Tree-Independence Number of P_5-Free Graphs with No Large Bicliques

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Abstract

The tree-independence number of a graph is the minimum, over all tree-decompositions of the graph, of the maximum size of an independent set contained in a bag. Graph classes of bounded tree-independence number have strong structural and algorithmic properties, but the parameter can be unbounded even in quite restricted classes. In particular, the presence of an induced biclique $K_{\ell,\ell}$ forces tree-independence number at least ℓ. This leads to the question whether large induced bicliques are the only obstruction to bounded tree-independence number in natural hereditary classes. A conjecture of Dallard, Krnc, Kwon, Milanič, Munaro, Štorgel, and Wiederrecht states that for all positive integers t and ℓ, $\{P_t, K_{\ell,\ell}\}$-free graphs have bounded tree-independence number. We prove this conjecture for $t = 5$ by showing that every $\{P_5, K_{\ell,\ell}\}$-free graph has tree-independence number at most 4ℓ. We also obtain related bounds for the weaker parameter of α-degeneracy.

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1 Introduction

For many graph parameters, one can identify natural substructures whose presence forces the parameter to be large. A basic question is whether, inside a hereditary graph class,[1] these obvious obstructions are the only reason for large parameter values. A classical example is χ-boundedness: which hereditary graph classes have chromatic number bounded in terms of clique number? This notion, introduced by Gyárfás [26], has been extensively studied; see, for instance, the survey [41].

Other examples include bounding treewidth in terms of clique number (see [10, 18]), bounding pathwidth in terms of clique number (see [28,30]), bounding degeneracy in terms of clique number (see [7]), bounding minimum degree in terms of chromatic number (see [27]), bounding minimum degree in terms of the largest size of a balanced biclique (see [23]), and bounding coloring number in terms of chromatic number (see [32]).

In this paper we study an analogous question for *tree-independence number* and *induced biclique number*. Tree-independence number was introduced independently by Yolov [46] (under the name *α-treewidth*) and by Dallard, Milanič, and Štorgel [19]. It is a relaxation of treewidth that can be bounded even on dense graph classes; at the same time, bounded tree-independence number still yields strong structural and algorithmic consequences (see [19, 34] and Section 2 for definitions). The *induced biclique number* of a graph G is the maximum integer ℓ such that G contains an induced subgraph isomorphic to $K_{\ell,\ell}$. Since induced biclique number is a lower bound on tree-independence number [19], it is natural to ask the following.

▶ **Question 1.1.** *For which classes of graphs is tree-independence number bounded from above in terms of induced biclique number?*

Before we discuss this question, let us introduce some terminology. Given a graph G and a set $\mathcal{F}$ of graphs, we say that G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{F}$; for a graph H, we write *H-free* for *$\{H\}$-free*. We denote the t-vertex path by P_t, the disjoint union of two graphs H and H' by $H + H'$, and the disjoint union of t copies of a graph H by tH.

Question 1.1 is meaningful already for quite restricted classes: for instance, grid graphs have induced biclique number at most 2, but unbounded tree-independence number [20]. On the positive side, tree-independence number is known to be bounded in terms of induced biclique number for P_4-free graphs [21], $(P_3 + P_1)$-free graphs [31], graph classes of bounded *induced matching treewidth* [1], and consequently also tP_2-free graphs, for every positive integer t. These results motivate the following refinement of Question 1.1.

[1] A graph class is *hereditary* if it is closed under taking induced subgraphs.

▶ **Question 1.2.** *For which graphs H is tree-independence number of H-free graphs bounded from above in terms of induced biclique number?*

A related question, asking for which graphs H the class of H-free graphs has bounded tree-independence number, was answered by Dallard, Milanič, and Štorgel [20]: this happens if and only if H is edgeless or an induced subgraph of P_3. Moreover, Question 1.2 is equivalent to asking for which graphs H and every positive integer ℓ the class of $\{H, K_{\ell,\ell}\}$-free graphs has bounded tree-independence number. A necessary condition is that H is a disjoint union of paths, by a result of Lozin and Razgon [38] on hereditary classes of bounded treewidth defined by finitely many forbidden induced subgraphs; see also [21].

It is open whether this necessary condition is also sufficient. Since every disjoint union of paths is an induced subgraph of a path, a positive answer would follow from the following conjecture.

▶ **Conjecture 1.3** (Dallard, Krnc, Kwon, Milanič, Munaro, Štorgel, and Wiederrecht [21]). *For every two positive integers t and ℓ, the class of $\{P_t, K_{\ell,\ell}\}$-free graphs has bounded tree-independence number.*

This conjecture is known to hold for $t \leq 4$ and every ℓ [21]. On the other hand, it is open already for $\ell = 2$ and every $t \geq 7$ [13], while the case $\ell = 1$ is trivial. One can also ask for analogous statements with an unbalanced biclique. For example, boundedness of tree-independence number is known to hold for $\{P_t, K_{1,\ell}\}$-free graphs [21] (see also [11]), whereas the case of $\{P_t, K_{2,\ell}\}$-free graphs is already open for all $t \geq 7$ and $\ell \geq 2$; it was only recently settled for $t \in \{5, 6\}$ and arbitrary ℓ [13].

Our focus. We study the first open case of Conjecture 1.3 with respect to t, namely $t = 5$. This is a natural next step in the context of Question 1.2, because P_5 contains each graph in $\{P_4, P_3 + P_1, 2P_2\}$ as an induced subgraph.

At first glance, one might expect P_5-free graphs to behave like a mild extension of P_4-free graphs, that is, cographs. In reality, the class is considerably richer. For instance, the complexity of Independent Set on P_5-free graphs remained open for more than twenty years, and its eventual polynomial-time solution required substantial new machinery [36]. Likewise, both the Erdős–Hajnal property and polynomial χ-boundedness for P_5-free graphs were established only very recently [39,40], after a long sequence of partial results [8,12,24,41–43]. These examples suggest that the class of P_5-free graphs is structurally rich but still sufficiently tame to admit strong decomposition theorems and algorithmic consequences.

Another indication of this intermediate behavior is that P_5-free graphs of bounded clique number have bounded *mim-width* [9], a parameter that is very useful algorithmically [5]. From this perspective, Conjecture 1.3 can be viewed as a biclique analogue of such boundedness phenomena. It is also worth noting that if one excludes both a clique and a biclique, then P_t-free graphs, for every fixed t, have bounded treewidth [3].

