Moderately Beyond Clique-Width: Reduced Component Max-Leaf and Related Parameters

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Abstract

Reduced parameters [BKW, JCTB '26; BKRT, SODA '22] are defined via contraction sequences. Based on this framework, we introduce the reduced component max-leaf, denoted by $\text{cml}^{\downarrow}$, where component max-leaf is the maximum number of leaves in any spanning tree of any connected component. Reduced component max-leaf is strictly sandwiched between clique-width and reduced bandwidth, it is bounded in unit interval graphs, and unbounded in planar graphs. We design polynomial-time algorithms for problems such as Maximum Independent Set, Maximum Clique, Maximum Induced d-Regular Subgraph, and Induced Disjoint Paths in graphs given with a contraction sequence witnessing low $\text{cml}^{\downarrow}$, unifying and extending tractability results for classes of bounded clique-width and unit interval graphs.

We get the following collapses in sparse classes of bounded $\text{cml}^{\downarrow}$: bounded maximum degree implies bounded treewidth, whereas $K_{t,t}$-subgraph-freeness implies strongly sublinear treewidth; we show the latter, more generally, for classes of bounded reduced cutwidth. We establish the former result by showing that graphs with bounded $\text{cml}^{\downarrow}$ admit balanced separators dominated by a bounded number of vertices. In contrast, there are graphs G of arbitrarily large girth and treewidth $\Theta(|V(G)|^{1/2})$ such that $\text{cml}^{\downarrow}(G) \leqslant 3$.

We then showcase an application of the reduced parameters to establishing non-transducibility results. We prove that for most reduced parameters $p^{\downarrow}$ (including reduced bandwidth), the family of classes of bounded $p^{\downarrow}$ is closed under first-order transductions. We then answer a question of [BKW '26] by showing that the 3-dimensional grids have unbounded reduced bandwidth. As the class of planar graphs (or any class of bounded genus) has bounded reduced bandwidth [BKW '26], this reproves a recent result [GPP, LICS '25; HJ, LICS '25] that planar graphs do not first-order transduce the 3-dimensional grids.

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1 Introduction

Contraction sequences[1] associate n-vertex graphs with sequences of n graphs of ever smaller vertex count, called *red graphs*. *Twin-width* [11] is the first parameter to be defined via contraction sequences, as the largest maximum degree of the red graphs, minimized over all contraction sequences. The so-called *reduced parameters*, introduced in [12, 10], generalize the link between maximum degree Δ and twin-width: For any graph parameter p, *reduced-p*, denoted by $p^\downarrow$, is the largest p over all red graphs of the contraction sequence, minimized among all contraction sequences. We mainly focus on graph parameters p that are monotone under taking subgraphs, reflecting the intuition that fewer red edges, which represent errors in a contraction sequence, should not make a red graph more complex.

Every graph admits a contraction sequence where each red graph consists of a star and isolated vertices. Thus, $p^\downarrow$ is mainly relevant for graph parameters p that are unbounded on the class of all stars. The interesting reduced parameters are thus less general than twin-width.[2] We write $p \sqsubset q$ if every class of bounded q has bounded p, but not vice versa.

If $p = \star$ is the size of a largest connected component, then $p^\downarrow$ is tied to clique-width [10, 2]. Thus, by the Courcelle–Makowsky–Rotics theorem [15] monadic second-order logic (with modular counting) can be efficiently model-checked in classes of bounded $\star^\downarrow$, whereas first-order model checking is tractable in classes of effectively[3] bounded $\Delta^\downarrow$, i.e., twin-width [11]. This suggests the following line of work:

> For parameters p with $\Delta \sqsubset p \sqsubset \star$, design polynomial algorithms for a strict superset of the problems definable in first-order logic on graphs of effectively bounded $p^\downarrow$.

A natural next target beyond first-order model checking is Maximum Independent Set, which on a graph G asks for $\max_{X \subseteq V(G)} |X|$ subject to the condition that X is an independent set. It is thus reasonable to expect that the superset contains this problem.

There are two main families of graphs with bounded twin-width on which Maximum Independent Set remains NP-hard: planar graphs [11, 24] and $f(n)$-subdivisions of n-vertex graphs with $2 \log n \leqslant f(n) \leqslant n^{O(1)}$ [3, 8]. In the work explicitly introducing the reduced parameters [12], the focus is on reduced bandwidth. The main result is that planar graphs have bounded reduced bandwidth; so do ($\geqslant n$)-subdivisions of n-vertex graphs. Since many problems, like Maximum Independent Set, remain NP-hard on planar graphs, the parameter p we consider cannot be as general as bandwidth.

An attempt is *stretch-width* [7], which, while strictly speaking not a reduced parameter, is strongly inspired by this framework. Classes of bounded stretch-width generalize those of bounded clique-width, and fittingly do not contain the class of planar graphs or of polynomial subdivisions. However it is currently open whether Maximum Independent Set can be solved in polynomial time in every class of bounded stretch-width (even with a witness of low stretch-width). Partial results comprise a $2^{\tilde{O}(n^{4/5})}$-time algorithm, and a polynomial-time algorithm when the maximum degree is also bounded.

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[1] See Section 2 for definitions and notation.
[2] As an abuse of language, we say that p is less general than q if classes of bounded p have bounded q.
[3] We say that a class has *effectively bounded* $p^\downarrow$ if there is a constant c and a polynomial-time algorithm that, given a graph G from the class, outputs a contraction sequence of G such that for every red graph H of the sequence, $p(H) \leqslant c$ holds.

In this paper, we introduce the *component max-leaf* parameter, denoted by cml, the least integer k such that every spanning tree of every connected component has at most k leaves, and its reduced counterpart $\mathrm{cml}^{\downarrow}$. Connected graphs have bounded component max-leaf if and only if they are subdivisions of graphs of bounded size (see for instance [37, Theorem 4.7]). We have bandwidth $\sqsubset$ cml $\sqsubset \star$, which sets $\mathrm{cml}^{\downarrow}$ in the "right" regime.

Since graphs of twin-width at most 2 have bounded $\mathrm{cml}^{\downarrow}$, unit interval graphs – which have twin-width at most 2 [9] and unbounded clique-width – testify that $\mathrm{cml}^{\downarrow}$ is *strictly* more general than clique-width. We actually prove that unit interval graphs have unbounded stretch-width, which is consistent with the conjecture that classes of bounded stretch-width have at most logarithmic clique-width [7].

We show in Theorem 11 that every graph G with $\mathrm{cml}^{\downarrow}(G) = k$ admits a balanced separator dominated by at most $k(k+2)$ vertices. This implies that every graph G with maximum degree Δ and $\mathrm{cml}^{\downarrow}(G) = k$ has treewidth $O(\Delta k^2)$. Since planar graphs of arbitrarily large treewidth admit arbitrarily large subdivided walls or their line graphs as induced subgraphs [1, Theorem 1.1], this establishes the following fact.

▶ **Theorem 1.** *A class $\mathcal{C}$ of planar graphs has bounded treewidth iff it has bounded* $\mathrm{cml}^{\downarrow}$.

