Parameterized Approximation of Rectangle Stabbing

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Abstract

In the Rectangle Stabbing problem, input is a set $\mathcal{R}$ of axis-parallel rectangles and a set $\mathcal{L}$ of axis-parallel lines in the plane. The task is to find a minimum size set $\mathcal{L}^* \subseteq \mathcal{L}$ such that for every rectangle $R \in \mathcal{R}$ there is a line $\ell \in \mathcal{L}^*$ such that ℓ intersects R. Gaur et al. [Journal of Algorithms, 2002] gave a polynomial time 2-approximation algorithm, while Dom et al. [WALCOM 2009] and Giannopoulos et al. [EuroCG 2009] independently showed that, assuming FPT $\neq$ W[1], there is no algorithm with running time $f(k)(|\mathcal{L}||\mathcal{R}|)^{O(1)}$ that determines whether there exists an optimal solution with at most k lines. We give the first parameterized approximation algorithm for the problem with a ratio better than 2. In particular we give an algorithm that given $\mathcal{R}$, $\mathcal{L}$, and an integer k runs in time $k^{O(k)}(|\mathcal{L}||\mathcal{R}|)^{O(1)}$ and either correctly concludes that there does not exist a solution with at most k lines, or produces a solution with at most $\frac{7k}{4}$ lines. We complement our algorithm by showing that unless FPT = W[1], the Rectangle Stabbing problem does not admit a $(\frac{5}{4} - \epsilon)$-approximation algorithm running in $f(k)(|\mathcal{L}||\mathcal{R}|)^{O(1)}$ time for any function f and $\epsilon > 0$.

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1 Introduction

In this paper, we consider the Rectangle Stabbing problem, which is a well-studied geometric covering problem defined as follows.

Rectangle Stabbing	**Parameter:** $k = |\mathcal{L}^*|$
Input: A set $\mathcal{R}$ of axis-parallel rectangles and a set $\mathcal{L}$ of axis-parallel lines in $\mathbb{R}^2$	
Goal: Compute a minimum subset $\mathcal{L}^* \subseteq \mathcal{L}$ such that $\mathcal{R}$ is stabbed by $\mathcal{L}^*$	

Here a line $\ell \in \mathcal{L}$ *stabs* a rectangle $R \in \mathcal{R}$ if ℓ and R have non-empty intersection. The problem was first considered by Hassin and Megiddo [14], who showed that the problem is NP-hard even when all the rectangles are horizontal line segments of unit length. In addition to being a very natural special case of the Set Cover problem, the Rectangle Stabbing problem generalizes the Optimal Discretization problem where input is a set of points in $\mathbb{R}^2$, each point colored with one of multiple colors, and the task is to divide the plane using the minimum number of axis-parallel lines such that no cell contains points of different colors [2, 20]. This has applications to fault tolerant sensor networks [2] and is also an idealized version of discretization problems considered in machine learning [11, 20, 26]. Tardos [25] proved that for every set $\mathcal{R}$ of axis-parallel rectangles and integer k, either there exists a subset $\mathcal{R}^*$ of $\mathcal{R}$ of size k such that no line in $\mathcal{L}$ stabs two rectangles in $\mathcal{R}^*$, or all the rectangles in $\mathcal{R}$ can be stabbed using at most $2k$ lines. However the proof of Tardos is non-constructive in the sense that it does not lead to a polynomial time algorithm that produces one of the two outcomes. In 2002, Gaur et al. [12] gave a polynomial time factor 2 approximation algorithm for Rectangle Stabbing based on an elegant rounding scheme for the relaxation of the most natural integer linear programming formulation of the problem. Subsequently Ben-David et al. [1] showed an integrality gap of $2 - \epsilon$ for every $\epsilon > 0$ for the considered integer linear programming formulation. Despite significant attention to related variants of the problem [2, 3, 8, 9, 16, 17, 19, 27] from the perspective of approximation algorithms, the factor 2 approximation of Gaur et al. [12] remains the state of the art.

The Rectangle Stabbing problem, and variants thereof, have also been considered from the perspective of parameterized algorithms, parameterized by the optimal value k. Dom et al. [6] (see also [7]) and Giannopoulos et al. [13] independently showed that, assuming FPT $\neq$ W[1], there is no algorithm with running time $f(k)n^{O(1)}$ that determines whether there exists an optimal solution with at most k lines. Here $n = |\mathcal{L}| + |\mathcal{R}|$, and the hardness results of [6, 13] apply even in the case when all the rectangles are squares of the same size. Giannopoulos et al. [13] additionally show that the hardness result applies when $\mathcal{R}$ is a set of disjoint, but not axis-parallel rectangles. Both Dom et al. [6] and Giannopoulos et al. [13] give $k^{O(k)}n^{O(1)}$ time algorithms for the case when $\mathcal{R}$ is a set of *disjoint* axis-parallel squares of the same size, and pose as an open problem whether the case of disjoint axis-parallel rectangles admits an algorithm with running time $f(k)n^{O(1)}$. This was resolved in the affirmative by Heggernes et al. [15], who gave an algorithm with running time $k^{O(k^2 \log k)}n^{O(1)}$ for this case.

The instances of Rectangle Stabbing that are equivalent to instances of Optimal Discretization have axis-parallel rectangles that may not be disjoint. Thus the algorithm of Heggernes et al. [15] does not imply an algorithm with running time $f(k)n^{O(1)}$ for the Optimal Discretization problem, where k is the solution size. An algorithm for Optimal Discretization with running time $k^{O(k^2 \log k)}n^{O(1)}$ was given by Kratsch et al. [20].

In this paper, we initiate the study of parameterized approximation algorithms (see the surveys [10, 18, 23] for an introduction to parameterized approximation) for the Rectangle Stabbing problem. Our main result is an algorithm with running time dependence on the solution size k which is better than what is possible (assuming FPT $\neq$ W[1]) for parameterized algorithms computing exact solutions, while simultaneously achieving an approximation ratio that is substantially better than that of the best known polynomial time approximation algorithm [12]. Specifically we show the following.

▶ **Theorem 1.** *There exists a $\frac{7}{4}$-approximation algorithm for Rectangle Stabbing with running time $k^{O(k)} \cdot n^{O(1)}$, where $n = |\mathcal{R}| + |\mathcal{L}|$.*

The algorithm of Theorem 1 works by first guessing some features of the optimal solution, including the number of horizontal and vertical lines. These features are used to greedily select at most k lines into the approximate solution, such that *(i)* the remaining rectangles not yet stabbed by the selected lines can be stabbed with only $\frac{3k}{4}$ additional lines, and *(ii)* the remaining instance can be solved in polynomial time by reduction to 2-SAT. We complement our main result by showing that the ratio of $\frac{7}{4}$ of Theorem 1 cannot be improved to a ratio arbitrarily close to one.

▶ **Theorem 2.** *Assuming that* FPT $\neq$ W[1]*, for any constant* $\epsilon > 0$ *and any function* f*, there does not exist a* $(\frac{5}{4} - \epsilon)$*-approximation algorithm for* Rectangle Stabbing *with running time* $f(k) \cdot n^{O(1)}$*, where* $n = |\mathcal{R}| + |\mathcal{L}|$.

Theorem 2 also holds even if only $\mathcal{R}$ is given as input and the set $\mathcal{L}$ is the set of all axis-parallel lines in the plane. The reduction of Theorem 2 can be viewed as a more robust (and therefore gap-preserving) version of the original W[1]-hardness reduction for Rectangle Stabbing of Giannopoulos et al. [13].

In Section 2 we define the notions needed for our proofs, and set up required notation. In Section 3 we prove Theorem 1, while in Section 4 we prove Theorem 2. We conclude with some open problems and future directions in Section 5.

2 Preliminaries

2.1 Basic notations

We write $\mathbb{N} = \{1, 2, \dots\}$ as the set of natural numbers, and write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a number $n \in \mathbb{N}$, we write $[n] = \{1, \dots, n\}$. For two real numbers a, b such that $a < b$ we denote by (a, b) the open interval $\{r \in \mathbb{R} \ : \ a < r < b\}$ and by $[a, b]$ the closed interval $\{r \in \mathbb{R} \ : \ a \leq r \leq b\}$. We also define $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z}$.

2.2 Strips and lines

Let x and y be real numbers. We denote by ℓ_h^y the *horizontal line* $\mathbb{R} \times \{y\}$. Similarly, we denote by ℓ_v^x the vertical line $\{x\} \times \mathbb{R}$. A horizontal (resp., vertical) *strip*, is a region in $\mathbb{R}^2$ of the form $\mathbb{R} \times (y^-, y^+)$ (resp., $(x^-, x^+) \times \mathbb{R}$), where $y^-, y^+ \in \mathbb{R}$ (resp., $x^-, x^+ \in \mathbb{R}$) and $y^- < y^+$ (resp., $x^- < x^+$). A *strip* refers to a horizontal or vertical strip. Two horizontal (resp., vertical) strips P, P' are *separated* by a horizontal (resp., vertical) line ℓ if P and P' are on opposite sides of ℓ. Let Γ be a set of disjoint horizontal (resp., vertical) strips and $\mathcal{L}$ be a set of horizontal (resp., vertical) lines. We say Γ is *separated* by $\mathcal{L}$ if (i) every pair of two (different) strips in Γ are separated by at least one line in $\mathcal{L}$ and (ii) $P \cap \ell = \emptyset$ for all $P \in \Gamma$ and all $\ell \in \mathcal{L}$. We say $\mathcal{L}$ is Γ*-structured* if there exists a bijective function $f : \mathcal{L} \to \Gamma$ such that $\ell \subseteq f(\ell)$ for all $\ell \in \mathcal{L}$. In other words, $\mathcal{L}$ is Γ-structured if every $P \in \Gamma$ contains exactly one line in $\mathcal{L}$ and these are the only lines in $\mathcal{L}$. A set $\mathcal{L}$ of horizontal (resp., vertical) lines cut the plane into $|\mathcal{L}| + 1$ horizontal (resp., vertical) strips; we use $\Gamma(\mathcal{L})$ to denote the set of these strips. Formally, $\Gamma(\mathcal{L})$ consists of the $|\mathcal{L}| + 1$ connected components of $\mathbb{R}^2 \backslash (\bigcup_{\ell \in \mathcal{L}} \ell)$.

2.3 Rectangles

All rectangles considered in this paper are axis-parallel. Specifically, we define a rectangle as a region in $\mathbb{R}^2$ of the form $[a, b] \times [c, d]$ for $a, b, c, d \in \mathbb{R}$ such that $a \leq b$ and $c \leq d$. We remark that all of our results easily extend to the case where all rectangles are nondegenerate, i.e. when line segments are excluded. A rectangle R is *stabbed* by a line ℓ if $R \cap \ell \neq \emptyset$, and is *stabbed* by a set $\mathcal{L}$ of lines if it is stabbed by some $\ell \in \mathcal{L}$. A set $\mathcal{R}$ of rectangles is *stabbed* by a set $\mathcal{L}$ of lines if every $R \in \mathcal{R}$ is stabbed by $\mathcal{L}$. For a set $\mathcal{R}$ of rectangles and a set $\mathcal{L}$ of lines, we denote by $\mathsf{opt}(\mathcal{R}, \mathcal{L})$ the minimum size of a subset of $\mathcal{L}$ that stabs $\mathcal{R}$; if such a subset does not exist (i.e., $\mathcal{R}$ is not stabbed by $\mathcal{L}$), we simply define $\mathsf{opt}(\mathcal{R}, \mathcal{L}) = \infty$.

2.4 One-Dimensional Greedy Stabbing

In the classic interval-stabbing (hitting-set) problem on the real line, we are given a collection of intervals and points, and we aim to choose the fewest points so that each interval is stabbed by at least one point. An optimal greedy strategy selects the rightmost point such that no interval lies entirely to its left, deletes all stabbed intervals, and repeats until no intervals remain. In a Rectangle Stabbing instance with only horizontal lines (resp. vertical lines), the same algorithm applies after projecting all rectangles onto segments on the y-axis (resp. x-axis), and projecting all lines to points on the y-axis (resp. x-axis). Clearly this algorithm runs in polynomial time, and we will frequently use it as the subroutine Stab.

3 Parameterized Approximation Algorithm

Consider a Rectangle Stabbing instance $(\mathcal{R}, \mathcal{L})$ with parameter k. Our goal is to compute a solution for $(\mathcal{R}, \mathcal{L})$ of size at most $\frac{7}{4}k$, assuming there is a solution for $(\mathcal{R}, \mathcal{L})$ of size at most k. Suppose $\mathcal{L} = \mathcal{H} \cup \mathcal{V}$ where $\mathcal{H}$ (resp., $\mathcal{V}$) consists of the horizontal (resp., vertical) lines in $\mathcal{L}$.

