Decomposing a Simple Polygon with Geodesic Unit-Balls

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Abstract

We consider covering and partitioning a simple polygon into pieces which either have unit geodesic radius or unit geodesic diameter, using the ℓ_2-metric for distances. There is no known method for finding an exact solution to these problems, even when the input size is constant, and the problem is known to be NP-hard in the case of polygons with holes. With this in mind, we instead devote our attention to developing simple approximation algorithms that run in polynomial time. For the radius problem, we present the first known approximation algorithms for both covering and partitioning, achieving a factor of 9. For the diameter problem, we are only able to give a positive result for the partition version of the problem, where we improve upon a complicated 72-approximation from Abrahamsen and Rasmussen [2], achieving a simple 15-approximation.

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1 Introduction

A fundamental problem which appears in many domains is that of describing large regions as a series of smaller, easier to process pieces. Depending on the application, there are varying definitions of small which are relevant. In some settings, we want pieces which have some simple property to work with, such as convex pieces or star-shaped pieces [23, 27], while in others, we care more about physical size, such as whether a piece fits inside of a square [2, 3, 1] or that this division is, in some sense, equitable [9, 25, 5]. There is also the concern of whether the pieces are allowed to overlap, in which case we are finding a *cover* of the region, or whether the pieces must be (interior) disjoint, in which case we are looking for a *partition* and whether we are covering a continuous domain or a discrete set of points [15, 18, 21, 6].

Our focus will be on two notions of *geodesic* smallness, specifically pieces which have unit geodesic *radius* or unit geodesic *diameter* (see Section 1.1 for formal definitions), and our region of interest will be a simple polygon P.

In the radius problem, we are finding a set of center points such that every point in P is within unit geodesic distance of at least one center. This notion of "small" is natural for facility location applications and is equivalent to the dual of the famous k-CENTER problem [33, 13, 30, 14, 11]: In k-CENTER, we have a fixed budget of k centers and want to instead minimize the maximum distance from a point to its nearest center. The problem of covering a *discrete* set of points in P with geodesic unit-balls admits a local search–based PTAS [28, 20], owing to the existence of planar supports [8]. In contrast, for covering (a continuum in) P, progress has been limited. Rabanca and Vigan gave a 2-approximation algorithm for covering *the boundary* of P in 2015 [31], and we are unaware of any subsequent improvements. Indeed, designing an algorithm to cover all of P was explicitly posed as an open problem by Abrahamsen and Rasmussen in 2022 [2].

The state of the art for the diameter variant, in which no two points within the same piece are at geodesic distance greater than one, is somewhat stronger. While no exact algorithms are known (even with exponential running time), Abrahamsen and Rasmussen achieve a 72-approximation for the partition version [2].

1.1 Preliminaries

Consider two points p and q on the plane. The *Euclidean distance* between them is denoted by $d(p,q)$. For $r \in \mathbb{R}$, let $B_r(p)$ denote the *Euclidean ball* of radius r centered at p, i.e., $B_r(p) := \{q \in \mathbb{R}^2 \mid d(p,q) \leq r\}$.

Throughout this paper, we use the term *polygonal domain* to denote polygons (possibly) with holes and *polygons* for simple polygons. For two points p and q in a polygonal domain P, let $\pi_P(p,q)$ denote a shortest path[1] between them within P. The *geodesic distance* $d_P(p,q)$ is defined as $|\pi_P(p,q)|$, the length of a shortest path between p and q within P . The *geodesic ball* of radius r centered at p is defined as $B_r^P(p) := \{q \in P \mid d_P(p,q) \leq r\}$. The *geodesic radius* of a polygonal domain P is the smallest r such that $B_r^P(c) = P$ for some $c \in P$. The *geodesic diameter* of P is the maximum geodesic distance between any two points in P.

A polygon Q is a *small geodesic radius piece* (hereafter, *small GR-piece*) if the geodesic radius of Q is at most 1. A collection of polygons $\mathcal{Q}$ is a *small geodesic radius cover* (hereafter, *small GR-cover*) of P if (i) each $Q \in \mathcal{Q}$ is a small GR-piece, and (ii) $\bigcup_{Q \in \mathcal{Q}} Q = P$. In particular, note that each $Q \in \mathcal{Q}$ is a subset of P. A small GR-cover $\mathcal{Q}$ is a *small GR-partition* if, for any distinct $Q, Q' \in \mathcal{Q}$, Q and Q' are interior disjoint. A *small geodesic diameter piece* (hereafter, *small GD-piece*), *small GD-cover*, and *small GD-partition* are defined similarly.

We are now ready to define our problems of interest. Given a polygonal domain P and an integer $k \in \mathbb{N}$, Small GR-Cover (resp. Small GR-Partition) asks whether there exists a small GR-cover (resp. small GR-partition) $\mathcal{Q}$ of P with $|\mathcal{Q}| \leq k$. Small GD-Cover and Small GD-Partition are similarly defined in terms of small GD-pieces.

It is known that Small GR-Cover is NP-Hard in polygonal domains [35]. Their construction can be readily adapted to show that the other three problems are also NP-Hard.

▶ **Theorem 1.1** (Theorem 10 in [35]). Small GR-Cover*,* Small GR-Partition*,* Small GD-Cover*, and* Small GD-Partition *are* NP-Hard *in polygonal domains.*

While it is not known whether these problems remain NP-Hard in simple polygons, there has been extensive work dating back nearly a century within the mathematics community on determining the minimum number of unit balls required to cover constant-complexity regions (such as squares, triangles or disks of larger radius). Despite this long history, even some very basic questions in this area remain unresolved; see, for example, [24, 34, 36, 16, 17, 2] and references therein. Recently, Abrahamsen and Stade showed that covering and partitioning polygons with pieces that fit inside a unit square is NP-Hard [3].

[1] Shortest paths between points within a polygon are unique; this might not hold in a polygonal domain.

1.2 Overview of results

As our first result, we establish in Section 2 that Small GR-Cover and Small GR-Partition are equivalent in the sense that they admit optimal solutions of the same size.

▶ **Theorem 1.2.** *Let $\mathcal{Q}$ and $\mathcal{Q}'$ denote an optimal small GR-cover and small GR-partition of a polygonal domain P respectively. Then, $|\mathcal{Q}| = |\mathcal{Q}'|$.*

Theorem 1.2 (along with the fact that we can *compute* a small GR-partition from a small GR-cover efficiently) allows us to focus on coverings without loss of generality. Building on this, we design (in Section 3) the first constant-factor approximation algorithms for Small GR-Cover and Small GR-Partition, thereby resolving an open question of Abrahamsen and Rasmussen [2]. We use $\widetilde{\mathcal{O}}(\cdot)$ to hide logarithmic factors, n to denote the number of vertices of P, and OPT for the optimal number of small pieces in the problem of interest.