We exhibit weakly sparse (i.e., without $K_{t,t}$ subgraph for some fixed integer t) classes of bounded $\mathrm{cml}^{\downarrow}$ and polynomially large treewidth. More precisely, for any $g \in \mathbb{N}$, there is an infinite family of n-vertex graphs G of girth at least g and treewidth $\Theta_g(\sqrt{n})$ such that $\mathrm{cml}^{\downarrow}(G) \leqslant 3$. On the other hand, we show that weakly sparse classes of bounded $\mathrm{cml}^{\downarrow}$ (even of bounded reduced cutwidth) have polynomial expansion. By a separator theorem of Plotkin, Rao, and Smith [31], graph classes of polynomial expansion have strongly sublinear separators. Combining this with the linear bound between separation number and treewidth due to Dvořák and Norin [19], every n-vertex subgraph of a graph from such a class has treewidth $O(n^{\beta})$ for some constant $\beta < 1$. This answers a question asked in [12].

▶ **Theorem 2.** *Weakly sparse classes of bounded reduced* $\mathrm{cml}^{\downarrow}$ *(even of bounded reduced cutwidth) have polynomial expansion.*

Exponential subdivisions of grids have unbounded $\mathrm{cml}^{\downarrow}$ by Theorem 1, but bounded stretch-width [7]. This gives the other direction for the incomparability of $\mathrm{cml}^{\downarrow}$ and stretch-width. In the full version, we show that these parameters are also incomparable to *mim-width* [36], another (moderate) generalization of clique-width lending itself to algorithmic meta-theorems [13, 5, 4].

We then design polynomial-time algorithms on graphs of effectively bounded $\mathrm{cml}^{\downarrow}$ for a family of problems containing Maximum Independent Set. Fix some finite set σ of non-negative integers. Let σ-Neighborhood be the problem, given a graph G, of finding a maximum subset $S \subseteq V(G)$ such that every $v \in S$ has a number of neighbors in S that is in σ. Setting $\sigma = \{0\}$ defines Maximum Independent Set, while $\sigma = \{1\}$ defines Maximum Induced Matching. This framework was first proposed by Telle and Proskurowski [35].

▶ **Theorem 3.** *There is an algorithm that, given an n-vertex graph G and a contraction sequence witnessing $\mathrm{cml}^{\downarrow}(G) \leqslant t$, solves σ-Neighborhood in time $n^{O(t(\alpha+1)^2)}$ with $\alpha = \max(\sigma)$.*

Bonnet et al. [9] proved that Maximum Independent Set can be solved in polynomial time on graphs of twin-width at most 2, provided that a contraction sequence of width 2 is given as part of the input. Since graphs of twin-width at most 2 have reduced component max-leaf at most 2, Theorem 3 generalizes this result. Bui-Xuan, Telle, and Vatshelle [13] showed that the σ-Neighborhood problem is polynomial time solvable on classes of effectively bounded mim-width, for any finite or cofinite set σ.

We also obtain a polynomial-time algorithm for the Induced Disjoint Paths problem on classes of effectively bounded $\mathrm{cml}^{\downarrow}$. Using this algorithm together with a simple reduction (see [26]), one can solve in polynomial time H-Induced Subdivision Containment, that is, the problem of finding an induced subdivision of a fixed graph H.

▶ **Theorem 4.** *There is an algorithm that, given an n-vertex graph G and a contraction sequence witnessing $\mathrm{cml}^{\downarrow}(G) \leqslant t$, solves Induced Disjoint Paths in time $n^{O(t)}$.*

A by-product of reduced parameters is that they provide a new tool to show that a class $\mathcal{C}$ does not first-order transduce another class $\mathcal{D}$. This is identified as a current gap [22]: "*the biggest bottleneck to obtaining a fine understanding of the first-order transduction quasi-order is the lack of a robust toolbox for proving non-transducibility results that extends beyond commonly studied transduction invariant properties.*" We show that for most reduced parameters $p^{\downarrow}$, the family of classes of bounded $p^{\downarrow}$ is closed under first-order transductions.

▶ **Theorem 5.** *For every monotone parameter p stable under powers and blowups, the family of classes of bounded $\mathrm{p}^{\downarrow}$ is closed under first-order transductions.*

One may check that the precondition of Theorem 5 is satisfied for each of the following parameters, giving a generic proof their stability under transduction (among them, only the case of twin-width [11] was previously known).

▶ **Corollary 6.** *The families of classes of bounded reduced bandwidth, reduced cutwidth, reduced pathwidth $+\,\Delta$, reduced treewidth $+\,\Delta$, and twin-width respectively, are each closed under applying first-order transductions.*

For any such parameter $p^{\downarrow}$, if $\mathcal{C}$ has bounded $p^{\downarrow}$ but $\mathcal{D}$ has unbounded $p^{\downarrow}$, this indeed proves that $\mathcal{C}$ does not transduce $\mathcal{D}$. For example, we reprove the result of [22, 25] with $p =$ bandwidth, $\mathcal{C} = \{\text{planar graphs}\}$ (or $\mathcal{C} = \{\text{graphs of genus } \leqslant g\}$), and $\mathcal{D} = \{\text{3-dimensional grids}\}$ since planar graphs (graphs of bounded genus) have bounded reduced bandwidth [12], but we show that 3-dimensional grids have unbounded reduced bandwidth (Theorem 15).

Figure 1 summarizes the context and content of the paper.

Admittedly, the only reduced parameter that does not fulfill the preconditions of Theorem 5 (nor its conclusion) is the one we introduce in the current paper: reduced component max-leaf. However this is also precisely what makes it suited to unify the polynomial-time algorithms for Maximum Independent Set and Maximum Induced Matching in classes of bounded clique-width and unit interval graphs. Indeed, unit interval graphs first-order transduce the class of grid graphs (induced subgraphs of grids), on which Maximum Induced Matching is NP-hard [16].

Bridging efficient first-order and monadic second-order model checking is an active line of work. We explore here the class-first side (similarly to [4, 7]); for the logic-first side, we refer the reader to [30, 34, 20, 6] and the references therein.

Organization. The proofs of the statements marked by ($*$) are deferred to the full version. In Section 3.1, we show Theorem 11, which in particular proves Theorem 1. In Section 3.2, we show that 3-dimensional grids have unbounded reduced bandwidth, which together with Theorem 5, proven in Section 5, reproves the result of [22]. In Section 4, we show Theorem 2. In Section 6, we establish Theorem 3. The proof of Theorem 4 is deferred to the long version.

