and $\mathbf{t}'_1$. If $\mathbf{x}$ hits two or more event lines simultaneously for $\mathbf{x} \in T$, we handle them one by one. We initialize a set E to be empty. For $\mathbf{x} \in T$, at most two type 2 edges appear and at most two type 2 edges disappear in $\mathsf{conv}(\mathbf{x})$. For the edge e disappearing in $\mathsf{conv}(\mathbf{x})$, if $e \in E_2$, we remove e from E_2. If not, we add e to E. For the edge e appearing in $\mathsf{conv}(\mathbf{x})$, if $e \in E$, we remove e from E. If not, we add e to E_2. This can be done in $O(1)$ time for each case. Let E'_2 be the set obtained from the algorithm. Thus, we can compute E'_2 in time linear in the number of event lines intersecting the segment.

We show that E'_2 is the set of type 2 edges of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}'_1))$. Observe that once the combinatorial structure changes, it does not revert. If e is a type 2 edge of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}'_1))$, then $e \in E'_2$, since e does not disappear for $\mathbf{x} \in T$. If e is not a type 2 edge of $\mathsf{conv}(C_0 \cup (C_1 + \mathbf{t}'_1))$, then e is irrelevant to the events in T or e disappears for some $\mathbf{x} \in T$. By the construction of the algorithm, $e \notin E'_2$. ◀

▶ **Lemma 24.** *Assume that $\mathsf{conv}(\mathbf{w})$ is given for every vertex $\mathbf{w}$ of a 1-simplex. $T_o(N, 2) = O(N + n_1) + T(N, 1)$.*

Proof. Let Q be the intersection of a line and a 2-simplex $\triangle$. Then Q is a line segment such that each end point lies on an edge of $\triangle$. We compute the edges of type 2 of $\mathsf{conv}(\mathbf{v})$ for an endpoint $\mathbf{v}$ of Q as follows. We compute $O(N)$ intersection points of event lines and the segment $\mathbf{wv}$ for a vertex $\mathbf{w}$ of $\triangle$. By Lemma 23, we can compute the edges of type 2 of $\mathsf{conv}(\mathbf{v})$ in $O(N)$ time. We can compute the minimum point $\mathbf{t}^*_Q$ on Q in $T(N, 1)$ time. By computing $\nabla\Psi_2$ at $\mathbf{t}^*_Q$, we determine the relative position of $\mathbf{t}^*$ with respect to Q in $\triangle$, which takes $O(n_1)$ time. Thus, we have $T_o(N, 2) = O(N + n_1) + T(N, 1)$. ◀

By Lemmas 10 and 24, we have $T(N,2) = O(N+n_1) + O(T(N,1)) + T(\lfloor N/2 \rfloor, m)$.

▶ **Theorem 25.** *Given a convex n_0-gon C_0 and a convex n_1-gon C_1 with $n_1 \le n_0$ in the plane, we can compute a translation vector $\mathbf{t}^* \in \mathbb{R}^2$ such that $\Psi(\mathbf{t}^*) \le \Psi(\mathbf{t})$ for all vectors $\mathbf{t} \in \mathbb{R}^2$, thereby minimizing the perimeter function Ψ, in $O(n_0 + n_1 \log^2(n_0+n_1))$ time.*

Proof. For $T(N,1)$, we compute event lines intersecting a 1-simplex with endpoints $\mathbf{v}_1$ and $\mathbf{v}_2$. Assume that the edges of type 2 of $\mathsf{conv}(\mathbf{v}_1)$ and $\mathsf{conv}(\mathbf{v}_2)$ are given. We apply the recursive decimation algorithm by Ahn et al. (Lemma 15 in [3]). Let $\mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1$, z be the line segment connecting $\mathbf{v}_1$ and $\mathbf{v}_2$, and $|z|$ be its length. Let $B(a)$ be the sum of length of edges of type 2 of $\mathsf{conv}(\mathbf{v}_1 + a\mathbf{v}_{21}/|z|)$ for $a \in [0,|z|]$. We abuse the notation and define $\mathsf{conv}(a) = \mathsf{conv}(\mathbf{v}_1 + a\mathbf{v}_{21}/|z|)$. We compute the set V of at most N values a such that the point $\mathbf{v}_1 + a\mathbf{v}_{21}/|z|$ lies on $z \cap H$ and is contained in the set of translations stored in H for $H \in \mathsf{E}$. This can be done in $O(N)$ time. Let $B^+(a) = \lim_{x \to a+} \nabla B(x)$ and $B^-(a) = \lim_{x \to a-} \nabla B(x)$. If $B^+(0) \ge 0$ or $B^-(|z|) \le 0$, we are done. So assume $B^+(0) > 0$ and $B^-(|z|) > 0$. Our goal is to find a value $a^* \in (0,|z|)$ such that $B^+(a^*) \ge 0$ and $B^-(a^*) \le 0$.

