From Coalgebraic Determinization to Belief Construction for Partial Observability

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Abstract

The *belief construction* is a fundamental technique for transforming partially observable systems to fully observable ones while preserving the relevant semantics. It plays a central role in the analysis of partially observable systems, in particular *partially observable Markov decision processes* (POMDPs), which is a central model in artificial intelligence and formal verification. In this paper, we develop a coalgebraic framework for the belief construction. To handle observations categorically, we lift a monad to slice categories and introduce a belief decomposition that reorganizes states according to their observations. This allows us to introduce a coalgebraic generalization of the belief construction, obtained by combining the belief decomposition with the coalgebraic determinization of Silva, Bonchi, Bonsangue, and Rutten. In this framework, we show that the semantics of a partially observable system coincides with that of the corresponding belief coalgebra. We then study when the latter further agrees with the semantics of its fully observable counterpart, and use this to identify conditions under which the semantics of a partially observable system coincides with that of the corresponding fully observable belief system. As a consequence, we recover the standard equivalence between POMDPs and belief MDPs, and obtain a new equivalence result for weighted transition systems with the semimodule monad.

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1 Introduction

The notion of observation is ubiquitous in artificial intelligence and formal verification, since in real-world settings it is often unrealistic to assume that we have precise information about the systems under analysis. *Partially observable Markov decision processes* (POMDPs) (e.g. [26]) provide a central model for incorporating partial observability, and hence for reasoning about imperfect information about systems.

▶ **Example 1.** We illustrate this with the POMDP c shown in Fig. 1.[1] A POMDP is a Markov decision process (MDP) equipped with an observation function assigning to each state an observation. In this example, there are five states $S := \{s_0, s_1, s_2, t_1, t_2\}$, and the observation function $\mathrm{obs}\colon S \to \{o, o_0, o_1, o_2\}$ is given by

$$\mathrm{obs}(s_0) := o_0, \quad \mathrm{obs}(s_1) = \mathrm{obs}(s_2) := o, \quad \mathrm{obs}(t_1) := o_1, \quad \mathrm{obs}(t_2) := o_2.$$

[1] While rewards are assigned to transitions in the example, our coalgebraic framework for POMDPs assigns rewards to states. This difference is not essential, in the sense that the former can be modelled by the latter by adding dummy states, and the converse direction is trivial.

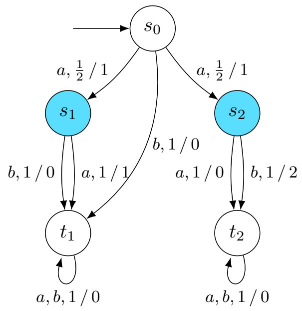

Figure 1 A POMDP c, where the states s_1 and s_2 are assigned the same observation o.

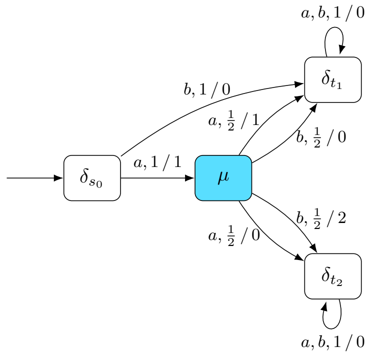

Figure 2 The reachable part of the belief MDP, where $\mu \coloneqq \frac{1}{2}\delta_{s_1} + \frac{1}{2}\delta_{s_2}$.

From each state, choosing an action a or b induces a probabilistic transition together with an immediate reward. For instance, from the initial state s_0, choosing action a leads to s_1 with probability $\frac{1}{2}$ and reward 1, and to s_2 with probability $\frac{1}{2}$ and reward 1.

The quantity of interest here is the maximal total expected reward over observation-based schedulers. An observation-based scheduler is a map $u \colon O^+ \to \{a, b\}$, where $O = \{o_0, o, o_1, o_2\}$ and O^+ is the set of nonempty finite sequences of observations, which selects the next action based only on the history of observations, rather than the underlying states.[2] For the POMDP c, the maximal total expected reward from s_0 is

$$\max_{u \in \{a,b\}^{O^+}} \mathrm{TER}_{c,u} = \frac{1}{2}(1+2) + \frac{1}{2}(1+0) = 2, \tag{1}$$

where $\mathrm{TER}_{c,u}$ is the total expected reward of c under the scheduler u. The maximum is attained by choosing action a first and then action b. Notably, if we instead regard the same system as a fully observable MDP, then the maximal total expected reward becomes $\frac{5}{2}$, since the choice of action can depend on the visited state (s_1 or s_2).

The belief construction for POMDPs (e.g. [27]) transforms a POMDP into an equivalent fully observable MDP, called its *belief MDP*. The states of this MDP are probability distributions over the original state space, called *beliefs*. Although the belief MDP of a finite-state POMDP is generally infinite-state, the construction reduces the original problem to one on MDPs, for which many effective abstraction and approximation techniques have been developed (e.g. [29, 11, 4, 23, 25]).

▶ **Example 2.** We present in Fig. 2 the belief MDP c^{Bel} of the POMDP c above; we write δ_s for the Dirac distribution at a state s. The figure shows only the reachable part from the initial belief δ_{s_0}. The maximal total expected reward from the initial belief δ_{s_0} is

$$\frac{1}{2}(1+2) + \frac{1}{2}(1+0) = 2, \tag{2}$$

which coincides with the maximal total expected reward (1) of the POMDP.

The above example illustrates that the usual belief construction for POMDPs is semantics-preserving: the maximal total expected reward of the original POMDP coincides with that of its belief MDP. This naturally raises the following two questions:

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[2] This presentation is equivalent to the more common presentation of schedulers as functions $O \times (\{a, b\} \times O)^* \to \{a, b\}$ since previously chosen actions are themselves determined by the preceding observations.

- How can belief constructions be defined uniformly for classes of partially observable systems that include POMDPs?
- How can we prove that such constructions preserve semantics?

In this paper, we answer these questions from a coalgebraic perspective. We model a partially observable system with initial states as *pointed partially observable coalgebras*. Concretely, such a coalgebra consists of a pair of an initial-state map $i\colon I \to S$ and a coalgebra $c := \langle \delta, \mathrm{obs} \rangle\colon S \to FTS \times O$, where $F\colon \mathbb{C} \to \mathbb{C}$ describes the one-step transition type, $T\colon \mathbb{C} \to \mathbb{C}$ is a monad describing the branching type, such as nondeterminism and probability, and O is an object of observations.

We then propose a generic construction, called the *coalgebraic belief construction*. Given a pointed coalgebra (i, c), the construction produces a new pointed coalgebra $\mathsf{Bel}(i, c)$, called its *belief coalgebra*. The idea is to abstract the ordinary belief update for POMDPs: one first applies the probabilistic transition to a belief, and then decomposes the resulting distribution into conditional beliefs indexed by observations. Categorically, the first step is provided by the coalgebraic determinization [28], while the second step is provided by a new structure that we call a *belief decomposition*.

To formulate the second step, we regard state spaces equipped with observations as objects of the slice category $\mathbb{C}/O$, and lift the monad T on $\mathbb{C}$ to a monad $\overline{T}_O$ on the slice category $\mathbb{C}/O$. The belief decomposition uses this lifted monad to describe the operation of splitting a T-structured state into components indexed by observations, generalizing the decomposition of a belief into conditional beliefs in the POMDP case.

Our main theorem states that the coalgebraic belief construction is correct: the semantics of a partially observable coalgebra coincides with that of its belief coalgebra. In our abstract setting, the belief coalgebra still carries an observation map, whereas the usual belief MDP of a POMDP is fully observable. We therefore further provide conditions under which the semantics of this belief coalgebra agrees with the semantics of its fully observable counterpart. This recovers the standard equivalence between a POMDP and its belief MDP.

We instantiate the framework with the nonempty powerset monad for nondeterminism, the distribution monad for probability, and the semimodule monad for weighted systems. These instances recover the usual belief construction for POMDPs and yield a new belief construction for partially observable weighted transition systems. As a corollary, we recover the decidability of termination for partially observable nondeterministic transition systems.

In summary, our contributions are as follows:

- a generic coalgebraic framework for belief constructions, based on monads lifted to slice categories and belief decompositions;
- a correctness theorem for the coalgebraic belief construction, showing that the semantics of a partially observable coalgebra coincides with that of its belief coalgebra;
- sufficient conditions under which partially observable coalgebras can be seen semantically as fully observable ones;
- concrete instances for nondeterministic, probabilistic, and weighted transition systems, recovering known belief constructions and yielding a new belief construction for partially observable weighted transition systems with the semimodule monad.

Structure. After recalling preliminaries in §2, we lift monads to slice categories in §3 and define the coalgebraic belief construction in §4. We introduce semantics in §5 and prove the correctness theorem in §6. We then compare partially observable coalgebras and their fully observable counterparts in §7, present examples in §8, discuss related work in §9, and conclude in §10.

2 Preliminaries

We recall preliminaries on coalgebras, as well as the coalgebraic framework for determinization introduced by Silva et al. [28], which has also been applied to trace semantics [19, 16, 7, 14] and graded semantics [13]. We refer the reader to [18, 2] for the background on coalgebras.

Given an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$, an F-*coalgebra* c is a morphism $c\colon S \to FS$, and a *coalgebra morphism* from $c\colon S \to FS$ to $c'\colon S' \to FS'$ is a morphism $f\colon S \to S'$ such that $c' \circ f = Ff \circ c$.

