# An Approximation Algorithm for 2-Vertex-Connectivity via Cycle-Restricted 2-Edge-Covers

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## Abstract

In the 2-Vertex-Connected Spanning Subgraph problem (2-VCSS), we are given an undirected graph $G$, and the objective is to find a 2-vertex-connected spanning subgraph $S$ of $G$ with the minimum number of edges. In the context of survivable network design, 2-VCSS is one of the most fundamental and well-studied problems. There has been active research on improving the approximation ratio of algorithms, and the current best ratio is $\frac{4}{3}$, achieved by Bosch-Calvo, Grandoni, and Jabal Ameli. In this paper, we improve the approximation ratio to $\frac{95}{72} + \varepsilon$ ($< 1.32$). The key idea in our algorithm is to introduce a 2-edge-cover without certain cycle components, and use it as an initial solution.

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## 1 Introduction

When designing networks in the real world, it is required not only that they be connected, but also that this connectivity be robust against the failure of some nodes or links. The survivable network design problem aims to minimize the network cost while ensuring robustness against such failures. One of the most fundamental models of robustness is vertex-connectivity. A graph $G = (V, E)$ is said to be 2-vertex-connected (2VC) if $G - \{v\}$ is connected for any vertex $v \in V$. In the 2-Vertex-Connected Spanning Subgraph problem (2-VCSS), given a graph $G = (V, E)$, the objective is to find an edge subset $S \subseteq E$ of minimum cardinality such that the subgraph $(V, S)$ is 2VC, if one exists.

It is easy to see that 2-VCSS is NP-hard, since a Hamiltonian cycle is an optimal solution whenever the input graph contains one. In addition, Czumaj and Lingas [7] have shown that 2-VCSS is APX-hard, hence there is no PTAS for 2-VCSS unless P = NP. A 2-approximation for 2-VCSS can be obtained by removing trivial ears from an open ear decomposition of the input graph, and various other approaches are also known. Khuller and Vishkin [20] presented the first non-trivial 5/3-approximation algorithm for 2-VCSS. This approximation ratio has been improved to 3/2 by Garg, Vempala, and Singla [12]. Cheriyan and Thurimella [5] also achieved 3/2 approximation by using different approaches; they used, as a lower bound on the optimal solution for 2-VCSS, a set of edges such that each vertex is incident to at least two edges (called a 2-edge-cover), and this technique has become the standard approach today. Heeger and Vygen [16] further improved this approximation ratio to 10/7. The current best approximation ratio is 4/3, achieved by Bosch-Calvo, Grandoni, and Jabal Ameli [3] through a refinement of a reduction of the input graph. Our main result is to improve the approximation ratio to $\frac{95}{72} + \varepsilon$ ($< 1.32$).

▶ **Theorem 1.** *For any constant $\varepsilon > 0$, there is a polynomial-time $\left(\frac{95}{72} + \varepsilon\right)$-approximation algorithm for 2-VCSS.*

Before describing our approach, we introduce a natural edge-connectivity version of the 2-VCSS, which is called the 2-Edge-Connected Spanning Subgraph problem (2-ECSS). In 2-ECSS, given a graph $G = (V, E)$, the objective is to find a minimum cardinality edge subset $S$ of $G$ such that the subgraph $(V, S \setminus \{e\})$ is connected for any edge $e \in E$; that is, $(V, S)$ is 2-edge-connected (2EC). As with 2-VCSS, 2-ECSS is APX-hard [7, 8], and no PTAS exists unless P = NP. Khuller and Vishkin [20] gave the first non-trivial 3/2-approximation algorithm for 2-ECSS, and various studies [1, 4, 11, 17, 18, 21, 24] have improved the approximation ratio. In recent years, competition to improve the approximation ratio has intensified. After Garg, Grandoni, and Jabal Ameli [11] achieved a 1.326-approximation in 2023, Kobayashi and Noguchi [21] presented a $(1.3 + \varepsilon)$-approximation algorithm that uses an algorithm for computing a maximum triangle-free 2-matching [2, 15, 22]. Then, Bosch-Calvo et al. [1] gave a 1.25-approximation algorithm, and Hommelsheim, Lindermayr, and Liu [17] further improved this ratio by a small margin, which currently stands as the best-known approximation ratio.

The approximation approaches for 2-VCSS and 2-ECSS are very similar, and the standard method is transforming a minimum 2-edge-cover into a feasible solution. Recall that a 2-edge-cover is a set of edges in which each vertex is incident to at least two edges. Since any feasible solution to 2-VCSS or 2-ECSS is clearly a 2-edge-cover, the size of a minimum 2-edge-cover can be used as a lower bound on their optimal values. By analyzing the number of edges added when transforming a minimum 2-edge-cover into a feasible solution, we can guarantee the approximation ratio of the algorithm.

Among several factors, two major contributors to recent progress in approximation algorithms for 2-ECSS are the introduction of triangle-free 2-edge-covers and smart reductions of the input graph.

A *triangle-free 2-edge-cover* is a 2-edge-cover that does not contain a triangle as a component. Since a triangle-free 2-edge-cover is closer to a feasible solution than a 2-edge-cover, by using a minimum triangle-free 2-edge-cover instead of a minimum 2-edge-cover as the initial solution, the approximation ratio for 2-ECSS is improved [21].

Reductions of the input graph aim to reduce the increase in the number of edges when transforming an initial solution into a feasible solution, by enlarging the flexibility in selecting edges during the transformation. The main reductions that have been used so far are the following:

1. Decomposing the input graph so that each resulting graph is nearly 3-vertex-connected [1, 11].
2. Contracting those 2EC subgraphs of the input graph in which many edges are already forced to be included in the solution (that is, subgraphs with little freedom in edge selection) [11, 18].

The first reduction for 2-ECSS, i.e., the decomposition into nearly 3-vertex-connected graphs has also been applied to 2-VCSS [3] (see also Subsection 2.2) and has contributed to improving the approximation ratio.

### Technical Challenges and Key Ideas

In this paper, we aim to improve the approximation ratio for 2-VCSS by using triangle-free 2-edge-covers and applying the second reduction above originally developed for 2-ECSS. However, contracting 2EC (or 2VC) subgraphs as a reduction of the input graph cannot be directly applied to 2-VCSS. This is because, unlike 2-edge-connectivity, 2-vertex-connectivity is not preserved under graph contraction, which presents the main difficulty in dealing with vertex-connectivity.