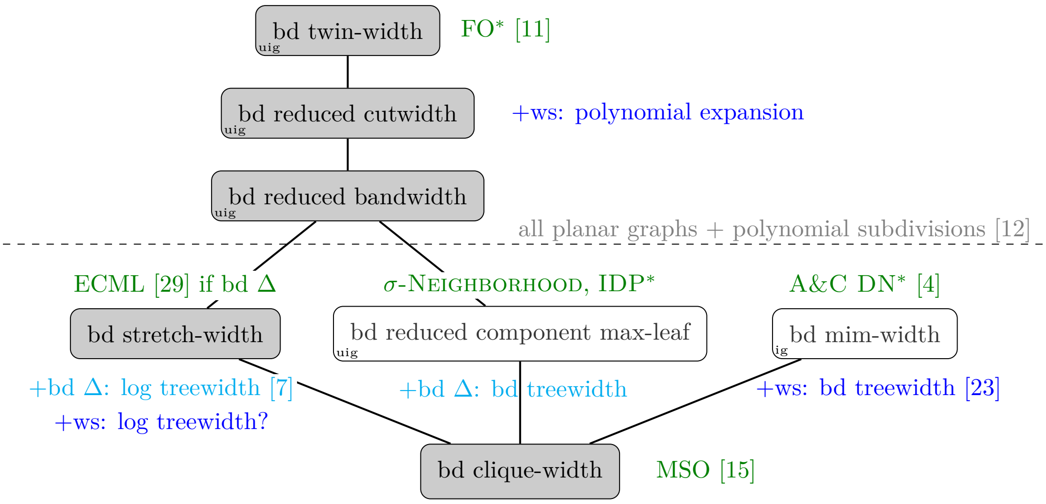

Figure 1 Hasse diagram of the handled families of classes. The shaded families are those closed under first-order transductions. We use the following abbreviations: "bd" → bounded, "ws" → weakly sparse, "log treewidth" → n-vertex graphs have treewidth $O(\log n)$, "uig" → contains the class of unit interval graphs, "ig" → contains the class of interval graphs (at the bottom-left corner of the boxes). We indicate in green the problems solvable in polynomial time, and refer the reader to the corresponding papers for the definition of the logics. The asterisks mean that a witness for low parameter is assumed by the algorithm.

2 Preliminaries

A *trigraph* is a triple $G = (V, E, R)$ where V is a finite set, and E and R are disjoint subsets of $\binom{V}{2}$. Elements of V are *vertices*, and elements of $E \cup R$ are *edges*. Edges in E are called *black*, and edges in R are called *red*. The *red graph* $\mathcal{R}(G)$ of G is (V, R). For a vertex v of G, let $B_G(v)$ denote the set of vertices that are adjacent to v by black edges, and let $R_G(v)$ denote the set of vertices that are adjacent to v by red edges. We also write $B_G[v] := B_G(v) \cup \{v\}$ and $R_G[v] := R_G(v) \cup \{v\}$. For $S \subseteq V(G)$, let $B_G(S) = \left(\bigcup_{v\in S} B_G(v)\right) \setminus S$ and $R_G(S) = \left(\bigcup_{v\in S} R_G(v)\right) \setminus S$. A subtrigraph H of a trigraph G is called *red-induced* if H is an induced subtrigraph of G and all edges of H are contained in $\mathcal{R}(G)$.

For a graph G and a vertex partition $\mathcal{P}$ of G, let $G/\mathcal{P}$ be the trigraph on the vertex set $\mathcal{P}$ such that for two parts $X, Y \in \mathcal{P}$,

- X and Y are adjacent by a black edge if X is complete to Y in G,
- X and Y are adjacent by a red edge if X and Y are not homogeneous, and
- X and Y are not adjacent if X is anti-complete to Y in G.

For a set of parts $\mathcal{Q} \subseteq \mathcal{P}$, we write $\bigcup \mathcal{Q} := \bigcup_{Z\in\mathcal{Q}} Z$.

For a graph G, a sequence $\mathcal{S} = \mathcal{P}_n, \dots, \mathcal{P}_1$ of partitions of the vertex set of G is a *contraction sequence* of G if $\mathcal{P}_n = \{\{v\} : v \in V(G)\}$, $\mathcal{P}_1 = \{V(G)\}$, and each $\mathcal{P}_i$ is obtained from $\mathcal{P}_{i+1}$ by merging two parts of $\mathcal{P}_{i+1}$.

For any graph parameter f, let *reduced-f* be the graph parameter $f^{\downarrow}$, where for every graph G, $f^{\downarrow}(G)$ is the minimum $k \in \mathbb{N}$ such that there is a contraction sequence $\mathcal{P}_n, \dots, \mathcal{P}_1$ of G where $f(\mathcal{R}(G/\mathcal{P}_i)) \leqslant k$ for each $i \in \{1, \dots, n\}$.

The *max-leaf* of a connected graph G, denoted by $\mathrm{ml}(G)$, is the minimum k such that every spanning tree of G has at most k leaves. The *component max-leaf* of G, denoted by $\mathrm{cml}(G)$, is the minimum k such that every connected component of G has max-leaf at most k. An edgeless graph G has $\mathrm{cml}(G) = 1$.

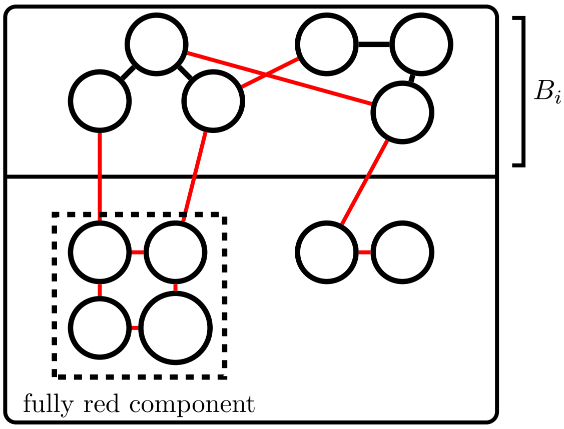

Figure 2 A fully red component in Lemma 9.

Our algorithms in Section 6 use dynamic programming over the collection of all *connected* induced subgraphs of a graph with small component max-leaf. The next two properties are crucial to establish the runtime of these algorithms.

▶ **Lemma 7.** *Let G be a graph with component max-leaf t, and let H be a connected subgraph of G. Then $|N_G(V(H))| \leqslant t$.*

Proof. For the sake of contradiction, assume that $|N_G(V(H))| \geqslant t+1$. Let T be a spanning tree of H. Let T' be a subgraph obtained from T by adding, for each vertex $v \in N_G(V(H))$, an edge between v and its neighbor in T. Since each vertex in $N_G(V(H))$ is attached to T by a single edge, T' is still a tree. Moreover, each vertex in $N_G(V(H))$ is a leaf in T'. Hence, T' has at least $t+1$ leaves. Finally T' can be extended to a spanning tree of the connected component of G containing H, which must still have at least $t+1$ leaves, a contradiction. ◀

▶ **Lemma 8** ($*$)**.** *Let t be a positive integer, and let G be a graph with component max-leaf t. The number of non-empty connected induced subgraphs of G is at most $|V(G)|^t$.*

3 Reduced parameters on subdivided walls and 3-dimensional grids

In Section 3.1, we prove that planar graphs have unbounded reduced component max-leaf. To prove it, we show that every weighted graph (G, w) with $\mathrm{cml}^{\downarrow}(G) \leqslant k$ admits a balanced separator dominated by at most $k(k+2)$ vertices of G. This implies that large subdivided walls or their line graphs have unbounded $\mathrm{cml}^{\downarrow}$.