Our results. Our main theorem confirms the case $t = 5$ of Conjecture 1.3 and answers Question 1.2 for the graph P_5.

▶ **Theorem 1.4.** *For every positive integer ℓ, every P_5-free graph with no induced $K_{\ell,\ell}$ has tree-independence number at most 4ℓ.*

In particular, tree-independence number of P_5-free graphs is bounded in terms of induced biclique number. Actually, the proof of Theorem 1.4 is algorithmic in the following strong sense: given a P_5-free graph and an integer ℓ, in polynomial time we can either find an induced $K_{\ell,\ell}$ or construct a tree-decomposition of independence number at most 4ℓ. In particular, the complexity of the algorithm depends only polynomially on ℓ.

We prove Theorem 1.4 by induction on the number of vertices. The key intermediate result is a bound on a weaker parameter, which is of independent interest.

For a graph G, the *α-degeneracy* of G is the smallest integer k such that every non-null induced subgraph H of G contains a vertex v with $\alpha(H[N[v]]) \leq k$. This parameter was considered previously in the literature [45]; the definition there uses open neighborhoods, but for our purposes the closed-neighborhood version is more convenient, and the two coincide on every graph with at least one edge. It is easy to see that α-degeneracy is a lower bound on tree-independence number; see Lemma 2.1. Thus, bounded α-degeneracy is a necessary condition for bounded tree-independence number, and it is natural to ask for a corresponding analogue of Conjecture 1.3.

▶ **Conjecture 1.5.** *For every two positive integers t and ℓ, the class of $\{P_t, K_{\ell,\ell}\}$-free graphs has bounded α-degeneracy.*

It follows from the aforementioned results on variants of Conjecture 1.3 that the analogues of Conjecture 1.5 with $K_{\ell,\ell}$ replaced by $K_{1,\ell}$ are known for all t, and with $K_{\ell,\ell}$ replaced by $K_{2,\ell}$ are known for $t \leq 6$; the latter remains open for all $\ell \geq 2$ and $t \geq 7$.

The parameter α-degeneracy leads to a companion version of Question 1.2. Indeed, every vertex in a balanced biclique $K_{\ell,\ell}$ has an independent neighborhood of size ℓ, so induced biclique number is a lower bound not only on tree-independence number, but also on α-degeneracy.

▶ **Question 1.6.** *For which graphs H is α-degeneracy of H-free graphs bounded from above in terms of induced biclique number?*

Equivalently, Question 1.6 asks for which graphs H and every positive integer ℓ the class of $\{H, K_{\ell,\ell}\}$-free graphs has bounded α-degeneracy. A necessary condition is that H is a forest, by combining a theorem of Kierstead and Penrice [33] on hereditary classes of bounded degeneracy with Ramsey's theorem; see also [37, Theorem 22].

Our first ingredient toward Theorem 1.4 proves Conjecture 1.5 for $t = 5$.

▶ **Theorem 1.7.** *For every $\ell \geq 2$, every non-null $\{P_5, K_{\ell,\ell}\}$-free graph has a vertex v such that $\alpha(N[v]) < 2\ell$.*

This is precisely the vertex deleted in the inductive proof of Theorem 1.4. Our approach in fact works in a more general setting. Let S_d denote the graph obtained from the d-leaf star $K_{1,d}$ by subdividing each edge once, so that $S_2 = P_5$.

▶ **Theorem 1.8.** *For every two integers $d, \ell \geq 2$, every non-null $\{S_d, K_{\ell,\ell}\}$-free graph has a vertex v such that $\alpha(N[v]) < d^2\ell + 2d\ell^{d-1}$.*

In particular, Theorem 1.8 answers Question 1.6 whenever H is an induced subgraph of S_d for some $d \geq 2$.

We also consider a complementary direction in which t is arbitrary but the forbidden biclique is fixed and unbalanced. In particular, we show that for every $\ell \geq 2$, the α-degeneracy of an n-vertex $\{P_t, K_{2,\ell}\}$-free graph is bounded by $(t-1)\ell \cdot \log n$. Actually, in the proof of this result we only use the fact that P_t-free graphs admit dominated balanced separators [4]. We say that a hereditary graph class $\mathcal{G}$ has *d-dominated balanced separators* if every graph $G \in \mathcal{G}$ contains a set $X \subseteq V(G)$ of size at most d such that every component of $G - N[X]$ has at most $\frac{1}{2}|V(G)|$ vertices. We prove the following more general statement.

▶ **Theorem 1.9.** *Let $d \geq 2$ be an integer and let $\mathcal{G}$ be a class with d-dominated balanced separators. Then, for every integer $\ell \geq 2$, every $K_{2,\ell}$-free graph from $\mathcal{G}$ with $n \geq 2$ vertices has a vertex v such that $\alpha(N[v]) \leq d\ell \cdot \log n$.*

Since the class of P_t-free graphs admits $(t-1)$-dominated balanced separators [4], Theorem 1.9 immediately implies the mentioned statement for $\{P_t, K_{2,\ell}\}$-free graphs. However, we know more classes with d-dominated balanced separators, and Theorem 1.9 applies to all of them.

Gartland and Lokshtanov conjectured (see [25]) that for every planar graph H, there exists an integer d such that graphs excluding H as an *induced minor* have d-dominated balanced separators. This conjecture is confirmed for some restricted graphs H, including paths [4, 26], cycles [16], and, more generally, wheels [15], as well as for some other cases (see [2, 14, 20]).

For a positive integer t, we denote by $S_{t,t,t}$ the graph obtained from the complete bipartite graph $K_{1,3}$ by subdividing each edge $t-1$ times, and by $W_{t\times t}$ the t-by-t hexagonal grid (also known as the $t \times t$-wall). Chudnovsky, Codsi, Lokshtanov, Milanič, and Sivashankar [14] showed that for every positive integer t there is an integer c_t such that if G is a $\{K_{t,t}, S_{t,t,t}\}$-free graph with $n \geq 2$ vertices that also excludes, as an induced subgraph, the line graphs of all subdivisions of $W_{t\times t}$, then the tree-independence number of G is at most $c_t \log^4 n$. This implies that every such graph has a vertex $v \in V(G)$ such that $\alpha(N[v]) \leq c_t \log^4 n$.