▶ **Definition 3.** *Let $F\colon \mathbb{C} \to \mathbb{C}$. For an object $I \in \mathbb{C}$, an $(I$-$)$*pointed coalgebra *is a pair (i, c) of a morphism $i\colon I \to S$ and an F-coalgebra $c\colon S \to FS$, often written as $(i, c)\colon I \to S \to FS$. A* morphism *$f$ between pointed coalgebras $(i, c)\colon I \to S \to FS$ and $(i', c')\colon I \to S' \to FS$ is a morphism $f\colon S \to S'$ such that $f \circ i = i'$ and $c' \circ f = Ff \circ c$.*

▶ **Definition 4.** *Let $T\colon \mathbb{C} \to \mathbb{C}$ be a monad and let $F\colon \mathbb{C} \to \mathbb{C}$ be a functor. A natural transformation $\lambda\colon TF \Rightarrow FT$ is a* distributive law *(of T over F) if $F(\eta_X) = \lambda_X \circ \eta_{FX}$ and $F(\mu_X) \circ \lambda_{TX} \circ T(\lambda_X) = \lambda_X \circ \mu_{FX}$ for each $X \in \mathbb{C}$.*

A distributive law allows one to combine the branching structure described by T with the one-step transition type described by F. It is known that distributive laws correspond bijectively to liftings of F to the category of Eilenberg–Moore algebras for T; see, e.g., [18].

▶ **Definition 5.** *Let $T\colon \mathbb{C} \to \mathbb{C}$ be a monad, $F\colon \mathbb{C} \to \mathbb{C}$ be a functor, and $\lambda\colon TF \Rightarrow FT$ be a distributive law. For a coalgebra $c\colon S \to FTS$, the* (coalgebraic) determinization *is the F-coalgebra given by $\mathrm{Det}(c) \coloneqq TS \xrightarrow{Tc} TFTS \xrightarrow{\lambda_{TS}} FTTS \xrightarrow{F\mu_S} FTS$.*

Intuitively, Tc applies the original transition structure pointwise to a T-structured state, λ exchanges the transition types T and F, and $F\mu_S$ flattens the resulting double T-structure. This construction generalizes familiar determinization procedures such as the powerset construction for nondeterministic automata.

▶ **Example 6.** Let T be the finite powerset monad $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ and F be the functor $(\_)^A \times \mathbf{2}\colon \mathbf{Set} \to \mathbf{Set}$ where A is a fixed finite set of actions and $\mathbf{2} = \{0, 1\}$. Then an FT-coalgebra $\langle \delta, \mathrm{acc}\rangle\colon S \to (\mathcal{P}S)^A \times \mathbf{2}$ is precisely a nondeterministic automaton: for each state s and action a, $\delta(s)(a)$ is the set of successors of s under a, and $\mathrm{acc}(s)$ indicates whether s is an accepting state. We define a distributive law $\lambda\colon \mathcal{P}((\_)^A \times \mathbf{2}) \Rightarrow (\mathcal{P}(\_))^A \times \mathbf{2}$ by $\lambda_X(U) \coloneqq \langle a \mapsto \bigcup_{(f,t)\in U} f(a), \bigvee_{(f,t)\in U} t\rangle$, for each $X \in \mathbf{Set}$ and $U \in \mathcal{P}(X^A \times \mathbf{2})$. Then $\mathrm{Det}(\langle \delta, \mathrm{acc}\rangle)$ is the standard powerset construction for nondeterministic automata. Its state space is $\mathcal{P}S$, the successor of a state U under an action a is the state $\bigcup_{s\in U} \delta(s)(a)$, and U is accepting if $\mathrm{acc}(s) = 1$ for some $s \in U$.

3 Slicing Monads

The coalgebraic determinization recalled in §2 applies to coalgebras of the form $c\colon S \to FTS$. For the belief construction, however, states are additionally equipped with observations. To incorporate observations categorically, we work in slice categories.

For an object $O \in \mathbb{C}$, recall that the slice category $\mathbb{C}/O$ has as objects morphisms $f\colon X \to O$, and as morphisms from $f\colon X \to O$ to $g\colon Y \to O$ morphisms $h\colon X \to Y$ satisfying $g \circ h = f$. Thus an object of $\mathbb{C}/O$ may be regarded as a state space X equipped with an observation map into O. By abuse of notation, for a morphism f in a slice category, we sometimes write f for its underlying morphism in $\mathbb{C}$.

In this section, we aim to lift the monad T on $\mathbb{C}$ to a monad on each slice category $\mathbb{C}/O$. We then study how these lifted monads behave under change of the observation object and how they relate to the original monad T via the forgetful functor $d_O\colon \mathbb{C}/O \to \mathbb{C}$.

We begin by recalling oplax monad morphisms.

▶ **Definition 7.** *Let $T = (T, \eta^T, \mu^T)$ be a monad on $\mathbb{C}$, and $S = (S, \eta^S, \mu^S)$ a monad on $\mathbb{D}$. An* (oplax) monad morphism *from $(\mathbb{C}, T)$ to $(\mathbb{D}, S)$ is a tuple (F, ψ) of a functor $F\colon \mathbb{C} \to \mathbb{D}$ and a natural transformation $\psi\colon FT \Rightarrow SF$ such that $\eta^S_{FX} = \psi_X \circ F(\eta^T_X)$ and $\mu^S_{FX} \circ S\psi_X \circ \psi_{TX} = \psi_X \circ F(\mu^T_X)$ for each $X \in \mathbb{C}$.*

▶ **Lemma 8** (Cf. [17, Sec. 2],[21])**.** *Consider functors $F\colon \mathbb{C}' \to \mathbb{C}, G\colon \mathbb{D}' \to \mathbb{D}$ and adjunctions $L \dashv R\colon \mathbb{D} \to \mathbb{C}$ and $L' \dashv R'\colon \mathbb{D}' \to \mathbb{C}'$. Assume that $LF = GL'$, and let $\alpha\colon FR' \Rightarrow RG$ be the mate of $LF = GL'$. Then $(F, \alpha L')$ is a monad morphism from $(\mathbb{C}', R'L')$ to $(\mathbb{C}, RL)$.*

We use Lem. 8 twice in this section: to obtain the base-change morphisms (Σ_u, θ_u) in Prop. 10 and the forgetful morphism (d, ι) in Prop. 12.

For the remainder of this section, we assume that $\mathbb{C}$ is a category with pullbacks. Let T be a monad on $\mathbb{C}$, and for each $f\colon X \to O$ in $\mathbb{C}$, we fix a pullback of Tf along η^T_O.

▶ **Definition 9.** *For each object $O \in \mathbb{C}$, we define an endofunctor $\overline{T}_O\colon \mathbb{C}/O \to \mathbb{C}/O$. For an object $f : X \to O$, the object $\overline{T}_O(f)\colon B_f \to O$ is defined by the pullback of Tf along η_O, as in the left diagram below. For a morphism $h\colon f \to g$ in $\mathbb{C}/O$, we define $\overline{T}_O(h)$ to be the unique morphism induced by universality of the pullback for B_g, as in the right diagram below.*

$$
\begin{tikzcd}
B_f \arrow[r, "\overline{T}_O f"] \arrow[d, "\iota_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & O \arrow[d, "\eta_O"] \\
TX \arrow[r, "Tf"'] & TO
\end{tikzcd}
\qquad
\begin{tikzcd}
B_f \arrow[r, dotted, "\overline{T}_O h"'] \arrow[rr, bend left, "\overline{T}_O f"] \arrow[d, "\iota_f"'] & B_g \arrow[r, "\overline{T}_O g"'] \arrow[d, "\iota_g"'] \arrow[dr, phantom, "\lrcorner", very near start] & O \arrow[d, "\eta_O"] \\
TX \arrow[r, "Th"] \arrow[rr, bend right, "Tf"'] & TY \arrow[r, "Tg"] & TO
\end{tikzcd}
\tag{3}
$$

▶ **Proposition 10.** *For each object $O \in \mathbb{C}$, the endofunctor $\overline{T}_O$ carries a canonical monad structure. Moreover, for each morphism $u : O \to O'$, the post-composition functor $\Sigma_u := u \circ (\_) : \mathbb{C}/O \to \mathbb{C}/O'$ extends to a monad morphism $(\Sigma_u, \theta_u) : (\mathbb{C}/O, \overline{T}_O) \to (\mathbb{C}/O', \overline{T}_{O'})$, where $\theta_u : \Sigma_u\overline{T}_O \Rightarrow \overline{T}_{O'}\Sigma_u$ is the natural transformation whose component at $f : X \to O$ is induced by universality of the pullback of η'_O along $T(u \circ f)$.*

We refer to Appendix C.1 for the proof.

▶ Remark 11. The result in Prop. 10 gives the following 2-categorical reformulation. Define a functor $\mathrm{Sl}\colon \mathbb{C} \to \mathbf{Cat}$ by $\mathrm{Sl}(O) := \mathbb{C}/O$ and $\mathrm{Sl}(u) := \Sigma_u$. Then the assignment $O \in \mathbb{C} \mapsto (\mathbb{C}/O, \overline{T}_O)$ with $(u\colon O \to O') \mapsto (\Sigma_u, \theta_u)$ defines a functor $\overline{T}\colon \mathbb{C} \to \mathbf{Mnd}$ whose composite with the forgetful 2-functor $\mathbf{Mnd} \to \mathbf{Cat}$ is Sl. Here $\mathbf{Mnd}$ is the 2-category of monads and oplax monad morphisms.

We sometimes omit the subscript O in $\overline{T}_O$ when it is clear from the context.

The forgetful functor $d\colon \mathbb{C}/O \to \mathbb{C}$ is also compatible with the monads $\overline{T}$ and T as below, see Appendix C.2 for the proof. Please note that $d\overline{T}(f) = B_f$ in (3) for each $f \in \mathbb{C}/O$.

▶ **Proposition 12.** *There is a monad morphism $(d, \iota)\colon (\mathbb{C}/O, \overline{T}) \to (\mathbb{C}, T)$ given by the morphisms $(\iota_f)_{f \in \mathbb{C}/O}$ defined in the pullback square on the left in (3).*

4 Coalgebraic Belief Construction

We introduce our coalgebraic belief construction, which transforms a coalgebra with an observation map into a coalgebra, called its belief coalgebra. The construction is based on the coalgebraic determinization (see §2) together with the monads $\overline{T}_O$ on slice categories introduced in §3.