In Section 3.2, we prove that 3-dimensional grids have unbounded reduced bandwidth.

3.1 Balanced separators and reduced component max-leaf

In a weighted graph (G, w), for $0 < a < 1$, say that a subset $S \subseteq V(G)$ is an a-balanced separator if each connected component C in $G - S$ has weight $w(C) \leqslant a \cdot w(G)$. Thus a $\frac{2}{3}$-balanced separator is the usual notion of balanced separator.

▶ **Lemma 9.** *For any weighted graph (G, w) with $\mathrm{cml}^{\downarrow}(G) \leqslant k$, there is a $(1 - \frac{1}{2k+3})$-balanced separator S of (G, w) contained in the open neighborhood of at most k vertices of G.*

Proof. Fix a contraction sequence $\mathcal{P}_n, \dots, \mathcal{P}_1$ witnessing that $\mathrm{cml}^{\downarrow}(G) \leqslant k$. For each $i \in [n]$, let $B_i \subseteq \mathcal{P}_i$ be the set of parts that have a black neighbor in $G/\mathcal{P}_i$, and $R_i = \mathcal{P}_i \setminus B_i$ the parts that only have red neighbors. The point is that any part $P \in B_i$ is contained in $N(v)$ for some vertex $v \in V(G)$, namely take v to be any vertex in the black neighbor of P. We call *fully red component* a connected component of $G/\mathcal{P}_i - B_i$; see Figure 2.

Let $\lambda = \frac{1}{2k+3}$, and consider the first partition $\mathcal{P}_i$ (i.e., the maximum i) such that there is

1. either a part $P \in B_i$ with weight $w(P) \geqslant \lambda w(G)$,
2. or a fully red component C in $G/\mathcal{P}_i - B_i$ with $w(\bigcup V(C)) \geqslant \lambda w(G)$.

In Case 1, P itself trivially is a $(1-\lambda)$-balanced separator since $w(V(G) \setminus P) \leqslant (1-\lambda)w(G)$, and P is in the neighborhood of a single vertex since it is in B_i. We can thus focus on Case 2.

We first bound the weight of the fully red component C.

▷ Claim 10. We have $\lambda w(G) \leqslant w(\bigcup V(C)) \leqslant (1-\lambda)w(G)$.

Proof. The first inequality is just the assumption on C. To prove the second one, we need to also consider the previous step $\mathcal{P}_{i+1}$. Say that $\mathcal{P}_i$ was obtained by merging $P_1, P_2 \in \mathcal{P}_{i+1}$ into $P := P_1 \cup P_2 \in \mathcal{P}_i$. Any black edge in $G/\mathcal{P}_{i+1}$ that was not incident to P_1 or P_2 remains in $G/\mathcal{P}_i$. It follows that any part in $R_i \setminus R_{i+1}$ is either P itself, or adjacent (by a red edge) to P in $G/\mathcal{P}_i$. In particular, $|R_i \setminus R_{i+1}| \leqslant k+1$.

Now define $X = V(C) \setminus R_{i+1} \subseteq R_i \setminus R_{i+1}$ the set of new red parts used in C. While P is not necessarily in X, the previous argument shows that $X \cup \{P\}$ is connected in $\mathcal{R}(G/\mathcal{P}_i)$. Thus by Lemma 7, $X \cup \{P\}$ has at most k neighbors in $\mathcal{R}(G/\mathcal{P}_i)$. Let $C_1, \dots, C_\ell$ be the connected components of $C - X$. Since C is connected in $\mathcal{R}(G/\mathcal{P}_i)$, X has a neighbor in each of $C_1, \dots, C_\ell$, and thus $\ell \leqslant k$. Note that each C_j was already a connected subset of R_{i+1}, hence is contained in a fully red component of $G/\mathcal{P}_{i+1}$.

To summarize, C consists of (1) at most $k+1$ parts of X, and (2) at most k components $C_1, \dots, C_\ell$, each of which is contained in a fully red component of $G/\mathcal{P}_{i+1}$. Each part of $\mathcal{P}_{i+1}$, and each fully red component of $G/\mathcal{P}_{i+1}$ has weight at most $\lambda w(G)$. Thus each of $C_1, \dots, C_\ell$ has weight at most $\lambda w(G)$, and so does each part of X, except possibly P (the only part in $\mathcal{P}_i \setminus \mathcal{P}_{i+1}$). The latter is the union of two parts of $\mathcal{P}_{i+1}$, hence $w(P) \leqslant 2\lambda w(G)$. Summing up, we obtain $w(\bigcup V(C)) \leqslant (2k+2)\lambda w(G) = (1-\lambda)w(G)$. ◁

This claim directly implies that $N_G(\bigcup V(C))$ is a $(1-\lambda)$-balanced separator in (G, w). Since C is a fully red component, the only neighbors of C are red neighbors. Since C is connected, Lemma 7 gives that $N_{G/\mathcal{P}_i}(V(C))$ consists of at most k parts of $\mathcal{P}_i$. Each of these k parts is in B_i, hence is dominated by a single vertex. Thus the neighborhood of $\bigcup V(C)$ is dominated by at most k vertices. Thus $N_G(\bigcup V(C))$ is a $(1-\lambda)$-balanced separator dominated by k vertices, as desired. ◀

While Lemma 9 only produces a 'slightly balanced' separator, we can apply this statement multiple times to obtain a $\frac{2}{3}$-balanced separator:

▶ **Theorem 11.** *For any weighted graph (G, w) with $\mathrm{cml}^{\downarrow}(G) \leqslant k$, there is a $\frac{2}{3}$-balanced separator S of (G, w) contained in the open neighborhood of at most $k(k+2)$ vertices of G.*

Proof. Define $a := 1 - \frac{1}{2k+3}$. Let W be the sum of the weights of all vertices in G. We construct a sequence of balanced separators $S_1, \dots, S_\ell$. Suppose they have already been constructed up to S_i (possibly $i = 0$). Let C_i be the connected component of $G - \bigcup_{j \leqslant i} S_j$ of maximum weight. Clearly any connected component but C_i has weight at most $\frac{1}{2}W$. If $w(C_i) \leqslant \frac{2}{3}W$, then $\bigcup_{j \leqslant i} S_j$ is a $\frac{2}{3}$-balanced separator, and we stop the process here. Otherwise, let S_{i+1} be an a-balanced separator in C_i with the restriction of w as weight, and continue the process.