They also showed that for every positive integer t, there exists an integer $d_t \geq 2$ such that the class of $S_{t,t,t}$-free graphs excluding the line graphs of all subdivisions of $W_{t\times t}$, is a class with d_t-dominated balanced separators [14]. This result, combined with Theorem 1.9, shows that every $\{K_{2,\ell}, S_{t,t,t}\}$-free graph with $n \geq 2$ vertices that also excludes, as an induced subgraph, the line graphs of all subdivisions of $W_{t\times t}$, has a vertex $v \in V(G)$ such that $\alpha(N[v]) \leq d_t \ell \log n$.

Progress since this work. Very recently, and independently of this work, Hajebi and Spirkl [29] established the validity of Conjecture 1.3 (and consequently of Conjecture 1.5) in full. In fact, they proved a stronger result implying in particular that for any finite set $\mathcal{F}$ of graphs, the class of $\mathcal{F}$-free graphs has bounded tree-independence number if and only if $\mathcal{F}$ contains a complete bipartite graph, a graph each component of which is a path or a subdivided claw, and the line graph of such a graph. While their approach is very general, they do not give explicit estimates on the obtained bounds on tree-independence number in terms of induced biclique number. They use Ramsey's theorem and similar tools that result in bounds that are at least exponential in the induced biclique number. On the other hand, our approach for Theorem 1.4 is tailored to the class of P_5-free graphs and results in bounds that are at most a constant multiplicative factor away from the optimal bounds.

2 Preliminaries

All graphs considered in this paper are finite, simple, and undirected. A graph is said to be *null* if it has no vertices. Let G be a graph. For a vertex $v \in V(G)$, the *(open) neighborhood* of v in G is the set $N_G(v)$ of vertices adjacent to v in G; the set $N_G(v) \cup \{v\}$, denoted by $N_G[v]$, is the *closed neighborhood* of v in G. For a set $X \subseteq V(G)$, we denote by $N_G[X]$ the set $\bigcup_{v\in X} N_G[v]$ and by $N_G(X)$ the set $N_G[X] \setminus X$. The *degree* of a vertex $v \in V(G)$ is denoted by $\deg_G(v)$ and defined as the cardinality of $N_G(v)$. In all these cases, we omit the index G whenever the graph is clear from the context. For $X \subseteq V(G)$, by $G[X]$ we denote the subgraph of G induced by the set X. Since we mainly work with induced subgraphs, we often identify an induced subgraph (in particular, a component) with its vertex set, e.g., we write that X is a connected set meaning that $G[X]$ is connected.

Given a graph G, we say that $X \subseteq V(G)$ *is complete to* $Y \subseteq V(G)$ if every vertex in X is adjacent to every vertex of Y. A *clique* in a graph G is a set of pairwise adjacent vertices; an *independent set* in G is a set of pairwise nonadjacent vertices. The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum size of an independent set in G. For a positive integer t, we denote by K_t the complete graph with t vertices. A *matching* in a graph G is a set of pairwise disjoint edges. An *induced matching* in a graph G is a matching M such that no edge of the graph joins two distinct edges in M. A graph G is *bipartite* if it admits a *bipartition*, that is, a pair of disjoint independent sets with union $V(G)$. For two positive integers m and n, the *complete bipartite graph* $K_{m,n}$ is a bipartite graph that admits a bipartition into parts of size m and n, respectively, that are complete to each other. A *vertex cover* in a graph G is a set S of vertices of G such that $V(G) \setminus S$ is an independent set. By König's theorem, in every bipartite graph, the maximum size of a matching equals the minimum size of a vertex cover.

Let G be a graph and let $\mathcal{T} = (T, \{B_t\}_{t\in V(T)})$ such that T is a tree and $\{B_t\}_{t\in V(T)}$ is a collection of *bags*, that is, subsets of $V(G)$ indexed by the nodes of T. For a vertex $v \in V(G)$, denote by $T(v)$ the subgraph of T induced by the nodes t of T such that the bag B_t contains v. We say that $\mathcal{T}$ is a *tree-decomposition* of G if the following conditions are satisfied: (i) for each vertex $v \in V(G)$, the set $T(v)$ is nonempty and connected, and (ii) for edge $uv \in E(G)$ it holds that $T(u) \cap T(v) \neq \emptyset$. The *independence number* of a tree-decomposition $\mathcal{T} = (T, \{B_t\}_{t\in V(T)})$ of a graph G, denoted by $\alpha(\mathcal{T})$, is the maximum, over all nodes t of T, of the independence number of the subgraph of G induced by the bag B_t. The *tree-independence number* of G, denoted by $\mathsf{tree}\text{-}\alpha(G)$, is the minimum independence number of a tree-decomposition of G.

The following lemma shows the relation between tree-independence number and α-degeneracy.[2]

▶ **Lemma 2.1** (❄). *For every graph G, the α-degeneracy of G is at most its tree-independence number.*

3 Bounding α-degeneracy

In this section we discuss α-degeneracy and prove Theorems 1.7–1.9.

3.1 P_5-free and S_d-free graphs

Recall that for $d \geq 2$, by S_d we denote the graph obtained from the d-leaf star $K_{1,d}$ by subdividing each edge once. Note that $S_2 = P_5$.

▶ **Lemma 3.1** (❄). *Let $d, \ell \geq 2$ be integers and let H be a $\{dK_2, K_{\ell,\ell}\}$-free bipartite graph with bipartition X, Y. Suppose that for every $x \in X$ it holds that $\deg(x) \geq \ell^{d-1}$. Then $|X| \leq \binom{d}{2}(\ell - 1)$.*

▶ **Lemma 3.2** (❄). *Let q, d be positive integers and let H be a bipartite graph with bipartition X, Y that contains a matching M covering X and for each $x \in X$ it holds that $\deg(x) \leq q$. If $|X| > 2(d - 1)q$, then H contains an induced matching with d edges.*

Now we show the following result, which in particular implies Theorems 1.7 and 1.8.

[2] Proofs of statements marked with (❄) are omitted and can be found in the full version of this paper [6].