We begin by introducing coalgebras equipped with observations. Here, we adopt a wide subcategory $\mathbb{O}$ of $\mathbb{C}$, that is, a subcategory containing all objects of $\mathbb{C}$, to specify admissible observation morphisms.

▶ **Definition 13.** *Let $\mathbb{C}$ be a cartesian category, $\mathbb{O}$ be a wide subcategory of $\mathbb{C}$, and G be an endofunctor on $\mathbb{C}$. A* partially observable G-coalgebra *(or* PO coalgebra *for short) is a morphism $\langle \delta, \mathrm{obs}\rangle : S \to GS \times O$, and a morphism from $\langle \delta, \mathrm{obs}\rangle : S \to GS \times O$ to $\langle \delta', \mathrm{obs}'\rangle : S' \to GS' \times O'$ is a pair of morphisms $f : S \to S'$ in $\mathbb{C}$ and $g : O \to O'$ in $\mathbb{O}$ such that $f : \delta \to \delta'$ is a G-coalgebra morphism and $g \circ \mathrm{obs} = \mathrm{obs}' \circ f$.*

*For an object $I \in \mathbb{C}$, an (I-)*pointed partially observable G-coalgebra *(or* pointed PO coalgebra*) is a pair (i, c) of a morphism $i : I \to S$ and a PO coalgebra $c : S \to GS \times O$, often written as $(i, c) : I \to S \to GS \times O$. A morphism $(f, g) : (i, c) \to (i', c')$ between pointed PO coalgebras is a morphism $(f, g) : c \to c'$ between PO coalgebras such that $f \circ i = i'$. We write $\mathrm{Coalg}^{\mathbb{O}}(G)$ for the category of partially observable G-coalgebras and $\mathrm{Coalg}^{\mathbb{O}}_I(G)$ for the category of I-pointed partially observable G-coalgebras.*

We say that $(i, \langle \delta, \mathrm{obs}\rangle) : I \to S \to GS \times O$ is *fully observable* if $S = O$ and $\mathrm{obs} = \mathrm{id}_S$.

▶ **Assumption 14.** *We fix the following data:*
- *$\mathbb{C}$ is a category with finite products and pullbacks,*
- *$\mathbb{O}$ is a wide subcategory of $\mathbb{C}$,*
- *$F : \mathbb{C} \to \mathbb{C}$ is a functor and $T : \mathbb{C} \to \mathbb{C}$ is a monad,*
- *$\lambda : TF \Rightarrow FT$ is a distributive law,*
- *I is an object of $\mathbb{C}$.*

Throughout the section, we work under Assumption 14 and restrict our attention to I-pointed partially observable FT-coalgebras $(i, \langle \delta, \mathrm{obs}\rangle) : I \to S \to FTS \times O$.

For instance, our framework specializes to POMDPs by taking $\mathbb{C} = \mathbf{Set}$, $F = (\_ \times \mathbb{R}_{\geq 0})^A$, and $T = \mathcal{D}$ where A is the set of actions and $\mathcal{D}$ is the finite distribution monad. Although our framework assumes deterministic observations $S \to O$, standard POMDPs with stochastic observations can be encoded by expanding the state space; see [12, Remark 1].

Our aim is to associate with each pointed PO coalgebra a pointed FT-coalgebra, called the belief coalgebra, whose state space is $d\overline{T}(\mathrm{obs})$, obtained by the pullback of η_O along $T(\mathrm{obs})$ as in (3). The following is a key ingredient for the construction of the belief coalgebra.

▶ **Definition 15** (belief decomposition)**.** *For each object $O \in \mathbb{O}$, define flat^O to be the natural transformation $\mu d \circ T\iota : Td\overline{T}_O \Rightarrow Td$ between functors of the type $\mathbb{C}/O \to \mathbb{C}$. A* belief decomposition *on T consists of a family of natural transformations $\alpha^O : Td \Rightarrow Td\overline{T}_O$ for each $O \in \mathbb{O}$ such that (i) each α^O is a section of flat^O, i.e. $\mathsf{flat}^O \circ \alpha^O = \mathrm{id}_{Td}$, and (ii) for each $u : O \to O'$ in $\mathbb{O}$, the following equality of natural transformations from $\mathbb{C}/O$ to $\mathbb{C}$ holds: $Td_{O'}\theta_u \circ \alpha^O = \alpha^{O'}\Sigma_u : Td_O \Rightarrow d_{O'}\overline{T}_{O'}\Sigma_u$, where we use $d_{O'}\Sigma_u = d_O$.*

Allowing observation objects to vary reflects the functoriality of the slice-category construction in the observation object. Accordingly, Condition (ii) requires the family of belief decompositions to be coherent under admissible changes of observation objects, see Appendix A for details. In all examples, we take $\mathbb{O}$ to be the wide subcategory of $\mathbb{C}$ consisting of all monomorphisms.

When no confusion arises, we omit the superscripts of α^O and flat^O. We now fix a belief decomposition α, which in turn induces the following coalgebraic belief construction.

▶ **Definition 16** (coalgebraic belief construction)**.** *Let $(i, \langle \delta, \mathrm{obs} \rangle) \colon I \to S \to FTS \times O$ be a pointed PO coalgebra, and write c for $\langle \delta, \mathrm{obs} \rangle$. We define its* belief coalgebra *by*

$$\mathsf{Bel}(i, \langle \delta, \mathrm{obs} \rangle) \coloneqq \big(\eta^{\overline{T}}_{\mathrm{obs}} \circ i, \langle c^{\mathsf{Bel}}, \overline{T}(\mathrm{obs}) \rangle\big) \colon I \to d\overline{T}(\mathrm{obs}) \to FT(d\overline{T}(\mathrm{obs})) \times O$$

where the transition part c^{Bel} is defined by the following composite:

$$c^{\mathsf{Bel}} \coloneqq d\overline{T}(\mathrm{obs}) \xrightarrow{\iota_{\mathrm{obs}}} TS \xrightarrow{\mathrm{Det}(\delta)} FTS \xrightarrow{F\alpha_{\mathrm{obs}}} FT(d\overline{T}(\mathrm{obs})).$$

Here, following the convention introduced at the beginning of §3, $\eta^{\overline{T}}_{\mathrm{obs}}$ denotes the underlying morphism in $\mathbb{C}$ of the unit morphism in $\mathbb{C}/O$.

The belief coalgebra is obtained by combining the determinization $\mathrm{Det}(\delta) \colon TS \to FTS$ with the inclusion morphism $\iota_{\mathrm{obs}} \colon d\overline{T}(\mathrm{obs}) \to TS$ and the belief decomposition $\alpha_{\mathrm{obs}} \colon TS \to T(d\overline{T}(\mathrm{obs}))$. More precisely, the morphism $\iota_{\mathrm{obs}} \colon d\overline{T}(\mathrm{obs}) \to TS$ first forgets the observation-indexed structure, then $\mathrm{Det}(\delta)$ applies the usual coalgebraic determinization, and finally α_{obs} reorganizes the resulting T-structured state according to the observation map obs.

The belief construction defined above extends to a functor on the category of pointed PO coalgebras, as follows. See Appendix C.3 for the proof.

▶ **Proposition 17.** *The assignment $(i, c) \mapsto \mathsf{Bel}(i, c)$ extends to a functor*

$$\mathsf{Bel} \colon \mathrm{Coalg}^{\mathbb{O}}_I(FT) \to \mathrm{Coalg}^{\mathbb{O}}_I(FT)$$

by mapping a morphism $(f, g) \colon (i, \langle \delta, \mathrm{obs} \rangle) \to (i', \langle \delta', \mathrm{obs}' \rangle)$ to $\big(d\overline{T}(f) \circ d(\theta_g)_{\mathrm{obs}}, g\big)$, where the first component is as below:

$$d\overline{T}(\mathrm{obs}) = d\Sigma_g \overline{T}(\mathrm{obs}) \xrightarrow{d(\theta_g)_{\mathrm{obs}}} d\overline{T}\Sigma_g(\mathrm{obs}) \xrightarrow{d\overline{T}(f)} d\overline{T}(\mathrm{obs}').$$

▶ **Example 18** (nondeterminism)**.** We use partially observable nondeterministic transition systems as a running example throughout the paper. Let T be the nonempty finite powerset monad $\mathcal{P}_\emptyset$ on **Set**, and $F \coloneqq (\_) + \{\checkmark\}$, where $\checkmark$ ($\notin S$) denotes the designated terminal state. We define a *distributive law* $\lambda \colon \mathcal{P}_\emptyset((\_) + \{\checkmark\}) \to \mathcal{P}_\emptyset(\_) + \{\checkmark\}$ by, for each $X \in \mathbf{Set}$, $\lambda_X(U) \coloneqq \checkmark$ if $U = \{\checkmark\}$, and $\lambda_X(U) \coloneqq U \cap X$ otherwise. We then define a *belief decomposition* $\{\alpha^O_f \colon (\mathcal{P}_\emptyset d)(f) \to (\mathcal{P}_\emptyset d\overline{\mathcal{P}_\emptyset})(f)\}_{f \colon X \to O}$ by

$$\alpha^O_f(U) \coloneqq \big\{ f^{-1}(o) \cap U \,\big|\, o \in O \text{ s.t. } f^{-1}(o) \cap U \neq \emptyset \big\}, \text{ for any } U \in \mathcal{P}_\emptyset(X).$$

See Appendix C.4 for the proof that α is indeed a belief decomposition. Given a pointed PO coalgebra $(i, \langle \delta, \mathrm{obs} \rangle) \colon I \to S \to \big(\mathcal{P}_\emptyset(S) + \{\checkmark\}\big) \times O$, its *belief coalgebra* $\mathsf{Bel}(i, \langle \delta, \mathrm{obs} \rangle) \colon I \to d\overline{\mathcal{P}_\emptyset}(\mathrm{obs}) \to \big(\mathcal{P}_\emptyset(d\overline{\mathcal{P}_\emptyset}(\mathrm{obs})) + \{\checkmark\}\big) \times O$ is induced by the distributive law and the belief decomposition, where the state object $d\overline{\mathcal{P}_\emptyset}(\mathrm{obs})$ is the set of $U \in \mathcal{P}_\emptyset(S)$ such that there exists $o \in O$ with $U \subseteq \mathrm{obs}^{-1}(o)$, and the transition part c^{Bel} is given by

$$c^{\mathsf{Bel}}(U) = \begin{cases} \big\{ V_U \cap \mathrm{obs}^{-1}(o) \,\big|\, o \in O \big\} \setminus \{\emptyset\} & \text{if } V_U \neq \emptyset, \\ \checkmark & \text{otherwise,} \end{cases}$$

where $V_U \coloneqq \bigcup_{s \in U \text{ s.t. } \delta(s) \neq \checkmark} \delta(s)$.