For $i < \ell$, since S_{i+1} is a-balanced in C_i, we have $w(C_{i+1}) \leqslant a \cdot w(C_i)$. On the other hand, we have $w(C_i) > \frac{2}{3}W$ (otherwise we would stop with $\ell = i$), hence

$$w(C_{i+1}) \leqslant w(C_i) - \frac{2}{3}(1-a)W = w(C_i) - \frac{2}{3(2k+3)}W.$$

So, for any $i \leqslant \ell$, we have $w(C_i) \leqslant W\left(1 - \frac{2i}{3(2k+3)}\right)$. For $i = k+2$, the right hand side is less than $\frac{2}{3}W$, hence the process must stop with $\ell \leqslant k+2$. Thus $\bigcup_{i \leqslant \ell} S_i$ is a $\frac{2}{3}$-balanced separator, contained in the open neighborhood of at most $k\ell \leqslant k(k+2)$ vertices. ◀

If the maximum degree is bounded, then Theorem 11 actually provides balanced separators of bounded size, proving that treewidth and reduced component max-leaf are functionally equivalent on bounded degree graphs:

▶ **Corollary 12.** *Any graph G with maximum degree Δ and* $\mathrm{cml}^{\downarrow} = k$ *has treewidth* $\leqslant 4\Delta k(k+2)$.

Proof. By Theorem 11, for any weight function, G has a balanced separator S in the open neighborhood of at most $k(k+2)$ vertices. The degree bound then gives $|S| \leqslant \Delta k(k+2)$. Thus G has $\frac{2}{3}$-balanced separators of size $\Delta k(k+2)$ for any weight function. By a classical result of Robertson and Seymour [33], this implies that G has treewidth at most $4\Delta k(k+2)$. ◀

It is known that planar graphs of arbitrarily large treewidth admit arbitrarily large subdivided walls or their line graphs as induced subgraphs [1, Theorem 1.1] (where wall subdivisions are simply called walls). By Corollary 12, large subdivided walls or their line graphs have unbounded $\mathrm{cml}^{\downarrow}$. Since every graph class of bounded treewidth has bounded $\mathrm{cml}^{\downarrow}$, Theorem 1 follows.

3.2 Reduced bandwidth of 3-dimensional grids

In this section, we relate the expansion of graphs to the structure of the red components in their contraction sequences. For any three functions $\tau_{\min}, \tau_{\max}, f$, we say that an n-vertex graph G is a $(\tau_{\min}, \tau_{\max}, f)$*-expander* if for every vertex subset $U \subseteq V(G)$ with $\tau_{\min}(n) \leqslant |U| \leqslant \tau_{\max}(n)$, we have $|N_G(U)| \geqslant f(|U|)$.

The next lemma has a rather technical statement, mainly because we want to use it in various settings. However, it can be simply summarized as follows: if for any $U \subseteq V(G)$ we have $|N_G(U)| \geqslant |U|^a$ for some $0 < a < 1$, then one can find, in any contraction sequence of bounded red degree, an arbitrary large red subgraph H such that $|V(H)| \geqslant \mathrm{diam}(H)^{\frac{a}{1-a}}$.

▶ **Lemma 13** ($*$). *Let f be such that $f(n) = \Omega(n^a)$ for some $0 < a < 1$, and $\tau_{\min}$ and $\tau_{\max}$ satisfy $\tau_{\max}(n)^{1-a} \geqslant (\log n + \tau_{\min}(n)) \log n$. Let r and d be integers, and let G be a $(\tau_{\min}, \tau_{\max}, f)$-expander without $K_{r,r}$ subgraph. If $\mathcal{P}_n, \ldots, \mathcal{P}_1$ is a contraction sequence of G of twin-width at most d, then at least one red graph $\mathcal{R}(G/\mathcal{P}_i)$ admits a connected subgraph H with $|V(H)| = \Omega(\mathrm{diam}(H)^{\frac{a}{1-a}})$ and $|V(H)| = \Omega((\log n)^a)$.*

Outline of the proof. We set $t := \lceil \max(\log n, \tau_{\min}(n)) \rceil$ with $|V(G)| = n$ which we assume large enough, and let $\mathcal{P}_s$ be the first partition with a part P of size at least t. We show that the red component of $G/\mathcal{P}_s$ containing P has large growth around P, that is, the red neighborhoods at a small distance from P are large. This fact allows us to exhibit the subgraph H from $G/\mathcal{P}_s$. ◀

The following is a corollary of the isoperimetric inequality of Loomis and Whitney [27].

▶ **Lemma 14** (∗). *The 3-dimensional grid is a $(1, x^{1/3} - 1, x^{2/3})$-expander. That is, in the $n \times n \times n$-grid, any set of vertices P of size less than n satisfies $|N(P)| \geqslant |P|^{2/3}$.*

▶ **Theorem 15.** *The reduced bandwidth of 3-dimensional grids is unbounded.*

Proof. By Lemma 14, the 3-dimensional grid is a $(1, \tau_{\max}, f)$-expander of bounded maximum degree for $f(x) = x^{2/3}$ and $\tau_{\max}(x) = x^{1/3} - 1$. Thus Lemma 13 yields connected subgraphs H in at least one red graph within any contraction sequence of the 3-dimensional grids (of increasing sizes) such that $|V(H)| = \Omega\left(\text{diam}(H)^{(2/3)/(1-2/3)}\right) = \Omega\left(\text{diam}(H)^2\right)$. However, connected graphs of bounded bandwidth have linear diameter in their number of vertices. ◀

4 Weakly sparse collapse of bounded reduced parameters

Within classes of bounded twin-width, it was shown that weak sparsity and bounded degeneracy are equivalent [8]. Furthermore, the treewidth of bounded-degree n-vertex graphs of bounded twin-width can be as large as $\Theta(n)$, as there are classes of cubic expanders with bounded twin-width [8]. In contrast, it was proven in [12] that no weakly sparse class of bounded reduced bandwidth can be an expander class.

In the full version, we lift this result to reduced cutwidth by showing that every weakly sparse class of bounded reduced cutwidth has polynomial expansion [28]. It was shown [18] that a subgraph-closed class has polynomial expansion if and only if it has *strongly sublinear separators*, i.e., all its n-vertex subgraphs have treewidth $O(n^{1-\varepsilon})$ for some $\varepsilon > 0$ depending on the class only. We prove here a more quantitative analogous statement for bounded reduced component max-leaf.

We first show a general statement where the $\text{cml}^{\downarrow}$ upper bound and the size of the excluded biclique subgraph may be superconstant.