Let $\mathcal{L}^* \subseteq \mathcal{L}$ be a (unknown) solution for $(\mathcal{R}, \mathcal{L})$ with $|\mathcal{L}^*| \leq k$. Define $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{H}$ and $\mathcal{V}^* = \mathcal{L}^* \cap \mathcal{V}$. Suppose $k_\mathsf{h} = |\mathcal{H}^*|$ and $k_\mathsf{v} = |\mathcal{V}^*|$. We first guess the numbers k_h and k_v. Without loss of generality, we assume that $k_\mathsf{h} \leq k_\mathsf{v}$. We shall compute a solution of size at most $2k_\mathsf{h} + \frac{3}{2}k_\mathsf{v}$, which is sufficient since $2k_\mathsf{h} + \frac{3}{2}k_\mathsf{v} \leq \frac{7}{4}(k_\mathsf{h} + k_\mathsf{v}) \leq \frac{7}{4}k$.

Our algorithm consists of multiple steps; below we detail each in its own subsection. During the algorithm, we shall generate various subsets $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}'_1, \mathcal{H}_2 \subseteq \mathcal{H}$ and $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$. The sets $\mathcal{H}_1, \mathcal{H}'_1, \mathcal{H}_2$ and $\mathcal{V}_1, \mathcal{V}_2$ together form our solution, while $\mathcal{H}_0$ and $\mathcal{V}_0$ are only for auxiliary uses. At a high level, the goal of the algorithm is to collect a set of vertical and horizontal strips that each contain exactly one line of the optimal solution, with the additional property that each rectangle intersects at most one vertical strip and at most one horizontal strip. Once we have this structure, we solve the problem in polynomial time via a reduction to 2-SAT. We start by computing a set of horizontal lines H_1 so that the remaining rectangles can be stabbed by a set of $O(k^2)$ vertical lines V_0 (section 3.1). Because V_0 is small, in Section 3.2 we can branch on the strips induced by V_0 to identify those that contain exactly one line of the optimal solution, while using additional lines (the set V_1) to handle the remaining strips. Then in Section 3.3, we remove redundant rectangles in order to achieve a horizontal stabbing set H_0 of size $O(k^2)$, enabling an analogous branching to get H'_1 in Section 3.4. Finally in Section 3.5, we get V_2, H_2 from the 2-SAT reduction.

The total runtime of the algorithm is $k^{O(k)}(|\mathcal{L}||\mathcal{R}|)^{O(1)}$. The exponential dependence on k appears only due to the two branching steps (Sections 3.2 and 3.4), both of which need to enumerate subsets of size at most k from a set of size $O(k^2)$. Thus, each of these steps results in a multiplicative overhead of $k^{O(k)}$. The remaining steps in Sections 3.1, 3.3, and 3.5 run in polynomial time.

3.1 Stabbing $\mathcal{R}$ using $k_\mathsf{h} + O(k^2)$ lines

Our first step is to compute k_h horizontal lines and $O(k^2)$ vertical lines that together stab $\mathcal{R}$ and satisfy certain nice properties. The horizontal lines that we select in this step are denoted by $\mathcal{H}_1$ and will be added to the final output, while the vertical lines selected are denoted by $\mathcal{V}_0$, and used as a candidate set for future selection processes.

Consider a subset $\mathcal{H}' \subseteq \mathcal{H}$ and let $y_1, \dots, y_r \in \mathbb{R}$ be the y-coordinates of the horizontal lines in $\mathcal{H}'$ where $y_1 < \cdots < y_r$. For convenience, set $y_0 = -\infty$. We say $\mathcal{H}'$ is *nicely positioned* with respect to another subset $\mathcal{H}'' \subseteq \mathcal{H}$ if for every $i \in [r]$, there exists a line in $\mathcal{H}''$ that is contained in the strip $\mathbb{R} \times (y_{i-1}, y_i]$. Observe that if $\mathcal{H}'$ is nicely positioned with respect to $\mathcal{H}''$, then $|\mathcal{H}'| \leq |\mathcal{H}''|$. The main goal of the first step is given in the following lemma.

▶ **Lemma 3.** *One can compute in $(|\mathcal{R}| + |\mathcal{L}|)^{O(1)}$ time $\mathcal{H}_1 \subseteq \mathcal{H}$ and $\mathcal{V}_0 \subseteq \mathcal{V}$ satisfying the following.*

(i) *$\mathcal{H}_1$ is nicely positioned with respect to $\mathcal{H}^*$. In particular, $|\mathcal{H}_1| \leq k_{\mathsf{h}}$.*

(ii) *$|\mathcal{V}_0| = O(k^2)$.*

(iii) *$\mathcal{R}$ is stabbed by $\mathcal{H}_1 \cup \mathcal{V}_0$.*

Proof. For two horizontal lines h and h', we define $\mathcal{R}(h, h') \subseteq \mathcal{R}$ as the subset consisting of all rectangles that are contained in the open strip in between h and h'. Suppose $\mathcal{H} = \{h_1, \dots, h_m\}$ where $h_1, \dots, h_m$ are sorted in ascending order by their y-coordinates. For convenience, let h_0 (resp., h_{m+1}) be a horizontal line below (resp., above) all rectangles in $\mathcal{R}$.

Algorithm 1 Greedy Horizontal Line Preselection.

$\mathcal{H}_1 \leftarrow \emptyset$
$i \leftarrow 0$
while $i \leq m$ **do**
  $j^* \leftarrow \max\{j \in [m+1] \backslash [i] : |\textsc{Stab}(\mathcal{R}(h_i, h_j), \mathcal{V})| \leq k_{\mathsf{v}}\}$
  if $j^* \leq m$ **then** $\mathcal{H}_1 \leftarrow \mathcal{H}_1 \cup \{h_{j^*}\}$
  $i \leftarrow j^*$
$\mathcal{R}' \leftarrow \{R \in \mathcal{R} : R \text{ is not stabbed by } \mathcal{H}_1\}$
$\mathcal{V}_0 \leftarrow \textsc{Stab}(\mathcal{R}', \mathcal{V})$
return $\mathcal{H}_1$ and $\mathcal{V}_0$

The algorithm for computing $\mathcal{H}_1$ and $\mathcal{V}_0$ is presented in Algorithm 1. It uses the one-dimensional greedy stabbing algorithm $\textsc{Stab}(\mathcal{R}, \mathcal{V})$ where we only use vertical lines in $\mathcal{V}$ to stab rectangles in $\mathcal{R}$ as a subroutine. Starting from h_0, repeatedly choose the furthest line h_{j^*} so that all rectangles between the current line and h_{j^*} can be stabbed by at most k_{v} vertical lines; add h_{j^*} to $\mathcal{H}_1$ and continue from there. Once no more h_j remain, let $\mathcal{R}'$ be the rectangles not yet stabbed, and compute $\mathcal{V}_0 = \textsc{Stab}(\mathcal{R}', \mathcal{V})$ by the usual greedy process.

It suffices to verify that $\mathcal{H}_1$ and $\mathcal{V}_0$ satisfy the three conditions. Condition (iii) is clear from our construction. To see condition (i), suppose $\mathcal{H}_1 = \{h_{i_1}, \dots, h_{i_p}\}$ where $i_1 < \cdots < i_p$. Let $i_0 = 0$ and $i_{p+1} = m+1$. We need to show that $\mathcal{H}^* \cap \{h_{i_{t-1}+1}, \dots, h_{i_t}\} \neq \emptyset$ for all $t \in [p]$. By construction, we have $\mathsf{opt}(\mathcal{R}(h_{i_{t-1}}, h_{i_t+1}), \mathcal{V}) > k_{\mathsf{v}}$ and thus there exists $R \in \mathcal{R}(h_{i_{t-1}}, h_{i_t+1})$ that is not stabbed by $\mathcal{V}^*$. So R must be stabbed by $\mathcal{H}^*$, which implies that $\mathcal{H}^*$ contains one of the lines $h_{i_{t-1}+1}, \dots, h_{i_t}$. To see condition (ii), observe that $\mathcal{R}' = \bigcup_{t=1}^{p+1} \mathcal{R}(h_{i_{t-1}}, h_{i_t})$. Since $\mathsf{opt}(\mathcal{R}(h_{i_{t-1}}, h_{i_t}), \mathcal{V}) \leq k_{\mathsf{v}}$ for all $t \in [p+1]$ by our construction, $|\mathcal{V}_0| = \mathsf{opt}(\mathcal{R}', \mathcal{V}) \leq \sum_{t=1}^{p+1} \mathsf{opt}(\mathcal{R}(h_{i_{t-1}}, h_{i_t}), \mathcal{V}) \leq k_{\mathsf{v}}(p+1)$. As $p = |\mathcal{H}_1| \leq k_{\mathsf{h}}$, we have $|\mathcal{V}_0| = O(k^2)$. ◀

3.2 Finding good vertical strips

We now have the sets $\mathcal{H}_1$ and $\mathcal{V}_0$ satisfying the conditions in Lemma 3. The next step aims to find a set $\Gamma_{\mathsf{v}} \subseteq \Gamma(\mathcal{V}_0)$ of vertical strips together with a subset of vertical lines $\mathcal{V}_1 \subseteq \mathcal{V}_0$ that separates Γ_{v}. The main property we want is that each strip in Γ_{v} contains exactly one line in the optimal solution $\mathcal{V}^*$, and these lines together with $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}^*$ stab $\mathcal{R}$. Specifically, we prove the following lemma.

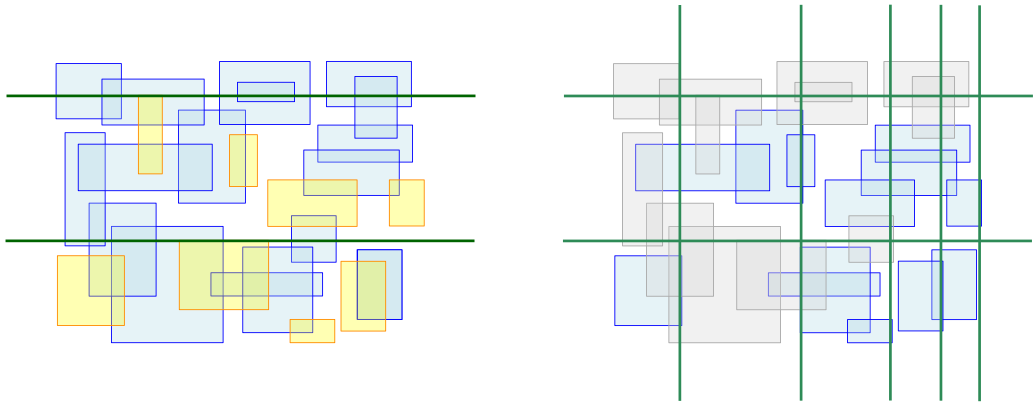

Figure 1 *The set of horizontal lines $\mathcal{H}_1$ and vertical lines $\mathcal{V}_0$ computed in Lemma 3.* On the left, sweeping upwards, we repeatedly place a horizontal line just before the number of vertically disjoint rectangles would exceed k. Here $k = 3$. On the right, the remaining rectangles (highlighted) after placing $\mathcal{H}_1$ are shown in blue, and stabbed by $\mathcal{V}_0$.

▶ **Lemma 4.** *There exist $\mathit{\Gamma}_{\mathsf{v}} \subseteq \mathit{\Gamma}(\mathcal{V}_0)$ and $\mathcal{V}_1 \subseteq \mathcal{V}_0$ satisfying the following.*

(i) $|\mathit{\Gamma}_{\mathsf{v}}| + |\mathcal{V}_1| \leq \frac{3}{2}k_{\mathsf{v}}$.

(ii) *$\mathit{\Gamma}_{\mathsf{v}}$ is separated by $\mathcal{V}_1$.*

(iii) *For each $P \in \mathit{\Gamma}_{\mathsf{v}}$, there exists exactly one line $v_P^* \in \mathcal{V}^*$ with $v_P^* \subseteq P$.*

(iv) *$\mathcal{R}$ is stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}^* \cup \{v_P^* : P \in \mathit{\Gamma}_{\mathsf{v}}\}$.*

Proof. For each strip $P \in \mathit{\Gamma}(\mathcal{V}_0)$, let $\partial P \subseteq \mathcal{V}_0$ denote the (at most) two vertical lines that bound P. We say P is *light* if P contains exactly one line in $\mathcal{V}^*$ and is *heavy* if P contains at least two lines in $\mathcal{V}^*$. Let $\mathit{\Gamma}' \subseteq \mathit{\Gamma}(\mathcal{V}_0)$ (resp., $\mathit{\Gamma}'' \subseteq \mathit{\Gamma}(\mathcal{V}_0)$) consist of all light (resp., heavy) strips. Suppose $\mathit{\Gamma}' = \{P_1, \ldots, P_r\}$ where $P_1, \ldots, P_r$ are sorted from left to right. Write $\mathit{\Gamma}'_0 = \{P_i : i \in [r] \text{ is even}\}$ and $\mathit{\Gamma}'_1 = \{P_i : i \in [r] \text{ is odd}\}$. Then we define $\mathit{\Gamma}_{\mathsf{v}} = \mathit{\Gamma}'_1$ and $\mathcal{V}_1 = (\mathcal{V}_0 \cap \mathcal{V}^*) \cup (\bigcup_{P \in \mathit{\Gamma}'_0} \partial P) \cup (\bigcup_{P \in \mathit{\Gamma}''} \partial P)$.