▶ **Theorem 3.3.** *Let $d, \ell \geq 2$ be integers, let G be an $\{S_d, K_{\ell,\ell}\}$-free graph, and let I be a maximum independent set in G. Then, for every $v \in I$, it holds that $\alpha(N[v]) < d^2\ell + 2d\ell^{d-1}$. Furthermore, for $d = 2$, the bound is $\alpha(N[v]) < 2\ell$.*

Proof. Let G and I be as in the statement of the theorem, and consider any $v \in I$. Let J be a largest independent set in $N[v]$. Clearly $J \cap I = \emptyset$, as $v \in I$ and v is adjacent to every vertex from J.

Let $I' = I \setminus \{v\}$ and consider the graph H induced by $J \cup I'$; note that it is bipartite with bipartition J, I'. Observe that $\alpha(H) \leq |I'| + 1$, as otherwise I is not a maximum independent set in G. Thus the size of a minimum vertex cover in H is at least $|J| - 1$. Consequently, by König's theorem, H has a matching M of size at least $|J| - 1$. Let J' be the set of vertices from J covered by M. Clearly $|J'| \geq |J| - 1$.

Let H' be the subgraph of H induced by $J' \cup I'$. It is bipartite and M is a matching in H' covering J'. Clearly H' is $K_{\ell,\ell}$-free, as it is an induced subgraph of G. Furthermore, H' is dK_2-free. Indeed, an induced matching with d edges in H', together with the vertex v, would induce a copy of S_d in G. We aim to bound the size of J. In general this is an easy application of Ramsey's theorem, but since we want to optimize the dependence on ℓ, we need a more careful argument.

The general case. Let J'_1 be the set of vertices in J' whose degree in H' is at least ℓ^{d-1}, and $J'_2 = J \setminus J'_1$. By Lemma 3.1 applied to the graph $H'[J'_1 \cup I']$ we observe that $|J'_1| \leq \binom{d}{2}(\ell-1) < d^2\ell$. On the other hand, as each vertex of J'_2 has degree at most $\ell^{d-1}-1$ in H', by Lemma 3.2 applied to the graph $H'[J'_2 \cup I']$ we observe that $|J'_2| \leq 2(d-1)(\ell^{d-1}-1) \leq 2d\ell^{d-1}$. Summing up, we obtain that $|J'| < d^2\ell + 2d\ell^{d-1}$ and thus $|J| \leq d^2\ell + 2d\ell^{d-1}$.

Better bound for $d = 2$. The above bound (if carefully inspected) only gives $|J| < 3\ell - 1$ for $d = 2$. Here we improve it to 2ℓ. For a vertex $x \in J'$, let $M(x)$ be the vertex matched with x by M. Towards contradiction suppose that $|J| \geq 2\ell$ and thus $|J'| \geq 2\ell - 1$. As H' is $2K_2$-free, there is an ordering $x_1, x_2, \ldots, x_{|J'|}$ of J' such that

$$N_{H'}(x_1) \subseteq N_{H'}(x_2) \subseteq \ldots \subseteq N_{H'}(x_{|J'|}) .$$

In particular, all vertices x_i for $i \geq \ell$ are adjacent to $M(x_1), \ldots, M(x_\ell)$. Thus, vertices

$$\{x_\ell, x_{\ell+1}, \ldots, x_{2\ell-1}\} \cup \{M(x_1), \ldots, M(x_\ell)\}$$

form an induced $K_{\ell,\ell}$ in G, a contradiction. ◀

3.2 Classes with dominated balanced separators

In what follows log denotes the logarithm with base 2.

▶ **Theorem 1.9.** *Let $d \geq 2$ be an integer and let $\mathcal{G}$ be a class with d-dominated balanced separators. Then, for every integer $\ell \geq 2$, every $K_{2,\ell}$-free graph from $\mathcal{G}$ with $n \geq 2$ vertices has a vertex v such that $\alpha(N[v]) \leq d\ell \cdot \log n$.*

Proof. First, for every positive $r \in \mathbb{R}$ we have $r \geq 1 + \ln r$. By letting $r = \frac{d-1}{d}$ we obtain

$$\frac{d-1}{d} \geq 1 + \ln \frac{d-1}{d} ,$$

thus

$$d - 1 \geq d + d \ln \frac{d-1}{d} .$$

Since for $r \leq 1$ we have $\log r \leq \ln r$, from there we observe that

$$d \log \frac{d-1}{d} + 1 \leq d \ln \frac{d-1}{d} + 1 \leq 0. \tag{1}$$

The proof of the theorem is by induction on n. If $n = 2$, then any vertex satisfies the claim. So let G be a $K_{2,\ell}$-free graph from $\mathcal{G}$ with $n > 2$ vertices and suppose that the result holds for all such graphs with fewer vertices.

If G is a complete graph, then for every $v \in V(G)$ it holds that $\alpha(N[v]) = 1$ and we are done. So we may assume that G is not complete. Let D be the set of universal vertices in G, that is, the set of vertices adjacent to all other vertices. Note that D is a clique in G and every vertex in D is adjacent to every vertex of $V(G) \setminus D$. Let $G' = G - D$. Since G is not complete, $n' := |V(G')| \geq 2$. Note that $G' \in \mathcal{G}$ as $\mathcal{G}$ is hereditary. Furthermore G' is $K_{2,\ell}$-free and has no universal vertices. We consider two cases.

Case 1. G' has a vertex x of degree at least $\frac{1}{d}n' - 1$. Note that every component of $G' - N[x]$ has at most $\frac{d-1}{d}n'$ vertices. Furthermore, there is at least one such component C, as x is not universal.

We apply induction to C, obtaining a vertex v such that

$$\begin{aligned}\alpha(N[v] \cap C) &\leq d\ell \log(|C|) \leq d\ell \log\left(\frac{d-1}{d}n'\right) \\ &= d\ell\left(\log \frac{d-1}{d} + \log n'\right) = d\ell \log \frac{d-1}{d} + d\ell \log n'.\end{aligned}$$

On the other hand, we observe that that $\alpha(N(v) \cap N(x)) \leq \ell - 1$, as G is $K_{2,\ell}$-free. Finally, $\alpha(N(v) \cap D) \leq 1$, as D induces a clique. Summing up, we obtain

$$\begin{aligned}\alpha(N[v]) &\leq \alpha(N[v] \cap C) + \alpha(N(v) \cap N(x)) + \alpha(N(v) \cap D) \\ &\leq d\ell \log \frac{d-1}{d} + d\ell \log n' + \ell - 1 + 1 \\ &\leq d\ell \log n' + \ell\left(d \log \frac{d-1}{d} + 1\right) \\ &\leq d\ell \log n' \leq d\ell \log n,\end{aligned}$$

as required, where the penultimate inequality follows from (1).