▶ **Theorem 1.3.** *Consider a simple polygon P with n vertices. There exists an algorithm running in $\widetilde{\mathcal{O}}(\text{OPT}^3 \cdot n^2)$ time that returns a small GR-cover of P with at most $9 \cdot \text{OPT}$ pieces. Moreover, there is an algorithm which returns a small GR-partition of P with the same running time and approximation guarantee.*

Along the way, we also obtain a significantly simpler proof that the greedy boundary cover of Rabanca and Vigan [31, Theorem 1] yields a 2-approximation. Our ideas are general enough to obtain a substantially improved approximation for Small GD-Partition, achieving a 15-approximation within the same running time. This not only improves the previously best-known factor of 72 [2, Table 1], but also leads to a conceptually simpler algorithm and analysis. We discuss this in Section 4.

▶ **Theorem 1.4.** *Consider a simple polygon P with n vertices. There is an algorithm running in $\widetilde{\mathcal{O}}(\text{OPT}^3 \cdot n^2)$ time that returns a small GD-partition of P with at most $15 \cdot \text{OPT}$ pieces.*

We conclude the paper in Section 5 by discussing avenues for further research.

▶ Remark 1.5. Our techniques also apply to the setting where we measure geodesic distances by the ℓ_1-metric, and can achieve matching approximations using a very similar argument. While there is not the same uncertainty of how to optimally cover even constant complexity convex regions with ℓ_1 disks (squares) as there is with ℓ_2 disks, the proof for NP-hardness for covering a *simple* polygon with pieces that fit in unit squares from [3] also applies to pieces with unit geodesic radius in the ℓ_1 metric by rotating the construction by 45 degrees.

2 Equivalence of Small GR-Cover and Small GR-Partition

We begin this section by proving our first (and simplest) result, Theorem 1.2. To this end, we first establish a useful lemma that will be used in its proof.

▶ **Lemma 2.1.** *Consider a small GR-cover $\mathcal{Q}$ of a polygonal domain P. There is a small GR-partition $\mathcal{Q}'$ of P with $|\mathcal{Q}'| = |\mathcal{Q}|$.*

Proof. Consider the centers $C_{\mathcal{Q}}$ of the geodesic disks used by $\mathcal{Q}$ to cover P. Construct the geodesic Voronoi diagram on $C_{\mathcal{Q}}$; let this be $\mathcal{Q}'$. Every point p in P is covered by at least one geodesic disk in $\mathcal{Q}$, so the closest disk center $c \in C_{\mathcal{Q}}$ to p has $d_P(c, p) \leq 1$. Thus, each Voronoi cell is a small GR-piece showing that $\mathcal{Q}'$ is a small GR-partition as required. ◀

Proof of Theorem 1.2. Consider an optimal small GR-cover $\mathcal{Q}$ and an optimal small GR-partition $\mathcal{Q}'$ of a polygonal domain P. Since all small GR-partitions are small GR-covers, $|\mathcal{Q}| \leq |\mathcal{Q}'|$. By Lemma 2.1, there is a small GR-partition $\mathcal{Q}''$ of P with $|\mathcal{Q}''| = |\mathcal{Q}|$. Therefore, $|\mathcal{Q}'| \leq |\mathcal{Q}''| = |\mathcal{Q}|$; showing that $|\mathcal{Q}| = |\mathcal{Q}'|$ as required. ◀

We note that a similar result was established in [1, Theorem 10] for the case of covering and partitioning polygonal domains with pieces that fit inside a unit square. We conclude this section with a useful remark.

▶ Remark 2.2. The geodesic Voronoi diagram of a simple polygon P for a set of m centers can be computed in $\mathcal{O}(n + m \log m)$ time [29, Theorem 8.1]. Since a small GR-cover $\mathcal{Q}$ of P can be efficiently represented by its centers $C_{\mathcal{Q}}$, we can compute a small GR-partition $\mathcal{Q}'$ of P with $|\mathcal{Q}'| = |\mathcal{Q}|$ from (the centers $C_{\mathcal{Q}}$ of) $\mathcal{Q}$ in $\mathcal{O}(n + |\mathcal{Q}| \log |\mathcal{Q}|)$ time.

3 Approximation Algorithm for Small GR-Cover and Partition

In this section, we prove Theorem 1.3. Throughout this section, OPT denotes the number of pieces in an optimal small GR-cover of the input (simple) polygon P. We first cover the boundary of P using at most $2 \cdot \text{OPT}$ pieces (we actually only define a set of centers; the pieces are simply the unit geodesic ball centered on this set), using an algorithm by Rabanca and Vigan [31]. Their correctness proof is quite involved; we simplify it substantially by invoking a result of Abrahamsen and Rasmussen [2] – see Lemma 3.2. We then place an additional $2 \cdot \text{OPT}$ centers to cover points "close" to ∂P. Although this redundancy may appear wasteful, it provides a crucial structural property (Lemma 3.6), which enables a simple greedy strategy for covering the interior using at most $5 \cdot \text{OPT}$ pieces. In Section 3.2, we show that all of these subroutines can be implemented in polynomial time, thereby establishing Theorem 1.3.

3.1 Algorithm description

We begin by stating a lemma concerning a partition of the boundary of a simple polygon, which is a slight generalization of a result by Abrahamsen and Rasmussen [2, Lemma 3]. Their proof immediately extends to our setting:

▶ **Lemma 3.1** (Lemma 3 of [2]). *Let $\mathcal{S}$ be a partition of a simple polygon P into connected pieces and let $\mathcal{S}_b = \{S \in \mathcal{S} \mid \partial S \cap \partial P \neq \emptyset\}$, i.e., $\mathcal{S}_b$ is the set of pieces in $\mathcal{S}$ which touch the boundary of P. Then, there is a partition of ∂P into contiguous intervals $\mathcal{I}$ such that each interval $I \in \mathcal{I}$ is a subset of some $S \in \mathcal{S}_b$ and $|\mathcal{I}| \leq \max(1, 2 \cdot |\mathcal{S}_b| - 2)$.*

We describe the algorithm SmallGRGreedyBoundaryCover, given by Rabanca and Vigan in [31, Section 2.2], to cover the boundary of P with small GR-pieces. Start from a vertex v of P and place disk centers greedily: beginning at v, cover as far along ∂P (walking counterclockwise) as possible such that the traversed contiguous interval I is contained in a unit geodesic disk. Place a center to cover I and repeat until returning to v. Let OPT' denote the optimal number of small GR-pieces required to cover ∂P – note that $\text{OPT}' \leq \text{OPT}$. Moreover, from (the proof of) Theorem 1.2, OPT' is also the optimal number of small GR-pieces required to *partition* ∂P. We now present a significantly simpler proof than that of [31] of the following lemma:

