Positional Determinacy with Colored Vertices: A 1-To-2-Player Lift

Raphaël Berthon
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Stéphane Le Roux
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Abstract

Positional determinacy of vertex-colored parity games was proved in the 1990s, which directly implies positional determinacy of edge-colored parity games. In 2006, it was shown that if a prefix-independent color-based objective ensures that every edge-colored two-player turn-based game is positionally determined, this objective is equivalent to a parity objective. We prove a similar result for vertex-colored games, namely that the following are equivalent for any prefix-independent objective W over a finite set of colors:

- W is positionally determined on all vertex-colored one-player games.
- W is positionally determined on all vertex-colored two-player games.
- W is equivalent to a parity objective on ordrerd pairs of colors.

We prove that finiteness of the color set is required for our equivalence to hold. Beyond this 1-to-2-player lift, the technique that we develop to handle the pairs of colors establishes a promising 2-way correspondence between edge-colored games and vertex-colored games.

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1 Introduction

Context. Game theory is applied to many fields such as economics, political science, evolutionary biology, and is used for model-checking processes in the industry [6]. While many game-theoretic models allow for multiple players and non-zero-sum payoffs, fundamental games in logic and computer science usually involve two players in a win-lose setting, where every play results in a win for exactly one player.

In model checking, these win-lose games are typically turn-based and played on a finite or infinite labeled graph, called an arena. Starting from an initial vertex, the player owning the current vertex selects an outgoing edge to reach the next vertex. This continues indefinitely, producing an infinite path. A player's strategy maps the history of visited vertices to her next move. Specifically, for every finite path ending at a vertex owned by that player, the strategy selects the edge to be followed. A pair of one strategy per player induces a unique infinite sequence of edges and vertices starting from the initial vertex, constituting the play.

General strategies can be complex to implement: in a play of infinite duration, the number of distinct histories is infinite; therefore, implementing such a strategy generally requires a mechanism to store and process a history of unbounded length. Much work has thus focused on strategies that require only finite memory, usually represented as a finite-state machine, or no memory at all. In the latter case, strategies map each position in the arena to a single outgoing edge, and are said to be positional (or memoryless).

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The objective of the game is defined independently of the arena, as a set of infinite words $W \subseteq \Gamma^\omega$, where Γ is a set of colors. Either vertices or edges are assigned colors from Γ, to obtain vertex-colored arenas (or games) or edge-colored arenas (or games). In both cases, every play generates an infinite sequence of colors called a trace. The player called Eve wins if this trace is in W, and Adam wins otherwise.

To state our results, we use the concept of *parity objective*: an infinite sequence of integers, chosen from a finite set called the set of priorities, satisfies the parity objective if the greatest integer occurring infinitely often is even. In [5], the authors considered *generalized parity objectives*: intuitively, they are defined on infinite sequences of colors but are similar to the parity objective via a *priority function* from colors to integers.

In the context of infinite-duration games, prefix-independence is an important property for win-lose objectives as well as for more general objective functions (like gain functions). An objective is prefix-independent, also called tail, if the winner of a play is unaffected by the addition or removal of any finite initial sequence of colors. This property is central to the analysis of infinite games: e.g. if there is a winning strategy from a given vertex, then there exists a winning strategy from every vertex from which the player can reach this given vertex.

An objective is said to be positionally determined on a class of arenas if a player with a winning strategy from a given vertex also has a positional one from that vertex. Colcombet and Niwiński showed in [5] that prefix-independent objectives that are positionally determined on both finite and infinite edge-colored arenas are exactly those equivalent to the parity objective. The case for vertex-colored games has been opened until now.

(a) Edge-colored hub-cycle arena. **(b)** Vertex-colored hub-cycle arena.

Figure 1 Two arenas where Eve controls all vertices.

▶ **Example 1.** Consider the objective $W = (a+b)^*(ab)^\omega$ on $\Gamma = \{a, b\}$, defining words that eventually alternate between a and b. It will be our running example throughout the paper.
Edge-colored case. In the edge-colored game defined by W and the arena from Figure 1a (which we will call a hub-cycle game), Eve, who controls the unique vertex, has a winning strategy, but no positional winning strategy: a non-positional strategy can alternate between edges a and b, producing the trace $(ab)^\omega$, but any positional strategy for Eve must choose a single edge $e \in \{a, b\}$ to play infinitely; this eventually results in either a^ω or b^ω.
Vertex-colored case. By contrast, W is positionally determined on the vertex-colored arena from Figure 1b. Eve can play the positional strategy visiting a from b and visiting b from a. Starting from a, this produces the infinite sequence $(ab)^\omega \in W$. We will later see that W is positionally determined on all vertex-colored arenas.

Contribution. In this article we distinguish two settings: generalized parity objectives defined on sequences of colors (as in [5]), and also those defined on sequences of ordered pairs of colors.

Furthermore, we identify a specific class of strongly connected arenas already occurring in [5], which we call *hub-cycle arenas*, that contain at most one vertex with more than one outgoing edge. In this article, colors are assigned either to the edges or to the vertices. A *hub-cycle game* is a one-player game played on such an arena. Figure 1 gives two examples of hub-cycle arenas.

Let Γ be a set of colors. Our main contributions characterize positional determinacy via these hub-cycle arenas:

1. If Γ is finite, the following are equivalent for any prefix-independent objective $W \subseteq \Gamma^\omega$:
   a. W is positionally determined on all vertex-colored hub-cycle (one-player) arenas.
   b. W is positionally determined on all vertex-colored two-player arenas.
   c. W is a generalized parity objective on pairs.

While 1-to-2-player lifts for edge-colored arenas already appear in the literature [8, 2, 1, 3], this is to our knowledge the first such result for vertex-colored arenas. We also prove that we cannot simply drop the assumption that Γ is finite. Several works consider half-positionality [11, 14, 3], where Eve has a positional winning strategy whenever she has a winning strategy. We discuss related works in more details in Section 7.

Outline. Section 2 provides preliminary definitions. Section 3 recalls the existing 1-to-2-player lift for edge-colored arenas. Vertex-colored arenas are addressed in Section 4 and Section 5 via reductions to generalized parity objectives on anchored words and then pairs. Section 6 compares our results with existing literature, while Section 7 proposes future directions. Finally, Section 8 offers concluding remarks.

2 Definitions

Since our results extends the results of [5], we broadly adopt their notational framework to facilitate the comparison with the original proofs and definitions.

Let $\mathbb{N} = \{0, 1, 2 \dots\}$ be the set of natural numbers. For $i, j \in \mathbb{N}$, $i \leq j$ we denote by $[i, j] \subseteq \mathbb{N}$ the set $\{i, i+1, \dots, j\}$. Let Γ be a non-empty, possibly infinite set. Let Γ^*, Γ^+, and Γ^ω respectively denote the sets of finite, non-empty finite, and infinite words over Γ, with ε representing the empty word. For $a \in \Gamma$, let $a^+ := \{a\}^+$. For $E \subseteq \Gamma^*$ and $W \subseteq \Gamma^* \cup \Gamma^\omega$, let EW denote the concatenation $\{uv \mid u \in E, v \in W\}$. The complement of $W \subseteq \Gamma^\omega$ is $\overline{W} := \Gamma^\omega \setminus W$. For $u \in \Gamma^+$, $u^\omega := uuu \dots \in \Gamma^\omega$, and for $E \subseteq \Gamma^+$, we take $E^\omega = \{u_0 u_1 u_2 \dots \mid \forall n \in \mathbb{N}, u_n \in E\}$.

▶ **Definition 2** (Arenas). *Let Γ be a non-empty, possibly infinite set of colors. A* ***two-player edge-colored arena*** *is a tuple $A = \langle V_E, V_A, E \rangle$, where the set of vertices $V = V_E \cup V_A$ is the disjoint union of the sets of vertices V_E and V_A, belonging to Eve and Adam, respectively. $E \subseteq V \times \Gamma \times V$ is the set of colored edges. We assume that every vertex $v \in V$ has at least one outgoing edge. The class of all such arenas is $\mathcal{A}_e$. An arena with either $V_E = \varnothing$ or $V_A = \varnothing$ is a* ***one-player*** *arena.*

The class $\mathcal{A}_v \subsetneq \mathcal{A}_e$ of ***vertex-colored arenas*** *contains all arenas satisfying the following constraint: for all $v, v', v'' \in V$ and $a, b \in \Gamma$, if $(v, a, v') \in E$ and $(v, b, v'') \in E$, then $a = b$. This amounts to a coloring on vertices.*

▶ **Definition 3** (Games and Plays). *A* ***game*** *is a tuple $G = \langle A, W \rangle$ where $A \in \mathcal{A}_e$ is an arena and $W \subseteq \Gamma^\omega$ is a winning objective. A* ***play*** *in G is an infinite sequence of edges $\pi = (v_0, a_0, v_1, a_1, \dots)$ such that $(v_i, a_i, v_{i+1}) \in E$ for all $i \in \mathbb{N}$. The resulting trace is the infinite word $w_\pi = a_0 a_1 \dots \in \Gamma^\omega$. Eve wins the play if $w_\pi \in W$; otherwise, Adam wins. We distinguish between* ***edge-colored*** *and* ***vertex-colored games*** *based on the class of arenas upon which they are defined, and also define* ***one-player games*** *that are played on a one-player arena.*