We address this issue by handling subgraphs with little freedom in edge selection at the stage of constructing the initial solution, rather than the reduction step. Specifically, we impose new constraints on the initial 2-edge-cover, which we call a *cycle-restricted 2-edge-cover*. A cycle-restricted 2-edge-cover is a 2-edge-cover in which there is no component that consists of a triangle, a certain 4-cycle (*4-cycle with an isolated pair*), or a certain 6-cycle (*6-cycle with an isolated triple*). If the initial solution contains such components, merging them into different components requires many additional edges, so imposing constraints to avoid them in advance helps improve the approximation ratio.

In this paper, our technical contributions are twofold: first, we introduce a cycle-restricted 2-edge-cover and show that an almost-minimum cycle-restricted 2-edge-cover can be computed in polynomial time (see Section 3); second, we demonstrate that the properties of cycle-restricted 2-edge-covers are effective in transforming it into a feasible solution of 2-VCSS (see the full version [23]).It is worth noting that it remains unclear whether we can compute an almost-minimum 2-edge-cover in which no component forms a triangle, 4-cycle, or 6-cycle. Therefore, focusing on 4-cycles with an isolated pair and 6-cycles with an isolated triple is key to our argument.

As in [3], the number of edges added when transforming the initial solution into a 2-edge-connected graph is analyzed using a *credit-based argument*. We first assign $\frac{23}{72}$ credits to each edge of the initial solution $S$. Then, by using these credits to pay for any increase in the number of edges of $S$, the size of $S$ remains within $1 + \frac{23}{72} = \frac{95}{72}$ times the initial size (see the full version for details).

### Related Work

Approximation algorithms have also been studied for the $k$-Edge-Connected Spanning Subgraph problem ($k$-ECSS). This problem generalizes 2-ECSS and asks for a spanning subgraph with a minimum number of edges such that the subgraph remains connected after the removal of any set of at most $k-1$ edges (see [5, 9, 10]). We can also consider the weighted version of 2-ECSS: given an edge-weighted graph $G$, the objective is to find a 2EC spanning subgraph of $G$ with minimum total weight. Jain [19] presented a 2-approximation algorithm for a more general problem, and for Weighted 2ECSS, it remains a major open problem whether there exists an approximation algorithm with a ratio strictly better than 2. The case in which edge weights are restricted to $\{0, 1\}$ is known as the Forest Augmentation problem, for which Grandoni, Jabal Ameli, and Traub [13] gave a 1.9973-approximation algorithm. The case in which the edges of weight 0 form a connected graph is called the (Weighted) Tree Augmentation problem, and the currently best known approximation ratio is $1.5 + \varepsilon$ for any $\varepsilon > 0$, achieved in a series of works by Traub and Zenklusen [25, 26].

Our algorithm for 2-VCSS uses a PTAS for a generalization of the triangle-free 2-matching problem. A *triangle-free 2-matching* is a 2-matching that contains no cycles of length three, and a complicated exact algorithm for finding a maximum triangle-free 2-matching was announced by Hartvigsen [14] in 1984; an improved version was recently published in [15].

A simple PTAS for the triangle-free 2-matching problem was proposed by Bosch-Calvo, Grandoni, and Jabal Ameli [2], and a simple analysis for this was given by Kobayashi and Noguchi [22]. The triangle-free 2-matching problem corresponds to the case where $k = 3$ in the $C_{\leq k}$-free 2-matching problem, which asks for a 2-matching of maximum size that does not contain cycles of length $k$ or less. The $C_{\leq k}$-free 2-matching problem has been studied due to its close relationship with Hamiltonian cycles. It has been shown that the $C_{\leq k}$-free 2-matching problem is NP-hard for the case of $k \geq 5$ by Papadimitriou (described by Cornuéjols and Pulleyblank [6]). Currently, the triangle-free 2-matching problem and its generalization, the $C_{\leq k}$-free 2-matching problem, are suggested to be related to 2-ECSS and 2-VCSS [2, 17, 22], and from this viewpoint as well, determining the complexity for the case $k = 4$ remains an important open question.

## 2 Preliminaries

A graph $G = (V, E)$ is said to be *2-vertex-connected* (abbreviated as *2VC*) if $G - \{v\}$ is connected for every $v \in V$ and $|V| > 3$. In the 2-Vertex-Connected Spanning Subgraph problem (2-VCSS), given a graph $G = (V, E)$, the objective is to find an edge subset $S \subseteq E$ of minimum cardinality such that the subgraph $(V, S)$ is 2VC, if one exists. We denote by $\mathsf{opt}(G)$ the optimal value of 2-VCSS on $G$.

In what follows, when the vertex set $V$ is clear from the context, we will often identify an edge subset $S \subseteq E$ with the induced subgraph $(V, S)$. For example, a connected component (simply called a *component*) of $S$ refers to a connected component of the subgraph $(V, S)$. A *path* $P$ (of length $k$) is defined as a sequence of distinct consecutive edges $u_1u_2, u_2u_3, \ldots, u_ku_{k+1}$ such that $u_i$ and $u_j$ are distinct for any $i, j$. We sometimes identify a path $P$ with the set of edges it contains, ignoring their order. We call a cycle $C = (v_1, v_2, \ldots, v_k)$ of length $k$ a *$k$-cycle*, and in particular, when its length is three, we call it a *triangle*. For a component or a cycle $C$, we denote its vertex set by $V(C)$ and its edge set by $E(C)$. For a vertex $v \in V$, we denote the set of vertices adjacent to $v$ by $N(v)$. For a vertex subset $W \subseteq V$, we denote the edge set between $W$ and $V \setminus W$ by $\delta(W)$. For a vertex subset $W \subseteq V$, let $G[W]$ denote the subgraph of $G$ induced by $W$.

### 2.1 $\mathcal{T}$-Free 2-Matching/2-Edge-Cover

In this subsection, we introduce the concept of a $\mathcal{T}$-free 2-edge-cover, which is required for constructing the initial solution. Let $G = (V, E)$ be a graph and $\mathcal{T}$ be a subset of triangles in $G$. An edge set $F$ is *$\mathcal{T}$-free* if no component of $F$ is a triangle in $\mathcal{T}$. An edge set $F$ is a *2-matching* if each vertex is incident to at most two edges in $F$. An edge set $F$ is a *2-edge-cover* if each vertex is incident to at least two edges in $F$. For a graph $G$ and a set of triangles $\mathcal{T}$ in $G$, we denote the maximum size of a $\mathcal{T}$-free 2-matching in $G$ by $\nu(G, \mathcal{T})$, and the minimum size of a $\mathcal{T}$-free 2-edge-cover in $G$ by $\rho(G, \mathcal{T})$. Kobayashi and Noguchi [22] proposed a PTAS for maximizing the size of a $\mathcal{T}$-free 2-matching.

