

Completeness for Probabilistic Boolean Tapes

Filippo Bonchi 

University of Pisa, Italy

Cipriano Junior Cioffo 

University of Pisa, Italy

Abstract

Probabilistic Boolean circuits have recently been proposed as a string-diagrammatic foundation for finite probabilistic programming. In this paper, we present a complete set of axioms for their semantics in terms of Markov kernels. Our approach is based on two intermediate results: completeness for *partial* Boolean circuits and completeness for probabilistic Boolean tapes, a diagrammatic language for rig categories.

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1 Introduction

Diagrammatic languages play a central role in computer science, with classical examples including Petri nets, graph rewriting systems [23, 29], and data- and control-flow diagrams [5], among others. More recently, the increasing importance of spatial structure in computation led Milner to move beyond the traditional term-based syntax of process calculi and introduce bigraphs [53]. Similar trends arise in quantum computing [63, 25, 1, 20, 3, 39] and probabilistic programming [49, 26, 62, 67, 51], where—unlike in classical computation—information cannot be freely copied or discarded. More broadly, the shift towards viewing data as a physical resource rather than a purely logical entity (see, e.g., [58, 37]) has driven the widespread adoption [4, 14, 16, 30, 31, 38, 55, 61, 13, 18, 44, 60, 12, 28, 27, 45, 54, 34, 33, 36, 67, 59, 35, 32] of *string diagrams* [47, 65] as a graphical syntax.

Formally, string diagrams are morphisms in the symmetric monoidal category freely generated by a monoidal signature Σ , which we denote by $\mathbf{Diag}_\Sigma$. The signature contains a basic set of generating symbols and complex diagrams are built via horizontal (;) and vertical ($\otimes$) composition. Such diagrams are typically interpreted in a monoidal category $\mathbf{C}$ via a map $\langle\!\langle\cdot\!\rangle\!\rangle: \mathbf{Diag}_\Sigma \rightarrow \mathbf{C}$ assigning semantics to each diagram. When $\langle\!\langle\cdot\!\rangle\!\rangle$ is a monoidal functor—i.e., it preserves both ; and $\otimes$ —the semantics is compositional, and the induced notion of semantic equality forms a congruence. Axiomatizing this congruence is a central challenge, as it enables equational reasoning directly at the level of diagrams.



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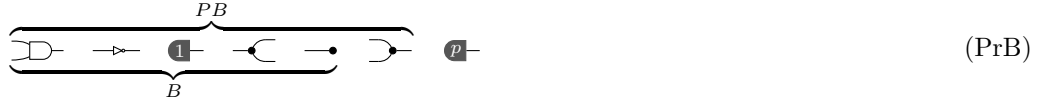
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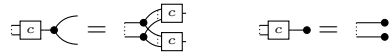
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In this paper, we provide a complete axiomatisation of *probabilistic Boolean circuits* [60]. These circuits are string diagrams generated by the following monoidal signature, where p belongs to $(0, 1)$.



The first three generators are the standard Boolean gates: $\curlywedge$ takes two inputs and returns their conjunction; $\neg$ maps an input to its negation; and $\mathbf{1}$ produces the constant 1. The generator $\curlyvee$ duplicates its input, while $\bullet$ discards it. The gate $\curlywedge$ can be seen as a dual of $\curlyvee$: it takes two inputs and returns their common value if they agree, and otherwise fails. Finally, $\textcircled{p}$ is a probabilistic generator that outputs 1 with probability p and 0 with probability $1 - p$. A circuit c with n inputs and m outputs denotes a function $\langle\!\langle c \rangle\!\rangle: 2^n \rightarrow \mathcal{D}_{\leq}(2^m)$, where $2 = \{0, 1\}$ and $\mathcal{D}_{\leq}(X)$ is the set of subdistributions over X . Categorically, the semantics $\langle\!\langle \cdot \rangle\!\rangle$ is a functor from $\mathbf{Diag}_{p_rB}$, the category of probabilistic circuits, to $\mathcal{KL}(\mathcal{D}_{\leq})$, the Kleisli category of the subdistribution monad (see e.g. [41]).

The brace below (PrB) highlights an important fragment: diagrams in $\mathbf{Diag}_B$ correspond to standard Boolean circuits, hence denote Boolean functions. As such, they can be freely copied and discarded, meaning respectively that the following equalities hold:



The brace above (P_B) identifies a larger fragment: diagrams in **Diag**_{P_B} denote Boolean *partial* functions. Indeed,  may fail to produce an output. In this setting, copying is still free, but discarding is not: for instance, ;  and   are not equal. Crucially, general probabilistic Boolean circuits cannot even be copied. For example, duplicating the result of a coin toss, as in ; , is not equivalent to tossing two independent coins, as in  .

While we refer the reader to [60] for a discussion of the practical relevance of this formalism, we emphasise here the advantages of diagrammatic syntax over traditional term-based representations. Standard Boolean identities fail in the probabilistic setting—for instance, $x \wedge x = x$ does not hold when $x \in \mathcal{D}(2)$. However, these laws hold in diagrammatic form, where copying and discarding are made explicit; see, e.g., (B5) in Figure 1. The latter collects the axioms of Boolean algebra which, as shown in [60], yields a complete axiomatisation of the Boolean fragment **Diag_B**.

Our first contribution is a complete axiomatisation of partial Boolean circuits, i.e., circuits in **Diag**_{PB}. The axioms extend those of Boolean algebra in Figure 1 with additional laws governing $\curvearrowright$, presented in Figure 3. Completeness –in Theorem 9– is crucial for the second contribution.

For the full language, rather than axiomatise $\mathbf{Diag}_{PrB}$ directly, we consider a more expressive formalism—*probabilistic Boolean tapes*—which, as illustrated in [9, Ex. 30], can faithfully express probabilistic Boolean circuits. Probabilistic Boolean tapes arise from a general construction applied to partial Boolean circuits. Concretely, by freely adding *convex biproducts* to $\mathbf{Diag}_{PB}$, we obtain a category $\mathbf{T}(\mathbf{Diag}_{PB})$ representing the syntax, together with a semantic map $\llbracket \cdot \rrbracket : \mathbf{T}(\mathbf{Diag}_{PB}) \rightarrow \mathcal{KL}(\mathcal{D}_{\leq})$ extending $\langle\langle \cdot \rangle\rangle : \mathbf{Diag}_{PB} \rightarrow \mathcal{KL}(\mathcal{D}_{\leq})$. Crucially, $\mathbf{T}(\mathbf{Diag}_{PB})$ is, like $\mathcal{KL}(\mathcal{D}_{\leq})$, a *rig category* [50, 46], featuring two monoidal structures: $\otimes$ and $\oplus$. Moreover, $\llbracket \cdot \rrbracket$ is a morphism of rig categories, i.e. it preserves $;$, $\otimes$ and $\oplus$. As a consequence, the semantics remains compositional. Finally, probabilistic Boolean tapes admit an intuitive graphical syntax, closely related to string diagrams. The reason

d'être of tape diagrams [11] is precisely to represent the three forms of composition— $\otimes$, $\oplus$, and $;$ —uniformly within two dimensions, rather than requiring an additional dimension as e.g. in [22].

The move from circuits to tapes allows us to reason not only about the monoidal product $\otimes$ in $\mathcal{KL}(\mathcal{D}_{\leq})$, but also about the operation $\oplus$, which plays a fundamental role in many probabilistic frameworks [67, 19, 51]. As discussed in Example 13, the presence of $\oplus$ provides a natural way of modelling probabilistic control and, perhaps more importantly, leads to a structurally transparent axiomatisation. In addition to the equalities induced by the tape construction in Figure 4, our axiomatisation comprises the three additional laws shown in Figure 5. The third of these is an implication expressing *cancellativity*. Although not purely equational, cancellativity is a standard reasoning principle that arises naturally in a wide variety of algebraic structures.

Relying on completeness for partial Boolean circuits and on a result from [7]—stated here as Corollary 18—characterising tape diagrams as stochastic matrices of string diagrams, we prove completeness for probabilistic Boolean tapes in Corollary 29. Consequently, as stated in Corollary 30, semantic equality of probabilistic Boolean circuits reduces to equality of their encodings as tapes.

Related Work. Figures 4 and 5 of [60] provide an axiomatisation of probabilistic Boolean circuits. Crucially, the induced equivalence is strictly coarser than semantic equality, i.e., the one induced by $\langle\!\langle - \rangle\!\rangle$. In fact, two diagrams c and d are equivalent in [60] if and only if $\langle\!\langle c \rangle\!\rangle \propto \langle\!\langle d \rangle\!\rangle$, where $\propto$ is the relation on $\mathcal{KL}(\mathcal{D}_{\leq})$ defined by $f \propto g$ whenever there exists $\lambda > 0$ such that $f(x)(y) = \lambda \cdot g(x)(y)$ for all $x \in X$ and $y \in Y$. Hence, several axioms in [60]—in particular F2 and F7—are unsound with respect to $\langle\!\langle \cdot \rangle\!\rangle$. Interestingly, as shown in Lemma 4, axiom F8 of [60] is derivable from our system.

To the best of our knowledge, partial Boolean circuits have not previously been axiomatised. In contrast, [60] gives a complete axiomatisation for the *causal* fragment, i.e., circuits generated by all gates in (PrB) except $\rhd$. In [7], the characterisation of tapes as stochastic matrices is used to derive an alternative axiomatisation for the same fragment. Despite exploiting the same result, the two axiomatisations differ substantially. Indeed, among all the axioms in Figures 3 and 5, (T3) is the only one that is even expressible with the tapes in [7].

Synopsis. Our presentation of probabilistic Boolean circuits is staged: we begin in Section 2 with **Diag_B**, then **Diag_{PrB}** in Section 3 and finally **Diag_{PrB}** in Section 4. Hence, the axiomatisation for partial Boolean circuits and its proof of completeness are illustrated in Section 3. Section 5 recalls from [9] probabilistic Boolean tapes as well as the encoding of circuits into tapes; Section 6 illustrates several general results from [7] about the tape construction. These are used in Section 7 to provide a complete axiomatisation of probabilistic Boolean tapes and prove its completeness. Appendix A contains additional material: in particular the axioms in Table 3 already appear in the form of tapes in Figure 4 in the main text.

An extended version of this paper, including full details of the proofs, is available in [6].

2 Boolean Circuits

We commence our exposition by recalling how Boolean circuits can be regarded as string diagrams.