▶ **Definition 4** (Strategies). *Let $A = \langle V_E, V_A, E\rangle$ be an arena. A* ***history*** *is a finite path $h = (v_0, a_0, v_1, a_1, \dots, v_n)$ such that $(v_i, a_i, v_{i+1}) \in E$ for all $0 \le i < n$. The set of all histories is $\mathcal{H}$, and $\mathcal{H}_X \subseteq \mathcal{H}$ denotes histories ending in $v_n \in V_X$ for player $X \in \{E, A\}$.*

A ***strategy*** *for player X is a function $\sigma_X : \mathcal{H}_X \to E$ such that for any $h = (\dots, v_n)$, $\sigma_X(h)$ is an edge $(v_n, a, v) \in E$. A play π is* ***consistent*** *with strategy σ_X if for every $i \in \mathbb{N}$ where $v_i \in V_X$, the transition satisfies $(v_i, a_i, v_{i+1}) = \sigma_X(v_0, a_0, \dots, v_i)$.*

A strategy σ_X is ***positional*** *if it depends only on the current vertex. It is represented as a function $\sigma_X : V_X \to E$ where $\sigma_X(v)$ is an outgoing edge from v. A strategy σ_X is* ***winning for player X*** *from an initial vertex v_0 if every play π starting at v_0 and consistent with σ_X is won by X. In this case, v_0 is a* ***winning vertex*** *for player X.*

▶ **Definition 5** (Positional determinacy). ***A game*** *$G = \langle A, W\rangle$ is positionally determined if, from any initial vertex $v_0 \in V$, one of the players has a positional winning strategy.*

An objective *$W \subseteq \Gamma^\omega$ is positionally determined on a class of arenas $\mathcal{A} \subseteq \mathcal{A}_e$ if for every arena $A \in \mathcal{A}$, the game $\langle A, W\rangle$ is positionally determined.*

▶ **Definition 6** (Prefix-independence). *An objective $W \subseteq \Gamma^\omega$ is* ***prefix-independent****, also called* ***uniform*** *or a* ***tail-objective*** *if $\Gamma W = W$.*

▶ Remark 7. For prefix-independent objectives, if a game is positionally determined for a player, then this player has a single positional strategy that wins from every vertex from where she can win. See Lemma 2.12 in [9] or Lemma 5 in [5] for a complete proof.

▶ **Definition 8** (Parity Objectives). *Let $n \in \mathbb{N}$.*

- *$W \subseteq [0, n]^\omega$ is a* ***parity objective*** *(of order n) if a word $u = u_0u_1 \cdots \in W$ iff $\limsup_{i\to\infty} u_i$ is even. We refer to $[0, n]$ as the set of priorities.*
- *$W \subseteq \Gamma^\omega$ is a* ***generalized parity objective*** *(of order n) if there exists a priority function $p : \Gamma \to \{0, \dots, n\}$ such that a word $u = u_0u_1 \cdots \in W$ iff $\limsup_{i\to\infty} p(u_i)$ is even.*
- *$W \subseteq \Gamma^\omega$ is a* ***generalized parity objective on pairs*** *(of order n) if there exists a priority function $p : \Gamma^2 \to \{0, \dots, n\}$ such that $u \in W$ iff $\limsup_{i\to\infty} p(u_i, u_{i+1})$ is even.*

▶ **Example 9.** We again consider the objective $W = (a + b)^*(ab)^\omega$ on $\Gamma = \{a, b\}$. This is a **generalized parity objective on pairs** of order 1. Indeed, we can define the priority function $p : \{a, b\}^2 \to \{0, 1\}$ such that $p(a, b) = p(b, a) = 0$ and $p(a, a) = p(b, b) = 1$.

3 Existing results on edge-colored games

For Γ a non-empty possibly infinite set of colors, and $W \subseteq \Gamma^\omega$, we define the set of winning finite cyclic words as $W_f := \{u \in \Gamma^+ \mid u^\omega \in W\}$.

We consider Γ, a possibly infinite set of colors. We recall the proof of the following, which includes a result demonstrated in Kopczyński's thesis [12], that extends [5] to one player games:

2. For any prefix-independent objective $W \subseteq \Gamma^\omega$, the following are equivalent:
  a. W is positionally determined on all edge-colored one-player arenas.
  b. W is positionally determined on all edge-colored two-player arenas.
  c. W is a generalized parity objective.

Sketch of proof. The implication 2b ⇒ 2a follows immediately as one-player arenas are a subclass of two-player arenas. The implication 2c ⇒ 2b, was established in the 1990s [7, 13, 15], showing that generalized parity objectives are positionally determined on vertex-colored arenas; this result extends directly to edge-colored arenas. In 2006, [5] proved the converse, i.e. 2b ⇒ 2c. Finally, 2a ⇒ 2c has been shown in 2008 by Kopczyński [12]. ◀

▶ **Example 10.** We come back to our running example, the objective $W = (a+b)^*(ab)^\omega$ on $\Gamma = \{a, b\}$. Example 1 shows that W is not positional on the edge-colored arena defined in Figure 1a. By the equivalence between 2b and 2c above, W is not a generalized parity objective.

The proof of our Lemma 20 is very similar to the one in [12], in particular to the following proposition:

▶ **Proposition 11.** *For any set of colors Γ, suppose $W \subseteq \Gamma^\omega$ is prefix-independent and positionally determined on edge-colored one-player arenas. Then, for any $L, L' \subseteq \Gamma^*$:*

$$\forall u \in L, \exists v \in L', uv \in W_f \iff \exists v \in L', \forall u \in L, uv \in W_f$$
$$\forall u \in L, \exists v \in L', uv \in \overline{W}_f \iff \exists v \in L', \forall u \in L, uv \in \overline{W}_f$$

4 Vertex-colored games: parity on words

Throughout the paper, we focus on a restricted class of one-player arenas that serves as a building block for our results.

▶ **Definition 12** (Hub-cycle arenas). *A* ***hub-cycle arena*** *is a strongly connected, one-player arena where at most one vertex has an out-degree strictly greater than one. A game is* ***hub-cycle*** *if its underlying arena is a hub-cycle arena, and can be edge-colored or vertex-colored.*

We find that positional determinacy for edge-colored two-player games is equivalent to its restriction to edge-colored hub-cycle games (which are one-player games), as stated by our characterization.

To study vertex-colored games, we begin by introducing a generalization of parity objectives, where priorities are assigned to finite words on colors instead of individual colors. For Γ a non-empty possibly infinite set of colors, and $W \subseteq \Gamma^\omega$, and for every letter $a \in \Gamma$, we define the set of winning finite cyclic words anchored at a as $W_{a,f} = \{u \in a\Gamma^* \mid u^\omega \in W\} \subseteq W_f$.

▶ **Definition 13** (Generalized parity objective on anchored words). *Let Γ be a non-empty set of colors and $W \subseteq \Gamma^\omega$. We say W is a* ***generalized parity objective on anchored words*** *if for every letter $a \in \Gamma$, there exists $n \in \mathbb{N}$ and a priority function $p_a : a\Gamma^* \to \{0, \dots, n\}$ such that the following holds:*

For every word $w \in \Gamma^\omega$ where a occurs infinitely often, let $aw_0, aw_1, aw_2, \dots$ (with $w_i \in \Gamma^*$) be any sequence of factors of w such that $w = xaw_0aw_1aw_2\dots$ for some $x \in \Gamma^*$. Then $w \in W$ if and only if $\limsup_{i\to\infty} p_a(aw_i)$ is even.

Note that the priority functions $(p_a)_{a\in\Gamma}$ of a generalized parity objective on anchored words do not specify membership in W for sequences where no colors occur infinitely often. If Γ is finite, they fully specify W, though. Also note that in general, these functions must be consistent with each other, e.g. when two colors occur infinitely often in a sequence.

▶ **Example 14.** We illustrate generalized parity objectives on anchored words with our running example $W = (a+b)^*(ab)^\omega$ on $\Gamma = \{a, b\}$. Define $p_a(au) = 0$ if $u \in b(ab)^*$ and $p_a(au) = 1$ otherwise; define $p_b(bu) = 0$ if $u \in a(ba)^*$ and $p_b(bu) = 1$ otherwise. If a word w contains infinitely many a, we present w as $w = xaw_0aw_1aw_2\dots$, and otherwise as $w = xbw_0bw_1bw_2\dots$. In both cases, the even priority 0 is only achieved by traces that eventually only alternate between a and b.

We add the condition 1d to our equivalence on vertex-colored games below. Condition 1d is a useful and informative intermediate step from condition 1a to condition 1c.

1. If Γ is finite, the following are equivalent for any prefix-independent objective $W \subseteq \Gamma^\omega$:
   a. W is positionally determined on all vertex-colored hub-cycle one-player arenas.
   b. W is positionally determined on all vertex-colored two-player arenas.
   c. W is a generalized parity objective on pairs.
   d. W is a generalized parity objective on anchored words.