▶ **Lemma 3.2** (Theorem 1 of [31]). SmallGRGreedyBoundaryCover *places at most $2 \cdot \text{OPT}' - 1$ disk centers.*

Consider an optimal partition $\mathcal{S}_b$ of ∂P with small GR-pieces; $|\mathcal{S}_b| = \textsc{opt}'$. Let $\mathcal{S}_i$ denote the collection of connected pieces of the region inside P that $\mathcal{S}_b$ leaves uncovered. Let $\mathcal{S} = \mathcal{S}_b \cup \mathcal{S}_i$. By Lemma 3.1, there is a partition of ∂P into contiguous intervals $\mathcal{I}$ where each $I \in \mathcal{I}$ is a small GR-piece and $|\mathcal{I}| \leq 2 \cdot |\mathcal{S}_b| - 2 \leq 2 \cdot \textsc{opt}' - 2$. Let $\mathcal{I}' = \{I'_1, I'_2 \dots I'_k\}$ denote the contiguous intervals constructed by SmallGRGreedyBoundaryCover. We will show that $|\mathcal{I}'| = k \leq |\mathcal{I}| + 1 \leq 2 \cdot \textsc{opt}' - 1$. Let $S' = \{s'_i \mid 1 \leq i \leq k\}$ where s'_i is the start-point of $I'_i \in \mathcal{I}'$; consider an $s'_j \in S'$ where $j \leq k-1$. For two points $a, b \in \partial P$, we use $\partial P[a, b]$ to denote the counterclockwise walk along ∂P from a to b. Since $\mathcal{I}$ partitions ∂P, there is an $I = \partial P[s_I, t_I] \in \mathcal{I}$ which contains s'_j. Recall that I is a small GR-piece; if $s'_{j+1} \in \partial P[s_I, t_I)$, then $I'_j = \partial P[s'_j, s'_{j+1}]$ would not be maximal walk, a contradiction. So:

▶ **Observation 3.3.** $s'_{j+1} \notin \partial P[s_I, t_I)$.

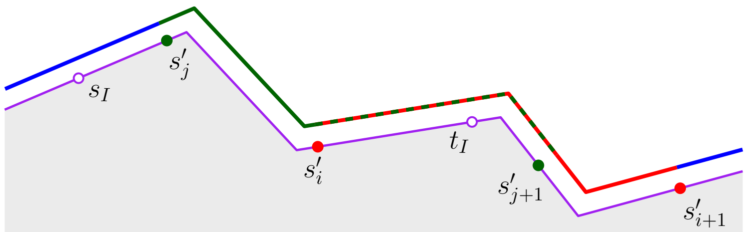

Figure 1 Illustration of Corollary 3.4. The boundary of P is marked purple; the gray region is outside P. If $s'_i, s'_j \in \partial P[s_I, t_I)$, then s'_{j+1} lies after s'_i. Therefore I'_i (marked red above ∂P for clarity) and I'_j (marked green) overlap and therefore $\{I'_\ell \mid i \leq \ell \leq j\}$ covers ∂P.

▶ **Corollary 3.4.** *No two points in $S' \setminus \{s'_k\}$ are in $\partial P[s_I, t_I)$, for any $I \in \mathcal{I}$.*

Proof. Assume s'_i and s'_j, with $1 \leq i < j \leq k-1$, are both in $\partial P[s_I, t_I)$, for some $I \in \mathcal{I}$. By Observation 3.3, $j \neq i+1$; moreover, $s'_{j+1} \notin \partial P[s_I, t_I)$ and therefore lies after s'_i along ∂P. Therefore, $\{I'_\ell \mid i \leq \ell \leq j\}$, a strict subset of $\mathcal{I}'$, covers ∂P, a contradiction. See Figure 1. ◀

We can now prove Lemma 3.2.

Proof of Lemma 3.2. By Corollary 3.4, at most one point in $S' \setminus \{s'_k\}$ is in each interval of $\mathcal{I}$. Therefore, $k - 1 \leq |\mathcal{I}| \leq 2 \cdot \textsc{opt}' - 2$; so $|\mathcal{I}'| = k \leq 2 \cdot \textsc{opt}' - 1$ as required. ◀

Let C'_b denote the centers placed by SmallGRGreedyBoundaryCover. We add additional (and, at first glance, redundant) centers to C'_b. Let $\mathcal{I}$ denote the collection of contiguous intervals considered by SmallGRGreedyBoundaryCover – clearly, $|C'_b| = |\mathcal{I}|$. At every endpoint of an interval of $\mathcal{I}$, we place a new geodesic disk center and add this to C'_b. We call this subroutine SmallGRAddRedundantCenters.

Algorithm 1 SmallGRAddRedundantCenters.

Input: A set of centers C'_b and a partition $\mathcal{I}$ of ∂P.
Output: A set of centers C_b.
$C_b \leftarrow C'_b$
for I in $\mathcal{I}$ **do**
    Append $I[0]$ to C_b
end for
return C_b

Let C_b denote the resulting set of centers. Clearly, $|C_b| = 2 \cdot |\mathcal{I}|$. Even though this is more centers than necessary to cover just the boundary of P, this cover gives some useful properties for constructing a cover of the interior. Let $\mathcal{Q}_b$ denote the collection of unit geodesic discs centered at C_b – i.e., $\mathcal{Q}_b = \{B_1^P(c) \mid c \in C_b\}$. Since $|\mathcal{Q}_b| = |C_b| = 2 \cdot |\mathcal{I}|$ and $|\mathcal{I}| \leq 2 \cdot \text{OPT}' \leq 2 \cdot \text{OPT}$ (by Lemma 3.2), we have:

▶ **Corollary 3.5.** $|\mathcal{Q}_b| \leq 4 \cdot \text{OPT}$.

We now show that $\mathcal{Q}_b$ not only covers ∂P but also all points that are "close" (in the Euclidean sense) to ∂P. Recall that $\pi_P(p, q)$ denotes the shortest path between p and q within P.

▶ **Lemma 3.6.** *$\mathcal{Q}_b$ covers every point of P that is a **Euclidean** distance at most $\frac{1}{2}$ from ∂P.*

Proof. Consider a point $p \in P$ which is within a distance of $\frac{1}{2}$ from ∂P. Consider $q \in \partial P$, a point on the boundary closest to p. We have $d(p, q) \leq \frac{1}{2}$ and that no point in the interior of $\overline{pq}$ is on ∂P – otherwise there is another point q' closer to p on ∂P. So, we have $\overline{pq} \subset P$. Since $\mathcal{I}$ covers ∂P, q lies in some contiguous interval $I \in \mathcal{I}$ of ∂P. Let $I = \partial P[s_I, t_I]$ and let $c_I \in C_b' \subseteq C_b$ be the geodesic disk center placed by SmallGRGreedyBoundaryCover for I. We have two cases (see Figure 2): $\overline{pq}$ intersects $\pi_P(s_I, c_I) \cup \pi_P(c_I, t_I)$ or it does not.