Case 2. G' has maximum degree smaller than $\frac{1}{d}n' - 1$. Let $X \subseteq V(G')$ be a set of size at most d such that every component of $G' - N[X]$ has at most $\frac{1}{2}n'$ vertices; such a set exists as $G' \in \mathcal{G}$ and $\mathcal{G}$ has d-dominated balanced separators. Furthermore, there is at least one component C of $G' - N[X]$, as $|N_{G'}[X]| \leq d \cdot (\frac{1}{d}n' - 1) < n'$. Again, we apply induction to C, obtaining a vertex v in C such that

$$\alpha(N[v] \cap C) \leq d\ell \log(|C|) \leq d\ell \log(n'/2) = d\ell \log n' - d\ell.$$

On the other hand, we observe that $\alpha(N(v) \cap N(X)) \leq d(\ell - 1)$. Indeed, suppose that there is an independent set $I \subseteq N(v) \cap N(X)$ of size at least $d(\ell - 1) + 1$. As $|X| \leq d$, by the pigeonhole principle, there is a vertex $x \in X$ such that $|I \cap N(x)| \geq \ell$, thus G contains an induced $K_{2,\ell}$, a contradiction. Summing up, we obtain

$$\begin{aligned}\alpha(N[v]) &\leq \alpha(N[v] \cap C) + \alpha(N(v) \cap N(X)) + \alpha(N(v) \cap D) \\ &\leq d\ell \log n' - d\ell + d(\ell - 1) + 1 \\ &\leq d\ell \log n' \leq d\ell \log n,\end{aligned}$$

as required. This completes the proof. ◀

4 Bounded tree-independence number for $\{P_5, K_{\ell,\ell}\}$-free graphs

In this section, we sketch the proof of Theorem 1.4.

▶ **Theorem 1.4.** *For every positive integer ℓ, every P_5-free graph with no induced $K_{\ell,\ell}$ has tree-independence number at most 4ℓ.*

The proof of the theorem is based on the following technical lemma.

▶ **Lemma 4.1 (❄).** *Let G be a $\{P_5, K_{\ell,\ell}\}$-free graph with at least two vertices, let $r \in V(G)$ be a vertex with $\alpha(N[r]) < 2\ell$, and let $\mathcal{T} = (T, \{B_t\}_{t \in V(T)})$ be a tree-decomposition of $G - r$ such that:*

(P1) *The independence number of each bag is at most 4ℓ.*

(P2) *Among tree-decompositions satisfying (P1), the number of vertex pairs $v, w \in N(r)$ with $v \neq w$ for which there exists $t \in V(T)$ such that $v, w \in B_t$, is maximized.*

Then, each pair of vertices in $N(r)$ belongs to a common bag.

Let us argue that Lemma 4.1 indeed implies Theorem 1.4.

Proof of Theorem 1.4 using Lemma 4.1. The proof is by induction on $|V(G)|$; the base case of $|V(G)| = 1$ is trivial. For $|V(G)| \geq 2$, we pick a vertex $r \in V(G)$ with $\alpha(N[r]) < 2\ell$ using Theorem 1.7. By the induction hypothesis, there is a tree-decomposition of $G - r$ such that the independence number of each bag is at most 4ℓ. In particular, we pick a tree-decomposition $\mathcal{T} = (T, \{B_t\}_{t \in V(T)})$ of $G - r$ that satisfies (P1) and (P2). By Lemma 4.1, each pair of vertices in $N(r)$ belongs to a common bag. By the Helly properties of trees, if for every pair of vertices $v, w \in N(r)$ the subtrees $T(v)$ and $T(w)$ have a common node, there is a node $t \in V(T)$ which is contained in $T(v)$ for every $v \in N(r)$. That is, the bag B_t contains all vertices of $N(r)$. Then we construct a tree-decomposition of G by adding to T a new leaf node t' adjacent to t and set its bag $B_{t'} = N[r]$. This is clearly a tree-decomposition of G witnessing $\mathsf{tree}\text{-}\alpha(G) \leq 4\ell$. ◀

Sketch of proof of Lemma 4.1

For the remainder of the section, we sketch the construction that enables us to prove Lemma 4.1. For the full proof, we refer the reader to the full version of this paper [6].

Setting

Let $G' \coloneqq G - r$ and let $\mathcal{T}$ be a tree-decomposition as in the premise of Lemma 4.1. For a vertex $v \in N(r)$, we define $N_{\overline{r}}(v) \coloneqq N(v) \setminus N[r]$.

Let $\mathcal{P}$ denote the set of ordered pairs (x, y), where x, y are distinct neighbors of r that are not in a common bag of $\mathcal{T}$. Note that x and y are nonadjacent for all $(x, y) \in \mathcal{P}$. To prove Lemma 4.1, we need to show that $\mathcal{P} = \emptyset$.

For a contradiction, suppose that $\mathcal{P}$ is nonempty. We say that a pair $(x, y) \in \mathcal{P}$ is *bad* if $\alpha(N_{\overline{r}}(x) \setminus N_{\overline{r}}(y)) \geq \ell$. We fix a specific pair $(x, y) \in \mathcal{P}$ as follows: If possible, we pick (x, y) to be a bad pair, and subject to this rule, we pick (x, y) that maximizes the distance between $T(x)$ and $T(y)$ in T.