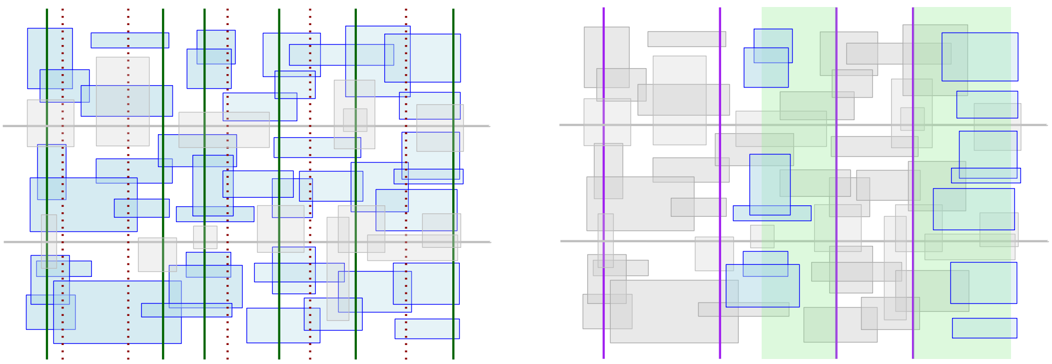

Figure 2 *The vertical lines $\mathcal{V}_1$ and strips Γ_v guaranteed by Lemma 4.* On the left, the vertical dashed lines represent the unknown lines of OPT, $\mathcal{V}^*$, while the solid green are those of $\mathcal{V}_0$. On the right, the strips of Γ_v are highlighted green. The vertical lines of $\mathcal{V}_1$ in purple include the boundaries of all heavy strips, and separate the strips of Γ_v.

We now verify the four conditions in the lemma. Conditions (ii) and (iii) follow readily from the construction. To see (iii), observe that $\mathit{\Gamma}_{\mathsf{v}}$ only contains strips in $\mathit{\Gamma}'$ (i.e., light strips), each of which contains exactly one line in $\mathcal{V}^*$ by definition. For (ii), consider a pair of strips $P_i, P_j \in \Gamma_{\mathsf{v}}$ where $i < j$. By construction, both i and j are odd, which implies $j \geq i+2$. Since $i+1$ is even, we have $P_{i+1} \in \Gamma_0'$ and thus $\partial P_{i+1} \subseteq \mathcal{V}_1$. The lines in ∂P_{i+1} then separate P_i and P_j.

For condition (iv), let $\mathcal{R}' \subseteq \mathcal{R}$ consist of rectangles not stabbed by $\mathcal{H}_1 \cup \mathcal{H}^*$. It suffices to show that every $R \in \mathcal{R}'$ is stabbed by $\mathcal{V}_1 \cup \{v_P^* : P \in \Gamma_{\mathsf{v}}\}$. Since R is not stabbed by $\mathcal{H}^*$, it must be stabbed by some $v \in \mathcal{V}^*$. If $v \in \mathcal{V}_0$, then $v \in \mathcal{V}_0 \cap \mathcal{V}^* \subseteq \mathcal{V}_1$ and thus $\mathcal{V}_1$ stabs R. Otherwise, v is contained in some strip $P \in \Gamma(\mathcal{V}_0)$. Then either $P \in \Gamma_1' = \Gamma_{\mathsf{v}}$ or $P \in \Gamma_0' \cup \Gamma''$. If $P \in \Gamma_{\mathsf{v}}$, the $v_P^* = v$ stabs R. If $P \in \Gamma_0' \cup \Gamma''$, then $\partial P \subseteq \mathcal{V}_1$. Note that at least one line in ∂P stabs R, because $R \cap P \neq \emptyset$ and $\mathcal{V}_0$ stabs R.

Finally, we verify condition (i), where we need $|\Gamma_{\mathsf{v}}| + |\mathcal{V}_1| \leq \frac{3}{2}k_{\mathsf{v}}$. We have $|\Gamma_{\mathsf{v}}| = |\Gamma_1'|$ and $|\mathcal{V}_1| \leq |\mathcal{V}_0 \cap \mathcal{V}^*| + 2(|\Gamma_0'| + |\Gamma''|)$. Note that $|\mathcal{V}_0 \cap \mathcal{V}^*| + |\Gamma'| + 2|\Gamma''| \leq |\mathcal{V}^*|$. Therefore,

$$|\Gamma_{\mathsf{v}}| + |\mathcal{V}_1| \leq |\Gamma_1'| + |\mathcal{V}_0 \cap \mathcal{V}^*| + 2(|\Gamma_0'| + |\Gamma''|) = |\mathcal{V}_0 \cap \mathcal{V}^*| + |\Gamma'| + 2|\Gamma''| + |\Gamma_0'| \leq |\mathcal{V}^*| + |\Gamma_0'|.$$

Note that $|\Gamma_0'| \leq |\Gamma_1'|$ and thus $|\Gamma_0'| \leq \frac{1}{2}|\Gamma'| \leq \frac{1}{2}|\mathcal{V}^*|$. So we have $|\Gamma_{\mathsf{v}}| + |\mathcal{V}_1| \leq \frac{3}{2}|\mathcal{V}^*| = \frac{3}{2}k_{\mathsf{v}}$. ◀

Since $\mathcal{H}^*$ and $\mathcal{V}^*$ are unknown to us, we cannot compute the sets Γ_{v} and $\mathcal{V}_1$ in the above lemma. However, we can afford guessing them. We have $|\Gamma(\mathcal{V}_0)| = |\mathcal{V}_0| + 1 = O(k^2)$ by (ii) of Lemma 3 and $|\Gamma_{\mathsf{v}}| + |\mathcal{V}_1| = O(k)$ by (i) of Lemma 4. Therefore, the total number of guesses for Γ_{v} and $\mathcal{V}_1$ is bounded by $k^{O(k)}$, and making the guess leads to a $k^{O(k)}$ overhead on the running time.

Note that $\{v_P^* : P \in \Gamma_{\mathsf{v}}\}$ is Γ_{v}-structured. The size of the set $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}^* \cup \{v_P^* : P \in \Gamma_{\mathsf{v}}\}$ is $|\mathcal{H}_1| + |\mathcal{V}_1| + k_{\mathsf{h}} + |\Gamma_{\mathsf{v}}|$, which is at most $2k_{\mathsf{h}} + \frac{3}{2}k_{\mathsf{v}}$ by (i) of Lemma 3 and (i) of Lemma 4. Therefore, (iv) of Lemma 4 implies the existence of a solution of size at most $2k_{\mathsf{h}} + \frac{3}{2}k_{\mathsf{v}}$ that consists of $\mathcal{H}_1 \cup \mathcal{V}_1$ together with a subset of $\mathcal{H}$ (i.e., $\mathcal{H}^*$) and a subset of $\mathcal{V}$ that is Γ_{v}-structured.

3.3 Removing redundant rectangles

In what follows, we shall focus on finding a solution of size at most $2k_{\mathsf{h}} + \frac{3}{2}k_{\mathsf{v}}$ that consists of $\mathcal{H}_1 \cup \mathcal{V}_1$ together with a subset of $\mathcal{H}$ and a Γ_{v}-structured subset of $\mathcal{V}$. As argued at the end of the last section, such a solution exists. To find such a solution, we shall first compute a good representative subset $\mathcal{K} \subseteq \mathcal{R}$ such that to solve the problem on $\mathcal{R}$, it suffices to solve the problem on $\mathcal{K}$. The desired property of $\mathcal{K}$ is that it can be stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1$ together with a set $\mathcal{H}_0 \subseteq \mathcal{H}$ of $O(k^2)$ additional horizontal lines.

▶ **Lemma 5.** *One can compute in $(|\mathcal{R}| + |\mathcal{L}|)^{O(1)}$ time $\mathcal{K} \subseteq \mathcal{R}$ and $\mathcal{H}_0 \subseteq \mathcal{H}$ satisfying the following.*

(i) $|\mathcal{H}_0| \leq O(k^2)$.

(ii) *For any $\mathcal{A} \subseteq \mathcal{H}$ with $|\mathcal{A}| \leq 2k$ and any Γ_{v}-structured $\mathcal{B} \subseteq \mathcal{V}$, $\mathcal{R}$ is stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{A} \cup \mathcal{B}$ if and only if $\mathcal{K}$ is stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{A} \cup \mathcal{B}$.*

(iii) *For each $R \in \mathcal{K}$, if R is stabbed by $\mathcal{H}$, then R is also stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}_0$.*

Proof. For a rectangle $R \in \mathcal{R}$ and a vertical strip P, we use $\mathsf{wid}_P(R)$ to denote the width of $R \cap P$, i.e., the length of the interval obtained by projecting $R \cap P$ to the x-axis. For $\mathcal{R}' \subseteq \mathcal{R}$ and a line ℓ, we use $\mathcal{R}' \Cap \ell$ to denote the subset of $\mathcal{R}'$ consisting of all rectangles stabbed by ℓ. As in the proof of Lemma 4, for each $P \in \Gamma_{\mathsf{v}}$, let $\partial P \subseteq \mathcal{V}_0$ consist of the (at most) 2 borderlines of P.

The algorithm for computing $\mathcal{K}$ and $\mathcal{H}_0$ is presented in Algorithm 2. Let $\mathcal{R}' \subseteq \mathcal{R}$ consist of the rectangles stabbed by $\mathcal{H}$ but not stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1$. The set $\mathcal{K}$ is obtained from $\mathcal{R}$ by removing a subset $\mathcal{X} \subseteq \mathcal{R}'$ as follows. Initially, $\mathcal{X} = \emptyset$. In each iteration, as long as there exists $\ell \in \partial P$ for some $P \in \mathit{\Gamma}_{\mathsf{v}}$ such that $\mathsf{opt}((\mathcal{R}'\backslash\mathcal{X}) \Cap \ell, \mathcal{H}) \geq 2k+2$, we add R^* to $\mathcal{X}$, where R^* is the rectangle in $(\mathcal{R}'\backslash\mathcal{X}) \Cap \ell$ that maximizes $\mathsf{wid}_P(R^*)$. We keep doing this until there does not exist such a line ℓ. This completes the construction of $\mathcal{X}$. We then set $\mathcal{K} = \mathcal{R}\backslash\mathcal{X}$ and define $\mathcal{H}_0 \subseteq \mathcal{H}$ as a minimum subset that stabs $\mathcal{R}'\backslash\mathcal{X}$. Here the sub-routine Stab is the same as the one in Algorithm 1.

Algorithm 2 Redundant Rectangle Elimination.

$\mathcal{R}' \leftarrow \{R \in \mathcal{R} : R$ is stabbed by $\mathcal{H}$ but not stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1\}$
$\mathcal{X} \leftarrow \emptyset$
while there exist $P \in \mathit{\Gamma}_{\mathsf{v}}$ and $\ell \in \partial P$ such that $\mathsf{opt}((\mathcal{R}'\backslash\mathcal{X}) \Cap \ell, \mathcal{H}) \geq 2k+2$ **do**
    $R^* = \arg\max_{R \in (\mathcal{R}'\backslash\mathcal{X}) \Cap \ell} \mathsf{wid}_P(R)$
    $\mathcal{X} \leftarrow \mathcal{X} \cup \{R^*\}$
$\mathcal{K} \leftarrow \mathcal{R}\backslash\mathcal{X}$
$\mathcal{H}_0 \leftarrow \text{Stab}(\mathcal{R}'\backslash\mathcal{X}, \mathcal{H})$
return $\mathcal{K}$ and $\mathcal{H}_0$

We now verify the three conditions in the lemma. Condition (iii) is clear from our construction. Indeed, if a rectangle $R \in \mathcal{K}$ is stabbed by $\mathcal{H}$ but not stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1$, then $R \in \mathcal{R}'\backslash\mathcal{X}$, which implies that R is stabbed by $\mathcal{H}_0$. Next we prove condition (i), for which we need $\mathsf{opt}(\mathcal{R}'\backslash\mathcal{X}, \mathcal{H}) = O(k^2)$. First, observe that $\mathsf{opt}((\mathcal{R}'\backslash\mathcal{X}) \Cap \ell, \mathcal{H}) \leq 2k+1$ for all $\ell \in \bigcup_{P \in \mathit{\Gamma}_{\mathsf{v}}} \partial P$, for otherwise the while loop of the algorithm cannot terminate. Since $|\bigcup_{P \in \mathit{\Gamma}_{\mathsf{v}}} \partial P| \leq 2|\mathit{\Gamma}_{\mathsf{v}}| = O(k)$ by (i) of Lemma 4, it follows that all rectangles in $\mathcal{R}'\backslash\mathcal{X}$ stabbed by $\bigcup_{P \in \mathit{\Gamma}_{\mathsf{v}}} \partial P$ can be stabbed by $O(k^2)$ lines in $\mathcal{H}$. So it suffices to consider the rectangles in $\mathcal{R}'\backslash\mathcal{X}$ that are not stabbed by $\bigcup_{P \in \mathit{\Gamma}_{\mathsf{v}}} \partial P$. Note that if a rectangle $R \in \mathcal{R}'\backslash\mathcal{X}$ is not stabbed by ∂P, then we have $R \cap P = \emptyset$, because $P \in \mathit{\Gamma}_{\mathsf{v}} \subseteq \mathit{\Gamma}(\mathcal{V}_0)$ and all rectangles in $\mathcal{R}'$ are stabbed by $\mathcal{V}_0$ according to (iii) of Lemma 3. Therefore, the rectangles in $\mathcal{R}'\backslash\mathcal{X}$ not stabbed by $\bigcup_{P \in \mathit{\Gamma}_{\mathsf{v}}} \partial P$ are just those disjoint from all strips in $\mathit{\Gamma}_{\mathsf{v}}$. By (iv) of Lemma 4, all such rectangles can be stabbed by $\mathcal{H}^*$. As $|\mathcal{H}^*| \leq k$, we have condition (i) in the lemma.