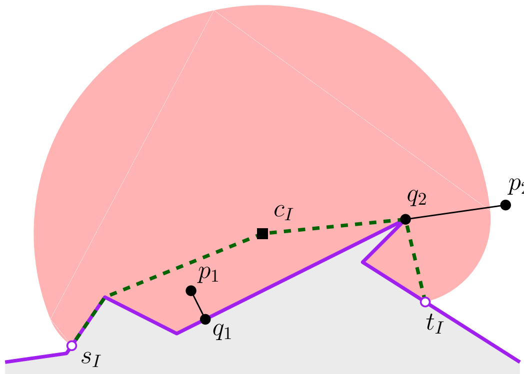

Figure 2 Illustration of Lemma 3.6. The boundary of P is drawn in purple, the gray region is outside P. The shortest paths from c_I to s_I and t_I are in dashed green. We have $p_1, p_2 \in P$ and they are at most an Euclidean distance of $\frac{1}{2}$ away from ∂P. The points $q_1, q_2 \in \partial P$ are the closest point on ∂P to p_1 and p_2. Since $\overline{p_1 q_1} \cap (\pi_P(s_I, c_I) \cup \pi_P(c_I, t_I)) = \emptyset$, $p_1 \in B_1^P(c_I)$. On the other hand, $\overline{p_2 q_2} \cap \pi_P(c_I, t_I) \neq \emptyset$; since this intersection point is closer to t_I than to c_I, $p_2 \in B_1^P(t_I)$.

▶ Case 3.6.1. Assume that $\overline{pq}$ does not intersect $\pi_P(s_I, c_I) \cup \pi_P(c_I, t_I)$. Then, p must lie inside the region[2] R within the closed curve $\pi_P(s_I, c_I) \cup \pi_P(c_I, t_I) \cup \partial P[t_I, s_I]$ by the Jordan curve theorem. Since I is contiguous, R is within the geodesic disk centered at c_I, and thus p is covered by $\mathcal{Q}_b$. ⌟

▶ Case 3.6.2. Assume that $\overline{pq}$ intersects $\pi_P(s_I, c_I) \cup \pi_P(c_I, t_I)$. We assume that there is a point $q' \in \overline{pq} \cap \pi_P(c_I, t_I)$; the other case is symmetric. Note that $d_P(c_I, t_I) \leq 1$ – i.e., the length of $\pi_P(c_I, t_I)$ is at most 1. Since $q' \in \pi_P(c_I, t_I)$, either $d_P(c_I, q') \leq \frac{1}{2}$ or $d_P(t_I, q') \leq \frac{1}{2}$. Since $d(q', p) \leq d(q, p) \leq \frac{1}{2}$, p is within a geodesic distance of 1 from either c_I or t_I. Since both c_I and t_I belong to C_b, it follows that p is covered by $\mathcal{Q}_b$. ⌟

These cases are exhaustive, so the claim follows. ◀

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[2] R is a degenerate weakly simple polygon as some of its sides could overlap – see Figure 2.

The above property implies the following corollary:

▶ **Corollary 3.7.** *Consider two points p and q in P which are not covered by $\mathcal{Q}_b$. If $d(p,q) \le 1$, $d_P(p,q) \le 1$.*

Proof. If $\overline{pq} \cap \partial P = \emptyset$, then $\pi_P(p,q) = \overline{pq}$; so $d_P(p,q) = d(p,q) \le 1$ as required. Otherwise, there is a $r \in \overline{pq} \cap \partial P$; so $d(r,p)$ or $d(r,q)$ is at most $\frac{1}{2}$ (since $|\overline{pq}| \le 1$), which, by Lemma 3.6, is a contradiction to the assumption that p and q are uncovered by $\mathcal{Q}_b$. ◀

With this in mind, we proceed to cover the interior of P with a greedy algorithm which we call GreedyInteriorCover. Consider the uncovered region $P' = P \setminus \cup_{Q \in \mathcal{Q}_b} Q$ and place a center at a point c in the interior of P'. Remove $B_1^P(c)$ from P' and iterate until all points in P' have been covered. Let C_i denote the centers placed by GreedyInteriorCover in P'. Define $\mathcal{Q}_i = \{B_1^P(c) \mid c \in C_i\}$ and let $\mathcal{Q} = \mathcal{Q}_b \cup \mathcal{Q}_i$.

Algorithm 2 GreedyInteriorCover.

Input: A simple polygon P, an uncovered region U, and a radius r.
Output: A set of centers C_i.
$C_i \leftarrow \emptyset$
$R \leftarrow U$
while $R \neq \emptyset$ **do**
    Find a point c in the interior of R
    Append c to C_i
    $R \leftarrow R \setminus B_r^P(c)$
end while
return C_i

▶ **Observation 3.8.** *For any two distinct centers c and c' in C_i, $d(c,c') > 1$.*

Proof. Since GreedyInteriorCover picks centers iteratively from the interior of the uncovered region, we have that $d_P(c,c') > 1$ for any two centers c and c' in C_i. The claim now follows from (the contrapositive of) Corollary 3.7. ◀

▶ **Observation 3.9.** *$\mathcal{Q}$ is a small GR-cover of P.*

Proof. Consider a point $p \in P$ which is not covered by $\mathcal{Q}_b$ – i.e, $p \in P'$. Since GreedyInteriorCover iterates until all points in P' are covered, p is covered by $\mathcal{Q}_i'$. ◀

To achieve the approximation ratio promised in Theorem 1.3, we prove that $|\mathcal{Q}| \le 9 \cdot \text{OPT}$. From Corollary 3.5, we already have $|\mathcal{Q}_b| \le 4 \cdot \text{OPT}$, so it remains to show the following:

▶ **Lemma 3.10.** $|\mathcal{Q}_i| \le 5 \cdot \text{OPT}$.