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A *monoidal signature* is a tuple $(\mathcal{S}, \Sigma, ar, coar)$ where $\mathcal{S}$ is a set of basic sorts, hereafter denoted by $A, B, \dots$, Σ is a set of generators, denoted by $s, t, \dots$, and $ar, coar: \Sigma \rightarrow \mathcal{S}^*$ assign to each symbol its arity and coarity, i.e., words over $\mathcal{S}$, denoted by $U, V, \dots$. We consider terms generated by the following context free grammar, where $A, B \in \mathcal{S}$ and $s \in \Sigma$:

$$c ::= id_A \mid id_1 \mid s \mid \sigma_{A,B}^{\otimes} \mid c; c \mid c \otimes c$$

The rules in Table 1 assigns a type $U \rightarrow V$ to terms. Well-typed terms, taken modulo the axioms in the same table, are the arrows of $(\mathbf{Diag}_{\Sigma}, \otimes, 1)$, the strict symmetric monoidal category freely generated by Σ . Its objects are words in $\mathcal{S}^*$. Arrows of $\mathbf{Diag}_{\Sigma}$ admit a graphical representation in terms of *string diagrams* [47, 65]. The grammar above can be depicted diagrammatically as follows:

$$c ::= A \text{---} A \mid \boxed{} \mid A \text{---} \boxed{s} \text{---} B \mid \begin{matrix} A \\ B \end{matrix} \text{---} \begin{matrix} B \\ A \end{matrix} \mid U \text{---} \boxed{c} \text{---} V \mid \begin{matrix} U \text{---} \boxed{c} \text{---} V \\ U' \text{---} \boxed{c} \text{---} V' \end{matrix}$$

The diagrammatic notation internalises the axioms in Table 1, yielding proofs that are both more concise and more intuitive. Throughout the paper, we will also rely on the term-based notation, which is often more succinct for inductive definitions.

A *monoidal theory* $\mathbb{T} = (\Sigma, E)$ consists of a monoidal signature Σ and a set E of pairs of arrows of $\mathbf{Diag}_{\Sigma}$ of the same type. Let $=_{\mathbb{T}}$ be the congruence closure (w.r.t. $;$ and $\otimes$) of E . We write $\mathbf{Diag}_{\mathbb{T}}$ for the monoidal category obtained as the quotient of $\mathbf{Diag}_{\Sigma}$ by $=_{\mathbb{T}}$ and $Q_{\mathbb{T}}: \mathbf{Diag}_{\Sigma} \rightarrow \mathbf{Diag}_{\mathbb{T}}$ for the monoidal functor mapping each diagram into its $=_{\mathbb{T}}$ equivalence class.

■ **Table 1** Typing rules (top) and axioms (bottom) for freely generated strict symmetric monoidal categories.

$id_1: 1 \rightarrow 1$	$id_A: A \rightarrow A$	$\sigma_{A,B}^{\otimes}: A \otimes B \rightarrow B \otimes A$
$\frac{ar(s) = U \quad coar(s) = V}{s: U \rightarrow V}$	$\frac{c: U \rightarrow V \quad d: V \rightarrow W}{c; d: U \rightarrow W}$	$\frac{c: U_1 \rightarrow V_1 \quad d: U_2 \rightarrow V_2}{c \otimes d: U_1 \otimes U_2 \rightarrow V_1 \otimes V_2}$
$(c; d); e = c; (d; e)$	$id_V; c = c; id_Q$	$id_1 \otimes c = c \otimes id_1$
$(c_1 \otimes c_2); (d_1 \otimes d_2) = (c_1; d_1) \otimes (c_2; d_2)$	$\sigma_{U,V}; \sigma_{V,U} = id_{U \otimes V}$	$(c \otimes d) \otimes e = c \otimes (d \otimes e)$
		$(s \otimes id_W); \sigma_{V,W} = \sigma_{U,W}; (id_W \otimes s)$

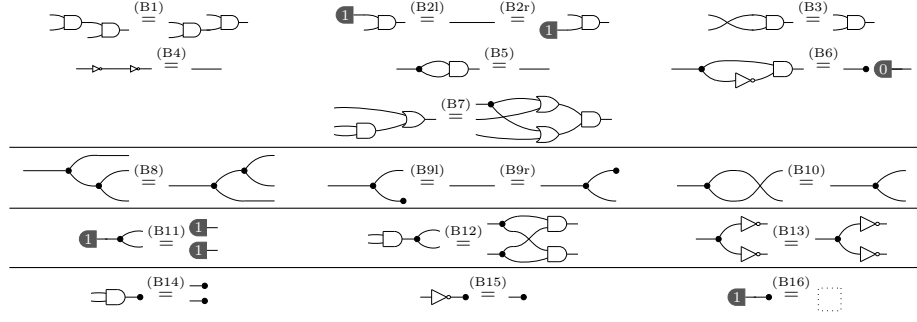
Consider the monoidal signature with a single sort, $\mathcal{S} \stackrel{\text{def}}{=} \{A\}$, and generators

$$B \stackrel{\text{def}}{=} \{\text{---}, \text{---}, \text{---}, \text{---}, \text{---}\}.$$

Arities and coarities are determined by the number of ports on the left and on the right: for instance, --- has arity $A^2 = AA$ and coarity A , while --- has arity A and coarity $A^0 = 1$.

The first three generators of B correspond to the operations and constants of Boolean algebras, namely $\wedge$, $\neg$, and 1 . The generator *copy*, denoted by --- , takes a Boolean signal as input and produces two identical outputs. The generator *discard*, denoted by --- , takes a Boolean signal as input and discards it. To make this formal, we interpret each generator $s \in B$ with arity A^n and coarity A^m as function $\langle\langle s \rangle\rangle_B: 2^n \rightarrow 2^m$ where 2 is the set of Booleans $\{0, 1\}$ and 2^0 is the singleton set $1 \stackrel{\text{def}}{=} \{\bullet\}$.

$$\begin{aligned} \langle\langle \text{---} \rangle\rangle_B: 2 \times 2 &\rightarrow 2 & \langle\langle \text{---} \rangle\rangle_B: 2 &\rightarrow 2 & \langle\langle \text{---} \rangle\rangle_B: 1 &\rightarrow 2 \\ (x, y) &\mapsto x \wedge y & x &\mapsto \neg x & \bullet &\mapsto 1 \\ \langle\langle \text{---} \rangle\rangle_B: 2 &\rightarrow 2 \times 2 & \langle\langle \text{---} \rangle\rangle_B: 2 &\rightarrow 1 & & \\ x &\mapsto (x, x) & x &\mapsto \bullet & & \end{aligned} \tag{1}$$



■ **Figure 1** The monoidal theory of Boolean algebras.

Such interpretation gives rise to a symmetric monoidal functor $\llbracket - \rrbracket_B$ from $(\mathbf{Diag}_B, \otimes, 1)$ to $(\mathbf{Set}, \otimes, 1)$. The latter is the category of sets and functions where the monoidal tensor is simply cartesian product. The functor $\llbracket - \rrbracket_B: (\mathbf{Diag}_B, \otimes, 1) \rightarrow (\mathbf{Set}, \otimes, 1)$ is defined on objects as $\llbracket A^n \rrbracket_B \stackrel{\text{def}}{=} 2^n$ and on arrows as

$$\llbracket id_A \rrbracket_B \stackrel{\text{def}}{=} id_2 \quad \llbracket id_1 \rrbracket_B \stackrel{\text{def}}{=} id_1 \quad \llbracket \sigma_{A,A}^\otimes \rrbracket_B \stackrel{\text{def}}{=} \sigma_{2,2}^\otimes \quad \llbracket c; d \rrbracket_B \stackrel{\text{def}}{=} \llbracket c \rrbracket_B; \llbracket d \rrbracket_B \quad \llbracket c \otimes d \rrbracket_B \stackrel{\text{def}}{=} \llbracket c \rrbracket_B \otimes \llbracket d \rrbracket_B \quad (2)$$

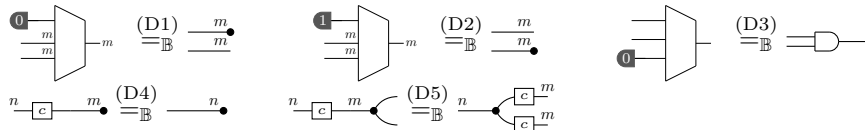
We regard diagrams as syntax and $\llbracket - \rrbracket_B$ as the map assigning semantics to syntax. Semantic equality—i.e., the congruence induced by $\llbracket - \rrbracket_B$ —is axiomatised by the laws in Figure 1. The equalities in the first block are the standard axioms of Boolean algebras, where the constant $\mathbf{0}$ and the *or* gate $\vee$ are defined as expected (see Figure 2). Note that, in the string-diagrammatic setting, the structural maps $\multimap$ and $\multimap$ must be made explicit, for instance in (B6) that is the usual law of non contradiction. The equalities in the second block state that $\multimap$ and $\multimap$ form a commutative comonoid. The equalities in the third and fourth blocks impose naturality of $\multimap$ and $\multimap$, respectively.

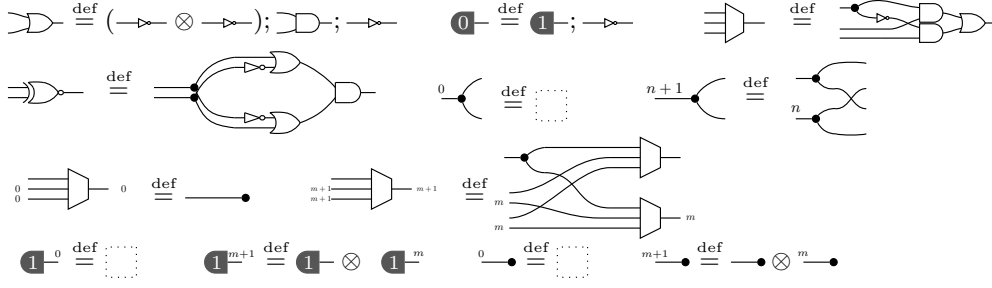
We write $\mathbb{B}$ for the monoidal theory consisting of the signature B and the axioms in Figure 1.

► **Theorem 1** (From [60]). *For all $c, d \in \mathbf{Diag}_B$, $\llbracket c \rrbracket_B = \llbracket d \rrbracket_B$ iff $c =_{\mathbb{B}} d$.*

We conclude this section by fixing some syntactic sugar, collected in Figure 2. The *xnor* gate $\Rightarrow$ takes two inputs and outputs 1 if they are equal and 0 otherwise. The *multiplexer* $\Rightarrow$ takes three inputs and returns one output: if the first input is 1, then it outputs the second input, otherwise the third input. The m -ary multiplexer $\Rightarrow^m$ works similarly, but it takes $2m + 1$ inputs: if the first input is 1, then it outputs the first m inputs, otherwise it outputs the second m inputs. Figure 2 also defines m -ary discard $\multimap^m$ and copier $\multimap^m$. Note that, for the sake of readability of string diagrams, we label wires by some natural number n as an abbreviation for A^n .

► **Lemma 2.** *Let c be a diagram in $\mathbf{Diag}_B[n, m]$. The following derived laws hold:*





■ **Figure 2** Definitions of or gate, $\mathbf{0}$ - and multiplexer (top); definitions of xnor gate and inductive definition of n -ary copier (middle); m -ary multiplexer (third row); $\mathbf{1}^m$ and n -ary dischargers (bottom).

3 Partial Boolean Circuits

Now consider the monoidal signature $PB \stackrel{\text{def}}{=} B \cup \{\curvearrowright\}$ obtained by adding to B the generator *cocopy* $\curvearrowright$. Intuitively, $\curvearrowright$ is the dual of $\curvearrowleft$: it compares two inputs and, if these are equal, it outputs that value; otherwise, it produces no output. Thus, $\curvearrowright$ does not denote a function but a *partial function*.