As in the edge-colored case, the implication 1b $\Rightarrow$ 1a is direct since hub-cycle games are a subclass of two-player games. We prove 1a $\Rightarrow$ 1d by progressing up to Lemma 27 using techniques similar to the edge-colored proof in [5]. Our proof holds even when Γ is infinite:

▶ **Theorem 15.** *Let Γ be a non-empty possibly infinite set of colors. If $W \subseteq \Gamma^\omega$ is prefix-independent and positionally determined on vertex-colored hub-cycle arenas, then W is a generalized parity objective on anchored words.*

We start with general results on languages, then show properties implied by prefix-independence and positional determinacy.

▶ **Lemma 16.** *Let $W \subseteq \Gamma^\omega$.*
1. $(\overline{W})_f = \Gamma^+ \setminus W_f$.
2. *Assume* $\forall u, v \in \Gamma^+, uv \in W_f \Rightarrow vu \in W_f$. *Then* $\forall u, v \in \Gamma^+, uv \in (\overline{W})_f \Leftrightarrow vu \in (\overline{W})_f$.

Proof.
1. Let $u \in \Gamma^+$. Then $u \in (\overline{W})_f$ iff $u^\omega \in \overline{W}$ iff $u^\omega \notin W$ iff $u \notin W_f$ iff $u \in \Gamma^+ \setminus W_f$. (Note that $\varepsilon \notin (\overline{W})_f \cup (\Gamma^+ \setminus W_f)$.)
2. Let $u, v \in \Gamma^+$ be such that $uv \in (\overline{W})_f$. So $uv \notin W_f$ by Lemma 16.1. By contraposition of the assumption, $vu \notin W_f$. Again by Lemma 16.1, $vu \in (\overline{W})_f$ (since $vu \in \Gamma^+$). This implication is an equivalence by symmetry. ◀

▶ **Lemma 17.** *Consider $W \subseteq \Gamma^\omega$ prefix-independent. Let $K, L \subseteq \Gamma^*$.*
1. $KL \subseteq W_f$ *iff* $LK \subseteq W_f$.
2. $KL \subseteq (\overline{W})_f$ *iff* $LK \subseteq (\overline{W})_f$.

Proof.
1. By symmetry it suffices to prove one implication. Let us assume that $KL \subseteq W_f$. Consider a word in LK, so the word can be written as vu with $v \in L$ and $u \in K$. By assumption $uv \in W_f$, so $(uv)^\omega \in W$ by definition of W_f, so $u(vu)^\omega \in W$, so $(vu)^\omega \in W$ by prefix independence of W, so $vu \in W_f$. This shows that $LK \subseteq W_f$.
2. The second result follows by applying Lemma 16.2 on Lemma 17.1. ◀

We use the following definition in the reminder of the section:

▶ **Definition 18.** *For $W \subseteq \Gamma^\omega$, if for all $a \in \Gamma$, we have $(W_{a,f})^\omega \subseteq W$ then we say that $W_{a,f}$ mixes well.*

▶ **Lemma 19.** *If $W \subseteq \Gamma^\omega$ is positionally determined on vertex-colored hub-cycle arenas, then $W_{a,f}$ mixes well. Symmetrically, for all $a \in \Gamma$, we have that $(\overline{W}_{a,f})$ mixes well.*

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a family with members in $W_{a,f} = W_f \cap a\Gamma^*$. We want to prove that $u_0 u_1 u_2 \cdots \in W$. Let $(v_n)_{n \in \mathbb{N}}$ be a family such that $\forall n \in \mathbb{N}, u_n = av_n$.

Consider a hub-cycle game controlled by Adam with hub a. For each $n \in \mathbb{N}$, there is a cycle starting and ending at a that traverses a sequence of states, resulting in v_n.

We first argue that Adam does not have any winning strategies: otherwise he would have a positional winning strategy. It would yield a play of the form $av_n = u_n$ for some n, contradiction since $u_n^\omega \in W$. Since Adam could yield the (non-positional) play $u_0u_1u_2\cdots \in \Gamma^\omega$, it follows that $u_0u_1u_2\cdots \in W$.

The second result holds directly since $\overline{W}$ also guarantees positional determinacy. Indeed, for any hub-cycle game G and objective W, Adam has a winning strategy for W in G iff Eve has a winning strategy for $\overline{W}$ in the dual hub-cycle game G' where Eve and Adam are inverted. It follows that $\overline{W}$ is positionally determined on G iff W is positionally determined on G', and this holds by assumption. ◀

▶ **Lemma 20.** *Let $W \subseteq \Gamma^\omega$ be a prefix-independent objective that is positionally determined on vertex-colored hub-cycle arenas. Then, for any $a, b \in \Gamma$ and $L, L' \subseteq \Gamma^*$:*

$$\forall u \in L, \exists v \in L', aubv \in W_f \iff \exists v \in L', \forall u \in L, aubv \in W_f$$
$$\forall u \in L, \exists v \in L', aubv \in \overline{W}_f \iff \exists v \in L', \forall u \in L, aubv \in \overline{W}_f$$

The statement and the proof of this lemma are similar to the ones of Proposition 11.

Proof. Consider the first equivalence $\forall u \in L, \exists v \in L', aubv \in W_f \iff \exists v \in L', \forall u \in L, aubv \in W_f$. The second equivalence follows by symmetry on $\overline{W}$, which is also prefix-independent and also guarantees positional determinacy by inverting Eve and Adam.

The right-to-left implication clearly holds. For the left-to-right implication, we proceed by contradiction. Assume: $\forall u \in L, \exists v \in L', aubv \in W_f$ and $\forall v \in L', \exists u \in L, aubv \notin W_f$.

Take $a = b$ and construct two sequences $(u_i)_{i\geq 0} \in L^{\mathbb{N}}$ and $(v_i)_{i\geq 0} \in (L')^{\mathbb{N}}$ by induction:

- Let $u_0 \in L$. By the first part of the assumption, $\exists v_0 \in L'$ such that $au_0av_0 \in W_f$.
- Given $v_i \in L'$, by the second part of the assumption, $\exists u_{i+1} \in L$ such that $au_{i+1}av_i \notin W_f$.
- Given $u_{i+1} \in L$, by the first part of the assumption, $\exists v_{i+1} \in L'$ such that $au_{i+1}av_{i+1} \in W_f$.

Consider the infinite word $w = au_0av_0au_1av_1au_2av_2\ldots$. Since W is prefix-independent, $w \in W$ iff any of its suffixes is in W. By construction, every block au_iav_i is in W_f. Since W is positionally determined in one-player arenas, Lemma 19 implies that the infinite concatenation of these winning blocks remains in the winning set: $(au_0av_0)(au_1av_1)(au_2av_2)\cdots \in W$

Now consider the word obtained by removing prefix au_0 from w: $w' = av_0au_1av_1au_2av_2\ldots$. By prefix-independence, $w \in W \iff w' \in W$. However, for all $i \geq 0$, by prefix-independence and by Lemma 17.2, $av_iau_{i+1} \in \overline{W}_f$ iff $au_{i+1}av_i \in \overline{W}_f$, which holds. Since $\overline{W}$ is also positionally determined in one-player arenas, applying Lemma 19 to these blocks gives $(av_0au_1)(av_1au_2)(av_2au_3)\cdots \in \overline{W}$.

This implies $w \in \overline{W}$ by prefix-independence. We have reached a contradiction where $w \in W$ and $w \in \overline{W}$. Therefore, the quantifier exchange must hold. ◀

▶ **Lemma 21.** *Let $W \subseteq \Gamma^\omega$ be a prefix-independent language that guarantees positional determinacy on all vertex-colored hub-cycle arenas. Let $a \in \Gamma$ and $K, L \subseteq W_{a,f}$. Then $KL \subseteq W_{a,f}$.*

Proof. Let $u \in K$ and $v \in L$. By Lemma 19, $W_{a,f}$ mixes well and we have $(uv)^\omega \in W$, so $uv \in W_f$. ◀

▶ **Lemma 22.** *Let $W \subseteq \Gamma^\omega$ be a prefix-independent language that is positionally determined on vertex-colored hub-cycle arenas. Let $u \in a\Gamma^*$ and $L \subseteq a\Gamma^*$ be such that $uL \subseteq W_f$. Then $\forall n \in \mathbb{N} \forall v \in L^{n+1}, u^{n+1}v \in W_f$.*

Proof. We proceed by induction on n.