We first establish two results about Euclidean disks that we will use to prove the above lemma. For a point $c \in \mathbb{R}^2$ and angles $0 \le \alpha < \beta \le 2\pi$, let $S(c,\alpha,\beta)$ denote the *sector* of $B_1(c)$ induced by the two radii that make angles α and β with the x–axis. Note that a sector is a closed set; in particular, it contains the two defining radii. A *sextant* $S'(c,\alpha)$ is the sector $S(c,\alpha,\alpha+\frac{\pi}{3})$. See Figure 3, *Left.*

▶ **Observation 3.11.** *For any two points p and q in an arbitrary sextant $S'(c,\alpha)$, $d(p,q) \le 1$.*

Proof. Since $d(c,p)$ and $d(c,q)$ are at most 1 and the angle between $\overline{cp}$ and $\overline{cq}$ is at most $\frac{\pi}{3}$, our claim follows. See Figure 3, *Left.* ◀

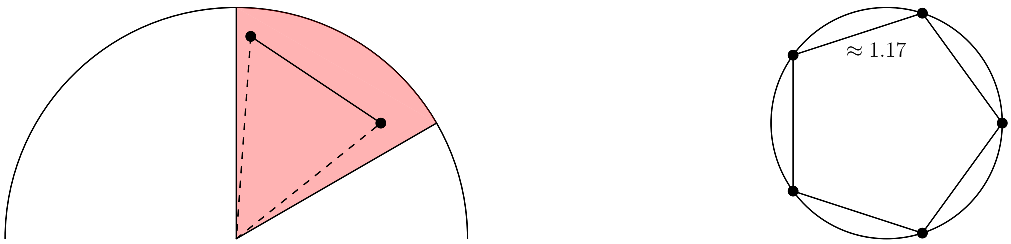

Figure 3 Left: The sextant $S'(c, \frac{\pi}{6})$ is in red. Two points inside a sextant are at most a distance 1 apart since they subtend an angle of at most $\frac{\pi}{3}$ between them at the center. Right: The distance between 5 equally spaced points along a unit circle is at least 1.17.

▶ **Corollary 3.12.** *There* ***does not*** *exist a subset $X \subseteq B_1(c)$ with $|X| \geq 6$ such that $d(p, q) > 1$ for any two points p and q in X.*

Proof. Assume that there exists such a set X with $|X| \geq 6$. Then, there exists two points $p, q \in X$ which subtend an angle of at most $\frac{\pi}{3}$ between them at c. This implies that $d(p, q) \leq 1$ from Observation 3.11, a contradiction. ◀

Note that Corollary 3.12 is tight – Let X denote the set of 5 equally spaced points along the unit circle; the distance between any two points in X is at least 1.17. See Figure 3, *Right*. We are now ready to prove Lemma 3.10.

Proof of Lemma 3.10. Consider an optimal small GR-cover $\mathcal{Q}^*$ of P and a small GR-piece $Q \in \mathcal{Q}^*$. Let c be the center of Q; we have $Q = B_1^P(c) \subseteq B_1(c)$. Let $X = C_i \cap B_1(c)$ – Note that $X \supseteq C_i \cap Q$. Moreover, every distinct pair of points in X are more than a distance 1 apart (Observation 3.8). From Corollary 3.12, we have that $|X| \leq 5$. Since every point in C_i is covered by some small GR-piece in $\mathcal{Q}^*$, we have $|C_i| \leq 5 \cdot |\mathcal{Q}^*|$; the result now follows. ◀

3.2 Runtime analysis

We now show that the three subroutines used by our algorithm SmallGRGreedyCover to construct $\mathcal{Q}$ – namely, SmallGRGreedyBoundaryCover, SmallGRAddRedundantCenters, and GreedyInteriorCover – can be executed in polynomial time.

Algorithm 3 SmallGRGreedyCover.

1: **Input:** A simple polygon P stored as a circular list.
2: **Output:** A small GR-cover $\mathcal{Q}$ of P.
3: $(C'_b, \mathcal{I}) \leftarrow$ SmallGRGreedyBoundaryCover(P)
4: $C_b \leftarrow$ SmallGRAddRedundantCenters$(C'_b, \mathcal{I})$
5: $\mathcal{Q}_b \leftarrow \{B_1^P(c) \mid c \in C_b\}$
6: $P' \leftarrow P \setminus \cup_{Q \in \mathcal{Q}_b} Q$
7: $C_i \leftarrow$ GreedyInteriorCover$(P, P', 1)$
8: $\mathcal{Q}_i \leftarrow \{B_1^P(c) \mid c \in C_i\}$
9: $\mathcal{Q} = \mathcal{Q}_b \cup \mathcal{Q}_i$
10: **return** $\mathcal{Q}$

The first two steps of SmallGRGreedyCover is to run SmallGRGreedyBoundaryCover and SmallGRAddRedundantCenters (Lines 3 and 4). SmallGRGreedyBoundaryCover was shown to run in polynomial time by Rabanca and Vigan in [31].

▶ **Lemma 3.13** (Section 2.2 in [31]). *Given a polygon P,* SmallGRGreedyBoundaryCover *returns C_b' and $\mathcal{I}$ in $\widetilde{\mathcal{O}}(|C_b'| + n)$ time.*

The running time of SmallGRAddRedundantCenters is immediate.

▶ **Lemma 3.14.** *Given a set of centers C_b' and a patition $\mathcal{I}$ of ∂P,* SmallGRAddRedundantCenters *returns C_b in $\mathcal{O}(|\mathcal{I}| + |C_b|)$ time.*

Next, we compute P' using $\mathcal{Q}_b'$ (Lines 5 and 6 in SmallGRGreedyCover). $\mathcal{Q}_b$ can be constructed in $\mathcal{O}(|C_b| \cdot n^2)$ using an algorithm by Borgelt, van Krevald, and Luo [7]. Since a geodesic ball is described by $\mathcal{O}(n)$ circular arcs and line segments, the arrangement of P and $\mathcal{Q}_b$ has complexity $\mathcal{O}(|C_b|^2 \cdot n^2)$; and can be computed in that time [19]. Thus, Line 6 takes $\mathcal{O}(|C_b|^2 \cdot n^2)$ time.

▶ **Lemma 3.15.** *Given an uncovered region $P' \subset P$ (as an overlay of $|C_b|$ geodesic disks and P),* GreedyInteriorCover *returns C_i in $\widetilde{\mathcal{O}}(|C_i| \cdot (|C_b| + |C_i|)^2 \cdot n^2)$ time.*

Proof. We can maintain R, the uncovered region through the course of the algorithm (Line 8 in GreedyInteriorCover) in $\widetilde{\mathcal{O}}((|C_b| + |C_i|)^2 \cdot n^2)$ time using an incremental construction of the arrangement. See, for example, [19, Theorem 28.4.2] and [4, Section 6]. Maintaining this arrangement also allows for us to check if R is empty (Line 5) in constant time in each iteration of the loop.