Let $(\mathbf{Par}, \otimes, 1)$ be the category of sets and partial functions, where $\otimes$ is the cartesian product of sets and, for all partial functions $f_1: X_1 \rightarrow Y_1$ and $f_2: X_2 \rightarrow Y_2$, $f_1 \otimes f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined for all $(x_1, x_2) \in X_1 \times X_2$ as

$$f_1 \otimes f_2(x_1, x_2) \stackrel{\text{def}}{=} \begin{cases} (f_1(x_1), f_2(x_2)) & \text{if } f_1(x_1) \neq \perp, f_2(x_2) \neq \perp \\ \perp & \text{otherwise} \end{cases} \quad (3)$$

Above and in the rest of the paper, we write $f(x) = \perp$ to say that f is undefined on x . The obvious injection $P: (\mathbf{Set}, \otimes, 1) \rightarrow (\mathbf{Par}, \otimes, 1)$ is easily proved to be a symmetric monoidal functor.

The generators in PB can all be interpreted as arrows of $\mathbf{Par}$: $\curvearrowright$ is interpreted as

$$\begin{aligned} \langle\langle \curvearrowright \rangle\rangle_{PB}: 2 \times 2 &\rightarrow 2 \\ (x, y) &\mapsto \begin{cases} x & \text{if } x = y \\ \perp & \text{else} \end{cases} \end{aligned} \quad (4)$$

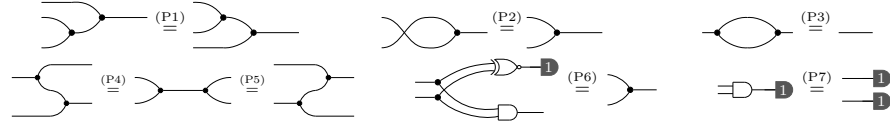
while generators $s \in B$ as $\langle\langle s \rangle\rangle_{PB} \stackrel{\text{def}}{=} P(\langle\langle s \rangle\rangle_B)$. The inductive extension, analogous to (2), of the above interpretation provides a symmetric monoidal functor $\langle\langle - \rangle\rangle_{PB}: \mathbf{Diag}_{PB} \rightarrow \mathbf{Par}$ mapping each diagram in its semantics. For example, the semantics of the following diagrams

$$\begin{aligned} \text{---} \mathbf{1} &\stackrel{\text{def}}{=} \text{---} \mathbf{1} \curvearrowright \bullet & \text{---} \mathbf{0} &\stackrel{\text{def}}{=} \text{---} \mathbf{0} \curvearrowright \bullet & \text{---} \mathbf{1} &\stackrel{\text{def}}{=} \text{---} \mathbf{1} \curvearrowright \mathbf{0} \end{aligned} \quad (5)$$

is illustrated below.

$$\begin{aligned} \langle\langle \text{---} \mathbf{1} \rangle\rangle_{PB}: 2 &\rightarrow 1 & \langle\langle \text{---} \mathbf{0} \rangle\rangle_{PB}: 2 &\rightarrow 1 & \langle\langle \text{---} \mathbf{1} \rangle\rangle_{PB}: 1 &\rightarrow 2 \\ x &\mapsto \begin{cases} \bullet & \text{if } x = 1 \\ \perp & \text{else} \end{cases} & x &\mapsto \begin{cases} \bullet & \text{if } x = 0 \\ \perp & \text{else} \end{cases} & \bullet &\mapsto \perp \end{aligned}$$

In this section, we present a complete axiomatization of the equality induced by $\langle\langle - \rangle\rangle_{PB}$. The axioms are illustrated in Figure 3: the equalities in the first row assert that $\curvearrowright$ is associative, commutative and idempotent; in the second row, (P4) and (P5) are the usual



■ **Figure 3** The monoidal theory of partial Boolean algebras.

Frobenius equalities ruling the interaction of $\rhd$ with $\lhd$; the remaining axioms are more peculiar: (P7) states that $x \wedge y = 1$ iff $x = y = 1$; (P6) provides a decomposition of $\rhd$ in terms of $\rhd$, $\rhd$ and $\lhd$. We write $\mathbb{PB}$ for the monoidal theory consisting of the signature PB and the axioms in Figures 1 and 3. Simple computations confirm that the axioms are sound.

► **Proposition 3 (Soundness).** *For all $c, d \in \mathbf{Diag}_{PB}$, if $c =_{\mathbb{PB}} d$, then $\langle\langle c \rangle\rangle_{PB} = \langle\langle d \rangle\rangle_{PB}$.*

The remainder of this section is devoted to proving the converse implication, namely completeness. Although this fact is not required for our proof, it is worth noting that the axiom F8 in [60]—corresponding to (D7) in the following lemma—can be derived within $\mathbb{PB}$.

► **Lemma 4.** *The following derived laws hold:*

$$\begin{array}{ccc} \text{---} \mathbf{1} \text{---} \mathbf{1} \text{---} & \stackrel{(D6)}{=}_{\mathbb{PB}} & \text{---} \mathbf{1} \text{---} \\ \text{---} \mathbf{1} \text{---} \mathbf{1} \text{---} & \stackrel{(D7)}{=}_{\mathbb{PB}} & \text{---} \mathbf{1} \text{---} \\ \text{---} \mathbf{1} \text{---} \mathbf{1} \text{---} & \stackrel{(D8l)}{=}_{\mathbb{PB}} & \text{---} \mathbf{1} \text{---} \mathbf{1} \text{---} \end{array}$$

The strategy for proving completeness is as follows. First, we show that any diagram c in $\mathbf{Diag}_{PB}$ can be decomposed into two Boolean circuits D_c and T_c in $\mathbf{Diag}_B$ (Proposition 6). We then appeal to the completeness result for Boolean circuits (Theorem 1).

► **Definition 5.** *For all $c \in \mathbf{Diag}_{PB}[A^n, A^m]$, the Boolean circuits $D_c \in \mathbf{Diag}_B[A^n, A]$ and $T_c \in \mathbf{Diag}_B[A^n, A^m]$ are inductively defined as*

$$\begin{array}{ll} D_c \stackrel{\text{def}}{=} n \text{---} \mathbf{1} \text{---} & T_c \stackrel{\text{def}}{=} c \text{ for all } c \in B \cup \{id_1, id_A, \sigma_{A,A}^{\otimes}\}; \\ D_{\rhd} \stackrel{\text{def}}{=} \text{---} \mathbf{1} \text{---} & T_{\rhd} \stackrel{\text{def}}{=} \text{---} \mathbf{1} \text{---} \\ D_{c;d} \stackrel{\text{def}}{=} \text{---} \mathbf{1} \text{---} & T_{c;d} \stackrel{\text{def}}{=} \text{---} \mathbf{1} \text{---} \\ D_{c \otimes d} \stackrel{\text{def}}{=} \text{---} \mathbf{1} \text{---} & T_{c \otimes d} \stackrel{\text{def}}{=} \text{---} \mathbf{1} \text{---} \end{array} \quad (6)$$

Intuitively, D_c and T_c represent, respectively, the domain and the total component of c . The diagram D_c returns 1 if the input of c lies in the domain of definition of c , and 0 otherwise. The diagram T_c returns the output of c when the input lies in its domain, and a vector of 0 otherwise. For instance, the domain of $\rhd$ is the xnor gate: it returns 1 if and only if the two inputs are equal. Its total part is given by the $\wedge$ gate: it returns 1 precisely when both inputs are 1. Similarly, the domain of $c \otimes d$ is given by the conjunction of the domains of c and d . Its total part returns the total parts of c and d when the domain evaluates to 1, and 0 otherwise.

► **Proposition 6.** *For all $c \in \mathbf{Diag}_{PB}$, it holds that $n \text{---} \boxed{c} \text{---} m =_{\mathbb{PB}} n \text{---} \begin{array}{c} \boxed{D_c} \\ \boxed{T_c} \end{array} \text{---} m$.*

Moreover, one can characterise exactly the image of $\mathbf{Diag}_{PB}$ through $\langle\langle - \rangle\rangle_{PB}$. Let $\mathbf{Par}_2$ be the full subcategory of $\mathbf{Par}$ where objects are powers of the set 2, i.e., 2^n for all $n \in \mathbb{N}$.

► **Proposition 10.** *The functor $\langle\langle - \rangle\rangle_{PB}: \mathbf{Diag}_{PB} \rightarrow \mathbf{Par}$ factors as*

$$\mathbf{Diag}_{PB} \xrightarrow{Q_{PB}} \mathbf{Diag}_{PB} \xrightarrow{\cong} \mathbf{Par}_2 \hookrightarrow \mathbf{Par}$$

where the rightmost functor is the obvious injection and the central arrow is an isomorphism.

Theorem 9 also provides a useful characterisation of diagrams in $\mathbf{Diag}_{PB}[1, A^n]$. We fix $\mathbb{1}^n \stackrel{\text{def}}{=} \mathbb{1}^-$; $\mathbb{1}^-: A \rightarrow A^n$ where $\mathbb{1}^-: A \rightarrow A^n$ is inductively defined as follows.

$$\mathbb{1}^- \stackrel{0}{\text{def}} \rightarrow \bullet \quad \mathbb{1}^- \stackrel{n+1}{\text{def}} \rightarrow \mathbb{1}^- \mathbb{1}^-$$

► **Lemma 11.** *For all $c \in \mathbf{Diag}_{PB}[1, A^n]$, either $c =_{PB} \mathbb{1}^n$ or $c =_{PB} b$ for some $b \in \mathbf{Diag}_B[1, A^n]$.*

4 Probabilistic Boolean circuits

We now recall probabilistic Boolean circuits from [60]. The signature is $PrB \stackrel{\text{def}}{=} PB \cup \{\mathbb{p}^- \mid p \in (0, 1)\}$. The gate $\mathbb{p}^-$ outputs a Boolean value, which is 1 with probability p and 0 with probability $1 - p$.

To provide a formal semantics, it is convenient to recall some key notions. We write $\mathcal{D}_{\leq}(X)$ for the set of all (finitely supported probability) *subdistributions* over a set X , namely, functions $d: X \rightarrow [0, 1]$ such that $\sum_{x \in X} d(x) \leq 1$, and $d(x) \neq 0$ for finitely many $x \in X$. For all $x \in X$, we write $\delta_x \in \mathcal{D}_{\leq}(X)$ for the Dirac distribution at x ; $\star \in \mathcal{D}_{\leq}(X)$ for the null subdistribution: $\star(x) \stackrel{\text{def}}{=} 0$ for all $x \in X$; for all $p \in (0, 1)$ and $d_1, d_2 \in \mathcal{D}_{\leq}(X)$, we write $d_1 +_p d_2$ for the subdistributions mapping each $x \in X$ into $p \cdot d_1(x) + (1 - p) \cdot d_2(x)$.