▶ **Lemma 16** (∗). *Let $0 < \varepsilon, \beta \leqslant \delta < \gamma < 1$ be reals, and n, k, t be integers such that $k, t \leqslant n^{\beta}$, and n is sufficiently large. Let G be an n-vertex graph with $\text{cml}^{\downarrow}(G) \leqslant k$ and no $K_{t,t}$ subgraph. Then G has a balanced separator of size at most n^z with $z := \max(1 + \beta + \delta - \gamma + \varepsilon,\ 1 - \delta + \beta + \varepsilon,\ \gamma + \beta + \varepsilon)$.*

From Lemma 16 we get that weakly sparse classes of $\text{cml}^{\downarrow}$ at most n^{β} with $\beta < 1/3$ admit strongly sublinear separator. More precisely:

▶ **Corollary 17.** *For every sufficiently small $\varepsilon > 0$ and sufficiently large n, every n-vertex graph G with $\text{cml}^{\downarrow}(G) \leqslant n^{\frac{1}{3}-2\varepsilon}$ and no $K_{n^{\frac{1}{3}-2\varepsilon},n^{\frac{1}{3}-2\varepsilon}}$ subgraph admits a balanced separator of size $n^{1-\varepsilon}$.*

Proof. We apply Lemma 16 with $\beta = \frac{1}{3} - 2\varepsilon$, $\delta = \frac{1}{3}$, $\gamma = \frac{2}{3}$. This gives a balanced separator of size n^z with $z = \max(1 + \frac{1}{3} - 2\varepsilon + \frac{1}{3} - \frac{2}{3} + \varepsilon,\ 1 - \frac{1}{3} + \frac{1}{3} - 2\varepsilon + \varepsilon,\ \frac{2}{3} + \frac{1}{3} - 2\varepsilon + \varepsilon) = 1 - \varepsilon$. ◀

Another consequence of Lemma 16 is that weakly sparse classes of bounded $\text{cml}^{\downarrow}$ have treewidth $O(n^{\frac{2}{3}+\varepsilon})$ for any $\varepsilon > 0$.

▶ **Corollary 18.** *Let $\mathcal{C}$ be a weakly sparse class of bounded $\text{cml}^{\downarrow}$. Then for every $\varepsilon > 0$, there is a constant $c := c_{\mathcal{C},\varepsilon}$ such that every n-vertex graph of $\mathcal{C}$ has treewidth at most $cn^{\frac{2}{3}+\varepsilon}$.*

Proof. Let k, t be integers such that $\mathcal{C}$ has no $K_{t,t}$ subgraph nor graphs of $\text{cml}^{\downarrow}$ more than k. In Lemma 16, one can then make $\beta + \varepsilon$ arbitrarily close to 0. Thus, for every sufficiently small $\varepsilon > 0$, every n-vertex induced subgraph of a graph of $\mathcal{C}$ admits a balanced separator

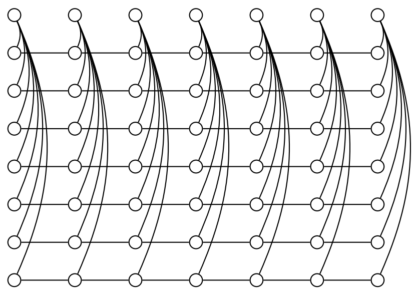

Figure 3 The 7×7 Pohoata–Davies grid.

of size $O(n^z)$ with $z := \max(1 + \delta - \gamma + \varepsilon,\ 1 - \delta + \varepsilon,\ \gamma + \varepsilon)$. We then choose $\delta = \frac{1}{3}$ and $\gamma = \frac{2}{3}$, so that $z = \frac{2}{3} + \varepsilon$. We conclude since separation number and treewidth are linearly tied [19]. ◀

The $n \times n$ Pohoata–Davies grid [32, 14] is illustrated in Figure 3. For every integer $g \geqslant 0$, let $G_{k,g}$ be the graph obtained from the $k \times k$ Pohoata–Davies by subdividing each edge of the k horizontal paths g times. Note that $G_{k,g}$ has girth $2(g+1)+4$ and treewidth k. Setting $n := |V(G_{k,g})|$, we have $n < k^2(1+g)$, so the treewidth of $G_{k,g}$ is larger than $\frac{1}{(1+g)^{1/2}}\sqrt{n}$. The reduced component max-leaf of $G_{k,g}$ is at most 3 due to the following contraction sequence, in which the only non-singleton connected component in each red graph is a path, plus optionally a pendant vertex attached to the penultimate vertex of the path. For each j going from 1 to k, for each i going from 1 to $k + g(k-1)$, contract the ith vertex of the first path with the i-th vertex of the j-th path. Then contract (in any order) vertices outside the red path with their black neighbor on the red path. Finally contract adjacent pairs of vertices along the red path, until it is reduced to a single vertex. This proves the following.

▶ **Proposition 19.** *For every positive integer g, there is an infinite family of graphs G of girth at least g and treewidth $\Omega_g(|V(G)|^{1/2})$ with* $\mathrm{cml}^{\downarrow}(G) \leqslant 3$.

5 Stability under transductions

For a graph $G = (V, E)$ and a parameter $k \in \mathbb{N}$, we define the following two operations:

power The *kth power* of G, denoted $\mathsf{pow}^k(G)$ is the graph with vertices V, and an edge xy for all pairs of vertices at distance at most k in G.

blowup The *kth blowup* of G, denoted $\mathsf{blow}^k(G)$, is obtained by duplicating each $x \in V$ into a clique $\{x_1, \dots, x_k\}$, and connecting every x_i and y_j whenever xy is an edge of G.

A graph parameter p is said to be stable under powers if there is some function f such that $p(\mathsf{pow}^k(G)) \leqslant f(p(G), k)$. Being stable under blowups is defined similarly.

Theorem 5 states that if p is a monotone parameter stable under powers and blowups, then reduced-p is stable under *first-order transductions*. In this section, we will prove this result only for *first-order interpretation*, which are the main operation in a transduction. See the full version of this paper for the complete proof of Theorem 5.

If $\phi(x, y)$ is a first-order formula with two free variables, then $\phi(G)$ is defined as the graph with the same vertex set $V(G)$, and where uv is an edge if and only if $G \models \phi(u, v)$ (i.e. G satisfies ϕ with x, y interpreted as u, v). The map $G \mapsto \phi(G)$ is called a first-order interpretation. In general, $\phi(G)$ is a directed graph; we may require $\phi(x, y)$ to be equivalent to $\phi(y, x)$, or replace it by $\phi(x, y) \vee \phi(y, x)$ to ensure $\phi(G)$ is in fact a graph.

▶ **Theorem 20.** *Let p be a monotone parameter stable under powers and blowups. Then reduced-p is stable under first-order interpretation, that is, for any first-order formula $\phi(x, y)$, there is a function f_ϕ such that for any graph G, $p^\downarrow(\phi(G)) \leqslant f_\phi(p^\downarrow(G))$.*

To prove Theorem 20, we will use a variant of contraction sequences defined using rank. In a graph G with adjacency matrix A_G, for two disjoint subsets $X, Y \subset V(G)$, the *rank* between X and Y in G, denoted by $\mathrm{rk}_G(X, Y)$, is the binary rank of $A_G[X, Y]$.

Consider a graph G, a partition $\mathcal{P}$ of $V(G)$, and a second graph $G_\mathcal{P}$ with vertex set $\mathcal{P}$. We say that $G_\mathcal{P}$ is a *rank-k error graph* for $(G, \mathcal{P})$ if for any part $P \in \mathcal{P}$, we have

$$\mathrm{rk}_G\left(P, \bigcup_{Q \notin N_{G_\mathcal{P}}[P]} Q\right) \leqslant k.$$

That is, the graph $G_\mathcal{P}$ specifies a number of "bad parts" which are neighbors of the part P, and after removing these bad parts, the rank between P and the remainder of G is at most k.