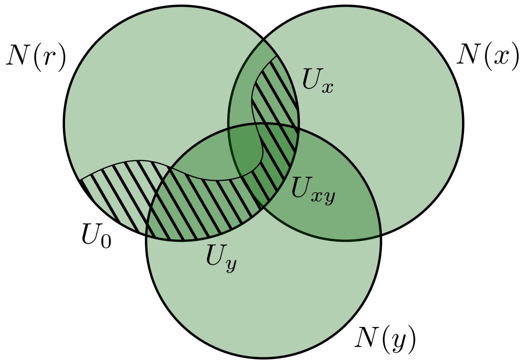
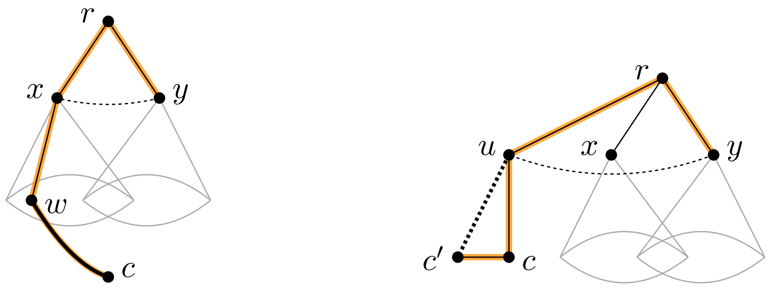


Figure 1 The green area denotes the set M. Striped area depicts the set U, i.e., vertices from $N(r)$ that have neighbors outside M.



(a) A copy of P_5 in Observation 4.2(a).

(b) A copy of P_5 in Observation 4.2(b).

Figure 2 The copies of P_5 from the proof of Observation 4.2.

Neighbors of r, x, and y

We denote $N(r) \cup N_{\overline{r}}(x) \cup N_{\overline{r}}(y)$ by M, see also Figure 1. Let U be the set of vertices $u \in N(r)$ that have a neighbor outside of $M \cup \{r\}$, that is,

$$U = \{u \in N(r) \mid N_{\overline{r}}(u) \setminus (N_{\overline{r}}(x) \cup N_{\overline{r}}(y)) \neq \emptyset\}.$$

Note that neither x nor y belongs to U. We further differentiate the vertices of U based on the connections between $u \in U$ and x, y, as follows:

$$U_0 := \{u \in U \mid u \notin N(x) \cup N(y)\}, \qquad U_x := \{u \in U \mid u \in N(x) \setminus N(y)\},$$
$$U_y := \{u \in U \mid u \in N(y) \setminus N(x)\}, \qquad U_{xy} := \{u \in U \mid u \in N(x) \cap N(y)\}.$$

▶ **Observation 4.2.** *Let C be a component of $G' - M$. Then, the following holds.*

(a) *$N(C) \subseteq U \cup (N_{\overline{r}}(x) \cap N_{\overline{r}}(y))$.*

(b) *C is complete to $N(C) \cap (U_0 \cup U_x \cup U_y)$.*

Proof. For (a), suppose that $c \in C$ has a neighbor w in, say, the set $N_{\overline{r}}(x) \setminus N_{\overline{r}}(y)$ (the other case is symmetric). Then G contains a copy of P_5 on vertices c, w, x, r, y, see Figure 2a.

For (b), towards a contradiction, assume that the component C is not complete to $N(C) \cap (U_0 \cup U_x \cup U_y)$, that is, there are two vertices $c, c' \in C$ and a vertex $u \in N(C) \cap (U_0 \cup U_x \cup U_y)$ such that u is adjacent to c, but nonadjacent to c'.

Moreover, we may assume that c is adjacent to c'. Indeed, as C is a component, there exists a path P from c to c' within C. The path starts from a vertex adjacent to u and ends in a vertex that is not adjacent to u. Hence, there are two consecutive vertices on P such that one is adjacent to u and the other is not.

We claim that G contains a P_5, leading to a contradiction. By symmetry, assume that $u \in U_0 \cup U_x$. Then u is not adjacent to y, and c', c, u, r, y is an induced P_5, see Figure 2b. ◀

Towards proving Lemma 4.1, we distinguish two cases, depending on whether the pair (x, y) is bad or not. The general outline of the proof in these two cases is similar, but the case that (x, y) is not a bad pair is significantly less involved.

Case 1: The pair (x, y) is not bad

By the choice of (x, y), we know that there are no bad pairs in $\mathcal{P}$ at all. We let t_x and t_y be the closest nodes of T such that $t_x \in T(x)$ and $t_y \in T(y)$. Note that t_x and t_y are distinct, since $(x, y) \in \mathcal{P}$ and uniquely defined since T is a tree.

We create a new tree-decomposition $\mathcal{T}' = (T', \{B'_t\}_{t \in V(T')})$ of G' satisfying (P1) such that all the pairs of vertices $v, w \in N(r)$ that are in a common bag in $\mathcal{T}$ are so in $\mathcal{T}'$ as well, and additionally x and y appear in a common bag in $\mathcal{T}'$. This gives a contradiction with property (P2) of $\mathcal{T}$.

Let us start by defining T', see Figure 3. We create a "master" copy T^M of T and another copy T^C of T for every component C of $G' - M$. For each $t \in V(T)$, we denote by t^M and t^C the copies of t in the respective copies T^M and T^C of T. Now, we create T' by taking the disjoint union of all these copies and adding edges $t_y^M t_y^C$ for every component C of $G' - M$.

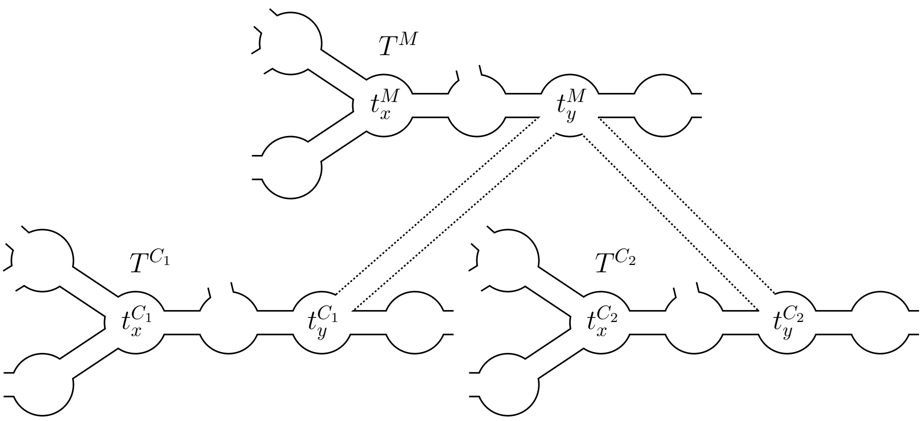


Figure 3 Tree T' obtained by connecting T^M and T^C for each component C (here we sketched just two components C_1 and C_2).