To find a point c in the interior of R (Line 6), we find the minimum enclosing (Euclidean) ball B of the vertices of R. Let v be one of the vertices of R on ∂B and let r be the radius of B. Consider the ball B' of radius r around v; $\partial B'$ intersects the interior of R in (possibly) disconnected arcs. Let c be the midpoint of one such arc[3]. Finding B takes linear (in the complexity of R) time; see, for example, [12, Chapter 4.7]. Since this complexity is bounded by $\mathcal{O}((|C_b| + |C_i|)^2 \cdot n^2)$ and there are $|C_i|$ iterations of the loop, the runtime follows. ◀

We now prove the main result of this section, Theorem 1.3 (repeated here for readability):

▶ **Theorem 1.3.** *Consider a simple polygon P with n vertices. There exists an algorithm running in $\widetilde{\mathcal{O}}(\textsc{opt}^3 \cdot n^2)$ time that returns a small GR-cover of P with at most $9 \cdot \textsc{opt}$ pieces. Moreover, there is an algorithm which returns a small GR-partition of P with the same running time and approximation guarantee.*

Proof. Consider the small GR-cover $\mathcal{Q}$ that SmallGRGreedyCover returns. From Corollary 3.5 and Lemma 3.10, we have that $|\mathcal{Q}| \leq 9 \cdot \textsc{opt}$. From Lemmas 3.13, 3.14 and 3.15, it is clear that its running time is dominated by the subroutine GreedyInteriorCover. The runtime guarantee follows since $|C_i| + |C_b| = |\mathcal{Q}| \leq 9 \cdot \textsc{opt}$.

To return a small GR-partition instead of a small GR-cover, we modify Lines 8 and 9 of SmallGRGreedyCover slightly: Compute the geodesic Voronoi diagram of P for the centers $C_b \cup C_i$ and return as $\mathcal{Q}$ the Voronoi cells instead (see Remark 2.2). By Theorem 1.2, we have that $|\mathcal{Q}| \leq 9 \cdot \textsc{opt}$; the running time is unaffected since the geodesic Voronoi diagram can be computed in $\mathcal{O}(n + \textsc{opt} \log \textsc{opt})$ time using [29, Theorem 8.1]. ◀

4 Approximation Algorithm for Small GD-Partition

Our goal in this section is to prove Theorem 1.4. Let OPT denote the number of pieces in an optimal small GD-partition of the input simple polygon P. Structurally, the algorithm we describe is similar to that in Section 3. We first partition the boundary ∂P into $2 \cdot \textsc{opt}$

[3] To bypass bit complexity issues, we assume the standard Real-RAM model.

pieces using an algorithm of Abrahamsen and Rasmussen [2] (Lemma 4.1). We then place at most $4 \cdot \textsc{opt}$ additional pieces to cover points "close" to ∂P, and finally cover the interior using GreedyInteriorCover. In contrast to Section 3, where the interior can be covered using at most $5 \cdot \textsc{opt}$ pieces, we obtain a bound of $9 \cdot \textsc{opt}$ pieces here (Lemma 4.12). We conclude in Section 4.2 that the running time of the algorithm is once again dominated by GreedyInteriorCover, thereby establishing Theorem 1.4.

4.1 Algorithm description

We begin by describing SmallGDGreedyBoundaryCover, the algorithm designed by Abrahamsen and Rasmussen [2, Sections 3.1 and 3.2] to cover the boundary of a simple polygon P with small GD-pieces. It is nearly identical to SmallGRGreedyBoundaryCover that we saw in the previous section: Start from a vertex v of P and cover as far along ∂P (walking counterclockwise) as possible such that the traversed contiguous interval I has geodesic diameter at most 1. From $I = \partial P[s_I, t_I]$, construct $\pi_P(t_I, s_I)$ to obtain the small GD-piece bounded by $\partial P[s_I, t_I]$ and $\pi_P(t_I, s_I)$. Repeat until returning to v.

Let $\mathcal{Q}'_b$ and $\mathcal{I}$ denote the collection of small GD-pieces and the partition of ∂P the above algorithm returns respectively.

▶ **Lemma 4.1** (Lemma 4 of [2]). $|\mathcal{Q}'_b| = |\mathcal{I}| \leq 2 \cdot \textsc{opt} - 1$.

Consider P', the region of P uncovered by $\mathcal{Q}'_b$. Note that P' is a (collection of) degenerate weakly simple polygon(s). since $\mathcal{I} \in \mathcal{O}(\textsc{opt})$, it follows that $|P'| \in \mathcal{O}(\textsc{opt} \cdot n)$.

▶ **Corollary 4.2.** *The perimeter of P' is at most* $2 \cdot \textsc{opt} - 1$.

Proof. Consider a small GD-piece $Q \in \mathcal{Q}'_b$. Let $I = \partial P[s_I, t_I]$ be the interval on ∂P that it corresponds to. Observe that $\partial Q \cap \partial P' \subseteq \pi_P(s_I, t_I)$ by construction. Since $|\pi_P(s_I, t_I)| \leq 1$ (Q is a small GD-piece), we infer that Q contributes a length of at most 1 to the perimeter of P'. As $|\mathcal{Q}'_b| \leq 2 \cdot \textsc{opt} - 1$ (see Lemma 4.1), the claim immediately follows. ◀

We will cover P' using geodesic balls of *radius* $\frac{1}{2}$ – Note that these have geodesic *diameter* at most 1 as required. To begin, we construct a cover $\mathcal{Q}''_b$ of the boundary of P': Start with a vertex v of P' and place centers at every $\frac{1}{2}$ interval along $\partial P'$. We call this subroutine SmallGDAddRedundantCenters.

Let C''_b denote the centers placed by SmallGDAddRedundantCenters and define $\mathcal{Q}''_b = \{B^{P'}_{\frac{1}{2}}(c) \mid c \in C''_b\}$. Corollary 4.3 is a direct consequence of Corollary 4.2.

▶ **Corollary 4.3.** $|\mathcal{Q}''_b| \leq 4 \cdot \textsc{opt}$.

We show a complementary lemma to Lemma 3.6: $\mathcal{Q}_b := \mathcal{Q}'_b \cup \mathcal{Q}''_b$ covers every point that is a Euclidean distance at most $\frac{1}{4}$ from ∂P. Then, Corollary 4.5 and Observation 4.6 follow from Lemma 4.4 just as Corollary 3.7 and Observation 3.8 did from Lemma 3.6.

▶ **Lemma 4.4.** *$\mathcal{Q}_b$ covers every point of P that is a* ***Euclidean*** *distance at most $\frac{1}{4}$ from ∂P.*

Proof. Consider a point $p \in P$ that is a Euclidean distance at most $\frac{1}{4}$ from ∂P. If $p \in P \setminus P'$, it must be covered by $\mathcal{Q}'_b$ by definition. Assume that $p \in P'$. Since $d(p, \partial P) \leq \frac{1}{4}$ and $\partial P'$ separates ∂P and P', $d(p, \partial P') \leq \frac{1}{4}$. Let $q \in \partial P'$ be the closest point to p on $\partial P'$ and let c be the closest center (where the distance is measured along $\partial P'$) in C''_b to q. Note that $\overline{pq} \subset P'$ – otherwise there is a point on $\partial P'$ closer to p than q. Therefore, we have that $d_{P'}(p, q) = d(p, q) \leq \frac{1}{4}$. Moreover, since SmallGDAddRedundantCenters places centers at every $\frac{1}{2}$ interval along $\partial P'$, $d_{P'}(c, q) \leq \frac{1}{4}$. This shows that the path from c to p through q has length at most $\frac{1}{2}$, and thus $p \in B^{P'}_{\frac{1}{2}}(c)$. So p is covered by $\mathcal{Q}''_b$ as required. ◀

Algorithm 4 SmallGDAddRedundantCenters.