The assignment $X \mapsto \mathcal{D}_{\leq}(X)$ extends to a monad $\mathcal{D}_{\leq}: \mathbf{Set} \rightarrow \mathbf{Set}$ (see e.g., [41]). Its Kleisli category is denoted by $\mathcal{KL}(\mathcal{D}_{\leq})$: objects are sets; morphisms $f: X \rightarrow Y$ are functions $X \rightarrow \mathcal{D}_{\leq}(Y)$. We often write $f(y \mid x)$ for $f(x)(y)$, as this number represents the probability that f returns y given the input x . Identities $id_X: X \rightarrow \mathcal{D}_{\leq}(X)$ map each $x \in X$ to δ_x . For two functions $f: X \rightarrow \mathcal{D}_{\leq}(Y)$ and $g: Y \rightarrow \mathcal{D}_{\leq}(Z)$, their composition in $\mathcal{KL}(\mathcal{D}_{\leq})$ is defined as $f; g(z \mid x) \stackrel{\text{def}}{=} \sum_{y \in Y} f(y \mid x) \cdot g(z \mid y)$.

This category carries a symmetric monoidal structure $(\mathcal{KL}(\mathcal{D}_{\leq}), \otimes, 1)$, where $\otimes$ is the cartesian product of sets and, for arrows $f: X_1 \rightarrow Y_1$ and $g: X_2 \rightarrow Y_2$, $f \otimes g: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined as

$$f \otimes g((y_1, y_2) \mid (x_1, x_2)) \stackrel{\text{def}}{=} f(y_1 \mid x_1) \cdot g(y_2 \mid x_2). \quad (7)$$

Finally, the symmetric monoidal functor $J: (\mathbf{Par}, \otimes, 1) \rightarrow (\mathcal{KL}(\mathcal{D}_{\leq}), \otimes, 1)$ is the identity on objects and associates to a partial function $f: X \rightarrow Y$ the function $J(f): X \rightarrow \mathcal{D}_{\leq}(Y)$, mapping x to $\delta_{f(x)}$, if $f(x)$ is defined and to the null distribution $\star$ otherwise.

The semantics is given by interpreting the generators in PrB as arrows in $\mathcal{KL}(\mathcal{D}_{\leq})$: $\mathbb{p}^-$ as

$$\begin{aligned} \langle\langle \mathbb{p}^- \rangle\rangle: 1 &\rightarrow 2 \\ \bullet &\mapsto \delta_1 +_p \delta_0 \end{aligned} \quad (8)$$

while generators $s \in PB$ as $\langle\langle s \rangle\rangle \stackrel{\text{def}}{=} J(\langle\langle s \rangle\rangle_{PB})$. Analogously to the previous sections, the inductive extension of the above interpretation yields a symmetric monoidal functor $\langle\langle - \rangle\rangle: \mathbf{Diag}_{PrB} \rightarrow \mathcal{KL}(\mathcal{D}_{\leq})$, assigning to each diagram its semantics. For instance,

$\langle\langle \text{p-} \otimes \text{p-} \rangle; \text{D-} \rangle = (\langle\langle \text{p-} \rangle \otimes \langle\langle \text{p-} \rangle \rangle; \langle\langle \text{D-} \rangle \rangle)$ which, by (8) and (7), is the $\mathcal{KL}(\mathcal{D}_{\leq})$ -arrows mapping $\bullet$ into $\delta_1 +_{p^2} \delta_0$, i.e., $\langle\langle \text{p}^2 \text{-} \rangle \rangle$. Similarly, one can check that $\langle\langle \text{p-}; \text{C-}; \text{D-} \rangle \rangle = \langle\langle \text{p-} \rangle \rangle$. This fact is also witnessed by (B5) in Figure 1.

A natural goal, mirroring the case of (partial) Boolean circuits, is to obtain a complete axiomatisation of semantic equality. Instead, we establish a complete axiomatisation for probabilistic Boolean tapes, a formalism into which circuits can be encoded.

5 From circuits to tapes

Example 30 in [9] encodes probabilistic Boolean circuits as *probabilistic Boolean tapes*. This relies on $\mathcal{KL}(\mathcal{D}_{\leq})$ having finite coproducts, which induce a second monoidal structure $(\mathcal{KL}(\mathcal{D}_{\leq}), \oplus, 0)$, where $\oplus$ is disjoint union of sets with units $0 \stackrel{\text{def}}{=} \{\}$; for $f: X_1 \rightarrow Y_1$ and $g: X_2 \rightarrow Y_2$, the arrow $f \oplus g: X_1 \oplus X_2 \rightarrow Y_1 \oplus Y_2$ is defined, for all $u \in X_1 \oplus X_2$, $v \in Y_1 \oplus Y_2$, by

$$f \oplus g(v \mid u) \stackrel{\text{def}}{=} \begin{cases} f(y_1 \mid x_1) & \text{if } u = \iota_1(x_1) \text{ and } v = \iota_1(y_1), \\ g(y_2 \mid x_2) & \text{if } u = \iota_2(x_2) \text{ and } v = \iota_2(y_2), \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Here ι_1 and ι_2 are the coproduct injections. In $\mathcal{KL}(\mathcal{D}_{\leq})$, $\otimes$ distributes over $\oplus$, so $(\mathcal{KL}(\mathcal{D}_{\leq}), \oplus, \otimes, 0, 1)$ forms a *rig category*, aka bimonoidal category [50]. Moreover, for each object X , the arrows

$$\begin{array}{ccccccc} \mathcal{R}_{\triangleleft X}: X \rightarrow X \oplus X & \downarrow_X: X \rightarrow 0 & \uparrow_X: 0 \rightarrow X & \triangleright_X: X \oplus X \rightarrow X & & & \\ x \mapsto \delta_{\iota_1(x)} +_p \delta_{\iota_2(x)} & x \mapsto \star & & \iota_i(x) \mapsto \delta_x & & & \end{array} \quad (10)$$

are central: $(\triangleright_X, \uparrow_X)$ and $(\mathcal{R}_{\triangleleft X}, \downarrow_X)$ form, respectively, a natural and coherent *monoid* and *co-pointed convex algebra*, namely they satisfy the laws in Tables 3d and 3e in Appendix A.

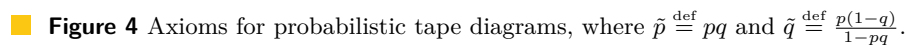
These structures are exploited in $\mathbf{T}(\mathbf{Diag}_{PB})$, the category of probabilistic Boolean tapes. Consider the following context-free grammar, where $p \in (0, 1)$, $U, V \in \mathcal{S}^*$, and c is an arrow in $\mathbf{Diag}_{PB}$:

$$\mathbf{t} ::= \mathcal{R}_{\triangleleft U} \mid \downarrow_U \mid \overline{c} \mid \uparrow_U \mid \triangleright_U \mid id_U \mid id_0 \mid \sigma_{U,V}^{\oplus} \mid \mathbf{t}; \mathbf{t} \mid \mathbf{t} \oplus \mathbf{t} \quad (11)$$

We restrict to terms typable by the rules of Table 3a. Types are arrows $P \rightarrow Q$, where $P, Q \in (\mathcal{S}^*)^*$ are viewed as sums $\bigoplus_{i=1}^n U_i$, with each $U_i = \bigotimes_{j=1}^{m_i} A_j$.

The category $\mathbf{T}(\mathbf{Diag}_{PB})$ has as set of objects $(\mathcal{S}^*)^*$. Arrows in $\mathbf{T}(\mathbf{Diag}_{PB})[P, Q]$ are terms of type $P \rightarrow Q$ modulo the axioms in Table 3, i.e., those of natural and coherent monoids, co-pcas and strict symmetric monoidal categories. By the latter axioms, $(\mathbf{T}(\mathbf{Diag}_{PB}), \oplus, 0)$ is a symmetric monoidal category. Define $P \otimes Q \stackrel{\text{def}}{=} \bigoplus_i \bigoplus_j U_i V_j$ for $P = \bigoplus_i U_i$ and $Q = \bigoplus_j V_j$; together with definition of $\mathbf{t}_1 \otimes \mathbf{t}_2$ in Table 4 in Appendix A, this yields another symmetric monoidal structure $(\mathbf{T}(\mathbf{Diag}_{PB}), \otimes, 1)$, which makes $(\mathbf{T}(\mathbf{Diag}_{PB}), \oplus, \otimes, 0, 1)$ a rig category. Crucially, there is a morphism of rig categories $\llbracket - \rrbracket: \mathbf{T}(\mathbf{Diag}_{PB}) \rightarrow \mathcal{KL}(\mathcal{D}_{\leq})$ assigning to each tape its semantics in $\mathcal{KL}(\mathcal{D}_{\leq})$:

$$\begin{array}{llllll} \llbracket \mathcal{R}_{\triangleleft A^n} \rrbracket \stackrel{\text{def}}{=} \mathcal{R}_{\triangleleft 2^n} & \llbracket \downarrow_{A^n} \rrbracket \stackrel{\text{def}}{=} \downarrow_{2^n} & \llbracket \overline{c} \rrbracket \stackrel{\text{def}}{=} J(\langle\langle c \rangle\rangle_{PB}) & \llbracket \uparrow_{A^n} \rrbracket \stackrel{\text{def}}{=} \uparrow_{2^n} & \llbracket \triangleright_{A^n} \rrbracket \stackrel{\text{def}}{=} \triangleright_{2^n} & \\ \llbracket id_{A^n} \rrbracket \stackrel{\text{def}}{=} id_{2^n} & \llbracket id_0 \rrbracket \stackrel{\text{def}}{=} id_0 & \llbracket \sigma_{A^n, A^m}^{\oplus} \rrbracket \stackrel{\text{def}}{=} \sigma_{2^n, 2^m}^{\oplus} & \llbracket \mathbf{s}; \mathbf{t} \rrbracket \stackrel{\text{def}}{=} \llbracket \mathbf{s} \rrbracket; \llbracket \mathbf{t} \rrbracket & \llbracket \mathbf{s} \oplus \mathbf{t} \rrbracket \stackrel{\text{def}}{=} \llbracket \mathbf{s} \rrbracket \oplus \llbracket \mathbf{t} \rrbracket & \end{array} \quad (12)$$

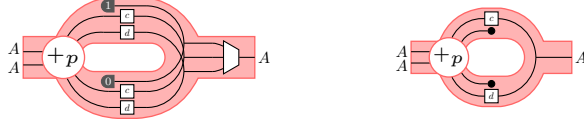

$$\begin{array}{lcl} \mathcal{E}(c) & \stackrel{\text{def}}{=} & \overline{c} \\ \mathcal{E}(\overline{p}) & \stackrel{\text{def}}{=} & \overline{p} \end{array} \quad \text{for } c \in PB \cup \{id_A, id_1, \sigma_{A,A}^{\otimes}\} \quad \left| \quad \begin{array}{lcl} \mathcal{E}(c; d) & \stackrel{\text{def}}{=} & \mathcal{E}(c); \mathcal{E}(d) \\ \mathcal{E}(c \otimes d) & \stackrel{\text{def}}{=} & \mathcal{E}(c) \otimes \mathcal{E}(d) \end{array} \right.$$
[illegible]

Consider the tape ${}^R\triangleleft_1; (\overline{\mathbf{1}} \oplus \overline{\mathbf{0}}); \triangleright_A: 1 \rightarrow A$ which is drawn as the diagram below.