At each iteration, we maintain an open interval (a_0, a_1), edges of type 2 of $\mathsf{conv}(a_0)$, and edges of type 2 of $\mathsf{conv}(a_1)$. Initially, $(a_0,a_1) = (0,|z|)$. Then, a_0 corresponds to $\mathbf{v}_1$, and a_1 corresponds to $\mathbf{v}_2$. In a recursive step, we compute the median a_2 of V in $O(|\mathsf{V}|)$ time. By Lemma 23, we can compute edges of type 2 of $\mathsf{conv}(a_2)$ in $O(N)$ time. By evaluating $B^+(a_2)$ and $B^-(a_2)$, we can find the relative position of a^* with respect to a_2, which takes $O(n_1)$ time. We remove the values in V on the opposite side of a^* relative to a_2. Since the size of V halves in each iteration, we can compute a^* in $O(N + n_1 \log N)$ time.

We have $T(N,1) = O(N + n_1 \log N)$, and $T(N,2) = O(N + n_1 \log N) + T(\lfloor N/2 \rfloor, 2) = O(N + n_1 \log^2 N)$. Since $N = O(n_0 + n_1)$, we can compute a 2-simplex $\triangle$ such that $\mathbf{t}^* \in \triangle$ and no event line in E intersects $\mathrm{int}(\triangle)$ in $T(|\mathsf{E}|, 2) = O(n_0 + n_1 \log^2(n_0+n_1))$ time. We can evaluate the function $\Psi(\mathbf{t})$ in $\triangle$ in $O(n_0+n_1)$ time. Thus, we can compute $\mathbf{t}_0^* \in \mathbb{T}$ satisfying $\Psi(\mathbf{t}_0^*) \le \Psi(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}$ in $O(n_0 + n_1 \log^2(n_0+n_1))$ time. ◀

A.2 More than two convex polygons

Since event surfaces of type 2 are quadric surfaces, we cannot apply the algorithm in Lemma 10 to $\mathbb{T}$ directly. We show that, for any $\mathbf{t}_0 \in \mathbb{R}^2$ and for any $\mathbf{a} \in \mathbb{T}$, event surfaces of type 2 can be expressed as a hyperplane in $\mathbb{T}(\mathbf{t}_0, \mathbf{a})$. Observe that $\mathbb{T}(\mathbf{t}_0,\mathbf{a})$ has the maximum possible dimension of affine subspaces of $\mathbb{T}$, in which event surfaces of type 2 can be expressed as a hyperplane. Then our goal is to compute a translation $\mathbf{t}_0^* \in \mathbb{T}$, such that $\Psi(\mathbf{t}_0^*) \le \Psi(\mathbf{t})$ for all $\mathbf{t}_0 \in \mathbb{T}(\mathbf{t}_0,\mathbf{a})$.

For a fixed unit vector $\mathbf{t}_0 \in \mathbb{R}^2$ and a fixed vector $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}) \in \mathbb{T}$ with $\mathbf{a}_i \in \mathbb{R}^2$ for $i \in [k-1]$, let $\mathbf{x} = (x_1, x_2, \ldots, x_{k-1})$ with $x_i \in \mathbb{R}$ for $i \in [k-1]$, and define $\mathbf{t}(\mathbf{x}) = (\mathbf{a}_1 + x_1\mathbf{t}_0, \ldots, \mathbf{a}_{k-1} + x_{k-1}\mathbf{t}_0)$. Let $\tilde{f}(\mathbf{x}) = \Psi(\mathbf{t}(\mathbf{x}))$ in $\mathbb{T}(\mathbf{t}_0,\mathbf{a})$.

▶ **Lemma 26.** *Given $\mathsf{conv}(\mathbf{w})$ for every vertex $\mathbf{w}$ of an m-simplex, we have $T_o(N,m) = O(n) + T(N, m-1)$.*

Proof. Let Q be the intersection of an $(m-1)$-flat and an m-simplex $\triangle$. Then Q is an $(m-1)$-dimensional convex polytope with $O(m) = O(1)$ vertices, each on an edge of $\triangle$. Then we can compute $\tilde{f}(\mathbf{v})$ and $\mathsf{conv}(\mathbf{v})$ for all vertices of Q in $O(n)$ time by evaluating one by one.

If Q is an $(m-1)$-simplex, $T_o(N,m) = O(n) + T(N,m-1)$ by Lemma 11. If Q is not an $(m-1)$-simplex, we triangulate Q, and then compute an $(m-1)$-simplex in the triangulation containing a translation minimizing $\tilde{f}(\mathbf{t})$ for all translations $\mathbf{t} \in Q$ by running the oracle for each $(m-2)$-face of the triangulation. Then $T_o(N,m) = O(n)+O(T_o(N,m-1))+T(N,m-1)$ by Lemma 11. Since $T(N,m-1) = O(n)+O(T_o(N,m-1))+T(\lfloor N/2 \rfloor, m-1)$ by Lemma 10, $T_o(N,m) = O(n) + T(N,m-1)$. ◀

By Lemmas 10 and 26, we have $T(N,m) = O(n) + O(T(N,m-1)) + T(\lfloor N/2 \rfloor, m)$.