▶ **Lemma 2** ( [22, Corollary 4.1]). *Let $\varepsilon > 0$. Let $G$ be a graph that may contain parallel edges and let $\mathcal{T}$ be a set of triangles in $G$ such that no edge of any triangle in $\mathcal{T}$ has a parallel edge.*[1] *Then, one can compute a $\mathcal{T}$-free 2-matching of size at least $(1 - \varepsilon)\nu(G, \mathcal{T})$ in polynomial time.*

[1] In the original statement of [22, Corollary 4.1], it is assumed that $G$ has no parallel edges. However, the argument also applies to graphs with parallel edges, provided that no edge of any triangle in $\mathcal{T}$ has a parallel edge; see Appendix A.

Moreover, in the concluding remarks of [21], Kobayashi and Noguchi proposed an algorithm that constructs a $(1+\varepsilon)$-approximate minimum triangle-free 2-edge-cover from a $(1-\varepsilon)$-approximate maximum triangle-free 2-matching, where "triangle-free" corresponds to the case in which $\mathcal{T}$ is the set of all triangles in the input graph. In fact, the same statement holds even when $\mathcal{T}$ is an arbitrary subset of triangles. Indeed, by replacing "triangle" with "triangle in $\mathcal{T}$" and "triangle-free" with "$\mathcal{T}$-free" in the proofs in [21], we obtain the following.

▶ **Lemma 3** ( [21, Proposition 10]). *For any $\varepsilon > 0$, given a $\mathcal{T}$-free 2-matching in $G$ of size at least $(1-\varepsilon)\nu(G,\mathcal{T})$, one can compute a $\mathcal{T}$-free 2-edge-cover in $G$ of size at most $(1+\varepsilon)\rho(G,\mathcal{T})$ in polynomial time.*

From Lemmas 2 and 3, we obtain a PTAS for computing a $\mathcal{T}$-free 2-edge-cover of minimum size.

▶ **Corollary 4.** *Let $\varepsilon > 0$. Let $G$ be a graph that may contain parallel edges and let $\mathcal{T}$ be a set of triangles in $G$ such that no edge of any triangle in $\mathcal{T}$ has a parallel edge. Then, one can compute a $\mathcal{T}$-free 2-edge-cover in $G$ of size at most $(1+\varepsilon)\rho(G,\mathcal{T})$ in polynomial time.*

## 2.2 Structured Graph

In the algorithm for 2-VCSS by Bosch-Calvo, Grandoni, and Jabal Ameli [3], they first reduce the problem to the case when the input graph is nearly 3-vertex-connected, where such a graph is called a *structured graph*.

▶ **Definition 5** (structured graph). *A 2VC simple graph $G = (V,E)$ is* structured *if it does not contain the following structures:*

- *(irrelevant edge) an edge $e = uv \in E$ such that $G - \{u,v\}$ is not connected (i.e., $\{u,v\}$ is a 2-vertex-cut of $G$);*
- *(non-isolating 2-vertex-cut) a 2-vertex-cut $\{u,v\} \subseteq V$ such that $G - \{u,v\}$ has either at least three components or exactly two components, each containing at least two vertices;*
- *(removable 5-cycle) a 5-cycle $C$ of $G$ with at least two vertices of degree 2 in $G$.*

▶ **Lemma 6** (Bosch-Calvo, Grandoni, and Jabal Ameli [3, Lemma 2.8]). *Given a constant $1 < \alpha \le \frac{3}{2}$, if there exists a polynomial-time algorithm for 2-VCSS in a structured graph $G$ that returns a solution of size at most $\max\{\mathsf{opt}(G), \alpha\cdot\mathsf{opt}(G) - 2\}$, then there exists a polynomial-time $\alpha$-approximation algorithm for 2-VCSS.*

One advantage of considering structured graphs is that they satisfy the following property, known as the *3-matching lemma*.

▶ **Lemma 7** ([3, Lemma 3.1]). *Let $G = (V,E)$ be a structured graph and $\{V_1, V_2\}$ be a partition of $V$ such that $|V_1| \ge |V_2| \ge 4$. There is a matching of size 3 between $V_1$ and $V_2$.*

Here, a *matching* between $V_1$ and $V_2$ is a set of edges in which one endpoint lies in $V_1$, the other lies in $V_2$, and no two edges share an endpoint.

When constructing an algorithm in structured graphs, it suffices to consider only the case where the number of vertices in the input graphs is at least 20 (and hence $\mathsf{opt}(G) \ge 20$); otherwise, we can compute an optimal solution by brute force. Since $\max\{\mathsf{opt}(G), \alpha\cdot\mathsf{opt}(G) - 2\} = \alpha\cdot\mathsf{opt}(G) - 2$ for $\mathsf{opt}(G) \ge 20$ and for $\alpha \ge \frac{95}{72}$, it suffices to show the following to obtain Theorem 1.

▶ **Lemma 8.** *Let $\varepsilon > 0$. Given a structured graph $G$ with at least 20 vertices, in polynomial time one can compute a feasible solution for 2-VCSS of size at most $\left(\frac{95}{72} + \varepsilon\right)\mathsf{opt}(G) - 2$.*

Therefore, in the following sections, we consider the input graph $G = (V,E)$ to be a structured graph with at least 20 vertices.

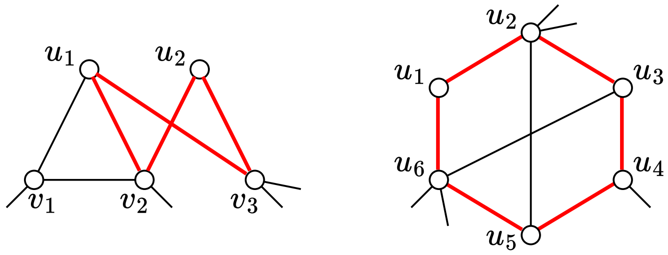


**Figure 1** A 4-cycle with an isolated pair $\{u_1, u_2\}$ and a 6-cycle with an isolated triple $\{u_1, u_3, u_5\}$.

## 3 Construction of an Initial Solution

In this section, we construct an initial solution for the 2-VCSS that is better than those used in previous algorithms. In Subsection 3.1, we introduce a *cycle-restricted* 2-edge-cover, whose minimum size provides a new lower bound on the optimal value for 2-VCSS. A cycle-restricted 2-edge-cover is defined as a 2-edge-cover that does not contain certain cycle components, and so any 2VC spanning subgraph is obviously cycle-restricted. We prove that a $(1 + \varepsilon)$-approximate solution of the minimum cycle-restricted 2-edge-cover can be computed in polynomial time, which means that we can construct an initial solution that is closer to the feasible solutions than a minimum 2-edge-cover. In Subsection 3.2, following the ideas in [3], we transform a cycle-restricted 2-edge-cover into a so-called "canonical" form without increasing the number of edges. Since we impose stronger constraints than those in [3], we refer to such a 2-edge-cover as *strongly canonical*.