By (12), its semantics $\llbracket \text{p} \rrbracket$ is $\mathbb{R}_{\perp 1}; (J(\llbracket \text{1} \rrbracket_{PB}) \oplus J(\llbracket \text{0} \rrbracket_{PB})); \triangleright_2$ which –by (1), (9) and (10)– is the function $\bullet \mapsto \delta_1 +_p \delta_0$. Note that this is exactly $\llbracket \text{p} \rrbracket$ as defined in (8). This simple observation is at the core of the encoding $\mathcal{E}(-): (\mathbf{Diag}_{PrB}, \otimes, 1) \rightarrow (\mathbf{T}(\mathbf{Diag}_{PB}), \otimes, 1)$ from [9] reported in Table 2.

The above result asserts that the encoding $\mathcal{E}(-)$ preserves the semantics. Thus, instead of axiomatising the equivalence induced by the semantics on probabilistic Boolean circuits, we rather axiomatise the one on tapes (Corollary 29) and then check equivalence of circuits by encoding them into tapes (Corollary 30). Before illustrating the axiomatisation for tapes and proving its completeness in Section 7, we recall some key results from [7] in the next section. We conclude this section by recalling from [9] an example comparing the expressivity of circuits and tapes.

► **Example 13.** In [60], probabilistic control is realised via a multiplexer, represented by the tape $\begin{array}{|c|} \hline \text{ } \\ \hline \end{array}$. Intuitively, when $\begin{array}{|c|} \hline p \\ \hline \end{array}$, $\begin{array}{|c|} \hline c \\ \hline \end{array}$ and $\begin{array}{|c|} \hline d \\ \hline \end{array}$ are connected, respectively, to the first, second, and third inputs of the multiplexer, the resulting output coincides with that of c with probability p and with that of d with probability $1 - p$. Formally, this behaviour is captured by the composite $(\begin{array}{|c|} \hline p \\ \hline \end{array} \otimes \begin{array}{|c|} \hline c \\ \hline \end{array} \otimes \begin{array}{|c|} \hline d \\ \hline \end{array}) ; \begin{array}{|c|} \hline \text{ } \\ \hline \end{array}$ which, by the definition of $\begin{array}{|c|} \hline p \\ \hline \end{array}$ in (13) and of $\otimes$ in Table 4, corresponds to the tape shown on the left below.



Although the two diagrams above exhibit similar behaviour, a crucial difference emerges when d (or symmetrically c) is instantiated as $\begin{array}{|c|} \hline \perp \\ \hline \end{array}$. In this case, the circuit on the left always fails: see (D7). By contrast, the circuit on the right still produces the output of c (respectively d) with probability p (respectively $1 - p$). Since the composition of any behaviour with a null behaviour via $\otimes$ necessarily results in a null behaviour, we believe that issues analogous to the one above are inherent in approaches relying exclusively on the monoidal product $\otimes$. The introduction of $\oplus$, on the other hand, offers a natural and expressive means of modelling probabilistic control.

6 On Freely Generated Convex Biproduct Categories

The category $\mathbf{T}(\mathbf{Diag}_{PB})$ of probabilistic Boolean tapes is an instance of a general construction, introduced in [8], which associates to any category $\mathbf{C}$ a category $\mathbf{T}(\mathbf{C})$. The latter is defined analogously to $\mathbf{T}(\mathbf{Diag}_{PB})$, but with $\mathcal{S}^*$ replaced by $Ob(\mathbf{C})$ and arrows of $\mathbf{Diag}_{PB}$ replaced by arrows of $\mathbf{C}$. Consequently, the objects of $\mathbf{T}(\mathbf{C})$ are elements of $Ob(\mathbf{C})^*$, and its arrows are equivalence classes of well-typed terms generated by the grammar in (11), where now $U, V \in Ob(\mathbf{C})$ and c ranges over arrows of $\mathbf{C}$.

In this section, we briefly recall from [8] the properties of $\mathbf{T}(\mathbf{C})$ that are needed for the completeness proof. Throughout, all categories are tacitly assumed to be locally small, and $\mathbf{Cat}$ denotes the category of locally small categories.

6.1 PCA-enriched categories

Recall from [68, 15] that a *pointed convex algebra* (pca) consists of a set X , a designated element $\star \in X$ and, for all p in the open real interval $(0, 1)$, a function $+_p: X \times X \rightarrow X$ such that, by fixing $\tilde{p} \stackrel{\text{def}}{=} pq$ and $\tilde{q} \stackrel{\text{def}}{=} \frac{p(1-q)}{1-pq}$, the following laws hold for all $x_1, x_2, x_3 \in X$.

$$(x_1 +_q x_2) +_p x_3 = x_1 +_{\tilde{p}} (x_2 +_{\tilde{q}} x_3) \quad x_1 +_p x_2 = x_2 +_{1-p} x_1 \quad x_1 +_p x_1 = x_1 \quad (14)$$

We denote by $\mathbf{PCA}$ the category of pcas and their morphisms, i.e., functions preserving $\star$ and $+_p$.

In any pca, $+_p$ is easily extended to any $p \in [0, 1]$ by fixing $x +_1 y \stackrel{\text{def}}{=} x$ and $x +_0 y \stackrel{\text{def}}{=} y$. This can be further extended for any $n \in \mathbb{N}$ and $p_1, \dots, p_n \in [0, 1]$ inductively as $\sum_{i=1}^0 p_i \cdot (-)_i \stackrel{\text{def}}{=} \star$ and $\sum_{i=1}^{n+1} p_i \cdot (-)_i \stackrel{\text{def}}{=} (-)_1 +_{p_1} \sum_{j=1}^n q_j \cdot (-)_j$ where $(-)_j = (-)_{i+1}$ and q_j is $\frac{p_{i+1}}{1-p_1}$ if $p_1 \neq 1$ and 0 otherwise. For $n = 1$, this is the *multiplication by a scalar* $p \in [0, 1]$, shortly defined as $p \cdot x \stackrel{\text{def}}{=} x +_p \star$.

The archetypical example of a pca is $\mathcal{D}_{\leq}(X)$ with $+_p$ and $\star$ defined as in Section 4. Such pca enjoys an additional property called cancellativity. A pca $(X, +_p, \star)$ is *cancellative* (or, in the terminology of [66], cancellative at $\star$) if for all $x, y \in X$ and $p \in (0, 1)$: $p \cdot x = p \cdot y \Rightarrow x = y$.

A category $\mathbf{C}$ is **PCA-enriched** if every homset carries a pca structure and composition of arrows is a pcas morphism, namely that the following equalities hold for all properly typed arrows e, f, g, h .

$$e; (f +_p g) = (e; f) +_p (e; g) \quad (f +_p g); h = (f; h +_p g; h) \quad f; \star = \star = \star; f \quad (15)$$

A functor F is **PCA-enriched** if it preserves the pca structure of each homset. **PCA-enriched** categories and functors form a category, denoted by **PCACat**.

Moreover, a monoidal category $(\mathbf{C}, \otimes, 1)$ is *monoidally enriched* over **PCA** if the following holds.

$$e \otimes (f +_p g) = (e \otimes f) +_p (e \otimes g) \quad (f +_p g) \otimes h = (f \otimes h) +_p (g \otimes h) \quad \star \otimes f = \star = f \otimes \star \quad (16)$$

The category $\mathcal{KL}(\mathcal{D}_{\leq})$ is monoidally enriched over **PCA**: for all $f, g: X \rightarrow Y$, $x \in X$, $y \in Y$ and $p \in (0, 1)$, $f +_p g(y|x) \stackrel{\text{def}}{=} p \cdot f(y|x) + (1-p) \cdot g(y|x)$ and $\star_{X,Y}(y|x) \stackrel{\text{def}}{=} 0$. Also $\mathbf{T}(\mathbf{C})$ is **PCA-enriched**: $f +_p g \stackrel{\text{def}}{=} \mathbb{R}_{\downarrow X}; (f \oplus g); \mathbb{D}_Y$ and $\star_{X,Y} \stackrel{\text{def}}{=} \mathbb{D}_X; \mathbb{I}_Y$. Such enrichment is monoidal whenever $(\mathbf{C}, \otimes, 1)$ is a symmetric monoidal category: see [9, Thm. 27].

6.2 Convex Biproduct Categories

We now recall the definition of convex biproduct categories from [8], which are suitable **PCA-enriched** categories where coproducts enjoy an additional universal property, similar to the one of products.

► **Definition 14.** Let X_1, X_2 be two objects of a **PCA-enriched** category $\mathbf{C}$. The convex product of X_1 and X_2 is an object Z with two arrows $\pi_1: Z \rightarrow X_1$ and $\pi_2: Z \rightarrow X_2$ satisfying the following property: for all $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 \leq 1$ and all arrows $f_1: A \rightarrow X_1$, $f_2: A \rightarrow X_2$, there exists a unique arrow $h: A \rightarrow Z$ such that $h; \pi_1 = p_1 \cdot f_1$ and $h; \pi_2 = p_2 \cdot f_2$.

Similarly, the convex product of n objects $X_1, \dots, X_n$ is an object Z with arrows $\pi_i: Z \rightarrow X_i$ for $i = 1, \dots, n$ satisfying the following property: for all $p_1, \dots, p_n \in [0, 1]$ where $\sum_{i=1}^n p_i \leq 1$ and arrows $f_i: A \rightarrow X_i$, there exists a unique arrow $h: A \rightarrow Z$ such that $h; \pi_i = p_i \cdot f_i$ for all $i = 1, \dots, n$. Observe that, by definition, the 0-ary convex product is a final object. Hereafter, we will denote the unique arrow h by $\langle f_1, \dots, f_n \rangle_{\vec{p}}$ where $\vec{p}$ is a compact notation for $p_1, \dots, p_n$.

► **Definition 15.** A convex biproduct category is a **PCA-enriched** category $\mathbf{C}$ with an object 0 which is both initial and final and, for every pair of objects X_1, X_2 , an object $X_1 \oplus X_2$ and morphisms $\pi_i: X_1 \oplus X_2 \rightarrow X_i$ and $\iota_i: X_i \rightarrow X_1 \oplus X_2$ such that $(X_1 \oplus X_2, \iota_1, \iota_2)$ is a coproduct, $(X_1 \oplus X_2, \pi_1, \pi_2)$ is a convex product and $\iota_i; \pi_j = id_{X_i}$ if $i = j$ and $\iota_i; \pi_j = \star_{X_i, X_j}$ if $i \neq j$.

A morphism of convex biproduct categories is a **PCA-enriched** functor $F: \mathbf{C} \rightarrow \mathbf{D}$ preserving finite coproducts. We write **CBCat** for the category of convex biproduct categories and their morphisms.

Note that the above definition is obtained from that of *category with finite biproducts* [52, 21] by just replacing products by convex products. Examples of convex biproduct categories include $\mathcal{KL}(\mathcal{D}_{\leq})$, its continuous analogue (see [8]) and, most importantly, $\mathbf{T}(\mathbf{C})$.

► **Theorem 16.** $\mathbf{T}(\mathbf{C})$ is a convex biproduct category. In particular, for every arrow $t: U \rightarrow \bigoplus_{i=1}^n U_i$ in $\mathbf{T}(\mathbf{C})$, there exist n arrows $t_i: U \rightarrow U_i$ such that $t = \langle t_1, \dots, t_n \rangle_{\vec{p}}$ for some $\vec{p} = p_1, \dots, p_n$.