▶ **Lemma 27.** *We can compute* $\mathbf{t}_0^* \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ *with* $\tilde{f}(\mathbf{t}_0^*) \leq \tilde{f}(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ *in* $O(n \log^{k-1} n)$ *time.*

Proof. For $T(N,1)$, we compute event hyperplanes intersecting a 1-simplex with endpoints $\mathbf{v}_1$ and $\mathbf{v}_2$. Assume that $\mathbf{v}_1\mathbf{v}_2$ are parallel to x_1-axis. We compute the set V of at most N points such that the point is the intersection of $\mathbf{v}_1\mathbf{v}_2 \cap H$ and is contained in the set of translations stored in H for $H \in \mathsf{E}$. This can be done in $O(N)$ time. We sort the points in V in increasing order of x_1-coordinate. Since we can evaluate $\tilde{f}(\mathbf{t})$ in $O(n)$ time, we can perform binary search on V in $O(n \log N)$ time. Then we obtain a line segment that no event hyperplane intersects. We can find the minimum point on the line segment by evaluating $\tilde{f}(\mathbf{t})$ in $O(n)$ time. Thus, $T(N,1) = O(n \log N)$.

We have $T(N,1) = O(n \log N)$, and $T(N,m) = O(n \log^{m-1} N) + T(\lfloor N/2 \rfloor, m) = O(n \log^m N)$. So, we can compute a $(k-1)$-dimensional simplex $\triangle$ such that $\mathbf{t}_0^* \in \triangle$ and no event hyperplane in E intersects $\mathrm{int}(\triangle)$ in $T(|\mathsf{E}|, k-1) = O(n \log^{k-1} n)$ time. We can evaluate the linear function $f(\mathbf{x})$ in $\triangle$ in $O(n)$ time. Thus, we can compute $\mathbf{t}_0^* \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ satisfying $f(\mathbf{t}_0^*) \leq f(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{T}(\mathbf{t}_0, \mathbf{a})$ in $O(n \log^{k-1} n)$ time. ◀

We now state an approximation algorithm.

▶ **Theorem 28.** *Given k convex polygons with n vertices in total for any fixed integer $k \geq 3$ in the plane and $\varepsilon \in (0,1)$, we can compute a translation $\mathbf{t}_{\mathsf{apx}}$ for the polygons in $O((n \log^{k-1} n)/\varepsilon^{k-1})$ time satisfying $\Psi(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Psi(\mathbf{t}^*)$ for any translation $\mathbf{t}^*$ minimizing the perimeter function Ψ.*

Proof. Assume C_0 has the diameter $D = \max_{C \in \mathsf{C}} d(C)$ and the topmost vertex of C_0 is at the origin. Let H be the set of $O(1/\varepsilon)$ horizontal lines $y = b$ for $b = -D + 2j\varepsilon D/\pi$ for each $j = 0, 1, \ldots, \lfloor \pi/\varepsilon \rfloor$. We run the algorithm in Theorem 15. By Lemma 27, we can compute a translation $\mathbf{t}_{\mathsf{apx}}$ in $O((n \log^{k-1} n)/\varepsilon^{k-1})$ time in total.

Let $\mathbf{t}^* = (\mathbf{t}_1^*, \mathbf{t}_2^*, \ldots, \mathbf{t}_{k-1}^*)$ be an optimal translation for Ψ. Let Q^* be an optimal convex hull of the k polygons by $\mathbf{t}^*$ and let $\mathsf{Q}_{\mathsf{apx}}$ be an optimal convex hull of the k polygons by $\mathbf{t}_{\mathsf{apx}}$. We have $w_\vartheta(\mathsf{Q}^*) \leq 2W_\vartheta$ for every $\vartheta \in [0, \pi)$ with little modification in Lemma 14. By the construction of H, there exist a line $\ell \in \mathsf{H}$ and a translation $\bar{\mathbf{t}}_i \in \mathbb{R}^2$ with $p_i + \bar{\mathbf{t}}_i \in \ell$ satisfying $x(\mathbf{t}_i^*) = x(\bar{\mathbf{t}}_i)$ and $|y(\mathbf{t}_i^*) - y(\bar{\mathbf{t}}_i)| \leq \varepsilon D/\pi$ for each $i \in [k-1]$. Observe that $(\bar{\mathbf{t}}_1, \bar{\mathbf{t}}_2, \ldots, \bar{\mathbf{t}}_{k-1})$ is in $\mathbb{T}(\mathbf{t}_0, \mathbf{t}^*)$, where $\mathbf{t}_0 = (0,1)$. Let Q be the set of points that are at distance at most $\varepsilon D/\pi$. Then $\mathsf{Q}_{\mathsf{apx}} \subseteq \mathsf{Q}$, and thus $\mathsf{peri}(\mathsf{Q}_{\mathsf{apx}}) \leq \mathsf{peri}(\mathsf{Q})$. Since $\mathsf{peri}(\mathsf{Q}) = \mathsf{peri}(\mathsf{Q}^*) + 2\pi(\varepsilon D/\pi) = \mathsf{peri}(\mathsf{Q}^*) + 2D\varepsilon$ [3] and $2D \leq \mathsf{peri}(C_0) \leq \Psi(\mathbf{t}^*)$, $\Psi(\mathbf{t}_{\mathsf{apx}}) - \Psi(\mathbf{t}^*) \leq \mathsf{peri}(\mathsf{Q}) - \mathsf{peri}(\mathsf{Q}^*) \leq 2D\varepsilon \leq \varepsilon\Psi(\mathbf{t}^*)$. ◀