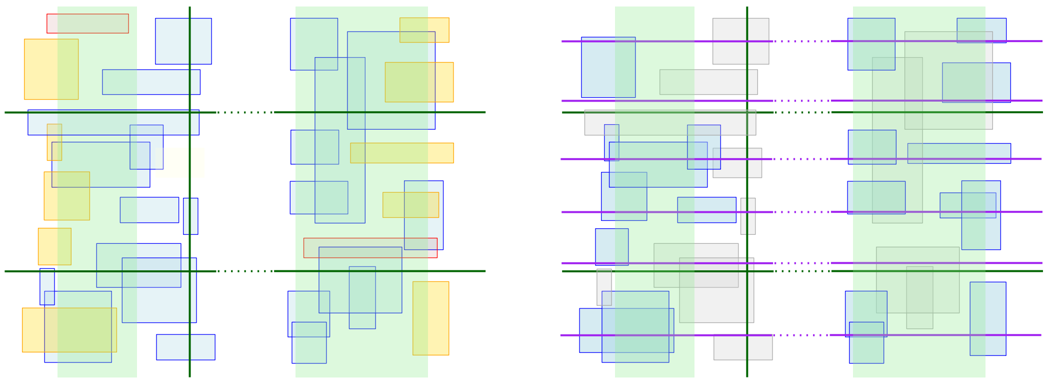

Figure 3 *The subset of rectangles $\mathcal{K}$ and horizontal lines $\mathcal{H}_0$ computed in Lemma 5.* On the left, for each vertical green strip of Γ_v, we remove the widest rectangle (red) that is part of a large set of horizontally disjoint rectangles (yellow) touching the strip boundary, since it must be stabbed vertically. The surviving rectangles make up $\mathcal{K}$. On the right, the set of purple lines $\mathcal{H}_0$ stab all remaining unstabbed rectangles.

Finally, we verify condition (ii). We focus on the "if" direction; the "only if" direction is trivial because $\mathcal{K} \subseteq \mathcal{R}$. Let $\mathcal{A} \subseteq \mathcal{H}$ with $|\mathcal{A}| \leq 2k$ and $\mathcal{B} \subseteq \mathcal{V}$ be $\mathit{\Gamma}_{\mathsf{v}}$-structured. Since $\mathcal{R} \setminus \mathcal{K} = \mathcal{X} \subseteq \mathcal{R}'$ and no rectangle of $\mathcal{R}'$ is stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1$, it suffices to show that $\mathcal{R}'$ is stabbed by $\mathcal{A} \cup \mathcal{B}$ if $\mathcal{R}' \backslash \mathcal{X}$ is stabbed by $\mathcal{A} \cup \mathcal{B}$. Assume $\mathcal{R}' \backslash \mathcal{X}$ is stabbed by $\mathcal{A} \cup \mathcal{B}$. Write $\mathcal{X} = \{R_1^*, \dots, R_r^*\}$ such that the rectangles are added to $\mathcal{X}$ in the order $R_r^*, \dots, R_1^*$ in Algorithm 2. We show by induction that $\mathcal{R}' \setminus \{R_{i+1}^*, \dots, R_r^*\}$ is stabbed by $\mathcal{A} \cup \mathcal{B}$ for all $i \in [r]_0$, which implies that $\mathcal{R}'$ is stabbed by $\mathcal{A} \cup \mathcal{B}$. The base case $i = 0$ clearly holds. Suppose $\mathcal{R}' \backslash \{R_i^*, \dots, R_r^*\}$ is stabbed by $\mathcal{A} \cup \mathcal{B}$. For induction, we need to show that R_i^* is also stabbed by $\mathcal{A} \cup \mathcal{B}$. By our assumption, at the point R_i^* is added to $\mathcal{X}$, we have $\mathcal{X} = \{R_{i+1}^*, \dots, R_r^*\}$. The reason for why we add R_i^* to $\mathcal{X}$ is that $\mathsf{opt}((\mathcal{R}' \backslash \{R_{i+1}^*, \dots, R_r^*\}) \Cap \ell, \mathcal{H}) \geq 2k + 2$ and $R_i^* = \arg\max_{R \in (\mathcal{R}' \backslash \{R_{i+1}^*, \dots, R_r^*\}) \Cap \ell} \mathsf{wid}_P(R)$ for some $P \in \mathit{\Gamma}_\mathsf{v}$ and $\ell \in \partial P$. Note that $\mathsf{opt}((\mathcal{R}' \backslash \{R_{i+1}^*, \dots, R_r^*\}) \Cap \ell, \mathcal{H}) < \infty$, because $\mathcal{R}'$ is stabbed by $\mathcal{H}$. Therefore, $\mathsf{opt}((\mathcal{R}' \backslash \{R_i^*, \dots, R_r^*\}) \Cap \ell, \mathcal{H}) \geq 2k + 1$. As $|\mathcal{A}| \leq 2k$ and $\mathcal{A} \subseteq \mathcal{H}$, we know that $(\mathcal{R}' \backslash \{R_i^*, \dots, R_r^*\}) \Cap \ell$ is not stabbed by $\mathcal{A}$. But $(\mathcal{R}' \backslash \{R_i^*, \dots, R_r^*\}) \Cap \ell$ is stabbed by $\mathcal{A} \cup \mathcal{B}$ thanks to our induction hypothesis, which implies the existence of a rectangle $R \in (\mathcal{R}' \backslash \{R_i^*, \dots, R_r^*\}) \Cap \ell$ that is stabbed by a line $v \in \mathcal{B}$. Note that v must be contained in some strip in $\mathit{\Gamma}_\mathsf{v}$, for $\mathcal{B}$ is $\mathit{\Gamma}_\mathsf{v}$-structured. We claim that $v \subseteq P$. Since $R \in \mathcal{R}'$, R is not stabbed by $\mathcal{V}_1$. Furthermore, because $R \cap P \neq \emptyset$ and $\mathit{\Gamma}_\mathsf{v}$ is separated by $\mathcal{V}_1$ according to (ii) of Lemma 4, we have $R \cap P' = \emptyset$ for all $P' \in \mathit{\Gamma}_\mathsf{v} \backslash \{P\}$. Therefore, if $v \subseteq P'$ for some $P' \in \mathit{\Gamma}_\mathsf{v} \backslash \{P\}$, R cannot be stabbed by v. This implies $v \subseteq P$. Using this fact, we show that v stabs R_i^*. As $R \in (\mathcal{R}' \backslash \{R_i^*, \dots, R_r^*\}) \Cap \ell$, R intersects ℓ and hence one side of $R \cap P$ is bounded by ℓ. For the same reason, one side of $R_i^* \cap P$ is also bounded by ℓ. By construction, $\mathsf{wid}_P(R_i^*) \geq \mathsf{wid}_P(R)$. Thus, if a vertical line contained in P stabs R, it also stabs R_i^*. In particular, v stabs R_i^*. It follows that $\mathcal{R}' \backslash \{R_{i+1}^*, \dots, R_r^*\}$ is stabbed by $\mathcal{A} \cup \mathcal{B}$. This completes the induction argument and implies condition (i). ◀

3.4 Finding good horizontal strips

Based on the discussion in the previous sections, we already have the sets $\mathcal{K}$, $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{V}_1$, and $\mathit{\Gamma}_\mathsf{v}$. The goal of the next step is somewhat similar to that in Subsection 3.2. This time we want to find a set $\mathit{\Gamma}_\mathsf{h}$ of *horizontal* strips each of which contains one line in $\mathcal{H}^*$, together with a set of horizontal lines in $\mathcal{H}_0$ separating $\mathit{\Gamma}_\mathsf{h}$. But the other desired properties for $\mathit{\Gamma}_\mathsf{h}$ are slightly different from those for $\mathit{\Gamma}_\mathsf{v}$ in Lemma 4, and thus we need a different proof as well. Recall that for each $P \in \mathit{\Gamma}_\mathsf{v}$, v_P^* is the (unique) vertical line in $\mathcal{V}^*$ with $v_P^* \subseteq P$. We prove the following Lemma.

▶ **Lemma 6.** *There exist $\mathit{\Gamma}_\mathsf{h} \subseteq \mathit{\Gamma}(\mathcal{H}_1 \cup \mathcal{H}_0)$ and $\mathcal{H}_1' \subseteq \mathcal{H}_0$ satisfying the following.*

(i) $|\mathcal{H}_1| + |\mathit{\Gamma}_\mathsf{h}| + |\mathcal{H}_1'| \leq 2k_\mathsf{h}$

(ii) *$\mathit{\Gamma}_\mathsf{h}$ is separated by $\mathcal{H}_1 \cup \mathcal{H}_1'$.*

(iii) *For each $Q \in \mathit{\Gamma}_\mathsf{h}$, there exists exactly one line $h_Q^* \in \mathcal{H}^*$ with $h_Q^* \subseteq Q$.*

(iv) *$\mathcal{K}$ is stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}_1' \cup \{h_Q^* : Q \in \mathit{\Gamma}_\mathsf{h}\} \cup \{v_P^* : P \in \mathit{\Gamma}_\mathsf{v}\}$.*

Proof. As in the proof of Lemma 4, for each strip $Q \in \mathit{\Gamma}(\mathcal{H}_0 \cup \mathcal{H}_1)$, let $\partial Q \subseteq \mathcal{H}_0 \cup \mathcal{H}_1$ consist of the (at most) two lines adjacent to Q. Also, we say a strip $Q \in \mathit{\Gamma}(\mathcal{H}_0 \cup \mathcal{H}_1)$ is *light* if Q contains exactly one line of $\mathcal{H}^*$ and *heavy* if Q contains at least two lines of $\mathcal{H}^*$. Let $\mathit{\Gamma}' \subseteq \mathit{\Gamma}(\mathcal{H}_0 \cup \mathcal{H}_1)$ (resp., $\mathit{\Gamma}'' \subseteq \mathit{\Gamma}(\mathcal{H}_0 \cup \mathcal{H}_1)$) consist of the light (resp., heavy) strips. We simply define $\mathit{\Gamma}_\mathsf{h} = \mathit{\Gamma}'$. The set $\mathcal{H}_1' \subseteq \mathcal{H}_0$ is defined as follows. First, we include in $\mathcal{H}_1'$ all lines in $(\mathcal{H}^* \cap \mathcal{H}_0) \cup (\bigcup_{Q \in \mathit{\Gamma}''} \partial Q)$. Suppose $\mathit{\Gamma}' = \{Q_1, \dots, Q_t\}$, where $Q_1, \dots, Q_t$ are sorted from bottom to top. For every $i \in [t-1]$ such that Q_i and Q_{i+1} are not separated by $\mathcal{H}_1 \cup (\mathcal{H}^* \cap \mathcal{H}_0) \cup (\bigcup_{Q \in \mathit{\Gamma}''} \partial Q)$, we add to $\mathcal{H}_1'$ an arbitrary line $h \in \mathcal{H}_0$ that separates Q_i and Q_{i+1}; such a line exists since $Q_i, Q_{i+1} \in \mathit{\Gamma}(\mathcal{H}_0 \cup \mathcal{H}_1)$ and $\mathcal{H}_1$ does not separate Q_i, Q_{i+1}.

We now verify the four conditions in the lemma. Conditions (ii) and (iii) directly follow from our construction. To see (iv), consider a rectangle $R \in \mathcal{K}$ that is not stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1$. If R is not stabbed by $\mathcal{H}^*$, then (iv) of Lemma 4 implies that R is stabbed by $\{v_P^* : P \in \mathit{\Gamma}_\mathsf{v}\}$.