- Case $n = 0$: let $v \in L$. Then $u^{n+1}v = uv \in W_f$ by assumption of the lemma.
- Let us assume that the claim holds for $n - 1 \geq 0$, and consider $v \in L^{n+1}$, so v can be written wv' with $w \in L$ and $v' \in L^n$. By IH, $u^n v' \in W_f$, so $v'u^n \in W_f$ by Lemma 17.1. By assumption of the lemma, $uw \in W_f$, so $uwv'u^n \in W_f$ by Lemma 21, so $u^{n+1}v = uu^n wv' \in W_f$ by Lemma 17.1 again. Therefore the claim holds for n. ◀

▶ **Lemma 23.** *Let $W \subseteq \Gamma^\omega$ be a prefix-independent language that is positionally determined on vertex-colored hub-cycle arenas. Then the following holds: $\forall a, b \in \Gamma, K, L \subseteq \Gamma^*, (\forall v \in L, \exists u \in K, aubv \in W_f) \Rightarrow \exists u \in K, \forall v \in L, aubv \in W_f$.*

Proof. Let $a, b \in \Gamma$ and $K, L \subseteq \Gamma^*$ be such that $\forall v \in L, \exists u \in K, aubv \in W_f$. So $\forall v \in L, \exists u \in K, bvau \in W_f$ by Lemma 17.1. So by Lemma 20, we have $\exists u \in K, \forall v \in L, bvau \in W_f$. Hence $\exists u \in K, \forall v \in L, aubv \in W_f$ by Lemma 17.1 again. ◀

▶ **Lemma 24.** *Let $W \subseteq \Gamma^\omega$ be a prefix-independent language that is positionally determined on vertex-colored hub-cycle arenas. Let $u \in a\Gamma^*$ and $L \subseteq a\Gamma^*$ be such that $uL \subseteq W_f$. Then $\exists n > 0,\ \forall v \in L^+, u^n v \in W_f$.*

Proof. Let $u' \in \Gamma^*$ be such that $u = au'$ and let $K := \{u'u^n \mid n \in \mathbb{N}\}$. Also let $L' \subseteq \Gamma^*$ be such that $L = aL'$, and let $L'' := L'L^*$.

By Lemma 22 we have $\forall v \in L^+ \exists n > 0, u^n v \in W_f$, i.e. $\forall y \in L'' \exists x \in K, axay \in W_f$. So $\exists x \in K \forall y \in L'', axay \in W_f$ by Lemma 23. Said otherwise, $\exists n > 0 \forall v \in L^+, u^n v \in W_f$. ◀

▶ **Lemma 25.** *Assume that $W \subseteq \Gamma^\omega$ is positionally determined on vertex-colored hub-cycle arenas. Let $u \in W_{a,f}$ and $L \subseteq a\Gamma^*$ be such that $uL \subseteq W_f$. Then $uL^* \subseteq W_f$.*

Our proof of this lemma involves a sequence of technical lemmas.

Proof. Let $E := \{n \geq 1 \mid u^n L^+ \subseteq W_f\}$, which is non-empty by Lemma 24.

- For all $n \in E$, we have $u^n L^+ \subseteq W_f$ by definition, so $uu^n L^+ \subseteq W_f$ since $u \in W_{a,f}$ and $W_{a,f}$ mixes well by Lemma 19, so $n + 1 \in E$.
- It suffices to show that $1 \in E$. Towards a contradiction, let us assume that $k := \min E > 1$. So let $v \in L^+$ be such that $u^{k-1}v \notin W_f$.
  - So $u^{k-1}v \in (\overline{W})_f$ by Lemma 16.1, so $vu^{k-1} \in (\overline{W})_f$ by Lemma 20, so $vu^{k-1}u^{k-1}v \in (\overline{W})_f$ by Lemma 19. So $u^{2k-2}vv \in (\overline{W})_f$.
  - Moreover, on the one hand $vv \in L^+$ since $v \in L^+$, and on the other hand $2 \leq k = \min E$, so $k \leq 2k - 2$, so $2k - 2 \in E$, so $u^{2k-2}vv \in W_f$ by definition of E, contradiction.

  So $\min E = 1$, i.e. $uL^+ \subseteq W_f$.

Therefore $uL^* \subseteq W_f$ since $u \in W_f$. ◀

We now build our generalized parity structure over the words in Γ^*.

Let $b \in \Gamma$. For all $u \in W_f$, let $g_b(u) := \{v \in (\overline{W})_{b,f} \mid uv \in W_f\}$. For all $u, u' \in W_f$, let us write $u \sqsubseteq_b u'$ if $g_b(u) \subseteq g_b(u')$, and $u \sqsubset_b u'$ if $u \sqsubseteq_b u'$ and $g_b(u) \neq g_b(u')$, and $u \sim_b u'$ if $g_b(u) = g_b(u')$.

Let $a \in \Gamma$. Let $\sqsubseteq_{a,b}$, $\sqsubset_{a,b}$, $\sim_{a,b}$ be the restrictions of $\sqsubseteq_b$, $\sqsubset_b$, $\sim_b$, respectively, to $W_{a,f}$.

▶ **Lemma 26.** *Let $W \subseteq \Gamma^\omega$ be a prefix-independent objective that is positionally determined on vertex-colored hub-cycle arenas. For all $a, b \in \Gamma$, the following holds:*

1. *$\sqsubseteq_{a,b}$ is a total preorder, i.e. a binary relation that is reflexive, transitive, and total.*
2. *$\sim_{a,b}$ is an equivalence relation with finite index, i.e. finitely many equivalence classes.*

Proof.

1. $\sqsubseteq_{a,b}$ is a preorder since $\subseteq$ is a preorder. Let us show that it is total. Towards a contradiction, let $u, u' \in W_{a,f}$ be incomparable, i.e. there exists $v \in g_b(u) \setminus g_b(u')$ and $v' \in g_b(u') \setminus g_b(u)$. So, on the one hand $uv, u'v' \in W_f$, and on the other hand $uv', u'v \notin W_f$. Thus, $vu, v'u' \in W_f$ by Lemma 17.1, so $vuv'u' \in W_f$ since $W_{b,f}$ mixes well by Lemma 19, so $uv'u'v$ by Lemma 17.2. But $uv'u'v \notin W_f$ since $(\overline{W})_{a,f}$ mixes well, again by Lemma 19, and we have a contradiction.
2. By Lemma 26.1 the classes of $\sim$ are totally ordered by the quotient order. So it suffices to show that there are no infinite strictly monotone sequences for $\sqsubseteq_{a,b}$. Towards a contradiction, let $u_0 \sqsubset_{a,b} u_1 \sqsubset_{a,b} u_2 \dots$ be such a sequence. For all $n \in \mathbb{N}$, let $v_n \in g_b(u_{n+1}) \setminus g_b(u_n)$, i.e. $u_n v_n \notin W_f$ and $v_n u_{n+1} \in W_f$. So on the one hand $(u_0v_0)(u_1v_1)(u_2v_2)\dots \notin W$ by Lemma 19, and on the other hand $(v_0u_1)(v_1u_2)(v_2u_3)\dots \in W$ again by Lemma 19, contradicting prefix-independence.
   In a similar way one can show that there are no infinite decreasing sequences. ◀

▶ **Lemma 27.** *Let $W \subseteq \Gamma^\omega$ be a prefix-independent objective that guarantees positional determinacy on all vertex-colored hub-cycle arenas. For all $a, b \in \Gamma$, there exists $n \in N$ and $B_1, \dots, B_{2n+1} \subseteq \Gamma^+$ such that :*

1. *The B_{2i} form a partition of $W_{a,f}$, and $B_{2i} \neq \emptyset$ for all $i < n$..*
2. *The B_{2i+1} are pairwise disjoint, their union is $(\overline{W})_{b,f}$, and $B_{2i+1} \neq \emptyset$ for all $i < n$.*
3. *For all $i \in \{1, \dots, n\}$ we have $B_{2i} \subseteq W_f$.*
4. *For all $i \in \{0, \dots, n\}$ we have $B_{2i+1} \subseteq (\overline{W})_f$.*
5. *For all $i \in \{1, \dots, n\}$ and $k \leq 2i$ we have $B_{2i}B_k \subseteq W_f$.*
6. *For all $i \in \{0, \dots, n\}$ and $k \leq 2i+1$ we have $B_{2i+1}B_k \subseteq (\overline{W})_f$.*
7. *Let $(u_n)_{n\in\mathbb{N}}$ be a family of members in $W_{a,f} \cup \overline{W}_{b,f}$. For all $n \in \mathbb{N}$ let $p_{a,b}(n) \in \mathbb{N}$ such that $u_n \in B_{p_{a,b}(n)}$. Then $u_0u_1 \dots \in W$ iff $\limsup_{n\in\mathbb{N}} p_{a,b}(n)$ is even.*

Proof. Let us define the B_i below.

- Let $B_2 \sqsubset_{a,b} B_4 \sqsubset_{a,b} \dots \sqsubset_{a,b} B_{2n}$ be the equivalence classes of $\sim_{a,b}$. They are finitely many by Lemma 26.2.
- Let $B_1 := g_b(B_2)$. (Note that $\forall u, v \in B_2, g_b(u) = g_b(v)$, hence the notation $g(B_2)$ used in the remainder.)
- For all $i \in \{1, \dots, n-1\}$ let $B_{2i+1} := g_b(B_{2i+2}) \setminus g_b(B_{2i})$
- Let $B_{2n+1} := (\overline{W})_{b,f} \setminus g_b(B_{2n})$.