Intuitively, a state of its belief coalgebra is a set of states of the original coalgebra that share the same observation. Its nondeterministic transition is defined over partitions of all possible transitions, where the partitions are induced by observations.

▶ Remark 19 (the design choice for nondeterminism). You may wonder whether we can define T and F by the powerset monad $T := \mathcal{P}$ and $F := \mathrm{id}$, respectively. This design choice is not suitable for us since it does not satisfy an assumption we will make in §6 for the correctness of the belief construction.

We conclude this section with two basic properties of the belief construction. We first consider the fully observable case, and show that the belief construction does not produce new states up to canonical isomorphism. We then make precise how the belief construction is related to ordinary coalgebraic determinization. See Appendix C.5 for the proof.

▶ **Proposition 20.** *For every FT-coalgebra $\delta\colon S \to FTS$, one has* $\mathsf{Bel}(\mathrm{id}_S, \langle\delta, \mathrm{id}_S\rangle) \cong (\mathrm{id}_S, \langle\delta, \mathrm{id}_S\rangle)\colon S \to S \to FTS \times S$.

▶ **Proposition 21.** *For each pointed PO coalgebra $(i, \langle\delta, \mathrm{obs}\rangle)\colon I \to S \to FTS \times O$, where $c = \langle\delta, \mathrm{obs}\rangle$, one has $F(\mathsf{flat}_{\mathrm{obs}}) \circ c^{\mathsf{Bel}} = \mathrm{Det}(\delta) \circ \iota_{\mathrm{obs}}$.*

Proof. This is immediate from the first condition in Def. 15. ◀

Thus, after collapsing the observation-wise decomposition via $F(\mathsf{flat}_{\mathrm{obs}})$, the belief transition agrees with the ordinary determinization along the morphism ι_{obs}.

5 Semantics of Pointed Partially Observable Coalgebras

One key property of the belief construction is that it preserves semantics, such as the maximal expected reward in the case of POMDPs. To make this statement precise, we first introduce a scheduler-based semantics of pointed PO coalgebras: for each scheduler, one obtains a semantics of the coalgebra under the scheduler, and the overall semantics is obtained by taking the join over all schedulers.

Accordingly, we restrict attention to partially observable $FT(\_)^A$-coalgebras

$$(i, \langle\delta, \mathrm{obs}\rangle)\colon I \to S \to (FTS)^A \times O,$$

where $A \in \mathbb{C}$ is a fixed object of actions. Intuitively, a scheduler specifies a choice of action based on a history of observations, as we make precise in the next subsections.

Before defining the semantics of pointed partially observable $FT(\_)^A$-coalgebras, we prepare several auxiliary notions.

▶ **Definition 22** (ordered object [3]). *An object $\Omega \in \mathbb{C}$ is an* ordered object *if, for each object $X \in \mathbb{C}$, the hom-set $\mathbb{C}(X, \Omega)$ is a complete lattice and each precomposition map preserves arbitrary joins. For an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$ and an ordered object Ω, an F-algebra $\tau\colon F\Omega \to \Omega$ is* monotone *if, for each object $X \in \mathbb{C}$, the map $\tau \circ F(\_)\colon \mathbb{C}(X, \Omega) \to \mathbb{C}(FX, \Omega)$ is monotone.*

We write $\bot_X$ and $\leq_X$ (or simply $\bot$ and $\leq$) for the bottom element and the order on $\mathbb{C}(X, \Omega)$, respectively.

▶ **Assumption 23.** *In the rest of the paper, we assume Assumption 14 and the following.*

- *The category $\mathbb{C}$ is a cartesian closed category with countable coproducts.*
- *The functor F and the monad T are strong. We write their strengths as: $\mathrm{st}^F_{X,Y}\colon X \times FY \to F(X \times Y)$ and $\mathrm{st}^T_{X,Y}\colon X \times TY \to T(X \times Y)$ for each $X, Y \in \mathbb{C}$.*

- *The distributive law λ satisfies $\lambda_{X\times Y} \circ T(\mathrm{st}^F) \circ \mathrm{st}^T = F\mathrm{st}^T \circ \mathrm{st}^F \circ X \times \lambda_Y$ for each $X, Y \in \mathbb{C}$.*
- *We fix an object $A \in \mathbb{C}$.*
- *We fix an ordered object $\Omega \in \mathbb{C}$ and a monotone algebra $\tau\colon FT\Omega \to \Omega$.*

Notation. We write $(\_)^*$ and $(\_)^+$ for the endofunctors $\coprod_{n\in\mathbb{N}}(\_)^n$ and $\coprod_{n\in\mathbb{N}_{>0}}(\_)^n$ on $\mathbb{C}$, respectively, and write nil for the canonical natural transformation $\mathrm{id} \Rightarrow (\_)^+$, and $\mathsf{cons}_O\colon O \times O^* \to O^+$ for the canonical isomorphism for each $O \in \mathbb{C}$. We write $f^\dagger$ for the adjoint transpose of a morphism f under the relevant adjunction $(\_) \times X \dashv (\_)^X$ (for $X \in \mathbb{C}$). We write ev for the components of its counit, i.e. evaluation morphisms, omitting subscripts when they are clear from the context. For each product $X_1 \times \cdots X_n$, the i-th projection ($i \in \{1, \cdots, n\}$) is denoted by π_i.

Under Assumption 23, we obtain a strength st^{FT} of FT given compositionally from the strength of T and F. Moreover, the distributive law of T over F induces a distributive law of T over $F(\_)^A$ by the following lemma. Hence the construction of §4 applies to $(i, \langle\delta, \mathrm{obs}\rangle)\colon I \to S \to (FTS)^A \times O$ simply by replacing F with $F(\_)^A$.

▶ **Lemma 24.** *Let $F\colon \mathbb{C} \to \mathbb{C}$, T be a strong monad on $\mathbb{C}$, and $\lambda\colon TF \Rightarrow FT$ be a distributive law. Then, there is a distributive law $\lambda'\colon T \circ (F(\_))^A \Rightarrow (F(\_))^A \circ T$ defined by*

$$\lambda'_X \coloneqq T((FX)^A) \xrightarrow{\widehat{\mathrm{st}}_{FX,A}} (TFX)^A \xrightarrow{(\lambda_X)^A} (FTX)^A,$$

where $\widehat{\mathrm{st}}_{FX,A}$ is the canonical map induced by the strength st^T.

▶ **Definition 25.** *For an FT-coalgebra $c\colon S \to FTS$, define a monotone map $\Phi_c\colon \mathbb{C}(S, \Omega) \to \mathbb{C}(S, \Omega)$ by $\Phi_c(f) \coloneqq \tau \circ FTf \circ c$. Then the semantics of an I-pointed FT-coalgebra $(i, c)\colon I \to S \to FTS$ is given as $\bigvee_{n\in\mathbb{N}} \Phi_c^n(\bot_S) \circ i$.*

▶ **Proposition 26.** *If $f\colon c \to c'$ is an FT-coalgebra morphism, then $\Phi_c^n(\bot_S) = \Phi_{c'}^n(\bot_{S'}) \circ f$ for each $n \in \mathbb{N}$. Consequently, for each morphism $f\colon (i, c) \to (i', c')$ between I-pointed coalgebras, the semantics of (i, c) is equal to that of (i', c'), that is,*

$$\bigvee_{n\in\mathbb{N}} \Phi_c^n(\bot_S) \circ i = \bigvee_{n\in\mathbb{N}} \Phi_{c'}^n(\bot_{S'}) \circ i'.$$

Proof. The first statement can be easily proved by induction on n. The second statement follows immediately from the first. ◀

▶ **Example 27** (nondeterminism)**.** We continue the example shown in Ex. 18. We define the ordered object Ω to be $\mathbf{B}$, where $\mathbf{B} \coloneqq \{\mathrm{t}, \mathrm{f}\}$ is the Boolean domain with the standard total order $\mathrm{f} < \mathrm{t}$. The order on $\mathbf{Set}(X, \mathbf{B})$ is given by the pointwise order. We define a monotone algebra $\tau\colon \mathcal{P}_\emptyset(\mathbf{B}) + \{\checkmark\} \to \mathbf{B}$ by $\tau(S) \coloneqq \wedge S$ for $S \in \mathcal{P}_\emptyset(\mathbf{B})$, and $\tau(\checkmark) = \mathrm{t}$. Given a coalgebra $c\colon S \to \mathcal{P}_\emptyset(S) + \{\checkmark\}$, the monotone map $\Phi_c\colon \mathbb{C}(S, \mathbf{B}) \to \mathbb{C}(S, \mathbf{B})$ is

$$\Phi_c(f)(s) = \begin{cases} \mathrm{f} & \text{if there exists } s' \in c(s) \cap S \text{ s.t. } f(s') = \mathrm{f}, \\ \mathrm{t} & \text{otherwise.} \end{cases}$$

For an I-pointed coalgebra $(i, c)\colon I \to S \to \mathcal{P}_\emptyset(S) + \{\checkmark\}$, the semantics $\bigvee_{n\in\mathbb{N}} \Phi_c^n(\bot_S) \circ i\colon I \to \mathbf{B}$ is then given by $\left(\bigvee_{n\in\mathbb{N}} \Phi_c^n(\bot_S) \circ i\right)(x) = \mathrm{t}$ if and only if all paths from $i(x)$ in c terminate (equivalently, reach the designated state $\checkmark$).