The *reduced-rank-p* of G is the smallest k such that the following holds: There is a contraction sequence $\mathcal{P}_n, \ldots, \mathcal{P}_1$ and a sequence of graphs $G_n, \ldots, G_1$, such that

1. G_i is a rank-k error graph for $(G, \mathcal{P}_i)$,
2. $p(G_i) \leqslant k$, and
3. $G_n, \ldots, G_1$ is monotone, i.e., for $i > j$, if PQ is an edge in G_i and P', Q' are the parts of G_j containing P and Q respectively, then $P'Q'$ is also an edge in G_j.

▶ **Lemma 21** ($*$)**.** *If p is a monotone parameter stable under blowups, then bounded reduced-rank-p is equivalent to bounded reduced-p.*

For a graph G and two subsets A, B of vertices, the *flip* $G \oplus (A, B)$ is the graph obtained by complementing the adjacency between A and B, i.e.

$$E(G \oplus (A, B)) = E(G) \triangle \{ab : a \in A, b \in B\}.$$

We call H a *t-flip* of G if there are $(A_i, B_i)_{i \in [t]}$ such that $H = \big(G \oplus (A_1, B_1)\big) \cdots \oplus (A_t, B_t)$.

The second tool we use is the following variant of Gaifman's Locality Theorem [21], which is implicitly in e.g. the proof of [17, Lemma 5.3].

▶ **Lemma 22** ($*$)**.** *For any first-order formula $\phi(x, y)$ and $t \in \mathbb{N}$, there are parameters $r, \ell \in \mathbb{N}$, where r only depends on ϕ, satisfying the following. Given G a graph and H a t-flip of G, there is a map $\lambda : V(G) \to [\ell]$ such that for any vertices x, y at distance more than r in H, whether or not $G \models \phi(x, y)$ depends only on $\lambda(x), \lambda(y)$.*

The core of the proof of Theorem 20 is the following.

▶ **Lemma 23.** *For any first-order formula $\phi(x, y)$ and $\Delta \in \mathbb{N}$, there are $r, \ell \in \mathbb{N}$ such that for any graph G and partition $\mathcal{P}$ of $V(G)$ with red graph $G_\mathcal{P} := \mathcal{R}(G/\mathcal{P})$, if $G_\mathcal{P}$ has maximum degree at most Δ, then $\mathsf{pow}^r(G)$ is a rank-ℓ error graph for $(\phi(G), \mathcal{P})$.*

Proof. Depending only on ϕ, Lemma 22 gives some $r \in \mathbb{N}$. Choose the number of flips $t = \Delta^r + 1$, which bounds the number of vertices in a ball of radius r in a graph with maximum degree Δ. Now, depending on ϕ and t, Lemma 22 gives a second parameter ℓ.

Fix now $G, \mathcal{P}$ such that the red graph $G_\mathcal{P}$ has maximum degree at most Δ. Consider any part $P \in \mathcal{P}$. Let $\mathcal{B} := \mathsf{Ball}^r_{G_\mathcal{P}}(P)$ be the parts of $\mathcal{P}$ within distance r of P in $G_\mathcal{P}$, and note that $|\mathcal{B}| \leqslant t$. Consider the graph G' obtained from G as follows: for each pair of parts $X, Y \in \mathcal{P}$, if X, Y are fully adjacent and either X or Y is in $\mathcal{B}$, then remove all edges between X and Y. For any fixed X, one can with a 1-flip remove all edges between X and all the parts Y that are fully adjacent to Y. The graph G' is obtained by doing such a 1-flip for each $X \in \mathcal{B}$, hence it is a t-flip of G.

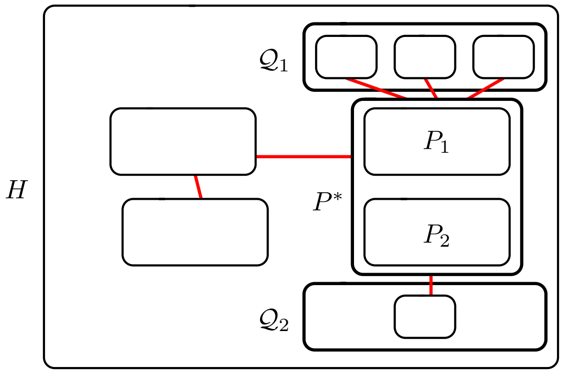

Figure 4 The partition $\mathcal{P}'$ is obtained from a partition $\mathcal{P}$ by merging P_1 and P_2 into P^*, and H is a connected red-induced subtrigraph of $G/\mathcal{P}'$ containing P^*. Each part in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ is adjacent to one of P_1 and P_2 by a black edge.

Let $R_P = V(G) \setminus \bigcup_{Q \in \mathcal{B}} Q$ denote the vertices outside $\mathcal{B}$. The graph G' is chosen so that $\mathsf{dist}_{G'}(P, R_P) > r$. Applying Lemma 22 to G and its flip G' yields a map $\lambda : V(G) \to [\ell]$ such that if $\mathsf{dist}_{G'}(x, y) > r$, then whether or not $G \models \phi(x, y)$ (or equivalently $xy \in E(\phi(G))$) only depends on $\lambda(x), \lambda(y)$. This is in particular the case whenever $x \in P$ and $y \in R_P$, which implies $\mathrm{rk}_{\phi(G)}(P, R_P) \leqslant \ell$.

Thus, for any $P \in \mathcal{P}$, if R_P is the set of vertices in parts at distance more than r from P in $G_{\mathcal{P}}$, then $\mathrm{rk}_{\phi(G)}(P, R_P) \leqslant \ell$. This precisely shows that the rth power of $G_{\mathcal{P}}$ is a rank-ℓ error graph for $\phi(G)$, as desired. ◀

Now, if p is a monotone graph parameter, stable under powers and blowups, and such that bounded p implies bounded degree, then Lemmas 21 and 23 together show that reduced-p is stable under first-order interpretation, proving Theorem 20 for p.

When bounded p does *not* imply bounded degree, Lemma 23 cannot be used as the degree assumption is not satisfied. In this case however (assuming again that p is monotone), the class of all stars has bounded p. Any graph has a contraction sequence in which every red graph is a star; this can be obtained by repeatedly merging the center of the star with any other part. Thus *all* graphs have bounded reduced-p, making Theorem 20 trivial for such p.

Finally, while Theorem 5 does not apply to stretch-width, the same techniques show:

▶ **Theorem 24** ($*$)**.** *Bounded stretch-width is stable under first-order transductions.*

6 Algorithmic applications of reduced component max-leaf

In this section, we prove Theorem 3. Let G be a graph and $\mathcal{P}$ be a vertex partition of G. For a subgraph H of G, let $\mathrm{trace}_{G,\mathcal{P}}(H)$ denote the set of all parts in $\mathcal{P}$ that contain a vertex of H. If G is clear from the context, we simply write it as $\mathrm{trace}_{\mathcal{P}}(H)$. For a family $\mathcal{Q}$ of vertex sets in G and $I \subseteq V(G)$, we write $I_{\mathcal{Q}} = I \cap \left(\bigcup_{X \in \mathcal{Q}} X\right)$.