Input: A region P'.
Output: A set of centers C''_b.
$C''_b \leftarrow \emptyset$
$s \leftarrow v_0$ ▷ v_i denotes the vertex $P[i]$
$i \leftarrow 1$ ▷ i holds the index of the vertex right after s
flag $\leftarrow$ FALSE ▷ Checks if we have walked through all of ∂P
while flag is FALSE **do**
  Append s to C''_b
  $j \leftarrow$ the index of the first vertex such that $|\partial P'[s, v_j]| > \frac{1}{2}$
  $t \leftarrow$ maximal point on $\overline{v_{j-1}v_j}$ such that $|\partial P'[s, t]| = \frac{1}{2}$
  $s \leftarrow t$
  $i \leftarrow j$
  If we have walked through all of ∂P, set flag $\leftarrow$ TRUE
end while
return C''_b

▶ **Corollary 4.5.** *Consider two points p and q in P which are not covered by $\mathcal{Q}_b$. If $d(p, q) \leq \frac{1}{2}$, $d_P(p, q) \leq \frac{1}{2}$.*

Let $P'' = P' \setminus \cup_{Q \in \mathcal{Q}''_b} Q$. We reuse GreedyInteriorCover to cover P''; we call GreedyInteriorCover with $U = P''$ and $r = \frac{1}{2}$ (Line 8, SmallGDGreedyPartition). C_i denotes the centers placed by GreedyInteriorCover in P''; $\mathcal{Q}_i := \{B^{P'}_{\frac{1}{2}}(c) \mid c \in C_i\}$. We have:

▶ **Observation 4.6.** *For any two distinct centers c and c' in C_i, $d(c, c') > \frac{1}{2}$.*

▶ **Observation 4.7.** *$\mathcal{Q}_b \cup \mathcal{Q}_i$ covers every point in P.*

▶ **Corollary 4.8.** *Consider a point $p \in P'$ and let $c \in C''_b \cup C_i$ be the center that is closest to p. Then, $d_P(c, p) \leq \frac{1}{2}$.*

Proof. From Observation 4.7, we infer that p is covered by a piece in $\mathcal{Q}''_b \cup \mathcal{Q}_i$. Since these are $\frac{1}{2}$-radius geodesic balls, our claim follows. ◀

Since we need to output a partition (and not just a cover), we compute the geodesic Voronoi diagram of $C''_b \cup C_i$ inside P' (see Remark 2.2). Let $\mathcal{Q}$ denote the union of these Voronoi cells and $\mathcal{Q}'_b$.

▶ **Lemma 4.9.** *$\mathcal{Q}$ is a small GD-partition of P.*

Proof. The pieces of $\mathcal{Q}'_b$ are interior disjoint from [2, Sections 3.1 and 3.2]. Since the remaining pieces are from a Voronoi partition of $P \setminus \mathcal{Q}'_b$, it follows that $\mathcal{Q}$ is a partition of P.

$\mathcal{Q}'_b$ contains small GD-pieces [2, Sections 3.1 and 3.2]; consider a piece $Q \in \mathcal{Q} \setminus \mathcal{Q}'_b$. Then, Q is a Voronoi cell of a center $c \in C''_b \cup C_i$. It follows from Corollary 4.8 that $Q \subseteq B^P_{\frac{1}{2}}(c)$. Q is therefore a small GD-piece. ◀

From Lemma 4.1 and Corollary 4.3, we have that $|\mathcal{Q}'_b| \leq 2 \cdot \text{OPT}$ and $|\mathcal{Q}''_b| \leq 4 \cdot \text{OPT}$ respectively; so to achieve the approximation guarantee promised in Theorem 1.4, we are left to show that $|\mathcal{Q}_i| \leq 9 \cdot \text{OPT}$. Note that we cannot use a similar argument as in Lemma 3.10: A small GD-piece does not necessarily need to fit inside a $B_{\frac{1}{2}}(c)$ for some center $c \in \mathbb{R}^2$; consider, for example, the equilateral triangle with side lengths 1. See Section 5 for further discussion. We make two observations before proving that $|\mathcal{Q}_i| \leq 9 \cdot \text{OPT}$:

▶ **Observation 4.10.** *Let Q be a small GD-piece of P. Then, Q fits inside a unit square.*

Proof. Let $x_{\min}$, $x_{\max}$, $y_{\min}$, and $y_{\max}$ denote the smallest and largest x and y-coordinates of points in Q. Since the diameter of Q is at most 1, $x_{\max} - x_{\min} \leq 1$ and $y_{\max} - y_{\min} \leq 1$. Our claim now follows. ◀

▶ **Observation 4.11.** *Consider a unit square S. There* ***does not*** *exist a subset $X \subseteq S$ with $|X| \geq 10$ such that $d(p,q) > \frac{1}{2}$ for any two points p and q in X.*

Proof. We partition S into 9 equal-sized subsquares $\mathcal{S} = \{S_1, S_2 \ldots S_9\}$. Note that the Euclidean diameter of a $S \in \mathcal{S}$ is $\frac{\sqrt{2}}{3} < \frac{1}{2}$. Therefore, each $S \in \mathcal{S}$ contains at most one point from X; this proves our claim. ◀

▶ **Lemma 4.12.** $|\mathcal{Q}_i| \leq 9 \cdot \text{OPT}$.

Proof. Consider an optimal small GD-partition $\mathcal{Q}^*$ of P and a small GD-piece $Q \in \mathcal{Q}^*$. Then, there is a unit square S such that $Q \subseteq S$ (see Observation 4.10). Let $X = C_i \cap S$; note that $X \supseteq C_i \cap Q$. Every distinct pair of points in X are a Euclidean distance $\frac{1}{2}$ apart (from Observation 4.6). So, from Observation 4.11, we have $|X| \leq 9$. Since every point in C_i is covered by some small GD-piece in $\mathcal{Q}^*$, we have $|C_i| \leq 9 \cdot |\mathcal{Q}^*|$; the result now follows. ◀

4.2 Runtime analysis

We refer to our algorithm to find a small GD-partition of a simple polygon as SmallGDGreedyPartition; we now prove that it runs in polynomial time.

Algorithm 5 SmallGDGreedyPartition.