By approximating the input polygons as in Theorem 16, we have the following theorem.

▶ **Theorem 29.** *Given k convex polygons with n vertices in total for any fixed integer $k \geq 3$ in the plane and $\varepsilon \in (0,1)$, we can compute a translation $\mathbf{t}_{\mathsf{apx}}$ for the polygons in $O(\varepsilon^{-1/2} \log n + \varepsilon^{1/2-k} \log^{k-1}(1/\varepsilon))$ time satisfying $\Psi(\mathbf{t}_{\mathsf{apx}}) \leq (1+\varepsilon)\Psi(\mathbf{t}^*)$ for any translation $\mathbf{t}^*$ minimizing the perimeter function Ψ.*

B Proof of Lemma 19

Proof. For a compact convex set $C \subset \mathbb{R}^2$, the set of points at distance at most $r > 0$ from C has area $\mathsf{area}(C) + r\mathsf{peri}(C) + \pi r^2$ [3], where $\mathsf{peri}(C)$ denotes the perimeter of C. We show an upper bound of $\mathsf{peri}(\mathsf{Q}^*)$ with respect to $\mathsf{peri}(C_0)$. Since C_0 has a diameter D, we have $2D \leq \mathsf{peri}(C_0) \leq \pi D$ [19, pg. 257, ex.7.17a]. By Lemma 14, there is a square with side length $2D$ containing Q^*. Since both the square and Q^* are convex, $\mathsf{peri}(\mathsf{Q}^*) \leq 8D$. Thus, we have $\mathsf{peri}(\mathsf{Q}^*) \leq 4\mathsf{peri}(C_0)$.

Let $\mathsf{Q}(\boldsymbol{\phi}) = \mathsf{conv}(C_0 \cup \bigcup_{i=1}^{k-1} C_i^*(\phi_i))$. Let $t = \arg\max_i \phi_i d(C_i)$, and let $X = \phi_t d(C_t)$. The distance from a point in $C_i^*(\phi_i)$ to C_i^* is at most $\phi_i d(C_i)/2 \leq X/2$ for every $i \in [k-1]$. Let Q' be the set of points that are at distance at most $X/2$ from Q^*. Then $\mathsf{Q}(\boldsymbol{\phi}) \subseteq \mathsf{Q}'$.

$$
\begin{aligned}
\mathsf{area}(\mathsf{Q}(\boldsymbol{\phi})) &\leq \mathsf{area}(\mathsf{Q}') = \mathsf{area}(\mathsf{Q}^*) + \frac{X}{2}\mathsf{peri}(\mathsf{Q}^*) + \pi\left(\frac{X}{2}\right)^2 \\
&\leq \mathsf{area}(\mathsf{Q}^*) + 2X\mathsf{peri}(C_0) + \frac{\pi X \phi_t D}{4} \\
&\leq \mathsf{area}(\mathsf{Q}^*) + 2X\pi D + \frac{\pi^2 X D}{24} \\
&\leq \mathsf{area}(\mathsf{Q}^*) + XD\left(2\pi + \frac{\pi^2}{24}\right) \\
&\leq \mathsf{area}(\mathsf{Q}^*) + 8XD = \mathsf{area}(\mathsf{Q}^*) + 8\phi_t d(C_t) D \\
&\leq \mathsf{area}(\mathsf{Q}^*) + \frac{DW}{2}\varepsilon \leq (1+\varepsilon)\mathsf{area}(\mathsf{Q}^*).
\end{aligned}
$$

The third and sixth inequalities come from $\phi_t \leq \min\left\{\frac{\varepsilon W}{16 d(C_t)}, \frac{\pi}{6}\right\}$. The last inequality holds by Lemma 17. ◀