1. The B_{2i} form a partition of $W_{a,f}$ by definition of $\sim_{a,b}$.
2. By construction the B_{2i+1} are pairwise disjoint and their union is $(\overline{W})_{b,f}$. For all $i < n$, each $B_{2i+1} = g_b(B_{2i+2}) \setminus g_b(B_{2i})$, which is non-empty since B_{2i+2} and B_{2i} are two different equivalence classes.
3. The domain of $\sim_{a,b}$ is $W_{a,f}$, so for all $i \in \{1, \dots, n\}$ we have $B_{2i} \subseteq W_{a,f}$.
4. The co-domain of g_b is $(\overline{W})_{b,f}$, so for all $i \in \{0, \dots, n\}$ we have $B_{2i+1} \subseteq (\overline{W})_{b,f}$.
5. Let $i \in \{1, \dots, n\}$ and $k \leq 2i$ (so $k \leq 2n$). Let us show that $B_{2i}B_k \subseteq W_f$.
   - If k is even, $B_k \subseteq W_{a,f}$. Since $B_{2i} \subseteq W_{a,f}$ too, $B_{2i}B_k \subseteq W_{a,f}$ by Lemma 17.1.
   - If k is odd, $k \leq 2n$ implies $B_k \subseteq g_b(B_{k+1}) \subseteq g_b(B_{2i})$ (since $k+1 \leq 2i$), so $B_{2i}B_k \subseteq W_f$ by definition of $g(B_{2i})$.
6. Let $i \in \{0, \dots, n\}$ and $k \leq 2i+1$. Let us show that $B_{2i+1}B_k \subseteq (\overline{W})_f$.
   - If k is odd, $B_k \subseteq (\overline{W})_f$. Since $B_{2i+1} \subseteq (\overline{W})_f$ too, $B_{2i+1}B_k \subseteq (\overline{W})_f$ by Lemma 17.2.
   - If k is even, then $k < 2i+1$ and $g(B_k) \subseteq g_b(B_{2i})$. So $B_{2i+1} \cap g_b(B_k) = \emptyset$, since $B_{2i+1} := g_b(B_{2i+2}) \setminus g_b(B_{2i})$. Thus $B_{2i}B_k \subseteq (\overline{W})_f$ by definition of $g(B_k)$.

7. Let $k = \limsup_{n\to\infty} p_{a,b}(n)$. By definition of the limit superior, there exists an index $N \in \mathbb{N}$ such that $p_{a,b}(n) \leq k$ for all $n \geq N$, and $p_{a,b}(n) = k$ for infinitely many $n \geq N$.
Because W is prefix-independent, the finite prefix $u_0 \dots u_{N-1}$ does not affect membership in W. Let $i_0 < i_1 < i_2 < \dots$ be the strictly increasing sequence of indices $\geq N$ where the maximum priority is reached, meaning $p_{a,b}(i_j) = k$.
We factor the infinite suffix $w = u_{i_0} u_{i_0+1} \dots$ into an infinite sequence of finite segments $w_0, w_1, w_2, \dots$ defined by $w_j = u_{i_j} u_{i_j+1} \dots u_{i_{j+1}-1}$
For each $j \in \mathbb{N}$, the segment w_j consists of exactly one element from B_k (which is u_{i_j}) followed by a finite, possibly empty, sequence of elements from blocks B_m with $m < k$. Let $B_{<k} = \bigcup_{m\leq k} B_m$. Then $w_j \in B_k(B_{<k})^*$.
We have two cases, depending on the parity of k. Theses cases are symmetric, so we only present the case for even k below.
If k is even, let $k = 2i$. By Point 5, we know that $B_{2i}B_m \subseteq W_f$ for all $m \leq 2i$. Consequently, $B_{2i}L \subseteq W_f$. Applying Lemma 25, we extend this to $B_{2i}L^* \subseteq W_f$. Therefore, $w_j \in W_f$. Since $u_{i_j} \in B_{2i} \subseteq W_{a,f}$, w_j starts with a, so $w_j \in W_{a,f}$. Because $W_{a,f}$ mixes well (by Lemma 19), the infinite concatenation $w = w_0 w_1 w_2 \dots$ belongs to W. Thus, $u_0 u_1 \dots \in W$.
Hence $u_0 u_1 \dots \in W$ if and only if $\limsup_{n\to\infty} p_{a,b}(n)$ is even. ◀

We can now conclude, and introduce an additional result that we use later.

Proof of Theorem 15. To obtain a generalized parity objective on anchored words as given in Definition 13, it suffices to take for every $a \in \Gamma$ the priority function $p_a = p_{a,a}$ built in Lemma 27. ◀

▶ **Lemma 28.** *Let Γ be a non-empty set of colors, and $W \subseteq \Gamma^\omega$ be a generalized parity objective on anchored words. Then W is positionally determined on vertex-colored hub-cycle arenas.*

Proof. Let G be a hub-cycle game where the vertices belong to Eve, with objective W, as in Definition 12. We assume that the vertices of the arena belong to Eve, in particular the only state with out-degree strictly greater than one, which we call the hub. The hub has some color $a \in \Gamma$. The case where it belongs to Adam can be done symetrically by replacing Adam and Eve in the remainder of the proof.

First, if Adam has a winning strategy, this strategy has to be positional since Adam does not control the hub, and it's the only vertex with more than one outgoing edge.

Second, assume Eve has a winning strategy in G. Then there exists $w \in W$ resulting from a play on G. Since G is strongly connected, it visits the hub colored with a infinitely often, hence $w = aw_0aw_1aw_2\dots$ where the finite words w_i are exactly the sequences that do not visit the hub, while every a is seen at the hub. For p_a the priority function on words associated to a, we take $k = \limsup_{i\to\infty} p_a(aw_i)$. Then there is some w_i such that $p_a(aw_i) = k$. The positional strategy playing $(aw_i)^\omega$ is winning. ◀

Although the following two results are not necessary for any of our proof, they provide additional context for the studied objectives.

▶ **Proposition 29.** *Let W be a generalized parity objective on anchored words. For all $a \in \Gamma$ and $u, v, w \in \Gamma^*$, if $auav, auaw \in W_f$ then $auavaw \in W_f$. Corollary (by symmetry): For all $a \in \Gamma$ and $u, v, w \in \Gamma^*$, if $auav, auaw \in \overline{W_f}$ then $auavaw \in \overline{W}_f$.*

Proof. Since the objective W (or more specifically its finitary version W_f) is stuctured as described in Lemma 27, each of the three words au, av, aw can be assigned a priority. If $auav, auaw \in W_f$, then the highest among these three priorities is even, which allows us to conclude. ◀

▶ **Corollary 30.** *Let Γ be a finite non-empty set of colors, and $W \subseteq \Gamma^\omega$ be a generalized parity objective on anchored words. The objective W is fully characterized by $W_f \cap \Gamma^{2|\Gamma|}$.*

Proof. Let $x \in \Gamma^*$ be such that $2|\Gamma| < |x|$. Since Γ is finite, there exists $a \in \Gamma$ that occurs at least three times in x. Then x is the circular permutation of some $auavaw$. Proposition 29 implies the following:

- If two or three words among $auav$, $auaw$, $avaw$ are in W_f, then $auavaw \in W_f$.
- Else $auavaw \notin W_f$.

Note that $|auav|, |auaw|, |avaw| < |x|$. ◀

5 Vertex-colored games: parity on pairs

Let Γ be a non-empty alphabet. We have shown that all objectives $W \subseteq \Gamma^*$ positionally determined on vertex-colored hub-cycle one-player arenas are generalized parity objectives on anchored words. We now consider objectives on pairs, and introduce the following. For a finite word $u = a_0a_1 \dots a_{n-1} \in \Gamma^+$, the **cyclic 2-factor set** is the set of consecutive pairs of colors: $F_2(u) = \{(a_i, a_{i+1 \pmod n}) \mid 0 \le i < n\}$. For an infinite word $w \in \Gamma^\omega$, the **limit 2-factor set** $F_2^\infty(w)$ consists of those consecutive pairs of colors $(x, y) \in \Gamma^2$ such that xy appears infinitely often as factor in w. Note that in both cases, pairs are ordered. We say $W \subseteq \Gamma^\omega$ is a Muller objective on pairs if there exists $M \subseteq 2^{\Gamma^2}$ such that for all $w \in \Gamma^\omega$, we have $w \in W$ if and only if $F_2^\infty(w) \in M$.

In this section, we will consider a finite Γ, and show 1d $\Rightarrow$ 1c in two steps. First, in Lemma 34, we show that if $W \subseteq \Gamma^\omega$ is a generalized parity objective on anchored words, then W satisfies a Muller objective on pairs. Second, using that generalized parity objectives on anchored words are positionally determined on vertex-colored hub-cycle arenas, as seen in Lemma 28, we then conclude with Lemma 35, showing that W is a generalized parity objective on pairs of colors.

Finally, for 1c $\Rightarrow$ 1b, we show Theorem 36: we consider any generalized parity objective on pairs. A vertex-colored game with this objective can be transformed into an edge-colored parity game by assigning to each edge q_1q_2 the priority $p(a, b)$, where a and b are the original vertex colors. Since the resulting edge-colored game is a parity game, it is positionally determined, and its winning strategies transfer back to the original vertex-colored setting.