Crucially, the assignment $\mathbf{C} \mapsto \mathbf{T}(\mathbf{C})$ extends to a functor $\mathbf{T}(-): \mathbf{Cat} \rightarrow \mathbf{CBCat}$ which is left adjoint to the forgetful functor $U: \mathbf{CBCat} \rightarrow \mathbf{Cat}$.

► **Theorem 17.** $\mathbf{T}(-): \mathbf{Cat} \rightarrow \mathbf{CBCat}$ is left adjoint to $U: \mathbf{CBCat} \rightarrow \mathbf{Cat}$.

In other words, $\mathbf{T}(\mathbf{C})$ is the convex biproduct category freely generated by $\mathbf{C}$. Theorems 1 and 2 in [7] illustrate that such adjunction is decomposed in the following two adjunctions, where $\mathbf{PCACat}$ denotes the category of **PCA**-enriched categories and **PCA**-enriched functors:

$$\begin{array}{ccccc} \mathbf{Cat} & \xrightleftharpoons[\perp]{(-)^+} & \mathbf{PCACat} & \xrightleftharpoons[\perp]{\mathbf{StMat}(-)} & \mathbf{CBCat} \\ & \xleftarrow{U} & & \xleftarrow{U} & \end{array} \quad (17)$$

► **Corollary 18.** For all categories $\mathbf{C}$, $\mathbf{T}(\mathbf{C})$ is isomorphic to $\mathbf{StMat}(\mathbf{C}^+)$ in $\mathbf{CBCat}$.

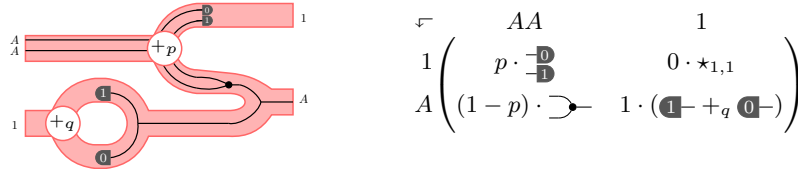
While it is not strictly necessary for our completeness proof, it is useful to provide some intuition about the two constructions in (17).

The leftmost adjunction is an instance of a general result about enriching over arbitrary algebraic theories: see [17, Prop. 6.4.7] or [69, Cor. 1]. For every category $\mathbf{C}$ one obtains **PCA**-enriched category $\mathbf{C}^+$: objects of $\mathbf{C}^+$ are those of $\mathbf{C}$; for all objects X, Y , the homset is defined as $\mathbf{C}^+[X, Y] \stackrel{\text{def}}{=} \mathcal{D}_{\leq}(\mathbf{C}[X, Y])$. For $d_1: X \rightarrow Y$ and $d_2: Y \rightarrow Z$, their composition $d_1; d_2: X \rightarrow Z$ is defined for all $h \in \mathbf{C}[X, Z]$ as $d_1; d_2(h) \stackrel{\text{def}}{=} \sum_{\{(f,g) \mid f;g=h\}} d_1(f) \cdot d_2(g)$; The identity $id_X: X \rightarrow X$ is given by δ_{id_X} .

The second adjunction is an adaptation of [52, Exercises VIII.2.5-6] which constructs a category of matrices from a category enriched over commutative monoids. Given a **PCA**-enriched category $\mathbf{C}$, $\mathbf{StMat}(\mathbf{C})$ is the category of *stochastic matrices* over $\mathbf{C}$. In a nutshell, objects of $\mathbf{StMat}(\mathbf{C})$ are words in $Ob(\mathbf{C})^*$, while arrows $M: \bigoplus_{k=1}^n U_k \rightarrow \bigoplus_{k=1}^m V_k$ are matrices with (j, i) -entries given by pairs (p_{ji}, f_{ji}) where $f_{ji} \in \mathbf{C}[U_i, V_j]$ and $p_{ji} \in [0, 1]$ satisfy $\sum_{j=1}^m p_{ji} \leq 1$. The composition of two morphisms is given by matrix multiplication where addition is given by the pca enrichment of $\mathbf{C}$ and multiplication by composition of arrows in $\mathbf{C}$.

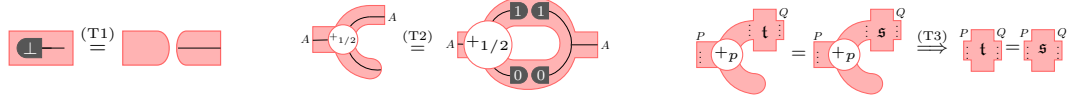
7 A Complete Axiomatisation for Probabilistic Boolean Tapes

Corollary 18 informs us that $\mathbf{T}(\mathbf{Diag}_{PB})$ is isomorphic to $\mathbf{StMat}(\mathbf{Diag}_{PB}^+)$. That is, probabilistic Boolean tapes are in bijective correspondence with stochastic matrices whose entries are subdistributions over diagrams in $\mathbf{Diag}_{PB}$. This correspondence—summarised in Figure 6—can be illustrated by the example below, where a tape (on the left) is paired with its associated matrix (on the right):



We now consider the axioms in $\mathbf{PB}$ and the category $\mathbf{Diag}_{PB}$, obtained as the quotient of $\mathbf{Diag}_{PB}$ by $\mathbf{PB}$. By applying Corollary 18 with $\mathbf{C} = \mathbf{Diag}_{PB}$, we obtain that $\mathbf{T}(\mathbf{Diag}_{PB})$ is isomorphic to $\mathbf{StMat}(\mathbf{Diag}_{PB})$. By Proposition 10, the latter is in turn isomorphic to $\mathbf{StMat}(\mathbf{Par}_2)$. Altogether:

$$\mathbf{T}(\mathbf{Diag}_{PB}) \cong \mathbf{StMat}(\mathbf{Diag}_{PB}^+) \cong \mathbf{StMat}(\mathbf{Par}_2^+). \quad (18)$$



■ **Figure 5** Axioms for tapes of partial Boolean circuits.

However, $\mathbf{StMat}(\mathbf{Par}_2^+)$ does not embed faithfully into $\mathcal{KL}(\mathcal{D}_{\leq})$: morphisms in $\mathbf{StMat}(\mathbf{Par}_2^+)$ carry strictly more information than those in $\mathcal{KL}(\mathcal{D}_{\leq})$. To see this, consider the tapes $\boxed{1}$ and $\boxed{0}$, together with their corresponding matrices in $\mathbf{StMat}(\mathbf{Par}_2^+)$:

$$\begin{array}{c} \begin{array}{c} \leftarrow \\ 2 \end{array} \begin{pmatrix} 1 \\ 0 \cdot \star_{1,2} \end{pmatrix} \quad \begin{array}{c} \leftarrow \\ 2 \end{array} \begin{pmatrix} 1 \\ 1 \cdot \langle \boxed{1} \rangle_{PB} \end{pmatrix}$$

The unique entry in the left matrix is the null subdistribution $\star \in \mathcal{D}_{\leq}(\mathbf{Par}[1, 2])$, whereas the entry in the right matrix is the Dirac distribution concentrated on the partial function $\langle \boxed{1} \rangle_{PB}: 1 \rightarrow 2$. These are two distinct morphisms in $\mathbf{Par}_2^+[1, 2]$. Nevertheless, one readily verifies that $\llbracket \boxed{1} \rrbracket$ and $\llbracket \boxed{0} \rrbracket$ coincide as morphisms in $\mathcal{KL}(\mathcal{D}_{\leq})$: both correspond to the arrow $\star_{1,2}: 1 \rightarrow 2$, i.e., the function $1 \rightarrow \mathcal{D}_{\leq}(2)$ sending the unique element $\bullet \in 1$ to the null subdistribution $\star \in \mathcal{D}_{\leq}(2)$.

7.1 Axiomatisation

To obtain a complete axiomatisation, we need to add to $\mathbb{PB}$ the axioms in Figure 5.

First, the axiom (T1) forces the equality of $\boxed{1}$ and $\boxed{0}$. Then, consider the tape diagram on the right-hand side of (T2): with probability $\frac{1}{2}$ it tests if the input is 1 and returns 1 and, with probability $\frac{1}{2}$, tests if the input is 0 and returns 0. The equality (T2) forces such tape to be equal to $\frac{1}{2} \cdot id_A$. Finally, axiom (T3) is an implication which asserts cancellativity: see Section 6.1.

From the theory $\mathbb{PB}$ together with the axioms in Figure 5, we generate a congruence $\sim$ on $\mathbf{T}(\mathbf{Diag}_{PB})$ via the inference rules below.

$$\begin{array}{c} \frac{t_1 \stackrel{(T1)}{=} t_2}{t_1 \sim t_2} \text{ (T1)} \quad \frac{t_1 \stackrel{(T2)}{=} t_2}{t_1 \sim t_2} \text{ (T2)} \quad \frac{}{t \sim t} \text{ (R)} \quad \frac{t_1 \sim t_2}{t_2 \sim t_1} \text{ (S)} \quad \frac{t_1 \sim t_2 \quad t_2 \sim t_3}{t_1 \sim t_3} \text{ (T)} \\ \frac{c =_{PB} d}{c \sim d} \text{ (PB)} \quad \frac{p \cdot t \sim p \cdot s}{t \sim s} \text{ (T3)} \quad \frac{t_1 \sim t_2 \quad s_1 \sim s_2}{t_1; s_1 \sim t_2; s_2} \text{ (;)} \quad \frac{t_1 \sim t_2 \quad s_1 \sim s_2}{t_1 \oplus s_1 \sim t_2 \oplus s_2} \text{ (}\oplus\text{)} \quad \frac{t_1 \sim t_2 \quad s_1 \sim s_2}{t_1 \otimes s_1 \sim t_2 \otimes s_2} \text{ (}\otimes\text{)} \end{array} \quad (19)$$

A straightforward verification shows that the axioms are sound.

► **Proposition 19 (Soundness).** *For all $s, t \in \mathbf{T}(\mathbf{Diag}_{PB})$, if $s \sim t$ then $\llbracket s \rrbracket = \llbracket t \rrbracket$.*

For the completeness proof, it is convenient to work with $\mathbf{T}(\mathbf{Diag}_{PB})$ rather than $\mathbf{T}(\mathbf{Diag}_{PB})$. To this end, consider the functor $\mathbf{T}(Q_{PB}): \mathbf{T}(\mathbf{Diag}_{PB}) \rightarrow \mathbf{T}(\mathbf{Diag}_{PB})$, obtained by applying $\mathbf{T}(-)$ to the quotient functor $Q_{PB}: \mathbf{Diag}_{PB} \rightarrow \mathbf{Diag}_{PB}$, i.e., the canonical functor identifying diagrams modulo $=_{PB}$. Moreover, let $\sim$ denote the congruence on $\mathbf{T}(\mathbf{Diag}_{PB})$ defined as $\sim$ but without the rule (PB). The following result shows that reasoning with $\sim$ suffices.