### 3.1 Cycle-Restricted 2-Edge-Cover

The structures that are forbidden to be contained in cycle-restricted 2-edge-covers are triangles as well as certain 4-cycles and 6-cycles. These 4-cycles and 6-cycles are referred to as 4-cycles with an *isolated pair* and 6-cycles with an *isolated triple*, respectively.

▶ **Definition 9** (isolated pair/triple). *A vertex $u \in V$ is* isolated *by a vertex triple $\{v_1, v_2, v_3\} \subseteq V \setminus \{u\}$ if $N(u) \subseteq \{v_1, v_2, v_3\}$. A pair or a triple of vertices $W \subseteq V$ is* isolated *if each vertex in $W$ is isolated by the same vertex triple $\{v_1, v_2, v_3\} \subseteq V \setminus W$.*

Note that, for a pair $\{u_1, u_2\}$ isolated by $\{v_1, v_2, v_3\}$ in a structured graph $G$, $N(u_1) \cup N(u_2) = \{v_1, v_2, v_3\}$; otherwise, $\{v_1, v_2, v_3\}$ would contain either a cut vertex or a non-isolating 2-vertex-cut. We call a 4-cycle $C$ a *4-cycle with an isolated pair* if $V(C)$ contains an isolated pair, and we call a 6-cycle $C$ a *6-cycle with an isolated triple* if $V(C)$ contains an isolated triple (see Figure 1 for examples). Since no two vertices in an isolated triple are adjacent, we can represent a 6-cycle with an isolated triple as $C = (u_1, \dots, u_6)$, where $\{u_1, u_3, u_5\}$ is isolated by $\{u_2, u_4, u_6\}$.

We now formally define a cycle-restricted 2-edge-cover.

▶ **Definition 10** (cycle-restricted 2-edge-cover). *A 2-edge-cover $S \subseteq E$ is* cycle-restricted *if $S$ satisfies the following conditions:*

- *No component of $S$ is a triangle.*
- *No component of $S$ is a 4-cycle with an isolated pair in $G$.*
- *For every 6-cycle $C$ with an isolated triple in $G$, we have $|\delta(V(C)) \cap S| \geq 2$.*

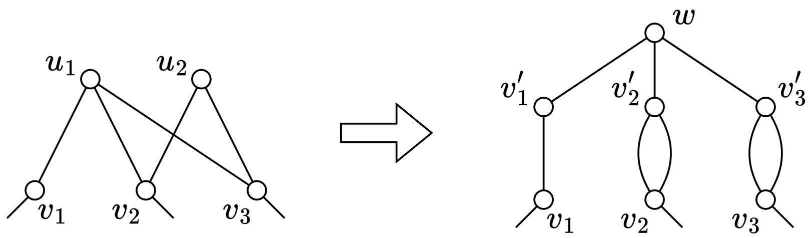


**Figure 2** Construction of $G'$.

It is clear that the edge set of any 2VC graph with at least seven vertices forms a cycle-restricted 2-edge-cover. Thus, the size of a minimum cycle-restricted 2-edge-cover in $G$ is a lower bound on $\mathsf{opt}(G)$. Let $\rho_c(G)$ denote the minimum size of a cycle-restricted 2-edge-cover in $G$.

One of our technical contributions is to present an algorithm for computing an almost-minimum cycle-restricted 2-edge-cover.

▶ **Lemma 11.** *Let $\varepsilon > 0$. Given a structured graph $G = (V, E)$, one can construct a cycle-restricted 2-edge-cover in $G$ of size at most $(1 + \varepsilon)\rho_c(G)$ in polynomial time.*

**Proof.** We reduce the problem to finding a minimum $\mathcal{T}$-free 2-edge-cover in a graph $G'$ obtained from $G$ as follows.

Take an inclusion-wise maximal set $\mathcal{C}$ of vertex-disjoint 6-cycles with an isolated triple in $G$. Then, take an inclusion-wise maximal set $\mathcal{D}$ of vertex-disjoint 4-cycles with an isolated pair in $G$ such that

- no 4-cycle in $\mathcal{D}$ contains a vertex in $\bigcup_{C\in\mathcal{C}} V(C)$; and
- for any distinct 4-cycles $C, C' \in \mathcal{D}$ with isolated pairs $\{u_1, u_2\}$ and $\{u_1', u_2'\}$, respectively, there is no edge in $G$ between $\{u_1, u_2\}$ and $\{u_1', u_2'\}$.

Let $\mathcal{P}$ be the set of isolated pairs, each corresponding to a 4-cycle in $\mathcal{D}$. For convenience, in what follows we mainly use $\mathcal{P}$ instead of $\mathcal{D}$. We transform the given graph $G$ as follows (see Figure 2 for Step 2):

1. Contract each 6-cycle $C \in \mathcal{C}$ into a single vertex $v_C$, removing any resulting self-loops but keeping parallel edges.
2. For each isolated pair $\{u_1, u_2\} \in \mathcal{P}$ and its adjacent vertices $\{v_1, v_2, v_3\}$,[2]
   a. remove $u_1$ and $u_2$, and edges incident to them,
   b. add new vertices $v_1', v_2', v_3'$, and $w$,
   c. add at most two parallel edges between $v_i$ and $v_i'$ to preserve the degree of $v_i$ for each $i \in \{1, 2, 3\}$, and
   d. add new edges $v_1'w, v_2'w$, and $v_3'w$.

Let $G' = (V', E')$ be the resulting graph, and let $\mathcal{T}$ be the set of triangles in $G'$ that do not contain the vertex $v_C$ for any $C \in \mathcal{C}$. Since $G$ is simple, any parallel edges contained in $G'$ have at least one endpoint that is either a vertex obtained by the contraction in Step 1 or a vertex added in Step 2b; therefore, none of these edges belongs to any triangle in $\mathcal{T}$. Hence, $G'$ and $\mathcal{T}$ satisfy the condition in Corollary 4.