We then introduce two scheduler-based semantics for pointed PO coalgebras. In §5.1, we define a semantics by carrying schedulers in the state space, and in §5.2, we give an equivalent formulation based on state histories.

5.1 Semantics based on Coalgebras Carrying Schedulers

For an object of observations O, a *scheduler* is a morphism $u\colon O^+ \to A$ in $\mathbb{C}$, assigning an action to each nonempty history of observations. One may also present schedulers as morphisms $O \times (A \times O)^* \to A$. In the deterministic setting, however, these two presentations are equivalent, because the past actions are recursively determined by the past observations.

As the history grows, the scheduler is consumed accordingly, and the remaining choice mechanism can be carried along as part of the state. This yields an FT-coalgebra on $S \times A^{O^*}$.

▶ **Definition 28.** *For a PO coalgebra $c = \langle \delta, \mathrm{obs}\rangle\colon S \to (FTS)^A \times O$, we define an FT-coalgebra $\mathsf{Sch}(c)\colon S \times A^{O^*} \to FT(S \times A^{O^*})$ as the composite*

$$S\times A^{O^*} \cong S\times A\times A^{O^+} \xrightarrow{\delta^\dagger\times\mathrm{id}} FTS\times A^{O^+} \xrightarrow{\mathrm{st}^{FT}} FT(S\times A^{O^+}) \xrightarrow{FT\langle\pi_1,\mathrm{ev}\circ(\mathrm{obs}\times\mathrm{id})\rangle} FT(S\times A^{O^*}).$$

Let $\mathrm{Coalg}^{\mathbb{O}}((FT(\_))^A)_{\mathrm{ctr}}$ be the category of partially observable $(FT(\_))^A$-coalgebras and morphisms that are contravariant in the observation part; that is, a morphism from $\langle \delta, \mathrm{obs}\rangle\colon S \to (FTS)^A \times O$ to $\langle \delta', \mathrm{obs}'\rangle\colon S' \to (FTS')^A \times O'$ is a pair of morphisms $f\colon S \to S'$ and $g\colon O' \to O$ such that $\delta' \circ f = (FTf)^A \circ \delta$ and $\mathrm{obs} = g \circ \mathrm{obs}' \circ f$.

The construction defined in Def. 28 extends to a functor

$$\mathsf{Sch}\colon \mathrm{Coalg}^{\mathbb{O}}((FT)^A)_{\mathrm{ctr}} \to \mathrm{Coalg}(FT)$$

with $\mathsf{Sch}(f, g) \coloneqq f \times A^{g^*}$ where $A^{g^*}\colon A^{O^*} \to A^{O'^*}$ is the morphism induced by g.

An objective $V^*_{i,c}$ of an I-pointed $FT(\_)^A$-coalgebra (i, c) is defined as the join of the semantics of the I-pointed FT-coalgebra $(\langle \mathrm{id}, (u \circ \mathsf{cons}_O)^\dagger \circ \mathrm{obs}\rangle \circ i, \mathsf{Sch}(c))\colon I \to S \times A^{O^*} \to FT(S \times A^{O^*})$ for all $u\colon O^+ \to A$:

$$V^*_{i,c} \coloneqq \bigvee_{u\colon O^+\to A,\ n\in\mathbb{N}} \Phi^n_{\mathsf{Sch}(c)}(\bot) \circ \langle \mathrm{id}, (u \circ \mathsf{cons}_O)^\dagger \circ \mathrm{obs}\rangle \circ i \ :\ I \to \Omega.$$

▶ **Example 29** (nondeterminism)**.** We continue Ex. 27. Consider a coalgebra $c = \langle \delta, \mathrm{obs}\rangle\colon S \to (\mathcal{P}_\emptyset(S) + \{\checkmark\})^A \times O$ and a scheduler $u\colon O^+ \to A$. For each $s \in S$, let $\tilde{u}_s\colon O^* \to A$ be $\lambda\vec{o}.u(\mathrm{obs}(s)\vec{o})$ where $\mathrm{obs}(s)\vec{o}$ is the sequence given by prepending $\mathrm{obs}(s)$ to $\vec{o}$. Then the composite $\Phi^n_{\mathsf{Sch}(c)}(\bot)\langle \mathrm{id}, (u \circ \mathsf{cons})^\dagger \circ \mathrm{obs}\rangle\colon S \to \mathbf{B}$ maps $s \in S$ to

$$\Phi^n_{\mathsf{Sch}(c)}(\bot)(s, \tilde{u}_s) = \begin{cases} \mathrm{t} & \text{if under } u\text{, all paths from } s \text{ reach } \checkmark \text{ in } c \text{ within } n \text{ steps,} \\ \mathrm{f} & \text{otherwise.} \end{cases}$$

Consequently, given an initial element $x \in I$, the objective $V^*_{i,c}$ is given by

$$V^*_{i,c}(x) = \begin{cases} \mathrm{t} & \text{if under some } u\colon O^+ \to A\text{, all paths from } i(x) \text{ eventually reach } \checkmark \text{ in } c\text{,} \\ \mathrm{f} & \text{otherwise.} \end{cases}$$

5.2 Semantics based on State-History Coalgebras

We now give an equivalent presentation of the objective $V^*_{i,c}$ in which the state-history is stored explicitly in the state space.

▶ **Definition 30.** *For a PO coalgebra $c = \langle \delta, \mathrm{obs}\rangle\colon S \to (FTS)^A \times O$ and a morphism $u\colon O \to A$, we define an FT-coalgebra $c_u\colon S \to FT(S)$ to be $\mathrm{ev} \circ \langle \delta, u \circ \mathrm{obs}\rangle$.*

Let ext_S be the canonical concatenation morphism $S^+ \times S \to S^+$ (or $S^+ \times S^* \to S^+$), and last: $S^+ \to S$ be the morphism sending a sequence to its last element.

▶ **Definition 31.** *Let* $(i, \langle \delta, \mathrm{obs} \rangle) \colon I \to S \to (FTS)^A \times O$ *be a pointed PO coalgebra, and* c *be the second part* $\langle \delta, \mathrm{obs} \rangle$*. We define a pointed PO coalgebra* $(\mathsf{nil} \circ i, \mathsf{Hist}(c)) \colon I \to S^+ \to \big(FT(S^+)\big)^A \times O^+$ *where* $\mathsf{Hist}(c) \coloneqq \langle f, \mathrm{obs}^+ \rangle$ *and* f *is the composite*

$$S^+ \xrightarrow{\langle \mathrm{id}, \delta \circ \mathrm{last} \rangle} S^+ \times (FTS)^A \xrightarrow{(\mathrm{st} \circ (\mathrm{id}_{S^+} \times \mathrm{ev}))^\dagger} \big(FT(S^+ \times S)\big)^A \xrightarrow{FT(\mathsf{ext})^A} \big(FT(S^+)\big)^A.$$

An objective $H^*_{i,c}$ of c can then be defined as the join of the semantics of the I-pointed FT-coalgebra $(\mathsf{nil} \circ i, \mathsf{Hist}(c)_u) \colon I \to S^+ \to FT(S^+)$ for all $u \colon O^+ \to A$:

$$H^*_{i,c} \coloneqq \bigvee_{u \colon O^+ \to A,\ n \in \mathbb{N}} \Phi^n_{\mathsf{Hist}(c)_u}(\bot) \circ \mathsf{nil}_S \circ i,$$

which is equal to $V^*_{i,c}$ as shown below. See Appendix C.6 for the proof.

▶ **Proposition 32.** *The equation* $V^*_{i,c} = H^*_{i,c}$ *holds.*

6 Correctness of the Belief Construction

We now turn to the main theorem of this paper: the correctness of the belief construction. We show that the belief construction preserves the semantics of pointed PO coalgebras, in the sense that for each pointed PO coalgebra $(i, c) \colon I \to S \to (FTS)^A \times O$, the equation $V^*_{i,c} = V^*_{\mathsf{Bel}(i,c)}$ holds.

Throughout the section, we fix a pointed PO coalgebra $(i, \langle \delta, \mathrm{obs} \rangle) \colon I \to S \to (FTS)^A \times O$ and write c for $\langle \delta, \mathrm{obs} \rangle$.

For a coalgebra $c \colon S \to GS$ of an endofunctor $G \colon \mathbb{C} \to \mathbb{C}$ and $n \in \mathbb{N}$, we write $c^n \colon S \to G^n S$ for the morphism defined inductively by $c^0 = \mathrm{id}_S$ and $c^{n+1} \coloneqq G(c^n) \circ c$. We use the same notation for algebras $\tau \colon G\Omega \to \Omega$. For each $n \geq 1$, let $\lambda^n \colon (FT)^n \Rightarrow F^n T^n$ denote the natural transformation obtained by repeatedly using the distributive law λ to move all occurrences of T to the right.

▶ **Lemma 33.** *We have the following equalities:*

1. $\big(d\overline{T}(\mathrm{obs}) \times A \xrightarrow{(c^{\mathsf{Bel}})^\dagger} FT(d\overline{T}(\mathrm{obs})) \xrightarrow{F(\mathsf{flat}_{\mathrm{obs}})} FTS\big)$
   $= \big(d\overline{T}(\mathrm{obs}) \times A \xrightarrow{\iota_{\mathrm{obs}} \times A} TS \times A \xrightarrow{\mathrm{st}^T} T(S \times A) \xrightarrow{T\delta^\dagger} TFTS \xrightarrow{\lambda_{TS}} FT^2 S \xrightarrow{F\mu^T_S} FTS\big).$
2. $\big(S \times A \xrightarrow{\eta^{\overline{T}}_{\mathrm{obs}} \times A} d\overline{T}(\mathrm{obs}) \times A \xrightarrow{(c^{\mathsf{Bel}})^\dagger} FT(d\overline{T}(\mathrm{obs})) \xrightarrow{F(\mathsf{flat}_{\mathrm{obs}})} FTS\big) = \big(S \times A \xrightarrow{\delta^\dagger} FTS\big).$

See [22] for the proof.