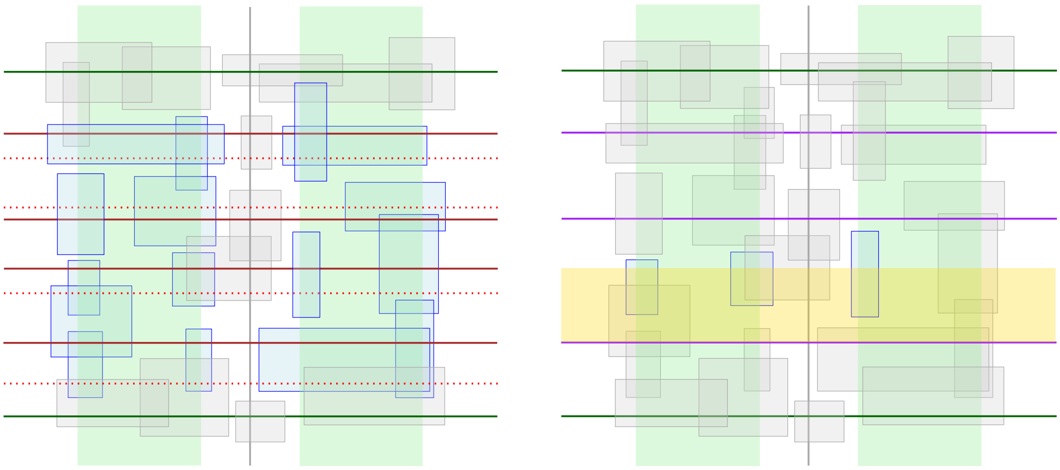

Figure 4 *The horizontal lines $\mathcal{H}'_1$ and strips Γ_h guaranteed by Lemma 6.* On the left, the horizontal dashed lines represent the unknown lines of OPT, $\mathcal{H}^*$, while the solid brown lines are those of $\mathcal{H}_0$ and the solid green lines are those of $\mathcal{H}_1$. On the right, the strips of Γ_h are highlighted yellow. The horizontal lines of $\mathcal{H}'_1$ in purple include the boundaries of all heavy strips and arbitrary lines between two consecutive light strips. They separate the strips of Γ_h. All the rectangles are now stabbed by either a strip or a line.

So assume R is stabbed by $\mathcal{H}^*$, and thus stabbed by $\mathcal{H}$ as well. Then (iii) of Lemma 5 implies that R is also stabbed by $\mathcal{H}_0$. Let $h \in \mathcal{H}^*$ be a line that stabs R. If $h \in \mathcal{H}_0$, then $h \in \mathcal{H}'_1$ and hence R is stabbed by $\mathcal{H}'_1$. Otherwise, since $h \notin \mathcal{H}_1$ because $\mathcal{H}_1$ does not stab R, $h \subseteq Q$ for some $Q \in \Gamma(\mathcal{H}_0 \cup \mathcal{H}_1)$. As $\mathcal{H}_0$ stabs R, we know that R is stabbed by ∂Q. Thus, if $Q \in \Gamma''$, then $\partial Q \subseteq \mathcal{H}'_1$ and R is stabbed by $\mathcal{H}'_1$. If $Q \notin \Gamma''$, we must have $Q \in \Gamma'$ since $h \subseteq Q$. Condition (iii) then implies $h = h^*_Q$ and hence R is stabbed by $\{h^*_Q : Q \in \Gamma_{\mathsf{h}}\}$.

Finally, we verify condition (i). We notice the inequality $|\mathcal{H}^* \cap \mathcal{H}_0| + |\Gamma'| + 2|\Gamma''| \leq |\mathcal{H}^*| = k_{\mathsf{h}}$. Let $\mathcal{A} = \mathcal{H}'_1 \backslash ((\mathcal{H}^* \cap \mathcal{H}_0) \cup (\bigcup_{Q \in \Gamma''} \partial Q))$. We have

$$|\mathcal{H}_1| + |\Gamma_{\mathsf{h}}| + |\mathcal{H}'_1| \leq |\mathcal{H}_1| + |\Gamma'| + (|\mathcal{H}^* \cap \mathcal{H}_0| + 2|\Gamma''| + |\mathcal{A}|) \leq k_{\mathsf{h}} + |\mathcal{H}_1| + |\mathcal{A}|.$$

Thus, to see (i), it suffices to show that $|\mathcal{H}_1| + |\mathcal{A}| \leq k_{\mathsf{h}}$. To this end, we use a charging argument as follows. Suppose $\mathcal{H}_1 = \{h_1, \dots, h_r\}$ where $h_1, \dots, h_r$ are sorted from bottom to top. Let y_i be the y-coordinate of h_i for $i \in [r]$. For convenience, write $y_0 = -\infty$. For each $i \in [r]$, we charge h_i to the topmost line in $\mathcal{H}^*$ whose y-coordinate is at most y_i. By (i) of Lemma 3, there exists at least one line in $\mathcal{H}^*$ whose y-coordinate is in the range $(y_{i-1}, y_i]$, and thus h_i is charged to a line whose y-coordinate is in $(y_{i-1}, y_i]$. Therefore, the lines in $\mathcal{H}_1$ are charged to different lines in $\mathcal{H}^*$. Next, recall how the lines in $\mathcal{A}$ are chosen. Every line $h \in \mathcal{A}$ corresponds to an index $i \in [t-1]$ such that Q_i and Q_{i+1} are not separated by $\mathcal{H}_1 \cup (\mathcal{H}^* \cap \mathcal{H}_0) \cup (\bigcup_{Q \in \Gamma''} \partial Q)$; we then charge h to $h^*_{Q_i}$, which is the (unique) line in $\mathcal{H}^*$ that is contained in Q_i. Again, the lines in $\mathcal{A}$ are charged to different lines in $\mathcal{H}^*$. Now every line in $\mathcal{H}_1$ and $\mathcal{A}$ is charged to a line in $\mathcal{H}^*$. Since $|\mathcal{H}^*| = k_{\mathsf{h}}$, to see $|\mathcal{H}_1| + |\mathcal{A}| \leq k_{\mathsf{h}}$, we only need to show that every line in $\mathcal{H}^*$ gets charged at most once. As aforementioned, the lines in $\mathcal{H}_1$ (resp., $\mathcal{A}$) are charged to different lines in $\mathcal{H}^*$. Thus, it suffices to exclude the case where a line $h \in \mathcal{H}^*$ gets charged by both $\mathcal{H}_1$ and $\mathcal{A}$. If the y-coordinate of h is greater than y_r, then only the lines in $\mathcal{A}$ can be charged to h. So suppose the y-coordinate of h is in the range $(y_{i-1}, y_i]$ for $i \in [r]$. If h is not the topmost line in $\mathcal{H}^*$ with y-coordinate in $(y_{i-1}, y_i]$, then again only the lines in $\mathcal{A}$ can be charged to h. So suppose h is the topmost line in $\mathcal{H}^*$

with y-coordinate in $(y_{i-1}, y_i]$. We claim that no line in $\mathcal{A}$ is charged to h. Assume there is a line in $\mathcal{A}$ charged to h, which is chosen to separate the strips Q_j and Q_{j+1}. We then have $h = h^*_{Q_j}$. Furthermore, Q_j and Q_{j+1} are not separated by $\mathcal{H}_1 \cup (\mathcal{H}^* \cap \mathcal{H}_0) \cup (\bigcup_{Q \in \Gamma''} \partial Q)$, and in particular not separated by $\mathcal{H}_1$. It follows that the y-coordinate of $h^*_{Q_{j+1}}$ is also in $(y_{i-1}, y_i]$. Since $h^*_{Q_{j+1}} \in \mathcal{H}^*$ and $h^*_{Q_{j+1}}$ is above $h^*_{Q_j}$, this contradicts the assumption that h is the topmost line in $\mathcal{H}^*$ with y-coordinate in $(y_{i-1}, y_i]$. Therefore, no line in $\mathcal{A}$ is charged to h. We can finally conclude that $|\mathcal{H}_1| + |\mathcal{A}| \leq k_{\mathsf{h}}$, which implies condition (i). ◀

Again, we cannot compute the sets Γ_{h} and $\mathcal{H}'_1$ in the above lemma, but we can afford guessing them. We have $|\Gamma(\mathcal{H}_1 \cup \mathcal{H}_0)| = |\mathcal{H}_1 \cup \mathcal{H}_0| + 1 = O(k^2)$ by (i) of Lemma 3 and (i) of Lemma 5. Also, $|\Gamma_{\mathsf{h}}| + |\mathcal{H}'_1| = O(k)$ by (i) of Lemma 6. Therefore, the total number of guesses for Γ_{h} and $\mathcal{H}'_1$ is bounded by $k^{O(k)}$, and making the guess leads to another $k^{O(k)}$ overhead on the running time.

3.5 Solving the problem by reducing to 2-SAT

Note that (iv) of Lemma 6 implies that $\mathcal{K}$ can be stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1$ together with a Γ_{h}-structured subset of $\mathcal{H}$ and a Γ_{v}-structured subset of $\mathcal{V}$. Thus, the last step of our algorithm is just to compute a Γ_{h}-structured $\mathcal{H}_2 \subseteq \mathcal{H}$ and a Γ_{v}-structured $\mathcal{V}_2 \subseteq \mathcal{V}$ such that $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1 \cup \mathcal{H}_2 \cup \mathcal{V}_2$ stabs $\mathcal{K}$. The size of $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1 \cup \mathcal{H}_2 \cup \mathcal{V}_2$ is $|\mathcal{H}_1| + |\mathcal{V}_1| + |\mathcal{H}'_1| + |\Gamma_{\mathsf{h}}| + |\Gamma_{\mathsf{v}}|$, which is at most $2k_{\mathsf{h}} + \frac{3}{2}k_{\mathsf{v}}$ by (i) of Lemma 4 and (i) of Lemma 6.

Let $\mathcal{K}' \subseteq \mathcal{K}$ consist of the rectangles *not* stabbed by $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1$. For each strip $P \in \Gamma_{\mathsf{v}}$ (resp., $Q \in \Gamma_{\mathsf{h}}$), define $\mathcal{V}_P \subseteq \mathcal{V}$ (resp., $\mathcal{H}_Q \subseteq \mathcal{H}$) as the subset consisting of all lines that are contained in P (resp., Q). Then our task is to pick one line from $\mathcal{V}_P$ for each $P \in \Gamma_{\mathsf{v}}$ (which forms $\mathcal{V}_2$) and pick one line from $\mathcal{H}_Q$ for each $Q \in \Gamma_{\mathsf{h}}$ (which forms $\mathcal{H}_2$) such that $\mathcal{K}'$ is stabbed by the picked lines. We show that this task can be reduced to solving a 2-SAT instance, which can be done in polynomial time. The key observation is the following.

▶ **Lemma 7.** *Each rectangle in $\mathcal{K}'$ intersects at most one strip in Γ_{h} and at most one strip in Γ_{v}.*

Proof. Let $R \in \mathcal{K}'$. Assume R intersects two (different) strips $P, P' \in \Gamma_{\mathsf{v}}$. By (ii) of Lemma 4, $\mathcal{V}_1$ separates Γ_{v} and in particular separates P and P'. Thus, $\mathcal{V}_1$ stabs R, contradicting the fact that $R \in \mathcal{K}'$. Similarly, assume R intersects two (different) strips $Q, Q' \in \Gamma_{\mathsf{h}}$. By (ii) of Lemma 6, $\mathcal{H}_1 \cup \mathcal{H}'_1$ separates Γ_{h} and in particular separates Q and Q'. Thus, $\mathcal{H}_1 \cup \mathcal{H}'_1$ stabs R, contradicting the fact that $R \in \mathcal{K}'$. ◀

Using Lemma 7, it is not hard to reduce the remaining task to a 2-SAT instance, which is polynomial time solvable. We get the following Lemma; the proof is included in the appendix.

▶ **Lemma 8.** *One can compute in $(|\mathcal{R}| + |\mathcal{L}|)^{O(1)}$ time a Γ_{h}-structured $\mathcal{H}_2 \subseteq \mathcal{H}$ and a Γ_{v}-structured $\mathcal{V}_2 \subseteq \mathcal{V}$ such that $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1 \cup \mathcal{H}_2 \cup \mathcal{V}_2$ stabs $\mathcal{K}$.*

Now we are ready to prove Theorem 1. We already computed $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1 \cup \mathcal{H}_2 \cup \mathcal{V}_2$ that stabs $\mathcal{K}$, and the time cost is $k^{O(k)} n^{O(1)}$ where $n = |\mathcal{R}| + |\mathcal{L}|$. Indeed, the guess of Γ_{v}, Γ_{h}, $\mathcal{V}_1$, $\mathcal{H}'_1$ results in the $k^{O(k)}$ factor, and all the other steps can be done in polynomial time. Note that

$$\begin{aligned}
|\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1 \cup \mathcal{H}_2 \cup \mathcal{V}_2| &= |\mathcal{H}_1| + |\mathcal{V}_1| + |\mathcal{H}'_1| + |\Gamma_{\mathsf{v}}| + |\Gamma_{\mathsf{h}}| \\
&\leq (|\mathcal{H}_1| + |\mathcal{H}'_1| + |\Gamma_{\mathsf{h}}|) + (|\mathcal{V}_1| + |\Gamma_{\mathsf{v}}|) \\
&\leq 2k_{\mathsf{h}} + \frac{3}{2}k_{\mathsf{v}} \leq \frac{7}{4}k,
\end{aligned}$$

by (i) of Lemma 4 and (i) of Lemma 6. Finally, it suffices to show that $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1 \cup \mathcal{H}_2 \cup \mathcal{V}_2$ stabs $\mathcal{R}$. Write $\mathcal{A} = \mathcal{H}'_1 \cup \mathcal{H}_2$ and $\mathcal{B} = \mathcal{V}_2$. We have $|\mathcal{A}| \leq 2k$ and $\mathcal{B}$ is Γ_{v}-structured. As $\mathcal{H}_1 \cup \mathcal{V}_1 \cup \mathcal{H}'_1 \cup \mathcal{H}_2 \cup \mathcal{V}_2$ stabs $\mathcal{K}$, (ii) of Lemma 5 implies that it also stabs $\mathcal{R}$. This completes the proof of Theorem 1.