[2] Note that one of $\{v_1, v_2, v_3\}$ is not contained in the 4-cycle in $\mathcal{D}$ corresponding to the isolated pair $\{u_1, u_2\}$. Such a vertex may be involved in a contraction in Step 1 or may belong to another 4-cycle in $\mathcal{D}$, but this does not affect the argument that follows.

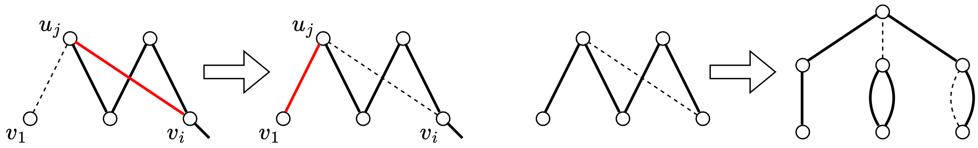


**Figure 3** Transformation of a cycle-restricted 2-edge-cover $F$ in $G$ into a $\mathcal{T}$-free 2-edge-cover $F'$ in $G'$.

We show that finding a minimum cycle-restricted 2-edge-cover in $G$ is equivalent to finding a minimum $\mathcal{T}$-free 2-edge-cover in $G'$ by establishing the following two claims.

▷ Claim 12. Given a cycle-restricted 2-edge-cover $F$ in $G$, a $\mathcal{T}$-free 2-edge-cover in $G'$ of size at most $|F| - 6|\mathcal{C}| + 2|\mathcal{P}|$ can be computed in polynomial-time. In particular, $\rho(G', \mathcal{T}) \leq \rho_c(G) - 6|\mathcal{C}| + 2|\mathcal{P}|$.

Proof. Suppose that for some isolated pair $\{u_1, u_2\} \in \mathcal{P}$ and its adjacent vertices $\{v_1, v_2, v_3\}$, both $u_1$ and $u_2$ are adjacent to only the same two vertices in $F$, namely $v_2$ and $v_3$. Since $F$ is cycle-restricted, no component of $F$ is a 4-cycle with an isolated pair; hence, $v_i$ has degree at least three in $F$ for some $i \in \{2, 3\}$. Let $u_j$ be a vertex adjacent to $v_1$ in $G$. Then, $(F \setminus \{u_j v_i\}) \cup \{u_j v_1\}$ is also a cycle-restricted 2-edge-cover of the same size as $F$ (see Figure 3 left). Therefore, by repeating this operation, we may assume that every isolated pair $\{u_1, u_2\} \in \mathcal{P}$ is adjacent to three vertices in $F$. Note that, since the isolated pairs in $\mathcal{P}$ are vertex-disjoint and nonadjacent by construction, replacing $F$ with $(F \setminus \{u_j v_i\}) \cup \{u_j v_1\}$ does not create a new pair in $\mathcal{P}$ that is adjacent to only two vertices in $F$.

After this preprocessing, we construct a 2-edge-cover $F'$ in $G'$ from $F$ using the following procedure (see Figure 3 right for (ii)):

**(i)** Set $F' := F \cap E'$.

**(ii)** For each isolated pair $\{u_1, u_2\} \in \mathcal{P}$ and its adjacent vertices $\{v_1, v_2, v_3\}$, apply the following procedure:

- **(ii-a)** for each $i \in \{1, 2, 3\}$, add $t_i$ parallel edges between $v_i$ and $v_i'$ to $F'$, where $t_i$ is the number of edges in $F$ between $v_i$ and $\{u_1, u_2\}$; and
- **(ii-b)** add two edges chosen from $v_1'w$, $v_2'w$, and $v_3'w$ to $F'$ so that each of $v_1', v_2', v_3'$, and $w$ has degree at least two in $F'$.

We show that (ii-b) is feasible. Consider the edge set $F'$ immediately after (ii-a). Since each of $v_1$, $v_2$, and $v_3$ is adjacent to $\{u_1, u_2\}$ by at least one edge in $F$ due to the preprocessing, each of $v_1'$, $v_2'$, and $v_3'$ has degree at least one in $F'$. In addition, the number of edges added in (ii-a) is equal to the number of edges between $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ in $F$. Thus, $F'$ contains at least four edges between $\{v_1, v_2, v_3\}$ and $\{v_1', v_2', v_3'\}$. Therefore, at most two vertices among $\{v_1', v_2', v_3'\}$ have degree exactly one in $F'$. Consequently, by adding two edges chosen from $v_1'w$, $v_2'w$, and $v_3'w$, we can make the degree of each vertex at least two.

Let us show that the resulting edge set $F'$ is a $\mathcal{T}$-free 2-edge-cover by using the cycle-restricted property of $F$. For each 6-cycle $C \in \mathcal{C}$, since at least two edges are incident to $V(C)$ in $F$, the vertex $v_C$ is incident to at least two edges in $F'$. Since $F$ does not have a triangle component and $G'$ does not contain a triangle involving the edges added in Step 2 of its construction, $F'$ is a $\mathcal{T}$-free 2-edge-cover.

We evaluate the size of $F'$ as follows. For a 6-cycle $C = (u_1, \ldots, u_6) \in \mathcal{C}$ with an isolated triple $\{u_1, u_3, u_5\}$, $F$ contains at least six edges incident to $u_1$, $u_3$, or $u_5$, and these edges are removed in (i). Taking into account that two edges were added for each pair in $\mathcal{P}$ in (ii), we have $|F'| \leq |F| - 6|\mathcal{C}| + 2|\mathcal{P}|$.

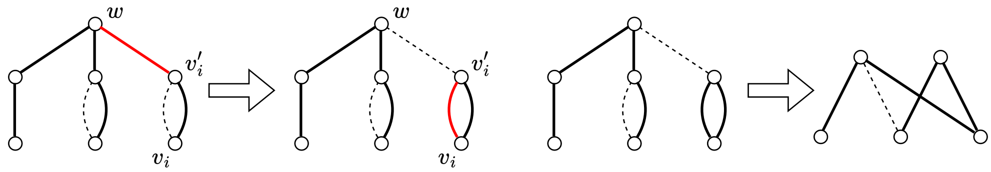


**Figure 4** Transformation of $\mathcal{T}$-free 2-edge-cover $F'$ in $G'$ to a cycle-restricted 2-edge-cover $F$ in $G$.