▶ **Example 31.** We have shown in Example 9 that $W = (a + b)^*(ab)^\omega$ on $\Gamma = \{a, b\}$ is a generalized parity objective on pairs. We have shown in Example 1 that it is positionally determined on the vertex-colored arena of Figure 1b. Proving 1c $\Rightarrow$ 1b will tell us it is determined on all vertex-colored two-player arenas.

▶ **Theorem 32.** *Let Γ be a finite non-empty set of colors. If $W \subseteq \Gamma^\omega$ is a generalized parity objective on anchored words, then W is a generalized parity objective on pairs.*

We introduce the following lemma to establish a closure property: rearranging an infinite word w such that a finite word u occurs infinitely often does not alter membership in W.

▶ **Lemma 33.** *Let Γ be a finite non-empty set of colors. Let $W \subseteq \Gamma^\omega$ be a generalized parity objective on anchored words. Let $w \in \Gamma^\omega$ be an infinite word, and $u \in \Gamma^*$ such that $F_2(u) \subseteq F_2^\infty(w)$. Then there exists w_u such that $F_2^\infty(w) = F_2^\infty(w_u)$, u appears infinitely often in w_u, and $w \in W$ iff $w_u \in W$.*

Proof. We proceed by considering every letter u_i successively, and reordering words beginning with u_i (without changing the acceptance condition given by Definition 13) until obtaining a word where $u = u_0u_1u_2\dots$ appears infinitely often.

Because all adjacent pairs (u_j, u_{j+1}) of the word u belong to $F_2^\infty(w)$, they each occur infinitely often in w. We can therefore find a finite prefix P_0 of w, after which only pairs in $F_2^\infty(w)$ appear, and all do infinitely often.

Hence the pair (u_0, u_1) occurs after P_0. After this occurrence, since the pair (u_1, u_2) also occurs infinitely often, some occurrence lies strictly later in the word. Iterating this argument for all consecutive pairs in u, we obtain a factorization of $w = w^0_{\text{seg}}w^1_{\text{seg}}w^2_{\text{seg}}\dots$, where each w^i_{seg} is of the form $w^i_{\text{seg}} = P^i_0 \cdot u_0u_1 \cdot P^i_1 \cdot u_1u_2 \cdot P^i_2 \dots u_{n-1}u_nP^i_n \cdot u_nu_0Q^i_0u_1Q^i_1\dots u_nQ^i_n$. To preserve some readability, subscripts are treated modulo the length of the sequence, meaning that the index following the last element wraps around to 0. For instance, u_{j+1} evaluates to u_0 when j is the final index. Each P^i_j is the factor between the chosen occurrence of $u_{j-1}u_j$ and the next occurrence of u_ju_{j+1}, and Q^i_j is the factor between the chosen occurrence of u_j and the next occurrence of u_{j+1}.

We highlight factors beginning with u_1 as follows: $w = (P^0_0u_0)\cdot(u_1P^0_1)\cdot(u_1u_2P^0_2u_2u_3P^0_3\dots u_{n-1}u_nP^0_nu_nu_0Q^0_0)\cdot(u_1Q^0_1u_2Q^0_2\dots u_nQ^0_nP^1_0u_0)\cdot(u_1P^1_1)\cdot(u_1u_2P^1_2u_2u_3P^1_3\dots u_{n-1}u_nP^1_nu_nu_0Q^1_0)\cdot u_1Q^1_1u_2Q^1_2\dots u_nQ^1_nP^2_0u_0\dots$

We now reorder those factors, defining $w_1 = (P^0_0u_0)\cdot(u_1u_2P^0_2u_2u_3P^0_3\dots u_{n-1}u_nP^0_nu_nu_0Q^0_0)\cdot(u_1P^0_1)\cdot(u_1Q^0_1u_2Q^0_2\dots u_nQ^0_nP^1_0u_0)\cdot(u_1u_2P^1_2u_2u_3P^1_3\dots u_{n-1}u_nP^1_nu_nu_0Q^1_0)\cdot(u_1P^1_1)\cdot u_1Q^1_1 u_2Q^1_2\dots u_nQ^1_nP^2_0u_0\dots$ Since W is a generalized parity objective on anchored words, we consider p_{u_1} as in Definition 13. Since the limit superior does not change by reordering, we have $w_1 \in W$ iff $w \in W$. We remark that in w_1, every P^i_0 is followed by the sequence $u_0u_1u_2$

We now do the same with words starting with u_2, with $w_1 = (P^0_0u_0u_1)\cdot(u_2P^0_2)\cdot(u_2u_3P^0_3\dots u_{n-1}u_nP^0_nu_nu_0Q^0_0u_1P^0_1u_1Q^0_1)\cdot(u_2Q^0_2\dots u_nQ^0_nP^1_0u_0u_1)\cdot(u_2P^1_2)\cdot(u_2u_3P^1_3\dots u_{n-1}u_nP^1_nu_nu_0Q^1_0u_1P^1_1u_1Q^1_1)\cdot u_2Q^1_2\dots u_nQ^1_nP^2_0u_0\dots$

And reorder similarly , defining $w_2 = (P^0_0u_0u_1)\cdot(u_2u_3P^0_3\dots u_{n-1}u_nP^0_nu_nu_0Q^0_0u_1P^0_1u_1Q^0_1)\cdot(u_2P^0_2)\cdot(u_2Q^0_2\dots u_nQ^0_nP^1_0u_0u_1)\cdot(u_2u_3P^1_3\dots u_{n-1}u_nP^1_nu_nu_0Q^1_0u_1P^1_1u_1Q^1_1)\cdot(u_2P^1_2)\cdot u_2Q^1_2\dots u_nQ^1_nP^2_0u_0\dots$ Considering p_{u_2} ensures $w_2 \in W$ iff $w_1 \in W$, and we remark that in w_2, every P^i_0 is followed by the sequence $u_0u_1u_2u_3$.

We proceed this way for every letter of u, moving each factor $u_{j+1}P^i_j$ to the right of the corresponding prefix, until reaching u_n and obtaining $w_n = P^0_0 \cdot (u_0u_1\dots u_n)\cdot(u_1P^0_1u_1Q^0_1u_2P^0_2u_2Q^0_2\dots u_nP^0_nu_nQ^0_n)\cdot P^0_0\cdot(u_0u_1\dots u_n)\dots$ where every P^i_0 is followed by u. Hence we name $w_u = w_n$. We have that $w_u \in W$ iff $w \in W$, that $F_2^\infty(w) = F_2^\infty(w_u)$, and that u appears infinitely often in w_u. ◀

▶ **Lemma 34.** *Let Γ be a finite non-empty set of colors. If $W \subseteq \Gamma^\omega$ is a generalized parity objective on anchored words, then W is a Muller objective on pairs.*

Proof. Assume that $W \subseteq \Gamma^\omega$ is a generalized parity objective on anchored words. Let $v, w \in \Gamma^\omega$, with $F_2^\infty(v) = F_2^\infty(w)$. Let $F = F_2^\infty(v)$. We show that $v \in W$ iff $w \in W$.

Let $a \in \Gamma$ be a color such that some pair $(a, b) \in F$. Since Γ is finite and the word is infinite, such an a exists and occurs infinitely often in both v and w. By Definition 13, the acceptance of both words is determined by a priority function p_a.

Consider any ordering $<$ on Γ. We recall that the radix (or shortlex) total order $<_{rad}$, is defined as follows: For $u, v \in \Gamma^*$, we have $u <_{rad} v$ iff $|u| < |v| \vee (|u| = |v| \wedge (\exists w, x, y \in \Gamma^*, a, b \in \Gamma : u = wax, v = wby, a < b))$.

We define a mapping $g : a\Gamma^* \to a\Gamma^*$ that maps every $u \in a\Gamma^*$ to the radix-minimal word u' such that we have both $p_a(u) = p_a(u')$ and $F_2(u) = F_2(u')$. Since the codomain of p_a and the number of subsets of Γ^2 are both finite, the image $S_{a,F} = \{g(u) \mid u \in a\Gamma^*, F_2(u) \subseteq F\}$ is a finite set of words.

Let U be a finite word formed by concatenating all elements in $S_{a,F}$. We have $F_2(U) = F$ and apply Lemma 33: we can reorder v into v_1 such that $v \in W$ iff $v_1 \in W$, $F_2^\infty(v_1) = F$, and the block U appears infinitely often in v_1. This forces every element of $S_{a,F}$ to appear infinitely often in v_1.

We factor v_1 into segments starting with a: $v_1 = xau_0au_1\dots$. We define $v_2 = xag(u_0)ag(u_1)\dots$. By Definition 13 of the generalized parity objective on anchored words, $v_1 \in W$ iff $v_2 \in W$ because $p_a(u_i) = p_a(g(u_i))$ for all i. Since U occurs infinitely often in v_1, the set of segments $\{ag(w_i)\}$ occurring infinitely often in v_2 is exactly $S_{a,F}$.