4 Hardness of Parameterized Approximation

In this section, we prove Theorem 2 by a gap preserving reduction from the Multicolored Clique problem, which is defined as follows.

> Multicolored Clique **Parameter:** k
> **Input:** An undirected graph G; A partition $\mathcal{P} = V_1, V_2, V_3, \ldots, V_k$ for $V(G)$ where $|V_1| = |V_2| = \cdots = |V_k| = r$.
> **Goal:** Find the maximum size clique C such that $\forall i \in [k], |C \cap V_i| \leq 1$.

A clique C is said to be *multicolored*, if for all $i \in [k]$ we have $|C \cap V_i| \leq 1$. We will rely on the following result of Chen et al. [4] (see also [24, 21, 22]).

▶ **Proposition 9.** *Assuming FPT $\neq$ W[1], for every function $h : \mathbb{N} \to \mathbb{N}$ such that $h(k) = k^{o(1)}$ there is no algorithm that takes as input a graph G and integer k, runs in time $f(k)|V(G)|^{O(1)}$ and either concludes that G has no clique of size at least k or produces a clique of size at least $k/h(k)$.*

Proposition 9 asserts hardness of the Clique problem, while we need hardness of the Multicolored Clique problem. Theorem 13.7 of [5] gives a parameterized reduction from Clique to Multicolored Clique. The proof of this theorem is an algorithm that given G, k constructs a graph G' with partition $V_1, \ldots, V_k$ of $V(G')$, such that *(i)* if G has clique of size k then G' has a multicolored clique of size k, and *(ii)* given a multicolored clique C' in G' with $|C'| = k$ one can construct in polynomial time a clique C in G with $|C| = |C'|$. The very short proof of *(ii)* never uses the assumption that $|C'| = k$, so it in fact shows that *(ii)'* given a multicolored clique C' in G' one can construct in polynomial time a clique C in G with $|C| = |C'|$. The construction of G' from G in Theorem 13.7 of [5] together with *(i)* and *(ii)'* imply the following.

▶ **Theorem 10.** *Assuming FPT $\neq$ W[1], for every function $h : \mathbb{N} \to \mathbb{N}$ such that $h(k) = k^{o(1)}$ there is no algorithm that takes as input a graph G, integer k and partition $V_1, V_2, \ldots V_k$ of $V(G)$, runs in time $f(k)|V(G)|^{O(1)}$ and either concludes that G has no multicolored clique of size at least k or produces a multicolored clique of size at least $k/h(k)$.*

4.1 Reduction from Multicolored Clique to Rectangle Stabbing

We now present a reduction that given a graph G, integers k and r and a partition of $V(G)$ into $V_1, V_2, \ldots, V_k$ such that $|V_i| = r$ for every $i \in [k]$ produces an instance of Rectangle Stabbing in polynomial time. Without loss of generality, for every $i \in [k]$ we have that $V_i = \{v_1^i, v_2^i, \ldots, v_r^i\}$. We will construct an instance $(\mathcal{R}, \mathcal{L})$ of Rectangle Stabbing with parameter $4k$. $\mathcal{L} = \{\ell_h^y \;:\; y \in \mathbb{Z}\} \cup \{\ell_v^x \;:\; x \in \mathbb{Z}\}$. In other words, $\mathcal{L}$ consists of all lines at integer positions. The set $\mathcal{R}$ of rectangles consists of three parts $\mathcal{R}_F, \mathcal{R}_A, \mathcal{R}_E$, that is $\mathcal{R} = \mathcal{R}_F \cup \mathcal{R}_A \cup \mathcal{R}_E$.

Before constructing each of the sets $\mathcal{R}_F, \mathcal{R}_A, \mathcal{R}_E$ we define a set of $4k$ strips, and briefly discuss the intuition behind the reduction. It is worth pointing out that in contrast with the strips used in the algorithm section, here the strips are *closed* in the sense that they include their bounding lines. For every $x \in [0, 2k-1]_{\mathbb{Z}}$ we define the vertical closed strip $\zeta_v^x = [2r+xr+1, 2r+xr+r] \times \mathbb{R}$ and the horizontal closed strip $\zeta_h^x = \mathbb{R} \times [2r+xr+1, 2r+xr+r]$. Since these are the only (closed or open) strips discussed in this reduction, we will simply refer to the closed strips above as strips.

The intention of the construction is that a solution of size $4k$ selects precisely one line from each of the above strips. This will be ensured by the rectangles in $\mathcal{R}_F$. For each $i \in [k]$, each of the solution lines in the strips ζ_v^{2i-2}, ζ_v^{2i-1}, ζ_h^{2i-2} and ζ_h^{2i-1} encodes the choice of a vertex in V_i in the multicolored clique. For each $i \in [k]$ we will add rectangles in the squares $\zeta_v^{2i-2} \cap \zeta_h^{2i-2}$, $\zeta_v^{2i-2} \cap \zeta_h^{2i-1}$, $\zeta_v^{2i-1} \cap \zeta_h^{2i-2}$ and $\zeta_v^{2i-1} \cap \zeta_h^{2i-1}$ that ensure that the four solution lines in the strips ζ_v^{2i-2}, ζ_v^{2i-1}, ζ_h^{2i-2} and ζ_h^{2i-1} all encode the *same* vertex in V_i. The set of these rectangles is $\mathcal{R}_E$. Finally, for every ordered pair $i, j \in [k]$ with $i \neq j$ we add rectangles in the square[1] $(\zeta_v^{2i-2} \cup \zeta_v^{2i-1}) \cap (\zeta_h^{2j-2} \cup \zeta_h^{2j-1})$ that ensure that, as long as the solution lines in (ζ_v^{2i-2} and ζ_v^{2i-1}) select the same vertex in V_i, and the solution lines in (ζ_h^{2j-2} and ζ_h^{2j-1}) select the same vertex in V_j, then the vertex selected in V_i and the vertex selected in V_j are adjacent in G. This set of rectangles is $\mathcal{R}_A$.

We leave the precise detailed construction of the instance to the appendix. A visualization of the construction is shown in Figure 5. The proof of correctness is included in the appendix.

We make two remarks about the construction in Theorem 2. First, note that the set $\mathcal{L}$ of lines produced by the construction in Section 4.1 is simply the set of all lines with integer coordinates. For a finite set of explicitly given lines one could restrict $\mathcal{L}$ to all vertical lines with integer position in the vertical strips $\{\zeta_v^x : x \in [0, 2k-1]_{\mathbb{Z}}\}$ and all horizontal lines with integer position in the horizontal strips $\{\zeta_h^x : x \in [0, 2k-1]_{\mathbb{Z}}\}$.

Second, the construction in Section 4.1 produces some degenerate rectangles (that is, rectangles of the form $[a, b] \times [c, d]$ where $a \leq b$, $c \leq d$ and either $a = b$ or $c = d$ or both equalities hold). This is simply for ease of notation, and degenerate rectangles are not necessary for the reduction to work. Indeed it is easy to see that replacing every rectangle $[a, b] \times [c, d]$ in $\mathcal{R}$ with the rectangle $[2a, 2b+1] \times [2c, 2d+1]$ to produce the set $\mathcal{R}'$, and keeping $\mathcal{L}$ as the set of all lines with integer positions gives an equivalent instance in the following sense. A subset $\mathcal{H}' \cup \mathcal{V}'$ of lines stabs $\mathcal{R}'$ if and only if the set $\{l_h^{\lfloor y/2 \rfloor} : l_h^y \in \mathcal{H}'\} \cup \{l_v^{\lfloor x/2 \rfloor} : l_v^x \in \mathcal{V}'\}$ stabs $\mathcal{R}$. Thus, algorithm $\mathcal{A}$ in the proof of Theorem 2 can be run on $\mathcal{R}'$ instead of $\mathcal{R}$, and the statement of Theorem 2 holds even for algorithms that require non-degenerate input rectangles.

5 Conclusion and future work

We have shown that Rectangle Stabbing admits a $\frac{7}{4}$-approximation algorithm with a runtime of $k^{O(k)} \cdot n^{O(1)}$, where $n = |\mathcal{R}| + |\mathcal{L}|$. The approximation ratio of the algorithm improves upon the best-known for polynomial-time algorithms, and the running time is faster than what can be achieved for exact parameterized algorithms, assuming FPT $\neq$ W[1]. Additionally, we showed that, assuming FPT $\neq$ W[1], no algorithm running in $f(k) \cdot n^{O(1)}$ time, for any computable function f, can achieve an approximation ratio better than $\frac{5}{4} - \varepsilon$, for any $\varepsilon > 0$.

In light of our work, a natural open problem is to find the tractability boundary of parameterized approximation for Rectangle Stabbing, in particular find the constant c such that there exists a parameterized c-approximation, while a parameterized c'-approximation

[1] Technically, this set is not a square, but a square with a cross cut out of it. This is an imprecision for the sake of simplifying notation in the intuitive explanation of the reduction.

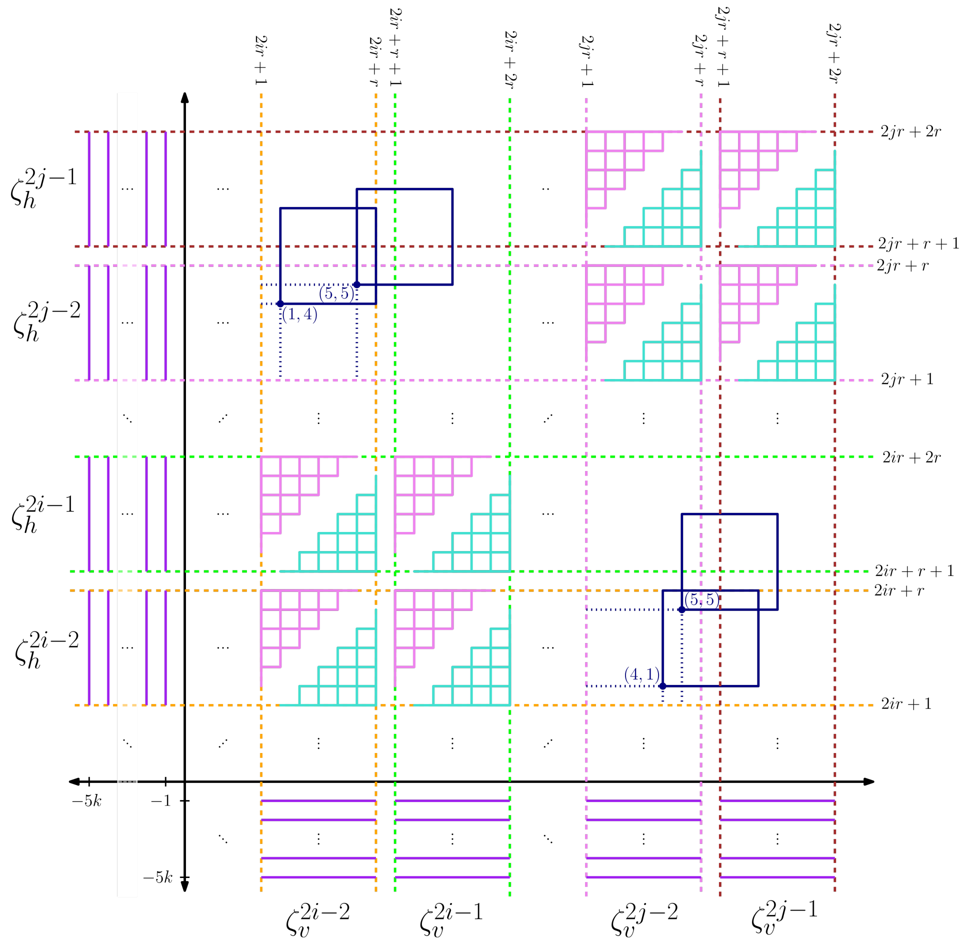

Figure 5 *Example snippet of the reduction construction from* Multicolored Clique *to* Rectangle Stabbing. The dark purple rectangles (line segments) with one negative coordinate are in $\mathcal{R}_F$. The dark blue rectangles intersecting $\zeta_v^{2i-2} \cap \zeta_h^{2j-2}$ and $\zeta_v^{2j-2} \cap \zeta_h^{2i-2}$ are in $\mathcal{R}_A$. The pink and cyan rectangles arranged like staircases are in $\mathcal{R}_E$.

Notice **in this example**, $r = 7$ and the bottom left endpoints of the rectangles intersecting $\zeta_v^{2i-2} \cap \zeta_h^{2j-2}$ have an offset of $(1,4)$ and $(5,5)$ from $(2ir+1, 2jr+1)$. These represent the non-edges (v_1^i, v_4^j), $(v_5^i, v_5^j) \notin E(G)$ for this example graph G.

for every $c' < c$ would imply FPT = W[1] or violate the (Gap) Exponential Time Hypothesis. Another interesting direction is to determine whether the approximation ratio of 2 is optimal when restricted to polynomial-time algorithms.