By applying this procedure to a minimum cycle-restricted 2-edge-cover in $G$, we obtain a $\mathcal{T}$-free 2-edge-cover of size at most $\rho_c(G) - 6|\mathcal{C}| + 2|\mathcal{P}|$. Hence, we have $\rho(G', \mathcal{T}) \leq \rho_c(G) - 6|\mathcal{C}| + 2|\mathcal{P}|$. $\lhd$

$\rhd$ Claim 13. Given a $\mathcal{T}$-free 2-edge-cover $F'$ in $G'$, a cycle-restricted 2-edge-cover in $G$ of size at most $|F'| + 6|\mathcal{C}| - 2|\mathcal{P}|$ can be computed in polynomial-time. In particular, $\rho_c(G) \leq \rho(G', \mathcal{T}) + 6|\mathcal{C}| - 2|\mathcal{P}|$.

Proof. For each $w \in V'$ added in Step 2b of the transformation from $G$ to $G'$, if $w$ is incident to three edges in $F'$, then choose $i \in \{1, 2, 3\}$ such that $G'$ has parallel edges between $v'_i$ and $v_i$ (such $i$ exists since there are at least four edges between $\{v_1, v_2, v_3\}$ and $\{v'_1, v'_2, v'_3\}$) and remove $v'_i w$ from $F'$. If this removal makes the degree of $v'_i$ equal to one, then add one edge $v'_i v_i$ to $F'$. This operation does not increase the size of $F'$ and it also preserves the property that $F'$ is a $\mathcal{T}$-free 2-edge-cover (see Figure 4 left). This operation ensures that $F'$ has exactly two edges incident to $w$, and at least four edges between $\{v'_1, v'_2, v'_3\}$ and $\{v_1, v_2, v_3\}$.

After this preprocessing, we construct a 2-edge-cover $F$ in $G$ from $F'$ using the following procedure (see Figure 4 right for (iii)):

**(i)** Set $F := F' \cap E$.

**(ii)** For each $C \in \mathcal{C}$, add the edge set of $C$ to $F$.

**(iii)** For each isolated pair $\{u_1, u_2\} \in \mathcal{P}$ and its adjacent vertices $\{v_1, v_2, v_3\}$, apply the following operation so that both $u_1$ and $u_2$ have degree at least two in $F$. For each $i \in \{1, 2, 3\}$,

 **(iii-a)** if $F'$ has two parallel edges between $v'_i$ and $v_i$, then add $u_1 v_i$ and $u_2 v_i$ to $F$, and

 **(iii-b)** if $F'$ has exactly one edge between $v'_i$ and $v_i$, then add either $u_1 v_i$ or $u_2 v_i$ to $F$.

We show that (iii) is feasible. Note that $F'$ has at least one edge between $v'_i$ and $v_i$ for each $i$, because at least two edges in $F'$ are incident to $v'_i$. The only nontrivial condition is that both $u_1$ and $u_2$ have degree at least two in $F$. Since $F'$ has at least four edges between $\{v'_1, v'_2, v'_3\}$ and $\{v_1, v_2, v_3\}$ due to the preprocessing, it contains two parallel edges between $v'_i$ and $v_i$ for some $i \in \{1, 2, 3\}$. Without loss of generality, assume $i = 3$; hence, $u_1 v_3$ and $u_2 v_3$ are added to $F$ in (iii-a). Since at least one of $v_1$ and $v_2$ (say $v_2$) is contained in the 4-cycle corresponding to $\{u_1, u_2\}$, $v_2$ is adjacent to both $u_1$ and $u_2$ in $G$. Therefore, if $u_j v_1$ is not added in $F$ for $j \in \{1, 2\}$, then we can add $u_j v_2$ to $F$ in (iii-a) or (iii-b). Then, the degrees of both $u_1$ and $u_2$ can be made at least two in $F$.

Let us show that the resulting 2-edge-cover $F$ is cycle-restricted. Assume to the contrary that $F$ contains a triangle component $C$. Since $F'$ is $\mathcal{T}$-free, by the definition of $\mathcal{T}$, $C$ must contain a vertex in some 6-cycle in $\mathcal{C}$ or some isolated pair in $\mathcal{P}$. However, any vertex in a 6-cycle in $\mathcal{C}$ or in an isolated pair in $\mathcal{P}$ lies in a component with at least five vertices in $F$, which is a contradiction. Thus, $F$ contains no triangle components.

For every 6-cycle $C \in \mathcal{C}$, since the vertex obtained by contracting $C$ is incident to at least two edges in $F'$, we have $|\delta(V(C)) \cap F| \geq 2$. Moreover, for every 6-cycle $C \notin \mathcal{C}$ with an isolated triple, since $C$ shares a vertex with some $C' \in \mathcal{C}$ by the maximality of $\mathcal{C}$, we have $|\delta(V(C)) \cap F| \geq |\delta(V(C)) \cap E(C')| \geq 2$ if $V(C) \neq V(C')$, and $|\delta(V(C)) \cap F| = |\delta(V(C')) \cap F| \geq 2$ if $V(C) = V(C')$. Therefore, $|\delta(V(C)) \cap F| \geq 2$ holds for every 6-cycle $C$ with an isolated triple.

For every isolated pair in $\mathcal{P}$, the component containing the pair has at least five vertices (the pair itself and its three adjacent vertices). This implies that no 4-cycle $C \in \mathcal{D}$ forms a component of $F$. For every 4-cycle $C \notin \mathcal{D}$ with an isolated pair, the maximality of $\mathcal{D}$ implies that there exists a 6-cycle $C' \in \mathcal{C}$ with $V(C) \cap V(C') \neq \emptyset$, or there exists an isolated pair $\{u_1, u_2\} \in \mathcal{P}$ with its adjacent vertices $\{v_1, v_2, v_3\}$ such that $V(C) \cap \{u_1, u_2, v_1, v_2, v_3\} \neq \emptyset$. Since both $V(C')$ and $\{u_1, u_2, v_1, v_2, v_3\}$ are contained in components with at least five vertices, $C$ does not form a single component in $F$ in either case. Taken together, $F$ is a cycle-restricted 2-edge-cover.

The size of $F$ obtained by this operation is at most $|F'| + 6|\mathcal{C}| - 2|\mathcal{P}|$. By applying this procedure to a minimum $\mathcal{T}$-free 2-edge-cover in $G'$, we obtain a cycle-restricted 2-edge-cover of size at most $\rho(G', \mathcal{T}) + 6|\mathcal{C}| - 2|\mathcal{P}|$. Hence, we have $\rho_c(G) \leq \rho(G', \mathcal{T}) + 6|\mathcal{C}| - 2|\mathcal{P}|$. ◁

Combining the two inequalities in Claims 12 and 13, we obtain $\rho_c(G) = \rho(G', \mathcal{T}) + 6|\mathcal{C}| - 2|\mathcal{P}|$. Moreover, these two claims imply that once either a minimum cycle-restricted 2-edge-cover in $G$ or a minimum $\mathcal{T}$-free 2-edge-cover in $G'$ is obtained, the other can be computed in polynomial time.