Now, let us define the bags B'_t for $t \in V(T')$. For each $t^M \in V(T^M)$, we do the following:

1. We initialize $B'_{t^M} := B_t \cap M$.
2. For each vertex $u \in U$, we extend the subtree $T^M(u)$ minimally so it contains vertices t_x^M and t_y^M. In other words, we add the vertex u to the bags of t_x^M and t_y^M, and to the minimum number of other bags so that $T^M(u)$ is connected.
3. We add the vertex x to the bag of every node on the path from t_x^M to t_y^M in T^M.

Now, consider a component C of $G' - M$. For each $t^C \in V(T^C)$, we set

$$B'_{t^C} := \big(B_t \cap N[C]\big) \cup \big(N(C) \cap (U \setminus U_{xy})\big).$$

This completes the definition of $\mathcal{T}'$. Note that for every $t^M \in V(T^M)$ we have $B'_{t^M} \subseteq M$.

▷ Claim 4.3. $\mathcal{T}'$ is a tree-decomposition of G'.

Proof. First, let us argue that for each vertex $v \in V(G')$, the set $T'(v)$ is nonempty. Consider the case $v \in M$, the other one is similar. Note that $T(v)$ is nonempty by the definition of $\mathcal{T}$, and the copies of nodes in $T(v)$ in the tree T^M are contained in $T'(v)$, as we were only adding vertices to the bags. This in particular means that $T'(v)$ is nonempty.

It is clear from the construction that for each v the set of bags where v appears is connected when restricted to either T^M or some T^C.

Observe that the vertices that do not belong solely to the bags of nodes in T^M or in T^C for some C are exactly those vertices v that belong to the neighborhood of C for some component C of $G' - M$. Thus, it remains to prove that for every such component C and every $v \in N(C)$, the set $T'(v)$ is connected. For this, it is enough to show that v belongs to the bags of both t^M_y and t^C_y, since these two nodes connect T^M and T^C.

By Observation 4.2(a), we have $v \in U \cup (N_{\overline{r}}(x) \cap N_{\overline{r}}(y))$.

If v is a common neighbor of x and y, so in particular if $v \in U_{xy} \cup (N_{\overline{r}}(x) \cap N_{\overline{r}}(y))$, we claim that v belongs to B_{t_y} in T. Indeed, since v is adjacent to both x and y, the set $T(v)$ must intersect both $T(x)$ and $T(y)$ so, as T is a tree, it must contain both t_x and t_y. Thus, v belongs to the bags of both t^M_y and t^C_y.

So, assume that $v \in U \setminus U_{xy}$. On one hand, v belongs to the bag of t^M_y, as it was explicitly added there in the second step of the definition of B'_{t^M} for $t^M \in V(T^M)$. On the other hand, v belongs to the bags of all nodes in T^C, so in particular to the bag of t^C_y.

Lastly, we need to argue that for each edge $uv \in E(G')$, there is a node $t \in V(T')$ such that $u, v \in B'_t$. Since T is a tree-decomposition of G', there is a node $t \in V(T)$ such that $u, v \in B_t$. If $u, v \in M$, then $u, v \in B'_{t^M}$. Otherwise, we may assume that u belongs to some component C of $G' - M$ and $v \in N[C]$. In this case we have $u, v \in B'_{t^C}$. ◁

▷ Claim 4.4. It holds that $\alpha(\mathcal{T}') \leq 4\ell$.

Proof. First, consider a node t^M of T^M. Let I be an independent set of G' contained in B'_{t^M}. If $I \subseteq B_t$, we have $|I| \leq 4\ell$ by the definition of $\mathcal{T}$. Hence, assume that I contains some vertex that was added either in the second or the third step of the construction of B'_{t^M}. Such a vertex is either x or some $u \in U$. Suppose first that $x \in I$. As $I \subseteq B'_{t^M} \subseteq M = N(r) \cup N_{\overline{r}}(x) \cup N_{\overline{r}}(y)$ and $I \cap N_{\overline{r}}(x) = \emptyset$, since $x \in I$, we have

$$|I| = |I \cap N(r)| + \left|I \cap \left(N_{\overline{r}}(y) \setminus N_{\overline{r}}(x)\right)\right| < 2\ell + (\ell - 1),$$

by the choice of r and since (y, x) is not a bad pair (as we are in the case that no pairs in $\mathcal{P}$ are bad).

So suppose that there is a vertex $u \in I$ such that $u \in U$. Note that we can assume that $T(u) \cap T(x) = \emptyset$ or $T(u) \cap T(y) = \emptyset$, as otherwise u must be present in both B_{t_x} and B_{t_y} and thus, was not added in the second step of the construction of B'_{t^M}. Consider the case $T(u) \cap T(x) = \emptyset$, the other one is similar. Since $u \in I$, we have $I \cap N_{\overline{r}}(u) = \emptyset$. Furthermore, since $T(u) \cap T(x) = \emptyset$, no bag of $\mathcal{T}$ contains both u and x, so $(x, u) \in \mathcal{P}$. This implies

$$|I| \leq |I \cap N(r)| + \left|I \cap \left(N_{\overline{r}}(x) \setminus N_{\overline{r}}(u)\right)\right| + \left|I \cap \left(N_{\overline{r}}(y) \setminus N_{\overline{r}}(x)\right)\right| < 2\ell + (\ell - 1) + (\ell - 1),$$

where $\left|I \cap \left(N_{\overline{r}}(x) \setminus N_{\overline{r}}(u)\right)\right| \leq \ell - 1$ follows from the fact that (x, u) is not a bad pair.