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A Appendix

A.1 Missing details of the 2-SAT construction for proof of Lemma 8

The variables of the 2-SAT instance are defined as follows. For each $P \in \mathit{\Gamma}_\mathsf{v}$ and each $v \in \mathcal{V}_P$, we define a variable $x_{P,\geq v}$, which is used to indicate whether the line we select from $\mathcal{V}_P$ is to the right of v (including v itself). Similarly, for each $Q \in \mathit{\Gamma}_\mathsf{h}$ and each $h \in \mathcal{H}_Q$, we define a variable $y_{Q,\geq h}$, which is used to indicate whether the line we select from $\mathcal{H}_Q$ is above h (including h itself). We need two types of clauses. The first type of clauses ensure that assignment to the variables is consistent in the sense that for each $P \in \mathit{\Gamma}_\mathsf{v}$ (resp., $Q \in \mathit{\Gamma}_\mathsf{h}$), the variables $x_{P,\geq v}$ (resp., $y_{Q,\geq h}$) correctly encodes the choice of one line in $\mathcal{V}_P$ (resp., $\mathcal{H}_Q$). Specifically, for each $P \in \mathit{\Gamma}_\mathsf{v}$ and two lines $v, v' \in \mathcal{V}_P$ such that v' is to the right of v, we introduce a 2-literal clause $x_{P,\geq v'} \rightarrow x_{P,\geq v}$, i.e., $(\neg x_{P,\geq v'}) \vee x_{P,\geq v}$. Also, for each $Q \in \mathit{\Gamma}_\mathsf{h}$ and two lines $h, h' \in \mathcal{H}_Q$ such that h' is above h, we introduce a 2-literal clause $y_{Q,\geq h'} \rightarrow y_{Q,\geq h}$, i.e., $(\neg y_{Q,\geq h'}) \vee y_{Q,\geq h}$. If an assignment to the variables satisfies these clauses, then the lines chosen from $\mathcal{V}_P$ (resp., $\mathcal{H}_Q$) is just the rightmost $v \in \mathcal{V}_P$ (resp., topmost $h \in \mathcal{H}_Q$) such that the variable $x_{P,\geq v}$ (resp., $y_{Q,\geq h}$) is true.

The second type of clauses ensure that each rectangle in $\mathcal{K}'$ is stabbed by the lines chosen by the variables. Consider a rectangle $R \in \mathcal{K}'$. We shall introduce $O(1)$ clauses for R such that the lines chosen stab R if and only if the clauses are satisfied. We know that R intersects at least one strip in $\mathit{\Gamma}_\mathsf{h} \cup \mathit{\Gamma}_\mathsf{v}$, because the rectangles in $\mathcal{K}'$ can be stabbed by a $\mathit{\Gamma}_\mathsf{h}$-structured subset of $\mathcal{H}$ together with a $\mathit{\Gamma}_\mathsf{v}$-structured subset of $\mathcal{V}$. We consider three cases: (i) R is disjoint from the strips in $\mathit{\Gamma}_\mathsf{h}$, (ii) R is disjoint from the strips in $\mathit{\Gamma}_\mathsf{v}$, and (iii) R intersects both strips in $\mathit{\Gamma}_\mathsf{h}$ and strips in $\mathit{\Gamma}_\mathsf{v}$. The first two cases are symmetric, and thus it suffices to consider case (i). Suppose R is disjoint from the strips in $\mathit{\Gamma}_\mathsf{h}$. Then by Lemma 7, there exists a unique $P \in \mathit{\Gamma}_\mathsf{v}$ with $R \cap P \neq \emptyset$, and R can only be stabbed by the vertical line chosen from $\mathcal{V}_P$. Let $v \in \mathcal{V}_P$ be the leftmost line that stabs R, and $v' \in \mathcal{V}_P$ be the leftmost line to the right of v that does not stab R. We then introduce two 1-literal clauses $x_{P,\geq v}$ and $\neg x_{P,\geq v'}$; clearly, R is stabbed by the line chosen from $\mathcal{V}_P$ if and only if both of $x_{P,\geq v}$ and $\neg x_{P,\geq v'}$ are true. (If the line v' does not exist, then we only need one clause $x_{P,\geq v}$.) Next, we consider case (iii). In this case, by Lemma 7, there exists a unique $P \in \mathit{\Gamma}_\mathsf{v}$ with $R \cap P \neq \emptyset$ and a unique $Q \in \mathit{\Gamma}_\mathsf{h}$ with $R \cap Q \neq \emptyset$. The rectangle R can be stabbed by the line chosen from $\mathcal{V}_P$ or the line chosen from $\mathcal{H}_Q$. Let $v \in \mathcal{V}_P$ (resp., $h \in \mathcal{H}_Q$) be the leftmost (resp., bottommost) line that stabs R, and $v' \in \mathcal{V}_P$ (resp., $h' \in \mathcal{H}_Q$) be the leftmost (resp., bottommost) line to the right of v (resp., above h) that does not stab R. Then R is stabbed if and only if $(x_{P,\geq v} \wedge (\neg x_{P,\geq v'})) \vee (y_{Q,\geq h} \wedge (\neg y_{Q,\geq h'}))$, which is equivalent to the conjunction of the four 2-literal clauses $x_{P,\geq v} \vee y_{Q,\geq h}$, $x_{P,\geq v} \vee (\neg y_{Q,\geq h'})$, $(\neg x_{P,\geq v'}) \vee y_{Q,\geq h}$, and $(\neg x_{P,\geq v'}) \vee (\neg y_{Q,\geq h'})$.

Based on the above discussion, we can construct a 2-CNF ϕ of polynomial length solving which gives us a Γ_{h}-structured subset of $\mathcal{H}$ and a Γ_{v}-structured subset of $\mathcal{V}$ which together stab $\mathcal{K}'$.

A.2 Precise Construction of the Hardness Reduction

We first describe $\mathcal{R}_F$, the set of rectangles that **f**orce at at least one line in each of the $4k$ strips.

For every $x \in [0, 2k-1]_{\mathbb{Z}}$ and every $q \in [-5k, -1]_{\mathbb{Z}}$ we define the rectangles

$$F_v^{x,q} = [2r + xr + 1, 2r + xr + r] \times \{q\}$$

and

$$F_h^{x,q} = \{q\} \times [2r + xr + 1, 2r + xr + r].$$

We set

$$\mathcal{R}_F = \{F_v^{x,q}, F_h^{x,q} \ : \ x \in [0, 2k-1]_{\mathbb{Z}}, \ q \in [-5k, -1]_{\mathbb{Z}}\}.$$

Next we describe $\mathcal{R}_A$, the rectangles encoding the **a**djacency matrix of G. For every $i, j \in [k]$ such that $i \neq j$, for every $p, q \in [r]$ such that $(v_p^i, v_q^j) \notin E(G)$, we define the rectangle

$$A_{p,q}^{i,j} = [2ir + p + 1, 2ir + r + p - 1] \times [2jr + q + 1, 2jr + r + q - 1].$$

Then we set

$$\mathcal{R}_A = \{A_{p,q}^{i,j} \ : \ (v_p^i, v_q^j) \notin E \text{ and } i \neq j\}.$$

Note that for every non-edge (v_p^i, v_q^j) of G with $i \neq j$, both $A_{p,q}^{i,j}$ and $A_{q,p}^{j,i}$ are rectangles in $\mathcal{R}_A$.

Finally we describe $\mathcal{R}_E$, the set of rectangles that ensures **e**quality of the vertices selected by the solution lines in the strips ζ_v^{2i-2}, ζ_v^{2i-1}, ζ_h^{2i-2} and ζ_h^{2i-1}. For every pair $x, y \in [0, 2k-1]_{\mathbb{Z}}$ such that $\lfloor x/2 \rfloor = \lfloor y/2 \rfloor$ and every integer $a \in [2, r]_{\mathbb{Z}}$ we define the "top left" rectangle

$$TL_a^{x,y} = [2r + xr + 1, 2r + xr + a - 1] \times [2r + yr + a, 2r + yr + r]$$

and the "bottom right" rectangle

$$BR_a^{x,y} = [2r + xr + a, 2r + xr + r] \times [2r + yr + 1, 2r + yr + a - 1].$$

We set

$$\mathcal{R}_E = \{TL_a^{x,y}, BR_a^{x,y} \ : \ x, y \in [0, 2k-1]_{\mathbb{Z}} \text{ s.t. } \lfloor x/2 \rfloor = \lfloor y/2 \rfloor, \ a \in [2, r]_{\mathbb{Z}}\}.$$

This concludes the construction of the instance of Rectangle Stabbing. A visualization of the construction is shown in Figure 5.

A.3 Proof of Correctness for the Hardness Reduction

▶ **Lemma 11.** *If there exists a multicolored clique of size k in G, then there exists a set of horizontal lines $\mathcal{H}$ and a set of vertical lines $\mathcal{V}$ such that $\mathcal{H} \cup \mathcal{V} \subseteq \mathcal{L}$, $|\mathcal{H} \cup \mathcal{V}| = 4k$ and $\mathcal{H} \cup \mathcal{V}$ stabs $\mathcal{R}$.*

Proof. Suppose that C is a multicolored clique of size k in G. For each $i \in [k]$ let r_i be such that $\{v^i_{r_i}\} = C \cap V_i$. For every $i \in [k]_\mathbb{Z}$ let $\hat{h}^{2i-2}$ be the line $\hat{h}^{2i-2} = \ell_h^{2r+(2i-2)r+r_i}$, $\hat{h}^{2i-1}$ be the line $\hat{h}^{2i-1} = \ell_h^{2r+(2i-1)r+r_i}$, $\hat{v}^{2i-2}$ be the line $\hat{v}^{2i-2} = \ell_v^{2r+(2i-2)r+r_i}$, and $\hat{v}^{2i-1}$ be the line $\hat{v}^{2i-1} = \ell_v^{2r+(2i-1)r+r_i}$. Observe that for every $x \in [0, 2k-1]_\mathbb{Z}$ we have $\hat{h}^x$ lies in the strip ζ_h^x while $\hat{v}^x$ is in the strip ζ_v^x. We set $\mathcal{H} = \{\hat{h}^x \ : \ x \in [0, 2k-1]_\mathbb{Z}\}$ and $\mathcal{V} = \{\hat{v}^x \ : \ x \in [0, 2k-1]_\mathbb{Z}\}$. Thus $|\mathcal{H} \cup \mathcal{V}| = 4k$. We now prove that all rectangles in $\mathcal{R}$ are stabbed by $\mathcal{H} \cup \mathcal{V}$.

First we verify that $\mathcal{R}_F$ is stabbed by $\mathcal{H} \cup \mathcal{V}$. For every $x \in [0, 2k-1]_\mathbb{Z}$ and every $q \in [-5k, -1]_\mathbb{Z}$ $F_v^{x,q}$ is stabbed by the vertical line $\hat{v}^x$ and $F_h^{x,q}$ is stabbed by the horizontal line $\hat{h}^x$.

Next we check that $\mathcal{R}_A$ is stabbed by $\mathcal{H} \cup \mathcal{V}$. Consider a rectangle $A^{i,j}_{p,q} \in \mathcal{R}_A$. Note that $i, j \in [k]$ and $i \neq j$. Since C is a clique we have that $r_i \neq p$ or $r_j \neq q$, and therefore $r_i \leq p-1$ or $r_i \geq p+1$ or $r_j \leq q-1$ or $r_j \geq q+1$. We check the four possible cases:

- If $r_i \geq p+1$ then $\hat{v}^{2i-2}$ stabs $A^{i,j}_{p,q}$,
- if $r_i \leq p-1$ then $\hat{v}^{2i-1}$ stabs $A^{i,j}_{p,q}$,
- if $r_j \geq q+1$ then $\hat{h}^{2j-2}$ stabs $A^{i,j}_{p,q}$, and
- if $r_j \leq q-1$ then $\hat{h}^{2j-1}$ stabs $A^{i,j}_{p,q}$.

Finally we verify that $\mathcal{H} \cup \mathcal{V}$ stabs $\mathcal{R}_E$. Let x and $y \in [0, 2k-1]_\mathbb{Z}$ be such that $\lfloor x/2 \rfloor = \lfloor y/2 \rfloor$, and a be an element of $[2, r]_\mathbb{Z}$. Let $i = \lfloor x/2 \rfloor = \lfloor y/2 \rfloor$. We now verify that the rectangles $TL_a^{x,y}$ and $BR_a^{x,y}$ are stabbed by $\mathcal{H} \cup \mathcal{V}$.

- If $r_i \geq a$ then $TL_a^{x,y}$ is stabbed by $\hat{h}^y$, and
- if $r_i \leq a-1$ then $TL_a^{x,y}$ is stabbed by $\hat{v}^x$.
- If $r_i \geq a$ then $BR_a^{x,y}$ is stabbed by $\hat{v}^x$, and
- if $r_i \leq a-1$ then $BR_a^{x,y}$ is stabbed by $\hat{h}^y$.