We find a $\mathcal{T}$-free 2-edge-cover $F' \subseteq E'$ of size at most $(1 + \frac{\varepsilon}{2})\rho(G', \mathcal{T})$ by Corollary 4. Then, we transform $F'$ into a cycle-restricted 2-edge-cover $F$ of size at most $|F'| + 6|\mathcal{C}| - 2|\mathcal{P}|$ by Claim 13. The size of $F$ can be evaluated as

$$\begin{aligned}
|F| &\leq |F'| + 6|\mathcal{C}| - 2|\mathcal{P}| \\
&\leq \left(1 + \frac{\varepsilon}{2}\right)\rho(G', \mathcal{T}) + 6|\mathcal{C}| - 2|\mathcal{P}| \\
&= \left(1 + \frac{\varepsilon}{2}\right)\left(\rho_c(G) - 6|\mathcal{C}| + 2|\mathcal{P}|\right) + 6|\mathcal{C}| - 2|\mathcal{P}| \\
&\leq \left(1 + \frac{\varepsilon}{2}\right)\rho_c(G) + \varepsilon|\mathcal{P}| \\
&\leq (1 + \varepsilon)\rho_c(G),
\end{aligned}$$

where we use $|\mathcal{P}| \leq \frac{|V|}{2} \leq \frac{\rho_c(G)}{2}$ in the last inequality. ◀

## 3.2 Strongly Canonical 2-Edge-Cover

In this subsection, we transform a cycle-restricted 2-edge-cover and impose additional structural restrictions to make it easier to handle, without increasing the number of edges.

We begin with some notation. In a 2-edge-cover, we call a component with at least seven edges *large*, and a component with at most six edges *small*. In a component $C$, we call a maximal 2VC subgraph with at least three vertices a *block*, and an edge whose removal disconnects $C$ a *bridge*. Note that every edge in $C$ is either a bridge or belongs to a block. If $C$ is not 2VC, we call it *complex*. In a complex component $C$, we call a block with only one cut vertex a *leaf-block*.

We now introduce a *strongly canonical 2-edge-cover*, which is in the same spirit as a *canonical 2-edge-cover* in [3] but is subject to more constraints than it.

▶ **Definition 14** (canonical 2-edge-cover [3]). *A 2-edge-cover $F$ is* canonical *if, in $F$, (1) every component with at most five edges is a cycle, and (2) each leaf-block in a complex component contains at least five vertices.*

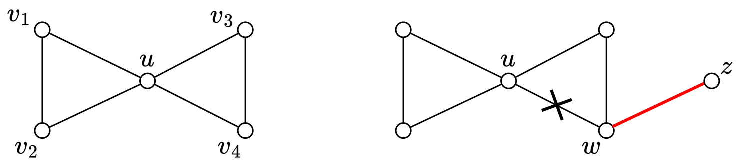


**Figure 5** Removing a bowtie.

▶ **Definition 15** (strongly canonical 2-edge-cover)**.** *A cycle-restricted 2-edge-cover $F$ is* strongly canonical *if, in $F$, (1) every small component is a cycle, and (2) each leaf-block in a complex component contains at least five vertices.*

The difference between these two concepts lies in components with six edges: in this paper, such components are treated as small components and restricted to cycles, whereas in [3] they are treated as large components and are not necessarily cycles. Although this difference is technical, it is essential for achieving an approximation ratio better than 4/3. In what follows, whenever we say that $F$ is a strongly canonical 2-edge-cover, it satisfies the cycle-restricted property as well. It is clear that a strongly canonical 2-edge-cover is also canonical.

We show how a strongly canonical 2-edge-cover can be obtained from a cycle-restricted 2-edge-cover without increasing its size; similar transformations have also been shown in [11].

▶ **Lemma 16.** *Given a cycle-restricted 2-edge-cover $F$ in a structured graph $G = (V, E)$, in polynomial time one can compute a strongly canonical 2-edge-cover $S$ in $G$ with $|S| = |F|$.*

**Proof.** We start with $S := F$. We show that if $S$ is a not strongly canonical 2-edge-cover, in polynomial time, we can construct a cycle-restricted 2-edge-cover of the same size that is lexicographically smaller than $S$ with respect to $(\mathrm{comp}(S), \mathrm{br}(S))$, where $\mathrm{comp}(S)$ is the number of components in $S$, and $\mathrm{br}(S)$ is the number of bridges in $S$.

1. Suppose $S$ contains a small component $C$ which is not a cycle. Then, the number of edges in $C$, which is at most six, is more than the number of vertices in $C$. Since $S$ contains no triangle component, the pair $(|V(C)|, |E(C)|)$ is $(4, 5)$, $(4, 6)$, or $(5, 6)$.
If $C$ is a cycle with at least one chord $e$, we can choose one edge $f \in \delta(V(C))$ and replace $S$ with $(S \setminus \{e\}) \cup \{f\}$. This operation decreases $\mathrm{comp}(S)$ without increasing the size.
Otherwise, $C$ must be either a bowtie or a $K_{2,3}$ (see Figures 5 and 6). Suppose that $C$ is a bowtie that has two triangles $(u, v_1, v_2)$ and $(u, v_3, v_4)$. Since $u$ is not a cut vertex of $G$, there is an edge $wz \in E \setminus S$ with $w \in \{v_1, v_2, v_3, v_4\}$ and $z \in V \setminus V(C)$. In this case, we can replace $S$ with $(S \setminus \{wu\}) \cup \{wz\}$. Suppose that $C$ is a $K_{2,3}$ with two sides $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$. Since $\{u_1, u_2\}$ is not a non-isolating 2-vertex cut of $G$, there is an edge $wz \in E \setminus S$ with $w \in \{v_1, v_2, v_3\}$ and $z \in V \setminus V(C)$. In this case, we can replace $S$ with $(S \setminus \{wu_1\}) \cup \{wz\}$. These operations decrease $\mathrm{comp}(S)$ without increasing the size.
2. Suppose $S$ contains a leaf-block $B$ with at most four vertices in some complex component $C$. Then, $B$ is a 3-cycle or a 4-cycle, possibly with chords. Let $(v_1, \ldots, v_k)$ be a cycle in $B$, where $k \in \{3, 4\}$, and assume without loss of generality that $v_1$ is the unique cut vertex of $B$ in $C$. When $k = 3$, since $v_1$ is not a cut vertex in $G$, there is an edge $wz \in E \setminus S$ with $w \in \{v_2, v_3\}$ and $z \in V \setminus V(B)$. When $k = 4$, since $\{v_1, v_3\}$ is not a non-isolating 2-vertex-cut of $G$, there is an edge $wz \in E \setminus S$ with $w \in \{v_2, v_4\}$ and $z \in V \setminus V(B)$. In either case, we can replace $S$ with $(S \setminus \{wv_1\}) \cup \{wz\}$. If $z$ belongs to a different component from $v_1$ in $S$, this operation decreases $\mathrm{comp}(S)$ without increasing the size. Otherwise, if $z$ belongs to the same component as $v_1$ in $S$, this operation decreases $\mathrm{br}(S)$ without increasing the size and $\mathrm{comp}(S)$.