The following lemma, especially its second statement, is a key result to show the correctness theorem. A standard way to prove such a preservation result would be to construct a coalgebra morphism, between $\mathsf{Sch}(c)$ and $\mathsf{Sch}(\langle c^{\mathsf{Bel}}, \overline{T}\mathrm{obs} \rangle)$. However, we do not expect such a morphism to exist in general. Instead, under the following assumptions $(\star)$:

1. the monotone algebra $\tau \colon FT\Omega \to \Omega$ is of the form $\rho \circ F\sigma$ for some monotone algebra $\rho \colon F\Omega \to \Omega$ and monotone Eilenberg–Moore algebra $\sigma \colon T\Omega \to \Omega$ such that $\sigma \circ T(\rho) = \rho \circ F(\sigma) \circ \lambda_\Omega$ and $\bot_{TX} = \sigma \circ T(\bot_X)$ for each $X \in \mathbb{C}$;
2. $\big(d\overline{T}(\mathrm{obs}) \xrightarrow{\iota_{\mathrm{obs}}} TS \xrightarrow{T\langle \mathrm{id}, \mathrm{obs} \rangle} T(S \times O)\big) = \big(d\overline{T}(\mathrm{obs}) \xrightarrow{\langle \iota_{\mathrm{obs}}, \overline{T}\mathrm{obs} \rangle} TS \times O \xrightarrow{\mathrm{st}^T} T(S \times O)\big),$

we show that the morphism $\mu \circ T(\mathrm{st}^T \circ (\iota_{\mathrm{obs}} \times \mathrm{id})) \colon T(d\overline{T}(\mathrm{obs}) \times A^{O^*}) \to T(S \times A^{O^*})$, becomes compatible with the transition structures after an application of FT and postcomposition with $F^2\mu \circ F\lambda$. See [22] for the proof.

▶ **Lemma 34.**
1. $\mathsf{Sch}(c) = F(\mu \circ T(\mathrm{st}^T \circ \iota_{\mathrm{obs}} \times \mathrm{id})) \circ \mathsf{Sch}(\langle c^{\mathsf{Bel}}, \overline{T}\mathrm{obs}\rangle) \circ (\eta^{\overline{T}}_{\mathrm{obs}} \times \mathrm{id})$.
2. *Under the assumption* ($\star$), *the following equation between morphisms of type* $FT(d\overline{T}(\mathrm{obs}) \times A^{O^*}) \to F^2T(S \times A^{O^*})$ *holds:*

$$\begin{aligned}&F^2\mu \circ F\lambda \circ FT(\mathsf{Sch}(c)) \circ F(\mu \circ T(\mathrm{st}^T \circ \iota_{\mathrm{obs}} \times \mathrm{id}))\\ &= F^2\mu \circ F\lambda \circ FTF(\mu \circ T(\mathrm{st}^T \circ \iota_{\mathrm{obs}} \times \mathrm{id})) \circ FT(\mathsf{Sch}(\langle c^{\mathsf{Bel}}, \overline{T}\mathrm{obs}\rangle)).\end{aligned}$$

▶ **Theorem 35.** *Under the assumption* ($\star$), *the equation* $V^*_{i,c} = V^*_{\mathsf{Bel}(i,c)}$ *holds.*

Proof. We prove that $\Phi^n_{\mathsf{Sch}(c)}(\bot) = \Phi^n_{\mathsf{Sch}(\langle c^{\mathsf{Bel}}, \overline{T}\mathrm{obs}\rangle)}(\bot) \circ (\eta^{\overline{T}}_{\mathrm{obs}} \times \mathrm{id})$ for each $n \geq 1$. Write $\overline{T}S$ for the object $d\overline{T}(\mathrm{obs})$. By definition, $\Phi^n_{\mathsf{Sch}(c)}(\bot) = \tau^n \circ (FT)^n(\bot) \circ \mathsf{Sch}(c)^n = \tau^n \circ (FT)^n(\bot) \circ FT(\mathsf{Sch}(c)^{n-1}) \circ \mathsf{Sch}(c)$. Hence by Lem. 34.1, it suffices to show the following equality between morphisms of type $FT(\overline{T}S \times A^{O^*}) \to \Omega$:

$$\begin{aligned}&\tau^n \circ (FT)^n\bot \circ FT(\mathsf{Sch}(c)^{n-1}) \circ F(\mu \circ T(\mathrm{st}^T \circ \iota_{\mathrm{obs}} \times \mathrm{id}))\\ &= \tau^n \circ (FT)^n\bot \circ FT(\mathsf{Sch}(\langle c^{\mathsf{Bel}}, \overline{T}\mathrm{obs}\rangle)^{n-1}),\end{aligned} \tag{4}$$

for each $n \geq 1$. The full proof of this equality is deferred to [22]. Here we only give a proof sketch. We proceed by induction on n. For the base case $n = 1$, the claim follows from the factorization assumption $\tau = \rho \circ F\sigma$, together with the equality $\sigma \circ T\bot_X \circ \mu_X = \sigma \circ T\bot_{TX}$ for each $X \in \mathbb{C}$. The inductive step uses $\sigma \circ T\bot = \bot$, $\sigma \circ \mu = \sigma \circ T\sigma$, and Lem. 34.2. ◀

▶ **Example 36** (nondeterminism)**.** We continue Example 29. By Thm. 35, it suffices to check the assumption ($\star$). Set $\rho\colon \mathbf{B} + \{\checkmark\} \to \mathbf{B}$ and $\sigma\colon \mathcal{P}_\emptyset(\mathbf{B}) \to \mathbf{B}$ by $\rho(b) = b$ for $b \in \mathbf{B}$ and $\rho(\checkmark) = \mathrm{t}$, and $\sigma(U) = \wedge U$, respectively. Clearly, we have $\tau = \rho \circ (\sigma + \{\checkmark\})$, and $\bot_{\mathcal{P}_\emptyset X} = \sigma \circ \mathcal{P}_\emptyset(\bot_X)$; note that the latter condition is not satisfied if we replace $\mathcal{P}_\emptyset$ with $\mathcal{P}$, and this is the reason why we employ this definition. See [22] for the details of the remaining conditions.

7 Comparing Semantics of Partially and Fully Observable Coalgebras

In this section, we compare the semantics of a pointed PO coalgebra and those of its fully observable counterpart, obtained by replacing the observation map with the identity. We show that the semantics of a pointed PO coalgebra is always bounded above by that of its fully observable counterpart, and we give two sufficient conditions under which the two semantics coincide: the first, given in Prop. 37, is simpler, whereas the second, given in Prop. 39, is weaker.

▶ **Proposition 37.** *For each pointed PO coalgebra* $(i, \langle\delta, \mathrm{obs}\rangle)\colon I \to S \to (FTS)^A \times O$, *it holds that* $V^*_{i,\langle\delta,\mathrm{obs}\rangle} \leq V^*_{i,\langle\delta,\mathrm{id}_S\rangle}$. *If* obs *is split mono, then* $V^*_{i,\langle\delta,\mathrm{obs}\rangle} = V^*_{i,\langle\delta,\mathrm{id}_S\rangle}$ *holds.*

This follows from the fact that $(\mathrm{id}_S, \mathrm{obs})\colon \langle\delta, \mathrm{obs}\rangle \to \langle\delta, \mathrm{id}_S\rangle$ is a morphism in the category $\mathrm{Coalg}^{\mathbb{O}}(FT(\_)^A)_{\mathrm{ctr}}$, together with functoriality of Sch and the preservation of the semantics under FT-coalgebra morphisms shown in Prop. 26. The split mono case is also proved similarly. The full proof is given in [22].

In §5, we defined the semantics

$$H^*_{i,c} = \bigvee_{u:O^+ \to A,\ n \in \mathbb{N}} \Phi^n_{\mathsf{Hist}(c)_u}(\bot) \circ \mathsf{nil}_S \circ i$$

as the join of the semantics of the pointed coalgebras $(\mathsf{nil}_S \circ i, \mathsf{Hist}(c)_u)$ for all $u\colon O^+ \to A$. By Prop. 26, the semantics of a pointed FT-coalgebra does not change under pointed coalgebra morphisms. Hence, for each u, one may compute the semantics of $(\mathsf{nil}_S \circ i, \mathsf{Hist}(c)_u)$ on any pointed subcoalgebra.

▶ **Lemma 38.** *For each morphism (f, g) from $c\colon S \to (FTS)^A \times O$ to $c'\colon S' \to (FTS')^A \times O'$ in $\mathrm{Coalg}^{\mathbb{O}}(FT(\_)^A)$ and each morphism $u\colon O' \to A$, the morphism f is also a morphism $f\colon c_{u\circ g} \to c'_u$ in $\mathrm{Coalg}(FT)$.*

The following proposition is based on the observation that, by Lem. 38 and Prop. 26, for each scheduler, the history-based semantics may be computed on a suitable pointed subcoalgebra of the corresponding resolved history coalgebra. See [22] for the proof.

▶ **Proposition 39.** *Let $(i, \langle\delta, \mathrm{obs}\rangle)\colon I \to S \to (FTS)^A \times O$ be an I-pointed PO coalgebra. Suppose that, for each $u\colon S^+ \to A$, there exists an I-pointed FT-subcoalgebra $m_u\colon (i_u, \delta_u) \rightarrowtail (\mathsf{nil} \circ i, \mathsf{Hist}(\langle\delta, \mathrm{id}\rangle)_u)$ such that there is $f_u\colon O^+ \to S^+$ satisfying $f_u \circ \mathrm{obs}^+ \circ m_u = m_u$. Then the equation $H^*_{i,\langle\delta,\mathrm{obs}\rangle} = H^*_{i,\langle\delta,\mathrm{id}\rangle}$ holds.*

Note that the split mono condition in Prop. 37 implies the condition in Prop. 39: if obs is split mono, then obs^+ is also, and hence $m_u \coloneqq \mathrm{id}_{(\mathsf{nil}\circ i, \mathsf{Hist}(\langle\delta,\mathrm{id}\rangle)_u)}$ satisfies the condition.