Applying the same procedure to w yields a word w_2 such that $w \in W$ iff $w_2 \in W$. The set of segments occurring infinitely often in w_2 is also exactly $S_{a,F}$. Both v_2 and w_2 are infinite concatenations of segments where the set of priorities $\{p_a(u) \mid u \text{ occurs infinitely often}\}$ is identical and equal to $\{p_a(s) \mid s \in S_{a,F}\}$, and we have:
$\limsup_{i\to\infty} p_a(\text{segments of } v_2) = \max\{p_a(s) \mid s \in S_{a,F}\} = \limsup_{i\to\infty} p_a(\text{segments of } w_2)$.

By the generalized parity condition on words, $v_2 \in W$ iff $w_2 \in W$. It follows that $v \in W$ iff $w \in W$. The membership depends only on F, defining a Muller objective on pairs $M = \{F \mid \max_{s \in S_{a,F}} p_a(s) \text{ is even}\}$. ◀

The proof of the following result is similar to the one in [15], but we slightly adapt the result to work with pairs of colors placed on vertices. We give it for the sake of completeness.

▶ **Lemma 35.** *Let Γ be a finite alphabet. If $W \subseteq \Gamma^\omega$ is positional on all vertex-colored hub-cycle arenas, and W is a Muller objective on pairs, then W is a generalized parity objective on pairs.*

Proof. Let $W \subseteq \Gamma^\omega$, and let $M \subseteq 2^{\Gamma^2}$ be the Muller objective on pairs such that $w \in W$ iff $F_2^\infty(w) \in M$. Let $U, V \in M$. Suppose there exists a letter $a \in \Gamma$ that appears infinitely often in both U and V. Specifically, let $(a, b) \in U$ and $(a, c) \in V$.

Since $U \in M$, there exists a word $u = (abu_2 \dots u_n)^\omega \in W$. Since $V \in M$, there also exists a word $v = (acv_2 \dots u_m)^\omega \in W$.

We consider the one-player game starting from an initial state a, belonging to player P_2, from where two actions are possible: the first action leads to a chain that reads $(abu_2 \dots u_n)$ before returning to the initial state, the second action leads to a chain that reads $(acv_2 \dots u_m)$ before returning to the initial state. Positional strategies will only yield u or v, which are both in W and thus winning for P_1, hence P_2 has no winning strategies. This means that alternating between the two actions is also winning for P_1, and so the word obtained this way $w = abu_2 \dots u_n \cdot acv_2 \dots v_m$ is in W. Since $F_2^\infty(w) = U \cup V$, we have $U \cup V \in M$.

Thus, M is closed under the union of sets that share a common letter in at least one pair.

We construct the priority function $p : \Gamma^2 \to \mathbb{N}$. To do so, let $F \subseteq 2^{\Gamma^2}$ be defined as $F = \{f \subseteq \Gamma^2 \mid \exists w \in \Gamma^\omega, F_2^\infty(w) = f\}$. We first define the tree Z associated with M and F. A node in Z is a set $S \in F$ by constructing it recursively:

- The root of Z is Γ^2.
- For any internal node $S \in Z$:
  - If $S \in M$, its children are the maximal sets $C \subsetneq S$ such that $C \in F \setminus M$.
  - If $S \notin M$, its children are the maximal sets $C \subsetneq S$ such that $C \in \cap M$.

Since W is positional, the children $C_1, C_2 \in M$ (resp. $\notin M$) of any node S do not share any $a \in \Gamma$ such that $(a,b) \notin C_1$ (resp. $\in C_1$) and $(a,c) \notin C_2$ (resp. $\in C_2$), otherwise their union $C_1 \cup C_2 \notin M$ (resp. $\in M$), which contradicts their maximality. Hence, all children are strictly disjoint.

Because the children are disjoint, every pair $e \in \Gamma^2$ belongs to a unique minimal node S_e in the tree Z (by subset inclusion).

We define the priority function $p : \Gamma^2 \to \mathbb{N}$ recursively from the leaves to the roots to satisfy a max-generalized parity condition. To do so, we first define p on F

- If S is a leaf, $p(S) = 0$ if $S \in M$, and $p(S) = 1$ if $S \notin M$.
- If $S \in M$ is an internal node, $p(S)$ is the smallest even integer strictly greater than $\max\{p(C) \mid C \text{ is a child of } S\}$.
- If $S \notin M$ is an internal node, $p(S)$ is the smallest odd integer strictly greater than $\max\{p(C) \mid C \text{ is a child of } S\}$.

For each pair $e \in \Gamma^2$, we set the transition priority $p(e) = p(S_e)$.

Consider any word $w \in \Gamma^\omega$ and let $U = F_2^\infty(w)$. Let S_U be the unique minimal node in Z such that $U \subseteq S_U$. Since U is not contained in any single child of S_U, U must contain at least one edge e that belongs to S_U but to no child of S_U. For this edge, the minimal containing node is $S_e = S_U$, so $p(e) = p(S_U)$.

For all other edges $e' \in U$, their minimal containing node is either S_U or a descendant of S_U. By construction, the priority of a node is strictly greater than the priorities of its descendants. Therefore, the maximum priority among all edges in U is exactly $p(S_U)$.

Since w traverses all edges in U infinitely often, $\limsup_{i\to\infty} p(w_i, w_{i+1}) = p(S_U)$. By the definition of $p(S_U)$, this maximum priority is even if and only if $S_U \in M$.

Finally, because U is contained in S_U and is not contained in any child of S_U, U must have the same membership status in M as S_U. If $S_U \in M$ and $U \notin M$, U would be an element of F not in M, meaning it would be contained in one of the children of S_U, contradicting the minimality of S_U. The symmetric argument holds if $S_U \notin M$. Thus, $U \in M$ iff $S_U \in M$.

We conclude that $w \in W$ iff $F_2^\infty(w) \in M$ iff $\limsup_{i\to\infty} p(w_i, w_{i+1}) \equiv 0 \pmod 2$, and so W is a generalized parity objective on pairs. ◀

We can now give the following:

Proof of Theorem 32. Let $W \subseteq \Gamma^\omega$ be a generalized parity objective on anchored words. Lemma 34 shows that W is a Muller objective on pairs, and since W is positionally determined on vertex-colored hub-cycle arenas by Lemma 28, we use Lemma 35 to conclude that W is a generalized parity objective on pairs. ◀

▶ **Theorem 36.** *Let $W \subseteq \Gamma^\omega$ be a generalized parity objective on pairs. Then W is prefix-independent and positionally determined on all vertex-colored two-player arenas.*

Proof. First, $W \subseteq \Gamma^\omega$ is prefix-independent because it is defined by a lim sup. Since the limit superior of a sequence is unaffected by finite prefixes, any change to a finite prefix of a word leaves its membership in W unchanged.

To show positional determinacy, let $G = \langle A, W \rangle$ be a vertex-colored game on arena $A = \langle V_E, V_A, E \rangle$. Since W is a generalized parity objective on pairs, there exists a priority function $p : \Gamma^2 \to \{0, \ldots, n\}$ such that $w_0 w_1 \cdots \in W$ iff $\limsup_{i\to\infty} p(w_i, w_{i+1})$ is even.

We consider the auxiliary alphabet Γ^2 and define $W_{\text{pair}} \subseteq (\Gamma^2)^\omega$ such that a word $u = (x_0, y_0)(x_1, y_1) \cdots \in W_{\text{pair}}$ iff $\limsup_{i\to\infty} p(x_i, y_i)$ is even. By Definition 8, this makes W_{pair} a generalized parity objective over Γ^2.

We construct the auxiliary edge-colored game $G_{\text{pair}} = \langle A_{\text{pair}}, W_{\text{pair}}\rangle$ where $A_{\text{pair}} = \langle V_E, V_A, E_{\text{pair}}\rangle$. We define the edges such that $(v,(a,b),v') \in E_{\text{pair}}$ iff $(v,a,v') \in E$ and b is the unique element of Γ such that for some $v'' \in V$, we have $(v',b,v'') \in E$.

By construction of E_{pair}, any valid play in G_{pair} produces a trace $u = (w_0,w_1)(w_1,w_2)\cdots \in (\Gamma^2)^\omega$. This trace belongs to W_{pair} iff $\limsup_{i\to\infty} p(w_i,w_{i+1})$ is even, which holds iff the underlying trace $w_0w_1\ldots$ belongs to W.

By Theorem 6 of [15], W_{pair} is positionally determined on all two-player edge-colored arenas. Hence, there exists a winning strategy $\sigma : V_E \to V$ on G_{pair} that depends only on the current vertex. Because G and G_{pair} share the same vertices and their winning conditions are equivalent over all valid plays, σ is also a positional winning strategy for G. Thus, W is positionally determined on all vertex-colored two-player arenas. ◀

6 Additional remarks

ω-regularity. When Γ is finite, the prefix-independent objectives characterizing positionally determined vertex-colored arenas are ω-regular.

▶ **Lemma 37.** *Let Γ be a finite non-empty set of colors. Languages $W \subseteq \Gamma^\omega$ defined by a generalized parity objective on pairs are ω-regular.*

Proof. Let $W \subseteq \Gamma^\omega$ be a language defined by a generalized parity objective on pairs. By definition, there exists a priority function $p : \Gamma \times \Gamma \to [0,n]$ such that a word $w = w_0w_1w_2\cdots \in \Gamma^\omega$ belongs to W iff the largest integer occurring infinitely often in the sequence $p(w_0,w_1), p(w_1,w_2),\ldots$ is even.