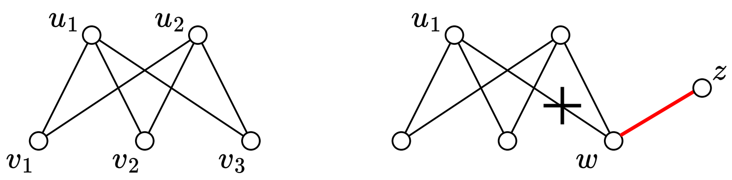


**Figure 6** Removing a $K_{2,3}$.

Note that the above operations preserve the cycle-restricted property. As long as $S$ does not satisfy Conditions (1) and (2) in Definition 15, at least one of the above operations is applicable, and each operation can be performed in polynomial time. Since each operation strictly decreases the lexicographical order of $(\mathrm{comp}(S), \mathrm{br}(S))$, we can transform $S$ into a strongly canonical 2-edge-cover of the same size in polynomial time. ◀

## 4 Transformation to a Feasible Solution: Overview

In the full version [23], we show how to transform this strongly canonical 2-edge-cover $S$, taken as the initial solution, into a feasible solution. Although our algorithm is based on the transformation proposed in the previous work [3], by leveraging the properties of cycle-restricted 2-edge-covers we are able to limit the increase in the number of edges compared with that transformation. Here, we briefly outline why each property of a cycle-restricted 2-edge-cover is useful in the transformation (see the full version for details).

- In [3], the authors handled 2-edge-covers that could contain triangle components and proposed an algorithm to remove such components without increasing the cost. This step was one of the reasons why the approximation ratio analysis became tight at 4/3. Since cycle-restricted 2-edge-covers do not contain triangle components, we do not need to perform such an operation in our algorithm.
- In [3], 6-cycles were treated as large components, and two credits were used to merge a 6-cycle with other components. This was also one of the factors that made the approximation ratio tight at 4/3. In a cycle-restricted 2-edge-cover, the most unfavorable type of 6-cycle component (namely, 6-cycles with an isolated triple) does not exist. Therefore, in our algorithm, 6-cycles can be handled using an analysis similar to that for 4- and 5-cycles.
- In [3, Section 3.1], removing 4-cycle components proved challenging. Indeed, some 4-cycle components were not fully removed at that stage, and underwent more complex operations later. In this paper, leveraging the fact that there is no 4-cycle components with an isolated pair in a cycle-restricted 2-edge-cover, we can remove all 4-cycle components within this stage, resulting in a much clearer analysis.

## 5 Concluding Remarks

In this paper, we presented an approximation algorithm for 2-VCSS that improves upon the previous best approximation ratio of $\frac{4}{3}$ to $\frac{95}{72} + \varepsilon$ for any $\varepsilon > 0$. This bound is tight only in the case of a transformation dealing with 6-cycles. Therefore, if it is possible to start with a strongly canonical 2-edge-cover containing few 6-cycle components, this bound can be further improved. Based on this observation, one possible approach to further improving the approximation ratio is to develop a transformation algorithm that performs efficiently when the initial 2-edge-cover contains many 6-cycle components.

The $\varepsilon$ factor in the approximation ratio $\frac{95}{72} + \varepsilon$ arises from the use of a PTAS, rather than an exact algorithm, for computing a maximum $\mathcal{T}$-free 2-matching. Therefore, if a maximum $\mathcal{T}$-free 2-matching can be computed exactly in polynomial time, this factor could be eliminated. It is not clear whether the polynomial-time algorithms for the triangle-free 2-matching problem [14, 15] can be extended to $\mathcal{T}$-free 2-matchings.

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## A Remarks on Lemma 2

In the original statement of Lemma 2, i.e., [22, Corollary 4.1], it is assumed that $G$ has no parallel edges. In particular, in its proof, this assumption is used to establish [22, Observation 1]. However, the argument also applies to graphs with parallel edges, provided that no edge of any triangle in $\mathcal{T}$ has a parallel edge. To complement this, we show that [22, Observation 1] holds under this assumption as well.

▶ **Lemma 17** ([22, Observation 1] under a weaker assumption). *Let $G$ be a graph graph that may contain self-loops and parallel edges, and let $\mathcal{T}$ be a subset of triangles in $G$ such that no edge of any triangle in $\mathcal{T}$ has a parallel edge. Let $M$ be a 2-matching, and $T$ be a triangle (possibly, $T \notin \mathcal{T}$) in $G$ such that $|M \cap E(T)| = 2$. Then, $M$ contains no triangle $T' \in \mathcal{T}$ such that $E(T') \cap E(T) \neq \emptyset$.*

**Proof.** Let $T = (u, v, w)$ be a triangle with edges $e_1 = uv$, $e_2 = vw$, and $e_3 = wu$ such that $\{e_1, e_2\} \subseteq M$ and $e_3 \notin M$. Assume to the contrary that $M$ contains a triangle $T' \in \mathcal{T}$ such that $E(T') \cap E(T) \neq \emptyset$. Since $M$ contains at most two edges incident to $v$, we have that $M \cap \delta(\{v\}) = \{e_1, e_2\}$, and hence $\{e_1, e_2\} \subseteq E(T')$. Therefore, $T'$ contains an edge $e'_3$ between $w$ and $u$. By the assumption, $e'_3$ has no parallel edge, implying that $e'_3 = e_3$. This contradicts the fact that $e_3 \notin M$. ◀