▶ Remark 40 (Reachable coalgebras). When $\mathbb{C} = \mathbf{Set}$, the equality $H^*_{i,\langle\delta,\mathrm{obs}\rangle} = H^*_{i,\langle\delta,\mathrm{id}\rangle}$ in Prop. 39 may be established pointwise in $x \in I$. Indeed, regarding $x \in I$ as a morphism $x\colon 1 \to I$, we have $H^*_{i,c}(x) = H^*_{i\circ x,c}(*)$, where $* \in 1$ is the unique element. Since joins in $\mathbf{Set}(I, \Omega)$ are computed pointwise, it therefore suffices, for each $x \in I$, to apply Prop. 39 to the 1-pointed coalgebra whose initial point is $i \circ x\colon 1 \to S$.

For fixed $x \in I$ and $u\colon S^+ \to A$, a canonical choice for the required subcoalgebra is the *reachable coalgebra* [1], a pointed coalgebra having no proper pointed subcoalgebra, whenever it exists. A reachable subcoalgebra of a given pointed coalgebra is also called its *reachable part*. This notion coincides with the usual reachable part of the graph induced by the given coalgebra when $\mathbb{C} = \mathbf{Set}$ and the functor under consideration preserves intersections [31]. There are several constructions of a reachable subcoalgebra of a given pointed coalgebra in the literature. For example, it is known that if $\mathbb{C}$ has intersections and a functor $G\colon \mathbb{C} \to \mathbb{C}$ preserves them, then each G-coalgebra has a unique reachable subcoalgebra, obtained as the intersection of all subcoalgebras [1]. Another construction of a reachable subcoalgebra is proposed in [31] via an iterative computation.

In all concrete applications of Prop. 39 in this paper, we work pointwise in $x \in I$ and take the subcoalgebra to be the reachable part.

These results are useful when comparing the semantics $V^*_{i,c}$ of a pointed PO coalgebra and that of the fully observable counterpart of the belief coalgebra. Thm. 35 yields the equality $V^*_{i,c} = V^*_{\mathsf{Bel}(i,c)}$, where the right-hand side is still formulated as the semantics of a pointed PO coalgebra. By applying Prop. 39 to the belief coalgebra $\mathsf{Bel}(i, c)$, we obtain a sufficient condition to ensure that the semantics of $\mathsf{Bel}(i, c)$ can be replaced by that of the coalgebra in the fully observable setting: $V^*_{i,c} = V^*_{\eta^{\overline{T}}_{\mathrm{obs}}\circ i, \left\langle c^{\mathsf{Bel}}, \mathrm{id}\right\rangle}$.

▶ **Example 41** (nondeterminism)**.** In the case of nondeterministic systems, the reachable part (cf. Remark 40) satisfies the condition in Prop. 39, since from an initial state, the mapping from paths over its history coalgebra under a scheduler $u\colon S^+ \to A$ to their observation sequences is injective by the definition of the belief construction. Since the belief coalgebra induced by a finite-state partially observable nondeterministic transition system is a finite-state labelled transition system, and since deciding the existence of a scheduler that ensures termination is clearly decidable for fully observable finite-state labelled transition systems, it follows from Thm. 35 that the termination problem for partially observable nondeterministic systems is decidable as well; see [22, Prop. 46] for the details.

8 Examples

We instantiate our framework with (1) partially observable Markov decision processes (POMDPs) and (2) partially observable weighted transition systems.

8.1 POMDP

Let T be the finitely supported distribution monad $\mathcal{D}$ on **Set**, and let $F := \_\times\mathbb{R}_{\geq 0}$, where the $\mathbb{R}_{\geq 0}$-component represents the reward assigned to state-action pairs. We define a *distributive law* $\lambda_X\colon \mathcal{D}(X\times\mathbb{R}_{\geq 0}) \to \mathcal{D}(X)\times\mathbb{R}_{\geq 0}$ by

$$\lambda_X(\nu) := \Big(\mathcal{D}(\pi_1)(\nu), \sum_{(x,r)\in\mathrm{supp}(\nu)} \nu(x,r)\cdot r\Big).$$

For a morphism $f\colon X\to O$, the object $d\overline{\mathcal{D}}(f)$ consists of beliefs $b\in\mathcal{D}(X)$ whose support is contained in a single fibre of f; concretely, there exists a (unique) $o\in O$ such that $\mathrm{supp}(b)\subseteq f^{-1}(o)$. Define a belief decomposition $\{\alpha^O_f\colon (\mathcal{D}d)(f)\to(\mathcal{D}d\overline{\mathcal{D}})(f)\}_{f\colon X\to O}$ by

$$\alpha^O_f(b)(b') := \begin{cases} Z_o(b) & \text{if } \exists o\in O.\ Z_o(b)>0 \text{ and } b' = \frac{1}{Z_o(b)}\cdot b\big|_o,\\ 0 & \text{otherwise,}\end{cases}$$

where $Z_o(b) := \sum_{x\in f^{-1}(o)} b(x)$ and $b\big|_o$ is defined by $b\big|_o(x) := b(x)$ if $f(x)=o$, and $b\big|_o(x) := 0$ otherwise. The morphism α^O_f decomposes the belief b into the family of conditional beliefs indexed by observations; see Appendix B.1 for the details of the belief decomposition α.

We define the ordered object Ω to be $[0,\infty]$ with the standard total order, and we equip each homset $\mathbf{Set}(X,[0,\infty])$ with the pointwise order. We define the two algebras $\rho\colon [0,\infty]\times\mathbb{R}_{\geq 0}\to[0,\infty]$ and $\sigma\colon\mathcal{D}([0,\infty])\to[0,\infty]$ by

$$\rho(r_1,r_2) := r_1+r_2,\ \text{and}\ \sigma(\nu) := \sum_{r\in\mathrm{supp}(\nu)}\nu(r)\cdot r.$$

The resulting algebra $\tau\colon\mathcal{D}([0,\infty])\times\mathbb{R}_{\geq 0}\to[0,\infty]$ obtained by ρ and σ is thus given by $\tau(\nu,r) = r+\sum_{r'\in\mathrm{supp}(\nu)}\nu(r')\cdot r'$.

Given $c\colon S\to\mathcal{D}(S)\times\mathbb{R}_{\geq 0}$, the monotone map $\Phi_c\colon\mathbb{C}(S,[0,\infty])\to\mathbb{C}(S,[0,\infty])$ is

$$\Phi_c(f)(s) = (\pi_2\circ c)(s) + \sum_{s'\in S}(\pi_1\circ c)(s)(s')\cdot f(s').$$

Now let $c = \langle\delta,\mathrm{obs}\rangle\colon S\to\big(\mathcal{D}(S)\times\mathbb{R}_{\geq 0}\big)^A\times O$. Then for each $u\colon O^+\to A$ and $n\in\mathbb{N}$, the composite $\Phi^n_{\mathsf{Sch}(c)}(\bot)\langle\mathrm{id},(u\circ\mathsf{cons})^\dagger\circ\mathrm{obs}\rangle\colon S\to[0,\infty]$ maps $s\in S$ to

$$\Phi^n_{\mathsf{Sch}(c)}(\bot)(s,\tilde{u}_s) = \text{the total expected reward from } s \text{ in } n \text{ steps under } u,$$

where $\tilde{u}_s := \lambda\vec{o}.u(\mathrm{obs}(s)\vec{o})\colon O^*\to A$. Consequently, given an initial element $x\in I$, the objective $V^*_{i,c}$ is given by

$$V^*_{i,c}(x) = \text{the maximum total expected reward from } i(x) \text{ over schedulers } u\colon O^+\to A.$$

The belief construction yields the usual belief MDP c^{Bel} together with an observation map $\overline{\mathcal{D}}(\mathrm{obs})$ sending each belief to the unique observation on whose fibre its support lies.

For the belief coalgebra $\mathsf{Bel}(i,c)\colon I \to d\overline{\mathcal{D}}(\mathrm{obs}) \to \big(\mathcal{D}(d\overline{\mathcal{D}}(\mathrm{obs}))\times\mathbb{R}_{\geq 0}\big)^A \times O$, the reachable part chosen as in Remark 40 for each initial element and scheduler satisfies the condition in Prop. 39. Because for each belief, action, and observation, there is at most one successor belief carrying that observation, every reachable belief history under a fixed scheduler u is uniquely determined by its observation history.

Consequently, for POMDPs, the semantics of the belief coalgebra agrees with the semantics of the corresponding fully observable belief MDP. Combining this with Thm. 35, we recover the standard equivalence between the semantics of a POMDP and that of its fully observable belief MDP.

8.2 Partially Observable Weighted Transition Systems

Finally, we illustrate our belief construction for partially observable weighted transition systems.

Let $T \coloneqq \mathcal{R}$ be the semimodule monad [8] that is induced by the standard semiring $(\mathbb{R}_{\geq 0}, +, \cdot)$ with the summation $+$ and the multiplication $\cdot$ on **Set**, and $F = \_ \times \mathbb{R}_{\geq 0}\colon \mathbf{Set} \to \mathbf{Set}$. We have a distributive law $\lambda\colon \mathcal{R}(\_ \times \mathbb{R}_{\geq 0}) \Rightarrow \mathcal{R}(\_) \times \mathbb{R}_{\geq 0}$ defined by

$$\lambda(\nu) \coloneqq \Big(\sum_{r\in\mathbb{R}_{\geq 0}} \nu(\_, r), \sum_{(x,r)\in X\times\mathbb{R}_{\geq 0}} \nu(x,r)\cdot r\Big).$$

For a morphism $f\colon X \to O$, the object $d\overline{\mathcal{R}}(f)$ consists of normalized finitely supported weight functions whose support is contained in a single fibre of f; that is, $\nu \in \mathcal{R}(X)$ such that $\sum_{x\in X}\nu(x) = 1$ and $\mathrm{supp}(\nu) \subseteq f^{-1}(o)$ for some (necessarily unique) $o \in O$. We define a belief decomposition $\{\alpha^O_f\colon (\mathcal{R}d)(f) \to (\mathcal{R}d\overline{\mathcal{R}})(f)\}_{f\colon X\to O}$ by

$$\alpha^O_f(\nu)(\mu) \coloneqq \begin{cases} Z_o(\nu) & \text{if } \exists o \in O.\ Z_o(\nu) > 0 \text{ and } \mu = \frac{1}{Z_o(\nu)}\cdot \nu\big|_o, \\ 0 & \text{otherwise,}\end{cases}$$

where $Z_o(\nu) \coloneqq \sum_{x\in f^{-1}(o)}\nu(x)$ and $\nu\big|_o(x) \coloneqq \nu(x)$ if $f(x) = o$, and $\nu\big|_o(x) \coloneqq 0$ otherwise. One can show that α is a belief decomposition by essentially the same argument as in §8.1.