1: **Input:** A simple polygon P stored as a circular list.
2: **Output:** A small GD-partition $\mathcal{Q}$ of P.
3: $(\mathcal{Q}'_b, \mathcal{I}) \leftarrow$ SmallGDGreedyBoundaryCover(P)
4: $P' \leftarrow P \setminus \cup_{Q \in \mathcal{Q}'_b} Q$
5: $C''_b \leftarrow$ SmallGDAddRedundantCenters(P')
6: $\mathcal{Q}''_b \leftarrow \{B^{P'}_{\frac{1}{2}}(c) \mid c \in C''_b\}$
7: $P'' \leftarrow P' \setminus \cup_{Q \in \mathcal{Q}''_b} Q$
8: $C_i \leftarrow$ GreedyInteriorCover$(P, P'', \frac{1}{2})$
9: $\mathcal{Q} \leftarrow \mathcal{Q}'_b \cup$ ComputeGeodesicVoronoi$(P', C''_b \cup C_i)$
10: **return** $\mathcal{Q}$

▶ **Lemma 4.13** (Lemma 5 of [2]). *Given a polygon P,* SmallGDGreedyBoundaryCover *returns $\mathcal{Q}'_b$ and $\mathcal{I}$ in $\mathcal{O}(n^2 \log n + |\mathcal{Q}_b|)$ time.*

Abrahamsen and Rasmussen [2] also construct P' and the above runtime includes this construction of P'. Lines 6 and 7 of SmallGDGreedyPartition take $\mathcal{O}(|C''_b|^2 \cdot n^2)$ time, see the discussion between Lemmas 3.14 and 3.15 in Section 3.2. Lemma 3.15 proves that GreedyInteriorCover takes $\widetilde{\mathcal{O}}(|C_i| \cdot (|C''_b| + |C_i|)^2 \cdot n^2)$ time while Remark 2.2 shows that Line 9 takes $\widetilde{\mathcal{O}}(|P'| + |C''_b \cup C_i|)$ time. We ascertain the runtime of SmallGDAddRedundantCenters below and then prove Theorem 1.4, restated here for convenience.

▶ **Observation 4.14.** *Given a region P',* SmallGDAddRedundantCenters *returns C''_b in $O(|P'| + |C''_b|)$ time.*

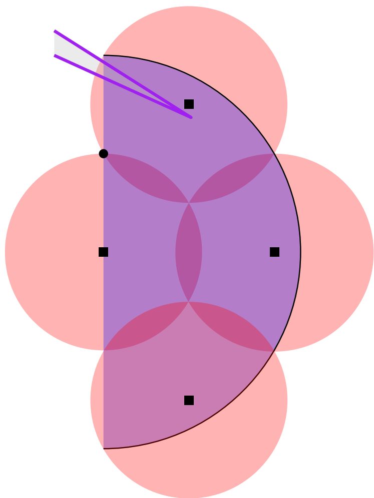

Figure 4 Four Euclidean unit balls (in red; centers marked with squares) cover a Euclidean half-ball of radius 2 (in blue). However, the half-ball cannot be covered by four geodesic unit balls; the point, marked with a disk, is not covered by the geodesic unit balls centered at these points, even though it is distance $> \frac{1}{2}$ from the boundary. The boundary of the polygon is given in purple and the gray region is outside the polygon.

▶ **Theorem 1.4.** *Consider a simple polygon P with n vertices. There is an algorithm running in $\widetilde{\mathcal{O}}(\text{OPT}^3 \cdot n^2)$ time that returns a small GD-partition of P with at most $15 \cdot \text{OPT}$ pieces.*

Proof. Consider $\mathcal{Q}$, the set that SmallGDGreedyPartition returns. From Lemma 4.9, $\mathcal{Q}$ is a small GD-partition of P and from Lemmas 4.1 and 4.12 and Corollary 4.3, we have $|\mathcal{Q}| \leq 15 \cdot \text{OPT}$. Since $|\mathcal{Q}'| \in \mathcal{O}(\text{OPT})$, $|P'| \in \mathcal{O}(\text{OPT} \cdot n)$. From Lemmas 4.13 and 3.15 and Observation 4.14 (and the discussion prior to Observation 4.14) it is clear that its running time of SmallGDGreedyPartition is dominated by the subroutine GreedyInteriorCover. The runtime guarantee follows since $|C_i| + |C_b| = |\mathcal{Q}| \leq 15 \cdot \text{OPT}$. ◀

5 Open Questions

We see two major directions for further research: improving the algorithms' running times and their approximation factors. The running times of both SmallGRGreedyCover and SmallGDGreedyPartition are dominated by the subroutine GreedyInteriorCover. It would be interesting to design a data structure that dynamically maintains the uncovered region (Line 8 in GreedyInteriorCover), perhaps drawing on recent work on dynamic orthogonal range searching data structures [10, 26]. This could potentially reduce the overhead incurred by repeatedly updating and querying the current uncovered region and get a runtime similar to the algorithm for Small GD-Partition in [2].

Turning to approximation guarantees, we note work by Biniaz, Liu, Maheshwari, and Smid [6, Section 2], where a 4-approximation is given for covering a *discrete* set of points in $\mathbb{R}^2$ using unit Euclidean disks. Their argument relies on the fact that a Euclidean half-ball of radius 2 can be covered by four unit balls. In our setting, however, an analogous statement does not hold for geodesic unit balls (see Figure 4), which presents a difficulty to improving our bounds for Small GR-Cover and Small GR-Partition via similar techniques.

To improve the approximation guarantee for Small GD-Partition, we attempted to extend the approach used in Lemma 3.10, but this did not yield a substantially better bound without introducing significant additional complexity. By Jung's theorem [22, 32], any region of diameter 1 is contained in a ball of radius $\frac{1}{\sqrt{3}}$. A natural idea is to partition such a ball into sectors of diameter at most $\frac{1}{2}$. However, this requires at least eight sectors: the regular 7-gon inscribed in a circle of radius $\frac{1}{\sqrt{3}}$ has side length $\frac{2}{\sqrt{3}}\sin(\frac{\pi}{7}) \approx 0.501 > \frac{1}{2}$. This already yields a lower bound of $7 \cdot \text{OPT}$ pieces (see the proof of Theorem 1.4 in Section 4). Moreover, this only accounts for the angular width of the sectors; their radial extent still exceeds $\frac{1}{2}$, necessitating further subdivision and incurring at least one additional piece. Altogether, these arguments suggest that achieving a bound significantly below $9 \cdot \text{OPT}$ via such arguments is unlikely. In recent work we (along with several other colleagues) design a PTAS for covering and partitioning polygonal domains with pieces that fit inside a unit square [1, Corollary 9]. It seems highly non-trivial to adapt this PTAS for the problems discussed here. Finally, we note that the existence of an approximation algorithm for Small GD-Cover remains open. Whether our methods (specifically, a suitable analogue of Theorem 1.2 or Lemma 4.1) can be extended to this problem warrants further investigation.

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