► **Proposition 20.** *For all $s, t \in \mathbf{T}(\mathbf{Diag}_{PB})$, $s \sim t$ iff $\mathbf{T}(Q_{PB})(s) \sim \mathbf{T}(Q_{PB})(t)$.*

Henceforth, we write $\mathbf{B}[1, A^n]$ for the set of n -ary Boolean vectors, i.e., the set of tapes in $\mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})[1, A^n]$ of the form $\vec{b} \stackrel{\text{def}}{=} \bigotimes_{i=1}^n b_i$, where each $b_i \in \{\boxed{0}, \boxed{1}\}$ if $n \neq 0$, and $\vec{b} \stackrel{\text{def}}{=} id_1$ when $n = 0$. Given $\vec{b} \in \mathbf{B}[1, A^n]$, we denote by $\overleftarrow{b}$ the tape $\bigotimes_{i=1}^n b'_i \in \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})[A^n, 1]$, where b'_i is defined as follows: $b'_i = \boxed{1}$ if $b_i = \boxed{0}$, and $b'_i = \boxed{0}$ if $b_i = \boxed{1}$.

We can now illustrate three key properties of $\sim$. First, axiom (T1) easily entails the following.

► **Lemma 21.** $\star_{A^0, A^n} \sim \boxed{1}^n$.

The second property crucially relies on axiom (T2) and on the monoidal enrichment in (16).

► **Lemma 22.** For all $n \in \mathbb{N}$, $\frac{1}{2^n} \cdot id_{A^n} \sim \sum_{\vec{b} \in \mathbf{B}[1, A^n]} \frac{1}{2^n} \cdot \overleftarrow{b}; \vec{b}$;

Note that for $n = 1$, the lemma above reduces to axiom (T2). Combining Lemma 22 and axiom (T3) we obtain the third key result.

► **Lemma 23.** Let $\mathfrak{s}, \mathfrak{t} \in \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})[A^n, A^m]$. If, for all $\vec{b} \in \mathbf{B}[1, A^n]$, $\vec{b}; \mathfrak{s} \sim \vec{b}; \mathfrak{t}$, then $\mathfrak{s} \sim \mathfrak{t}$.

7.2 Completeness

We denote by $\mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})_{\sim}$ the quotient of $\mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})$ by $\sim$, and by $Q_{\sim}: \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}}) \rightarrow \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})_{\sim}$ the corresponding quotient functor. By Propositions 19 and 20, $\llbracket - \rrbracket: \mathbf{T}(\mathbf{Diag}_{PB}) \rightarrow \mathcal{KL}(\mathcal{D}_{\leq})$ factors as

$$\mathbf{T}(\mathbf{Diag}_{PB}) \xrightarrow{\mathbf{T}(Q_{\mathbb{PB}})} \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}}) \xrightarrow{Q_{\sim}} \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})_{\sim} \xrightarrow{I} \mathcal{KL}(\mathcal{D}_{\leq})$$

for some functor I . Proving completeness amounts to showing that I is faithful.

Now, observe that $J: \mathbf{Par} \rightarrow \mathcal{KL}(\mathcal{D}_{\leq})$ from Section 3 restricts to $J_2: \mathbf{Par}_2 \rightarrow \mathcal{KL}(\mathcal{D}_{\leq})$. Since $\mathcal{KL}(\mathcal{D}_{\leq})$ is a convex biproduct category, by the two adjunctions in (17), there exists a functor

$$J'_2: \mathbf{StMat}(\mathbf{Par}_2^+) \rightarrow \mathcal{KL}(\mathcal{D}_{\leq}).$$

Consider the diagram below, where the vertical isomorphisms are those in (18). Such diagram commutes, as stated by the following result where H is the composition of the left vertical isomorphisms with J'_2 .

$$\begin{array}{ccc} \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}}) & \xrightarrow{Q_{\sim}} & \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})_{\sim} \\ \cong \downarrow & & \downarrow I \\ \mathbf{StMat}(\mathbf{Diag}_{\mathbb{PB}}^+) & & \\ \cong \downarrow & & \\ \mathbf{StMat}(\mathbf{Par}_2^+) & \xrightarrow{J'_2} & \mathcal{KL}(\mathcal{D}_{\leq}) \end{array}$$

► **Lemma 24.** $I \circ Q_{\sim} = H$.

We first observe that H is faithful for subdistributions of Boolean vectors.

► **Lemma 25.** For all $\mathfrak{t}, \mathfrak{t}' \in \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})[A^0, A^m]$, if $H(\mathfrak{t}) = H(\mathfrak{t}')$, then $\mathfrak{t} = \mathfrak{t}'$.

Moreover, by Lemma 11, we have the following normal forms for tapes $t: A^0 \rightarrow A^m$.

► **Lemma 26.** *For all $t \in \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})[A^0, A^m]$, there exist, $p, q \in [0, 1]$ and for $i = 1, \dots, n$, $p_i \in (0, 1)$ such that $\sum_{i=1}^n p_i = 1$, and $\vec{b}_i \in \mathbf{B}[1, A^m]$ such that*

$$t = \left(\sum_{i=1}^n p_i \cdot \vec{b}_i \right) +_p (\star_{A^0, A^m} +_q \text{[Diagram]}).$$

The previous two results, combined with Lemma 21 allow us to prove the following key property.

► **Lemma 27.** *For all $t, t' \in \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})[A^0, A^m]$, if $H(t) = H(t')$, then $t \sim t'$.*

► **Theorem 28.** *$I: \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})_{\sim} \rightarrow \mathbf{StMat}(\mathcal{KL}(\mathcal{D}_{\leq})_2)$ is faithful.*

Proof. Let $[t]_{\sim}, [s]_{\sim} \in \mathbf{T}(\mathbf{Diag}_{\mathbb{PB}})_{\sim}[A^n, A^m]$ be such that $I([t]_{\sim}) = I([s]_{\sim})$. Then:

$$\begin{aligned} I([t]_{\sim}) = I([s]_{\sim}) &\implies H(t) = H(s) && \text{(Lemma 24)} \\ &\implies \text{for all } \vec{b} \in \mathbf{B}[1, A^n], H(\vec{b}); H(t) = H(\vec{b}); H(s) \\ &\implies \text{for all } \vec{b} \in \mathbf{B}[1, A^n], H(\vec{b}; t) = H(\vec{b}; s) && \text{(Functoriality)} \\ &\implies \text{for all } \vec{b} \in \mathbf{B}[1, A^n], \vec{b}; t \sim \vec{b}; s && \text{(Lemma 27)} \\ &\implies t \sim s && \text{(Lemma 23)} \end{aligned}$$

Hence, $[t]_{\sim} = [s]_{\sim}$. The above derivation proves that I is faithful for arrows of type $A^n \rightarrow A^m$. For arrows of type $A^k \rightarrow \bigoplus_{i=1}^n A^{m_i}$ one can rely on convex products (Definition 14) and reduce to the previous case. While for arrows of arbitrary type $\bigoplus_{j=1}^o A^{k_j} \rightarrow \bigoplus_{i=1}^n A^{m_i}$, one can easily rely on the universal property of coproducts. See [6] for all details. ◀

► **Corollary 29 (Completeness).** *For all $s, t \in \mathbf{T}(\mathbf{Diag}_{PB})$, if $\llbracket s \rrbracket = \llbracket t \rrbracket$ then $s \sim t$.*

► **Corollary 30.** *For all $c, d \in \mathbf{Diag}_{PB}$, $\langle\langle c \rangle\rangle = \langle\langle d \rangle\rangle$ iff $\mathcal{E}(c) \sim \mathcal{E}(d)$.*

8 Conclusions and Future Work

We introduced an axiomatisation of *partial* Boolean circuits (Figures 1 and 3) and proved it complete (Theorem 9). Extending this, we gave laws of *probabilistic Boolean tapes* (Figures 4 and 5) and, using a general result from [7], established their completeness (Corollary 29). Combined with the encoding in [9], this allows checking semantic equality of *probabilistic Boolean circuits* [60] (Corollary 30).

In Example 13, it is shown that the combined use of $\oplus$ and $\otimes$ allows probabilistic tapes to realise probabilistic control effectively. Extending tapes with traces for $\oplus$ —following the approach of [10], which provides a unifying account of several program logics [42, 56, 2, 24]—would enable the representation of iteration and, potentially, provide algebraic foundations for probabilistic program logics [48, 57]. Indeed, such extended tapes can encode probabilistic regular expressions from [64]. For instance, the expression $i; c^{[p]}; t$ is depicted as the tape on the left below.



This extension is technically feasible since, as shown in [43], $\mathcal{KL}(\mathcal{D}_{\leq})$ is a traced monoidal category satisfying the *uniformity principle* [40] as well as the axiom on the right above. Combining this axiom with the laws introduced in this paper may lead to a proof principle analogous to *martingales*.

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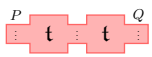
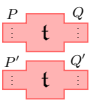
A Additional figures and tables

Table 3 Type system, syntactic sugar and axioms for $\mathbf{T}(\mathbf{C})$.