Thus all rectangles in $\mathcal{R}$ are stabbed by $\mathcal{H} \cup \mathcal{V}$, concluding the proof. ◀

▶ **Lemma 12.** *There exists a polynomial time algorithm that takes as input $G, k, V_1, \ldots, V_k$, $\mathcal{R}$, $\mathcal{L}$, a real $\epsilon > 0$, a set $\mathcal{H}$ of horizontal lines, and a set $\mathcal{V}$ of vertical lines such that $|\mathcal{H} \cup \mathcal{V}| \leq 5k - \epsilon k$ and $\mathcal{H} \cup \mathcal{V}$ stabs $\mathcal{R}$, and outputs a multicolored clique C of size at least ϵk in G.*

Proof. Let $\mathcal{H}$ be a set of horizontal lines and $\mathcal{V}$ be a set of vertical lines such that $|\mathcal{H} \cup \mathcal{V}| \leq 5k - \epsilon k$ and $\mathcal{H} \cup \mathcal{V}$ stabs $\mathcal{R}$.

▷ Claim 13. For every $x \in [0, 2k-1]_\mathbb{Z}$, $\mathcal{H}$ contains at least one line in the strip ζ_h^x and $\mathcal{V}$ contains at least one line in the strip ζ_v^x.

Proof. Suppose for contradiction that there exists an $x \in [0, 2k-1]_\mathbb{Z}$ such that $\mathcal{H}$ does not contain any lines in the strip ζ_h^x. For every $q \in [-5k, -1]_\mathbb{Z}$ the rectangle $F_h^{x,q}$ is not stabbed by $\mathcal{H}$, since every horizontal line stabbing $F_h^{x,q}$ is in ζ_h^x. Thus the set $\{F_h^{x,q} \ : \ q \in [-5k, -1]_\mathbb{Z}\}$ is stabbed by $\mathcal{V}$. But then $|\mathcal{V}| \geq 5k$, because no vertical line stabs two rectangles in $\{F_h^{x,q} \ : \ q \in [-5k, -1]_\mathbb{Z}\}$. This contradicts that $|\mathcal{H} \cup \mathcal{V}| \leq 5k - \epsilon k$. This shows that for every $x \in [0, 2k-1]_\mathbb{Z}$, $\mathcal{H}$ contains at least one line in the strip ζ_h^x. The proof that $\mathcal{V}$ contains at least one line in the strip ζ_v^x is symmetric. ◁

Let A be the subset of all integers $i \in [k]$ such that each strip ζ_h^{2i-2}, ζ_h^{2i-1}, ζ_v^{2i-2} and ζ_v^{2i-1} contains precisely one line from $\mathcal{H} \cup \mathcal{V}$. We claim that $|A| \geq \epsilon \cdot k$. Aiming towards a contradiction, suppose $|A| < \epsilon \cdot k$. Then, by Claim 13 for each $i \in [k]$ we have that the strips $\{\zeta_h^{2i-2}, \zeta_h^{2i-1}, \zeta_v^{2i-2}, \zeta_v^{2i-1}\}$ contain at least 4 lines from $\mathcal{H} \cup \mathcal{V}$ if $i \in A$ and at least 5 lines from $\mathcal{H} \cup \mathcal{V}$ if $i \notin A$. It follows that $|\mathcal{H} \cup \mathcal{V}| \geq 4|A| + 5(k - |A|) = 5k - |A| > 5k - \epsilon k$, contradicting $|\mathcal{H} \cup \mathcal{V}| \leq 5k - \epsilon k$. Hence $|A| \geq \epsilon \cdot k$.

For every $i \in A$ we define the integers $v\langle i,-\rangle$, $v\langle i,+\rangle$, $h\langle i,-\rangle$, $h\langle i,+\rangle$ as the unique integers in $[r]$ such that

- $\ell_v^{2ir+v\langle i,-\rangle} \in \mathcal{V}$ and lies in ζ_v^{2i-2},
- $\ell_v^{2ir+r+v\langle i,+\rangle} \in \mathcal{V}$ and lies in ζ_v^{2i-1},
- $\ell_h^{2ir+h\langle i,-\rangle} \in \mathcal{H}$ and lies in ζ_h^{2i-2},
- $\ell_h^{2ir+r+h\langle i,+\rangle} \in \mathcal{H}$ and lies in ζ_h^{2i-1}.

▷ Claim 14. For every $i \in A$ we have $v\langle i,-\rangle = h\langle i,+\rangle$.

Proof. Suppose for contradiction that $v\langle i,-\rangle > h\langle i,+\rangle$. Then $v\langle i,-\rangle \geq 2$. Since $\lfloor(2i-2)/2\rfloor = \lfloor(2i-1)/2\rfloor$, we have that $TL_{v\langle i,-\rangle}^{2i-2,2i-1}$ is a rectangle in $\mathcal{R}_E$. Since $TL_{v\langle i,-\rangle}^{2i-2,2i-1} \subseteq \zeta_v^{2i-2} \cap \zeta_h^{2i-1}$, $\ell_v^{2ir+v\langle i,-\rangle}$ is the unique line in $\mathcal{V}$ which lies in ζ_v^{2i-2}, and $\ell_h^{2ir+r+h\langle i,+\rangle}$ is the unique line in $\mathcal{H}$ which lies in ζ_h^{2i-1}, it follows that $TL_{v\langle i,-\rangle}^{2i-2,2i-1}$ is not stabbed by $\mathcal{H}\cup\mathcal{V}-\{\ell_v^{2ir+v\langle i,-\rangle}, \ell_h^{2ir+r+h\langle i,+\rangle}\}$. However, the largest x-coordinate of a point in $TL_{v\langle i,-\rangle}^{2i-2,2i-1}$ is

$$2r + (2i-2)r + v\langle i,-\rangle - 1 < 2ir + v\langle i,-\rangle,$$

and therefore $\ell_v^{2ir+v\langle i,-\rangle}$ does not stab $TL_{v\langle i,-\rangle}^{2i-2,2i-1}$. Furthermore, the smallest y-coordinate of a point in $TL_{v\langle i,-\rangle}^{2i-2,2i-1}$ is

$$2r + (2i-1)r + v\langle i,-\rangle = 2ir + r + v\langle i,-\rangle > 2ir + r + h\langle i,+\rangle,$$

and therefore $\ell_h^{2ir+r+h\langle i,+\rangle} \in \mathcal{H}$ does not stab $TL_{v\langle i,-\rangle}^{2i-2,2i-1}$. This contradicts that $TL_{v\langle i,-\rangle-1}^{2i-2,2i-1}$ is stabbed by $\mathcal{H}\cup\mathcal{V}$, so we conclude that $v\langle i,-\rangle \leq h\langle i,+\rangle$.

Next suppose for contradiction that $v\langle i,-\rangle < h\langle i,+\rangle$. Then $h\langle i,+\rangle \geq 2$. Since $\lfloor(2i-2)/2\rfloor = \lfloor(2i-1)/2\rfloor$, we have that $BR_{h\langle i,+\rangle}^{2i-2,2i-1}$ is a rectangle in $\mathcal{R}_E$. Again $BR_{h\langle i,+\rangle}^{2i-2,2i-1}$ is not stabbed by $\mathcal{H}\cup\mathcal{V}-\{\ell_v^{2ir+v\langle i,-\rangle}, \ell_h^{2ir+r+h\langle i,+\rangle}\}$. However the largest y-coordinate of $BR_{h\langle i,+\rangle}^{2i-2,2i-1}$ is

$$2r + (2i-1)r + h\langle i,+\rangle - 1 < 2ir + r + h\langle i,+\rangle,$$

and so $\ell_h^{2ir+r+h\langle i,+\rangle}$ does not stab $BR_{h\langle i,+\rangle}^{2i-2,2i-1}$ Furthermore the smallest x-coordinate of $BR_{h\langle i,+\rangle}^{2i-2,2i-1}$ is

$$2r + (2i-2)r + h\langle i,+\rangle = 2ir + h\langle i,+\rangle > 2ir + v\langle i,-\rangle,$$

and so $\ell_v^{2ir+v\langle i,-\rangle}$ does not stab $BR_{h\langle i,+\rangle}^{2i-2,2i-1}$. This contradicts that $BR_{h\langle i,+\rangle}^{2i-2,2i-1}$ is stabbed by $v\langle i,-\rangle \leq h\langle i,+\rangle$, so we conclude that $v\langle i,-\rangle \geq h\langle i,+\rangle$. We have now shown that $v\langle i,-\rangle \leq h\langle i,+\rangle$ and that $v\langle i,-\rangle \geq h\langle i,+\rangle$, which implies that $v\langle i,-\rangle = h\langle i,+\rangle$, concluding the proof of the claim. ◁

Proofs symmetrical to that of Claim 14 show $v\langle i,-\rangle = h\langle i,-\rangle$, $v\langle i,+\rangle = h\langle i,-\rangle$, and $v\langle i,+\rangle = h\langle i,+\rangle$ as well. Hence we obtain that for every $i \in A$ we have

$$v\langle i,-\rangle = v\langle i,+\rangle = h\langle i,-\rangle = h\langle i,+\rangle.$$

For each $i \in A$ we define $r\langle i\rangle = v\langle i,-\rangle$ and $C = \{v_{r\langle i\rangle}^i \ : \ i \in A\}$. Clearly $|C| = |A| \geq \epsilon k$ and so it remains to show that C is in fact a clique in G. Suppose for contradiction that there exist distinct $i,j \in A$ such that $(v_{r\langle i\rangle}^i, v_{r\langle j\rangle}^j)$ is not an edge of G. Consider the rectangle $A_{r\langle i\rangle,r\langle j\rangle}^{i,j}$. By construction it is not stabbed by any line with integer coordinates which is not in one of the strips ζ_v^{2i-2}, ζ_v^{2i-1}, ζ_h^{2j-2}, or ζ_h^{2j-2}. Thus, no lines in $\mathcal{H} \cup \mathcal{V} - \{\ell_v^{2ir+r\langle i\rangle}, \ell_v^{2ir+r+r\langle i\rangle}, \ell_h^{2jr+r\langle j\rangle}, \ell_h^{2jr+r+r\langle j\rangle}\}$ stab $A^{i,j}_{r\langle i\rangle,r\langle j\rangle}$. However the rectangle $A^{i,j}_{r\langle i\rangle,r\langle j\rangle}$ is defined precisely so it is contained in the interior of the rectangle defined by the four lines $\{\ell_v^{2ir+r\langle i\rangle}, \ell_v^{2ir+r+r\langle i\rangle}, \ell_h^{2jr+r\langle j\rangle}, \ell_h^{2jr+r+r\langle j\rangle}\}$. Hence $A^{i,j}_{r\langle i\rangle,r\langle j\rangle}$ is not stabbed by any of $\{\ell_v^{2ir+r\langle i\rangle}, \ell_v^{2ir+r+r\langle i\rangle}, \ell_h^{2jr+r\langle j\rangle}, \ell_h^{2jr+r+r\langle j\rangle}\}$, and therefore it is not stabbed by $\mathcal{H} \cup \mathcal{V}$ yielding the desired contradiction.

Both the construction of A from the input and the construction of C from A clearly take polynomial time. This concludes the proof. ◀

We are now ready to prove Theorem 2.

Proof of Theorem 2. Suppose that there exists an $\epsilon > 0$, function $f : \mathbb{N} \to \mathbb{N}$, and an algorithm $\mathcal{A}$ which takes as input an instance $(\mathcal{R}, \mathcal{L})$, an integer k, runs in time $f(k) \cdot n^{O(1)}$, where $n = |\mathcal{R}| + |\mathcal{L}|$, and either concludes that no subset of at most k lines of $\mathcal{L}$ stabs $\mathcal{R}$ or finds a set $\mathcal{L}^* \subseteq \mathcal{L}$ of size at most $\left(\frac{5}{4} - \epsilon\right) k$ that stabs $\mathcal{R}$.

We describe an approximation algorithm for MULTICOLORED CLIQUE. Given an instance G, k and partition $V_1, V_2, \ldots V_k$ of $V(G)$, the algorithm constructs the instance $(\mathcal{R}, \mathcal{L}, 4k)$ of RECTANGLE STABBING using the reduction described in Section 4.1. It then runs the algorithm $\mathcal{A}$ on this instance. If $\mathcal{A}$ returns that $\mathcal{R}$ cannot be stabbed with at most $4k$ lines from $\mathcal{L}$, the algorithm returns that G does not contain a multicolored clique of size k. This is correct by Lemma 11. If $\mathcal{A}$ returns a set of at most $\left(\frac{5}{4} - \epsilon\right) k$ lines that stab $\mathcal{R}$, the algorithm applies Lemma 12 to produce a multicolored clique C in G of size at least ϵk. The running time of the algorithm is upper bounded by $f(4k)n^{O(1)}$. Hence, by applying Theorem 10 with $h(k) = 1/\epsilon$ to the resulting algorithm for MULTICOLORED CLIQUE we obtain that FPT = W[1]. ◀