For $a,b \in \Gamma$, let $M_{a,b} = \{w \in \Gamma^* \mid k{+}1 = |w|,\ w_0 = a,\ w_1 = b,\ \forall i \in [0,k{-}1],\ p(w_i,w_{i+1} \leq p(a,b)$ and $p(w_k,a) \leq p(a,b)\}$ contain all words starting with ab and with priorities always less or equal to $p(a,b)$. This language is regular: it is recognized by a finite automaton with states $\Gamma \cup \{f, a_{init}, b_{next}\}$, where f is the only accepting state. From the initial state a_{init}, the first transition must reach b_{next}. Subsequent transitions between any $\gamma,\gamma' \in \Gamma$ are allowed iff $p(\gamma,\gamma') \leq p(a,b)$. Finally, a transition to f is permitted from any γ such that $p(\gamma,a) \leq p(a,b)$.

Since every word w must contain infinitely often some pair ab such that $p(a,b)$ is the maximal priority seen infinitely often in w, we then take:

$$W = \Gamma^* \cdot \bigcup_{\substack{i\in[0,n]\\ i \text{ even}}} \bigcup_{\substack{a,b\in\Gamma\\ p(a,b)=i}} (M_{a,b})^\omega \tag{1}$$

Since these languages are obtained by binary concatenation, finite union, and ω-power of regular languages, they are ω-regular. ◀

Relation with games on finite arenas. We compare our setting with Theorem 2 of [8], that considers quantitative objectives represented by a binary relation $\sqsubseteq$. It proves that $\sqsubseteq$ is positionally determined on **finite** edge-colored one-player games iff $\sqsubseteq$ is positionally determined on finite edge-colored two-player games iff $\sqsubseteq$ is **selective and monotone**.

Prefix-independence implies monotonicity, but our running example $(a+b)^*(ab)^\omega$ on $\Gamma = \{a,b\}$ is not selective.

▶ **Lemma 38.** *There exists $W \subseteq \Gamma^\omega$ prefix-independent and positionally determined on all vertex-colored one-player arenas, such that its associated relation $\sqsubseteq$ is not selective.*

Proof. Let $W = (a+b)^*(ab)^\omega$ on $\Gamma = \{a, b\}$, which is the generalized parity condition such that $p(a,a) = p(b,b) = 1$ and $p(a,b) = p(b,a) = 0$, hence by Theorem 32 it is prefix-independent and positionally determined on all vertex-colored two-player arenas. W is monotone: prefix-independence ($x \in W \iff ax \in W$) ensures $x \sqsubseteq y \Rightarrow ax \sqsubseteq ay$.

However, W is not selective: $W \cap \{a\}^\omega = \emptyset, W \cap \{b\}^\omega = \emptyset,$ yet $W \cap \{a,b\}^\omega \neq \emptyset$ shows that W is not closed under the union of sub-alphabets. ◀

Infinite alphabet. We have proved our characterization with generalized parity objectives on pairs only when considering a finite sets of colors. For an arbitrary set of colors, this characterization does not hold any more. The proof relies on parity games with infinite colors defined in [10] by Grädel and Walukiewicz to create a chain of strictly increasing priorities of arbitrary length.

▶ **Lemma 39.** *Let $\Gamma = \mathbb{N}$. There exists an objective $W \subseteq \Gamma^\omega$ prefix-independent and positionally determined on all vertex-colored two-player arenas, such that for every generalized parity function on pairs W_p, there exists a word $w \in \Gamma^\omega$ such that $w \in W \iff w \notin W_p$.*

Proof. We define the objective W based on an extension of the min generalized parity condition to an infinite set of priorities. $u = u_0 u_1 \cdots \in W \subseteq \Gamma^\omega$ iff $\liminf_{i\to\infty} u_i \neq \emptyset \Rightarrow \liminf_{i\to\infty} u_i$ is even. Hence, a word is in W if either the set of priorities appearing infinitely often is empty, or otherwise if its minimum element is even.

W is prefix-independent, since the limit inferior is independent of any finite prefix of w. Positional determinacy on all vertex-colored two-player arenas is non-trivial, and has been shown in [10].

Let W_p be a generalized parity function on pairs of order n (hence it has n priorities in $[0,n]$). To show it does not captures W, we consider for each $k \in [0, n+1]$ the word $w_k = (k, k+1, \ldots, n+1, n+1, n, \ldots, k+1)^\omega$. The set of elements of Γ^2 traversed infinitely often by w_k is $F_2^\infty(w_n) = \{(i, i+1), (i+1, i) \mid k \leq i \leq n\} \cup \{(n+1, n+1)\}$. Note that for all n, $F_2^\infty(w_{k+1}) \subsetneq F_2^\infty(w_k)$.

By definition of W, $w_k \in W$ if and only if k is even. For the generalized parity condition on pairs W_p, $w_k \in W_p$ iff the priority $M_n = \max_{k \leq i \leq n}(p(w_i, w_{i+1}))$ is even.

Since for all $k \in \mathbb{N}$ we have $F_2^\infty(w_{k+1}) \subsetneq F_2^\infty(w_k)$, we have $M_{k+1} \leq M_k$. If W_p captures W, then for every k, M_k must have the same generalized parity as k, hence $M_k \not\equiv M_{k+1} \pmod 2$. These two conditions imply that $M_{k+1} < M_k$ for all $n \in [0, n+1]$.

The sequence of priorities $(M_k)_{k\in[0,n+1]}$ must be strictly decreasing. This contradicts the assumption that the range of p is $[0,n]$. Hence, no such generalized parity function on pairs W_p can represent W. ◀

7 Related works and future directions

In 2005, Gimbert and Zielonka [8] established a foundational 1-to-2-player lift for quantitative objectives represented by a preference relation $\sqsubseteq$. They proved that $\sqsubseteq$ is positionally determined on finite one-player edge-colored arenas iff it is positionally determined on finite two-player edge-colored arenas iff $\sqsubseteq$ is selective and monotone. Here, quantitative objectives assign non-binary payoffs to plays, generalizing standard qualitative win-lose objectives. This raises an *open question* for our framework: Is $\sqsubseteq$ positionally determined on finite one-player vertex-colored arenas iff it is positionally determined on finite two-player vertex-colored arenas iff $\sqsubseteq$ is selective and monotone on pairs?

Finite-memory strategies offer a natural generalization of positional determinacy while remaining finitely representable, retaining some simplicity in terms of description and implementation. Characterizations of finite-memory determinacy extending the work of [8] have been obtained for finite games [2] and infinite games [1]. This raises the *open question* whether our pair-based framework can bridge edge and vertex-colored games for finite-memory determinacy.

Determinacy that is positional or finite-memory for one player only (called half-positional determinacy, etc) has also been extensively studied [11, 14, 3]. In this framework, our notion of objectives on pairs might be adapted to establish a correspondence between edge-colored and vertex-colored games.

A recent work by Colcombet and Idir [4] considers the ω-regular objectives that are Eve-positional (Eve has a positional winning strategy whenever she has a winning strategy). For edge-colored finite games, they give a characterization of this property.

Finally, lifting the finiteness assumption in our second equivalence remains an open problem. As discussed previously, resolving this is particularly relevant for capturing parity games with infinitely many priorities [10].

8 Conclusion

Further remarks. The finiteness of Γ is used only to obtain the Muller objective on pairs in Lemma 35: since the first step of the proof, Theorem 15, with a generalized parity objective on anchored words, remains valid for infinite color sets, we emphasize this intermediate result in the body of the article. Furthermore, the finiteness assumption in the second step is essential: parity games on infinitely many natural numbers are positionally determined [10], yet they generally cannot be reduced to parity objectives on pairs of colors from consecutive vertices, as shown in Lemma 39. Finally, we establish in Section 6 that our characterization with generalized parity objectives on pairs of colors from consecutive vertices is ω-regular.

Discussion. Our results establish a 1-to-2-player lift, as the characterization of determinacy for two-player games reduces to the study of one-player hub-cycle games. This starts clarifying the relationship between edge and vertex games. Although any vertex-colored game can be viewed as an edge-colored game via a direct injection, the class of edge-colored games is strictly richer. This is reflected in the objectives: generalized parity objectives on a single color form a strict subclass of generalized parity objectives on pairs of colors. For instance, the objective $(a+b)^*(a^\omega + b^\omega)$ can be defined as an objective on pairs, but not on a single color.

Positionally determined objectives on vertex-colored arenas are strictly more expressive than those on edge-colored arenas. Indeed, vertex-colored games allow more complex objectives to remain positionally determined by effectively using the vertex color to track transitions. While we can convert a vertex-colored game with an objective on pairs into a standard edge-colored game, as in Theorem 36, this incurs a quadratic blowup in the color domain, since the priority function has domain Γ on single colors, but Γ^2 on pairs of colors.

Our equivalence thus establishes a two-way correspondence, between the vertex setting and the edge setting from [5, 12], via generalized parity objectives. Future works could strengthen this result.

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