$\begin{array}{c} \mathbb{R}_{\triangleleft U}: U \rightarrow U \oplus U \quad \mathbb{L}_U: U \rightarrow 0 \quad \sigma_{U,V}^{\oplus}: U \oplus V \rightarrow V \oplus U \quad \triangleright_U: U \oplus U \rightarrow U \quad \mathbb{I}_U: 0 \rightarrow U \\ id_0: 0 \rightarrow 0 \quad id_U: U \rightarrow U \quad \frac{c: U \rightarrow V}{\underline{c}: U \rightarrow V} \quad \frac{t: P \rightarrow Q \quad s: Q \rightarrow R}{t; s: P \rightarrow R} \quad \frac{t: P_1 \rightarrow Q_1 \quad s: P_2 \rightarrow Q_2}{t \oplus s: P_1 \oplus P_2 \rightarrow Q_1 \oplus Q_2} \end{array}$			
(a) Typing rules.			
$\begin{array}{c} \mathbb{R}_{\triangleleft 0} \stackrel{\text{def}}{=} id_0 \quad \mathbb{L}_0 \stackrel{\text{def}}{=} id_0 \quad \mathbb{L}_{U \oplus P} \stackrel{\text{def}}{=} \mathbb{L}_U \oplus \mathbb{L}_P \quad \triangleright_0 \stackrel{\text{def}}{=} id_0 \quad \mathbb{I}_0 \stackrel{\text{def}}{=} id_0 \quad \mathbb{I}_{U \oplus P} \stackrel{\text{def}}{=} \mathbb{I}_U \oplus \mathbb{I}_P \\ \mathbb{R}_{\triangleleft U \oplus P} \stackrel{\text{def}}{=} (\mathbb{R}_{\triangleleft U} \oplus \mathbb{R}_{\triangleleft P}); (id_U \oplus \sigma_{U,P} \oplus id_P) \quad \triangleright_{U \oplus P} \stackrel{\text{def}}{=} (id_U \oplus \sigma_{P,U} \oplus id_P); (\triangleright_U \oplus \triangleright_P) \end{array}$			
(b) Inductive definition of monoid and co-pca structures.			
$\begin{array}{c} (f; g); h = f; (g; h) \quad id_P; f = f = f; id_Q \quad id_0 \oplus f = f = f \oplus id_0 \quad (f \oplus g) \oplus h = f \oplus (g \oplus h) \\ (f_1 \oplus f_2); (g_1 \oplus g_2) = (f_1; g_1) \oplus (f_2; g_2) \quad \sigma_{P,Q}; \sigma_{Q,P} = id_{P \oplus Q} \quad (s \oplus id_R); \sigma_{Q,R} = \sigma_{P,R}; (id_R \oplus s) \\ \underline{id_P} = id_P \quad \underline{c; d} = \underline{c}; \underline{d} \quad (\text{Tape}) \end{array}$			
(c) Axioms for strict symmetric monoidal categories and Tape axioms.			
$\begin{array}{c} (id_P \oplus \triangleright_P); \triangleright_P = (\triangleright_P \oplus id_P); \triangleright_P \quad (\triangleright\text{-as}) \quad (\mathbb{I}_P \oplus id_P); \triangleright_P = id_P \quad (\triangleright\text{-un}) \\ \mathbb{I}_0 = id_0 \quad \triangleright_0 = id_0 \quad (\mathbb{I}_0, \triangleright_0\text{-coh}) \quad \sigma_{P,P}; \triangleright_P = \triangleright_P \quad (\triangleright\text{-sym}) \\ \mathbb{I}_{P \oplus Q} = \mathbb{I}_P \oplus \mathbb{I}_Q \quad (\mathbb{I}\text{-coh}) \quad \triangleright_{P \oplus Q} = (id_P \oplus \sigma_{Q,P} \oplus id_Q); (\triangleright_P \oplus \triangleright_Q) \quad (\triangleright\text{-coh}) \\ \mathbb{I}_P; f = \mathbb{I}_Q \quad (\mathbb{I}\text{-nat}) \quad \triangleright_P; f = (f \oplus f); \triangleright_Q \quad (\triangleright\text{-nat}) \end{array}$			
(d) Axioms for natural and coherent monoids in a strict monoidal category.			
$\begin{array}{c} \mathbb{R}_{\triangleleft P}; (\mathbb{Q}_{\triangleleft P} \oplus id_P) = \tilde{\mathbb{R}}_{\triangleleft P}; (id_P \oplus \tilde{\mathbb{Q}}_{\triangleleft}) \quad (\mathbb{R}_{\triangleleft}\text{-as}) \quad \tilde{p} \stackrel{\text{def}}{=} pq \quad \tilde{q} \stackrel{\text{def}}{=} \frac{p(1-q)}{1-pq} \\ \mathbb{R}_{\triangleleft P}; \triangleright_P = id_P \quad (\mathbb{R}_{\triangleleft}\text{-idem}) \quad \mathbb{R}_{\triangleleft P} \sigma_{P,P} = {}^{1-\mathbb{R}_{\triangleleft P}} \quad (\mathbb{R}_{\triangleleft}\text{-sym}) \\ \mathbb{R}_{\triangleleft 0} = id_0 \quad (\mathbb{R}_{\triangleleft 0}\text{-coh}) \quad \mathbb{L}_0 = id_0 \quad (\mathbb{L}_0\text{-coh}) \\ \mathbb{L}_{P \oplus Q} = \mathbb{L}_P \oplus \mathbb{L}_Q \quad (\mathbb{L}\text{-coh}) \quad \mathbb{R}_{\triangleleft P \oplus Q} = (\mathbb{R}_{\triangleleft P} \oplus \mathbb{R}_{\triangleleft Q}); (id_P \oplus \sigma_{P,Q} \oplus id_Q) \quad (\mathbb{R}_{\triangleleft}\text{-coh}) \\ f; \mathbb{L}_Q = \mathbb{L}_P \quad (\mathbb{L}\text{-nat}) \quad f; \mathbb{R}_{\triangleleft Q} = \mathbb{R}_{\triangleleft P}; (f \oplus f) \quad (\mathbb{R}_{\triangleleft}\text{-nat}) \end{array}$			
(e) Axioms for natural and coherent co-pcas in a strict monoidal category.			

Table 4 Inductive definitions of left distributor $\delta_{P,Q,R}^l$ and $\otimes$ -symmetry $\sigma_{P,Q}^{\otimes}$ (top); monomial and polynomial whiskerings (center); definition of $\otimes$ (bottom).

$\delta_{P,Q,R}^l: P \otimes (Q \oplus R) \rightarrow (P \otimes Q) \oplus (P \otimes R)$			$\sigma_{P,Q}^{\otimes}: P \otimes Q \rightarrow Q \otimes P$, with $P = \bigoplus_i U_i$		
$\delta_{0,Q,R}^l \stackrel{\text{def}}{=} id_0$	$\delta_{U \oplus P', Q, R}^l \stackrel{\text{def}}{=} (id_U \otimes (Q \oplus R)) \oplus \delta_{P', Q, R}^l; (id_U \otimes Q \oplus \sigma_{U \otimes R, P' \otimes Q}^{\oplus} \oplus id_{P' \otimes R})$		$\sigma_{P,0}^{\otimes} \stackrel{\text{def}}{=} id_0$	$\sigma_{P, V \oplus Q'}^{\otimes} \stackrel{\text{def}}{=} \delta_{P, V, Q'}^l; (\bigoplus_i \sigma_{U_i, V}^{\otimes} \oplus \sigma_{P, Q'}^{\otimes})$	
$\mathbb{L}_U(id_0) \stackrel{\text{def}}{=} id_0$	$\mathbb{L}_U(t_1 \oplus t_2) \stackrel{\text{def}}{=} \mathbb{L}_U(t_1) \oplus \mathbb{L}_U(t_2)$	$\mathbb{R}_U(id_0) \stackrel{\text{def}}{=} id_0$	$\mathbb{R}_U(t_1 \oplus t_2) \stackrel{\text{def}}{=} \mathbb{R}_U(t_1) \oplus \mathbb{R}_U(t_2)$		
$\mathbb{L}_U(\underline{c}) \stackrel{\text{def}}{=} \underline{id_U \otimes c}$	$\mathbb{L}_U(t_1; t_2) \stackrel{\text{def}}{=} \mathbb{L}_U(t_1); \mathbb{L}_U(t_2)$	$\mathbb{R}_U(\underline{c}) \stackrel{\text{def}}{=} \underline{c \otimes id_U}$	$\mathbb{R}_U(t_1; t_2) \stackrel{\text{def}}{=} \mathbb{R}_U(t_1); \mathbb{R}_U(t_2)$		
$\mathbb{L}_U(\sigma_{V,W}^{\oplus}) \stackrel{\text{def}}{=} \sigma_{\mathbb{L}_U V, \mathbb{L}_U W}^{\oplus}$		$\mathbb{R}_U(\sigma_{V,W}^{\oplus}) \stackrel{\text{def}}{=} \sigma_{\mathbb{R}_U V, \mathbb{R}_U W}^{\oplus}$			
$\mathbb{L}_U(\mathbb{R}_{\triangleleft V}) \stackrel{\text{def}}{=} \mathbb{R}_{\triangleleft \mathbb{L}_U V}$	$\mathbb{L}_U(\mathbb{L}_V) \stackrel{\text{def}}{=} \mathbb{L}_{\mathbb{L}_U V}$	$\mathbb{R}_U(\mathbb{R}_{\triangleleft V}) \stackrel{\text{def}}{=} \mathbb{R}_{\triangleleft \mathbb{R}_U V}$	$\mathbb{R}_U(\mathbb{L}_V) \stackrel{\text{def}}{=} \mathbb{L}_{\mathbb{R}_U V}$		
$\mathbb{L}_U(\triangleright_V) \stackrel{\text{def}}{=} \triangleright_{\mathbb{L}_U V}$	$\mathbb{L}_U(\mathbb{I}_V) \stackrel{\text{def}}{=} \mathbb{I}_{\mathbb{L}_U V}$	$\mathbb{R}_U(\triangleright_V) \stackrel{\text{def}}{=} \triangleright_{\mathbb{R}_U V}$	$\mathbb{R}_U(\mathbb{I}_V) \stackrel{\text{def}}{=} \mathbb{I}_{\mathbb{R}_U V}$		
$\mathbb{L}_0(t) \stackrel{\text{def}}{=} id_0$	$\mathbb{L}_{W \oplus S'}(t) \stackrel{\text{def}}{=} \mathbb{L}_W(t) \oplus \mathbb{L}_{S'}(t)$	$\mathbb{R}_0(t) \stackrel{\text{def}}{=} id_0$	$\mathbb{R}_{W \oplus S'}(t) \stackrel{\text{def}}{=} \delta_{P, W, S'}^l; (\mathbb{R}_W(t) \oplus \mathbb{R}_{S'}(t)); \delta_{Q, W, S'}^{l-1}$		
$t_1 \otimes t_2 \stackrel{\text{def}}{=} \mathbb{L}_P(t_2); \mathbb{R}_S(t_1) \quad (\text{for } t_1: P \rightarrow Q, t_2: R \rightarrow S)$					

Symbol	Diagram	Matrix
$\mathbb{P}_{\triangleleft U}$		$\begin{matrix} \swarrow & U \\ U & \left(\begin{array}{c} p \cdot u \text{ --- } u \\ (1-p) \cdot u \text{ --- } u \end{array} \right) \\ \swarrow & U \end{matrix}$
$\mathbb{I}_U$		$\begin{matrix} \swarrow & U \\ U & \left(\begin{array}{c} \\ \end{array} \right) \\ \swarrow & \end{matrix}$
$\overline{c}$		$\begin{matrix} \swarrow & U \\ V & \left(\begin{array}{c} 1 \cdot u \text{ --- } c \text{ --- } v \end{array} \right) \\ \swarrow & \end{matrix}$
$\mathbb{I}_U$		$\begin{matrix} \swarrow & U \\ \left(\begin{array}{c} \\ \end{array} \right) \\ \swarrow & \end{matrix}$
$\mathbb{P}_{\triangleright U}$		$\begin{matrix} \swarrow & U & U \\ U & \left(\begin{array}{cc} 1 \cdot u \text{ --- } u & 1 \cdot u \text{ --- } u \end{array} \right) \\ \swarrow & \end{matrix}$
id_U		$\begin{matrix} \swarrow & U \\ U & \left(\begin{array}{c} 1 \cdot u \text{ --- } u \end{array} \right) \\ \swarrow & \end{matrix}$
id_0		$\begin{matrix} \swarrow & \\ \left(\begin{array}{c} \end{array} \right) \\ \swarrow & \end{matrix}$
$\sigma_{U,V}^{\oplus}$		$\begin{matrix} \swarrow & U & V \\ V & \left(\begin{array}{cc} 0 \cdot \star_{U,V} & 1 \cdot v \text{ --- } v \\ 1 \cdot u \text{ --- } u & 0 \cdot \star_{V,U} \end{array} \right) \\ \swarrow & \end{matrix}$
$\mathbf{t}; \mathbf{t}$		matrix multiplication
$\mathbf{t} \oplus \mathbf{t}$		direct sum of matrices

■ **Figure 6** Dictionary of correspondences.