Finally, consider a component C of $G' - M$ and a node t^C in T^C. Let $I \subseteq B'_{t^C}$ be an independent set. Again, if $I \subseteq B_t$, we have $|I| \leq 4\ell$ by the assumption on $\mathcal{T}$. Hence, assume that I contains some of the new vertices from $N(C) \cap (U \setminus U_{xy})$. However, the component C is complete to these vertices by Observation 4.2(b). Thus, $I \subseteq N(C) \cap (U \setminus U_{xy}) \subseteq N(r)$, so $|I| < 2\ell$ by the choice of r. ◁

To complete the proof sketch of Lemma 4.1 for the case that (x, y) is not a bad pair, let us argue that the number of pairs of vertices in $N(r)$ that appear in a single bag increased, compared to $\mathcal{T}$. Consider a pair $x', y' \in N(r)$ of distinct vertices that appear in the bag of some node of $\mathcal{T}$, say $x', y' \in B_t$. Note that $x', y' \in M$ and thus, $x', y' \in B'_{t^M}$. Additionally, due to the third step of the construction of B'_{t^M}, we have $x, y \in B'_{t_y^M}$. Thus, by (P2), this contradicts the choice of $\mathcal{T}$ and completes the proof sketch of Lemma 4.1 in this case.

Case 2: The pair (x, y) is bad

Now let us briefly discuss the case when (x, y) is a bad pair.

In order to get a contradiction to the existence of such x, y and hence to prove Lemma 4.1 in Case 2, we construct a tree-decomposition $\mathcal{T}' = (T', \{B'_t\}_{t \in V(T)})$ that satisfies (P1) and is superior to $\mathcal{T}$ with respect to (P2). As before, T' is composed of a "master" subtree T^M to which we join subtrees T^D reminiscent of T^C from Case 1. However, the construction of the bags is more involved compared to Case 1, as we do not have a bound on $\alpha(N_{\overline{r}}(x) \setminus N_{\overline{r}}(y))$. Thus, we cannot move vertices of U as freely within the bags of nodes of T^M.

To be more specific, we construct the bags of nodes in T^M in two steps. First, we define intermediate bags B''_{t^M} by intersecting the bag B_t with M, adding some of the vertices from U to the bag. We also add the vertex x to some bags, in order to ensure that the vertices x and y appear in a common bag. This yields an intermediate decomposition $\mathcal{T}^M$ of $G'[M]$, which is similar to the final tree-decomposition from Case 1, restricted to T^M.

Unlike in the previous case, however, there might be vertices c from some component C of $G' - M$ whose neighborhood within M is *not* contained in a single bag of $\mathcal{T}^M$, which makes it difficult to cover the edges between c and $N(c)$ in T^C (as in the previous case). Hence, as the second step, we define bags B'_{t^M} by extending the bags B''_{t^M} by such vertices c, so that we cover the problematic edges in the bags of nodes in T^M instead. Let us denote by $\mathcal{C}$ the set of components of $G' - M$ and define $M' \coloneqq M \cup \bigcup_{C \in \mathcal{C}} C'$, where C' are the vertices from C added to the bags of T^M. We show that $\mathcal{T}^{M'} \coloneqq (T^M, \{B'_t\}_{t \in V(T^M)})$ is actually a tree-decomposition of $G'[M']$.

The role of the nodes of T^D is very similar to the role of T^C from Case 1. For each component D of $G' - M'$, we create a copy T^D of T whose bags will be representing the adjacencies within D and between D and $N(D)$, yielding a tree-decomposition $\mathcal{T}^D$ of the graph with vertex set $N[D]$ formed by edges that have at least one endpoint in D. Moreover, the choice of nodes by which we join T^D to T^M is slightly more elaborate compared to Case 1, and possibly different for different components D.

The construction and proof use more structural properties of the analyzed sets, as well as many more omitted details. We refer to the full version of this paper [6] for the full construction and proof of this case.

5 Algorithmic consequences

Let us conclude the paper by discussing some algorithmic consequences of our results. First, we observe that the proof of Theorem 1.4 can be easily turned into a polynomial-time algorithm. The only non-obvious part is that on the way we need to find largest independent sets in various induced subgraphs of the input graph, e.g., to find the vertex r with small $\alpha(N[r])$ or to decide whether bad pairs exist. However, finding a largest independent set in a P_5-free graphs can be done in polynomial time [36]. Thus, we obtain the following algorithmic version of Theorem 1.4.

▶ **Theorem 5.1** (algorithmic version of Theorem 1.4). *There is an algorithm that, given a P_5-free graph G and an integer ℓ, in time polynomial in $|V(G)|$ and ℓ returns one of the following outputs:*

- *an induced $K_{\ell,\ell}$ in G, or*
- *a tree-decomposition of G with independence number at most 4ℓ.*

In particular, the running time depends only polynomially on ℓ. Thus, the algorithm yields a polynomial-time constant-factor-approximation algorithm for computing tree-independence number (and a corresponding decomposition) in P_5-free graphs. Indeed, we run the algorithm for $\ell = 1, 2, \ldots$ and consider the smallest value for which the algorithm returns the second outcome, i.e., a decomposition of tree-independence number $k^* \leq 4\ell$. As the input graph G contains $K_{\ell-1,\ell-1}$ as an induced subgraph, we conclude that

$$\ell - 1 \leq \mathsf{tree}\text{-}\alpha(G) \leq k^* \leq 4\ell \leq 4(\mathsf{tree}\text{-}\alpha(G) + 1).$$

We note that for general graphs, the existence of essentially any non-trivial polynomial-time approximation algorithm for computing tree-independence number (or even any FPT-approximation, when parameterized by the tree-independence number) is excluded by known hardness results [17].

Conversely, the algorithm from Theorem 5.1 can be seen as a polynomial-time constant-factor-approximation algorithm for the problem of computing the induced biclique number in a P_5-free graph: this time we consider the largest value of ℓ for which the algorithm returns the second outcome. The existence of a tree-decomposition with independence number at most $4(\ell + 1)$ is an evidence that the induced biclique number of G is at most $4(\ell + 1)$.

While the problem of computing the induced biclique number received some attention [22, 44], we are not aware of any results in P_5-free graphs. We believe that this is an interesting question to investigate.

Finally, let us recall that graphs of bounded tree-independence number have very useful algorithmic properties [19,34]. Furthermore, the complexity of many such algorithms depends only single-exponentially on the tree-independence number [35]. Thus, the bounds on the tree-independence number obtained in this paper can be used to obtain quasipolynomial-time (resp., subexponential-time) algorithms for various problems in P_5-free graphs of polylogarithmic (resp., sublinear) induced biclique number.

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