Proof of Theorem 3. Let $\mathcal{P}_n, \ldots, \mathcal{P}_1$ be a given contraction sequence of G witnessing that $\mathrm{cml}^{\downarrow}(G) \leqslant t$. Let $G_i = G/\mathcal{P}_i$ for each i. For $i \in [n]$, let $\mathcal{C}(i)$ be the collection of all tuples (H, S, η) such that H is a non-empty connected red-induced subtrigraph of G_i, S is a subset of $V(G)$ such that

- $|S \cap P| \leqslant \alpha$ for every $P \in R_{G_i}(V(H)) \setminus B_{G_i}(V(H))$, and
- $|S \cap P| = 0$ for every $P \in V(G_i) \setminus \big(R_{G_i}(V(H)) \setminus B_{G_i}(V(H))\big)$,

$\eta : S \to \{0, 1, \ldots, \alpha\}$ is a function. For $(H, S, \eta) \in \mathcal{C}(i)$, a vertex subset I of G is *valid* with respect to (i, H, S, η) if (1) $\mathrm{trace}_{\mathcal{P}_i}(I) = V(H)$, (2) for every $v \in I$, $|N_G(v) \cap (I \cup S)| \in \sigma$, and (3) for every $v \in S$, $|N_G(v) \cap I| = \eta(v)$.

Note that H is always non-empty, and by the first condition, any valid set is non-empty. Moreover, from the second condition, $G[I]$ has maximum degree at most α. We recursively store a maximum valid vertex set I of G with respect to (i, H, S, η). For $(H, S, \eta) \in \mathcal{C}(i)$, let $\Phi(i, H, S, \eta)$ be a largest valid vertex set. If there is no valid vertex set, then we assign $\Phi(i, H, S, \eta) = \perp$. Observe that $\Phi(1, G_1, \emptyset, \eta)$ is the required solution where η is the empty function. When $i = n$, G_n has no red edges, and it is straightforward to assign $\Phi(i, H, S, \eta)$.

We assume that $i < n$, and for every $(H, S, \eta) \in \mathcal{C}(i+1)$, $\Phi(i+1, H, S, \eta)$ has been recorded. Let P_1 and P_2 be the parts of $\mathcal{P}_{i+1}$ that are merged into P^* in $\mathcal{P}_i$. Let $(H, S, \eta) \in \mathcal{C}(i)$.

There are two cases that are relatively easy to handle:

- $P^* \in V(G_i) \setminus \big(V(H) \cup (R_{G_i}(V(H)) \setminus B_{G_i}(V(H)))\big)$, and
- $P^* \in R_{G_i}(V(H)) \setminus B_{G_i}(V(H))$ and $S \cap P^* = \emptyset$.

In these cases, merging operation did not affect the H part, and we can set $\Phi(i, H, S, \eta) = \Phi(i + 1, H, S, \eta)$. Thus, we may assume that neither case occurs. See Figure 4.

We use the following concept to capture the possible positions that contain a bounded number of vertices from a partial solution. For each $j \in [2]$, let $\mathcal{Q}_j = B_{G_{i+1}}(P_j) \cap (V(H) \setminus \{P^*\})$. A pair $(\mathcal{O}, U)$ of a subset $\mathcal{O} \subseteq \{P_1, P_2\} \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ and a vertex subset $U \subseteq \bigcup_{Q \in \mathcal{O}} Q$ is a *σ-separator* with respect to $(P_1, P_2, \mathcal{Q}_1, \mathcal{Q}_2)$ if the following conditions hold:

- U contains at most α vertices from each part in $\mathcal{O}$,
- if $\mathcal{O} \cap \mathcal{Q}_i \neq \emptyset$ for some i, then $\{P_i\} \cup \mathcal{Q}_i \subseteq \mathcal{O}$, $|U_{\mathcal{Q}_i}| \leqslant \alpha$, $|U \cap Q| \geqslant 1$ for every $Q \in \mathcal{Q}_i$,
- $G[U]$ has maximum degree α, and
- for every part $Q \in \{P_1, P_2\} \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ that is not in $\mathcal{O}$ and every part $Q' \in \{P_1, P_2\} \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ where Q and Q' are linked by a black edge in $G/\mathcal{P}$, it holds that $Q' \in \mathcal{O}$ and $U \cap Q' = \emptyset$.

For a given σ-separator, we construct a valid subset corresponding to it. Let $(\mathcal{O}, U)$ be a σ-separator with respect to $(P_1, P_2, \mathcal{Q}_1, \mathcal{Q}_2)$ such that if $P^* \in R_{G_i}(V(H)) \setminus B_{G_i}(V(H))$, then $\{P_1, P_2\} \subseteq \mathcal{O}$ and $U \cap P^* = S \cap P^*$.

Let F be the subtrigraph of G_{i+1} induced by $(V(H) \setminus \{P^*\}) \cup \{P_1, P_2\}$. Roughly speaking, we guess a possible partial solution on the σ-separator and then propagate this information to the parts corresponding to the components of $F - \mathcal{O}$. One can observe that each component of $F - \mathcal{O}$ is a red-induced subtrigraph of G_{i+1}, and that the guessed set on the σ-separator contains at most α vertices in each part. Thus, by combining the information from $\Phi(i + 1, \cdot, \cdot, \cdot)$, we can obtain $\Phi(i, H, S, \eta)$.

One can verify that $|\mathcal{C}(i)| \leqslant n^{(\alpha+1)t} \cdot (\alpha+1)^{\alpha t}$ and the number of possible σ-separators with respect to a given $(P_1, P_2, \mathcal{Q}_1, \mathcal{Q}_2)$ is at most $4n^{4\alpha}$. Also, the number of possible propagations from a guessed set on a fixed σ-separator is at most $(\alpha + t + 2)^{\alpha^2(t+4)}$. So, for fixed $i \in [n]$, $\Phi(i, H, S, \eta)$ for all $(H, S, \eta) \in \mathcal{C}(i)$ can be computed in time $O(|\mathcal{C}(i)| \cdot n^{4\alpha} \cdot (\alpha + t + 2)^{\alpha^2(t+4)})$, and this algorithm runs in time

$$O\big(n^{(\alpha+1)t+4\alpha+1} \cdot (\alpha + 1)^{\alpha t} \cdot (\alpha + t + 2)^{\alpha^2(t+4)}\big) = n^{O((\alpha+1)^2 t)},$$

where $\alpha \leqslant n$ can be assumed because it suffices to consider σ only up to n. ◀

For Induced Disjoint Paths, we proceed along a similar line of argument. A key point is that if two parts P_1 and P_2 are adjacent by a black edge, then either the solution does not intersect one of P_1 and P_2, or it intersects each of P_1 and P_2 in at most two vertices. This naturally leads to a concept similar to a σ-separator.

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