We take the ordered object Ω to be $[0,\infty]$ with the standard total order, and we equip each $\mathbf{Set}(X,[0,\infty])$ with the pointwise order. We define the two algebras $\rho\colon [0,\infty]\times\mathbb{R}_{\geq 0} \to [0,\infty]$ and $\sigma\colon \mathcal{R}([0,\infty]) \to [0,\infty]$ by $\rho(r_1, r_2) \coloneqq r_1 + r_2$, and $\sigma(\nu) \coloneqq \sum_{r\in[0,\infty]}\nu(r)\cdot r$, for any $\nu \in \mathcal{R}([0,\infty])$. The resulting algebra $\tau\colon \mathcal{R}([0,\infty])\times\mathbb{R}_{\geq 0} \to [0,\infty]$ is then given by $\tau(\nu, r) = r + \sum_{r'\in[0,\infty]}\nu(r')\cdot r'$. Given a coalgebra $c\colon S \to \mathcal{R}(S)\times\mathbb{R}_{\geq 0}$, we regard c as a weighted transition system where $\pi_1(c)\colon S \to \mathcal{R}(S)$ describes the weight assigned to transitions, and $\pi_2(c)\colon S \to \mathbb{R}_{\geq 0}$ describes the weight to terminate immediately; see Appendix B for the details. We remark that we can show that the assumption ($\star$) holds by almost the same argument as in §8.1.

The monotone map $\Phi_c\colon \mathbb{C}(S,[0,\infty]) \to \mathbb{C}(S,[0,\infty])$ is then given by

$$\Phi_c(f)(s) = (\pi_2\circ c)(s) + \sum_{s'\in S}(\pi_1\circ c)(s)(s')\cdot f(s').$$

Given a coalgebra $c = \langle\delta, \mathrm{obs}\rangle\colon S \to \big(\mathcal{R}(S)\times\mathbb{R}_{\geq 0}\big)^A \times O$ and a scheduler $u\colon O^+ \to A$, the composite $\Phi^n_{\mathsf{Sch}(c)}(\bot)\langle \mathrm{id}, (u\circ\mathsf{cons})^\dagger\circ\mathrm{obs}\rangle\colon S \to [0,\infty]$ maps $s \in S$ to

$$\Phi^n_{\mathsf{Sch}(c)}(\bot)(s, \tilde{u}_s) = \text{the sum of weights over all terminating paths from } s \text{ in } n \text{ steps under } u.$$

where $\tilde{u}_s \coloneqq \lambda\vec{o}.u(\mathrm{obs}(s)\vec{o})\colon O^* \to A$. See Appendix B for the definitions of terminating paths and their weights. Consequently, given an initial element $x \in I$ in I-pointed coalgebra $(i, c)\colon I \to S \to \big(\mathcal{R}(S) \times \mathbb{R}_{\geq 0}\big)^A \times O$, the objective $V^*_{i,c}$ is given by

$$V^*_{i,c}(x) = \text{the maximum sum of weights over terminating paths from } i(x) \text{ over } u.$$

By Thm. 35 and the same argument as in §8.1, the objective $V^*_{i,c}$ coincides with that of the fully observable counterpart of $\mathsf{Bel}(i, c)$.

9 Related Work

The work most closely related to ours is the recent work by Baltieri, Torresan, and Nakai [5], which proposes a coalgebraic approach to POMDPs in order to capture notions of behavioural equivalence in a systematic manner. They also propose a generalized determinization of POMDPs 'a la [28]. This differs from our belief construction: their generalized determinization yields deterministic systems, whereas our belief coalgebras remain effectful, reflecting the stochastic nature of belief MDPs. Compared with their work, our novelty lies in providing a unified coalgebraic belief construction and proving the coincidence of two semantics: one for partially observable systems and the other for belief coalgebras. Such an equivalence between partially observable systems and their belief coalgebras has not been established in [5].

Bonchi, Sokolova, and Vignudelli [10] propose a determinization for systems combining nondeterministic and probabilistic choices. In their work, the monad of convex subsets of distributions plays a crucial role. Subsequently, Goy and Petrisan showed that this monad arises from a weak distributive law of the powerset monad over the finite distribution monad. Goy [15] showed that the belief-state transformer of probabilistic automata [9] can be derived naturally from weak distributive laws, and Turkenburg et al. [30] developed the notion of invertible steps [24] induced by weak distributive laws to show the preservation and reflection of bisimilarity. Future work is to investigate whether our belief construction can be extended to these generalized determinization constructions arising from weak distributive laws.

Bezhanishvili, Cupke, and Panangaden [6] study minimization in the category of compact Hausdorff spaces to support the minimization of belief automata for POMDPs. In this paper, we study the construction of belief coalgebras from pointed PO coalgebras and the equivalence of two semantics that were not considered in [6].

Jacobs and Sokolova [20] study the notion of schedulers coalgebraically, and they propose a coalgebraic treatment of schedulers for nondeterministic systems through a strong monad map. Our treatment of strategies is different and we regard executions under strategies as stateful computations (see Def. 28).

10 Conclusion

We propose a coalgebraic belief construction for partially observable systems, including POMDPs, and show the equivalence of two semantics: that of a partially observable system and that of its belief coalgebra, under a mild assumption. As future work, we would like to support randomized strategies, which are commonly used in POMDPs. More generally, supporting effectful strategies itself would be an interesting direction for future work.

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A An Indexed-Categorical View of Def. 15

▶ Remark 42. By precomposing the functor Sl from Remark 11 with the inclusion functor $\mathbb{O} \hookrightarrow \mathbb{C}$, we may regard it as a functor $\mathbb{O} \to \mathbf{Cat}$. Let $\Delta\mathbb{C}\colon \mathbb{O} \to \mathbf{Cat}$ be the constant functor at $\mathbb{C}$, and let $d\colon \mathrm{Sl} \Rightarrow \Delta\mathbb{C}$ be the indexed functor whose component at O is $d\colon \mathbb{C}/O \to \mathbb{C}$. The monads T on $\mathbb{C}$ and $\overline{T}$ on slice categories induce the indexed functor $T\colon \Delta\mathbb{C} \Rightarrow \Delta\mathbb{C}$ and the lax indexed functor $\overline{T}\colon \mathrm{Sl} \Rightarrow \mathrm{Sl}$, respectively. With this notation, the second condition in Def. 15 says exactly that the family $\alpha = (\alpha^O)_{O\in\mathbb{O}}$ forms a modification $\alpha\colon Td \Rightarrow Td\overline{T}$. Likewise, the family $\mathsf{flat} = (\mathsf{flat}^O)_{O\in\mathbb{O}}$ forms a modification $\mathsf{flat}\colon Td\overline{T} \Rightarrow Td$. Hence a belief decomposition structure on T can be seen simply as a modification $\alpha\colon Td \Rightarrow Td\overline{T}$ that is a section of flat.

B Omitted Definitions and Proofs in §8

B.1 Omitted Definitions and Proofs in §8.1

We first see that $(\alpha^O\colon \mathcal{D}d \Rightarrow \mathcal{D}d\overline{\mathcal{D}})_O$ is a belief decomposition.

▶ **Proposition 43.** *The data* $(\alpha^O\colon \mathcal{D}d \Rightarrow \mathcal{D}d\overline{\mathcal{D}})_O$ *is a belief decomposition.*

Proof. We first see that α^O is a natural transformation. Let $h\colon f \to g$ and $b \in \mathcal{D}(X)$. We have

$$
\begin{aligned}
&\big((\mathcal{D}d\overline{\mathcal{D}})(h) \circ \alpha^O_f\big)(b)(b') \\
&= \begin{cases} \sum_{x \in f^{-1}(o)} b(x) & \text{if } \exists o \in O.\ \sum_{x \in f^{-1}(o)} b(x) > 0 \text{ and } b' = \frac{1}{\sum_{x \in f^{-1}(o)} b(x)} \cdot B(o), \\ 0 & \text{otherwise,} \end{cases} \\
&\big(\alpha^O_g \circ (\mathcal{D}d)(h)\big)(b)(b') \\
&= \begin{cases} \sum_{y \in g^{-1}(o)} \mathcal{D}(h)(b)(y) & \text{if } \exists o \in O.\ C(o) > 0 \text{ and } b'(y) = \frac{1}{C(o)} \cdot \mathcal{D}(h)(b)\big|_o, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum_{x \in f^{-1}(o)} b(x) & \text{if } \exists o \in O.\ \sum_{x \in f^{-1}(o)} b(x) > 0 \text{ and } b' = \frac{1}{\sum_{x \in f^{-1}(o)} b(x)} \cdot \mathcal{D}(h)(b)\big|_o, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum_{x \in f^{-1}(o)} b(x) & \text{if } \exists o \in O.\ \sum_{x \in f^{-1}(o)} b(x) > 0 \text{ and } b' = \frac{1}{\sum_{x \in f^{-1}(o)} b(x)} \cdot B(o), \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

where $B(o)(y) = \sum_{x \in f^{-1}(o) \cap h^{-1}(y)} b(x)$ and $C(o) = \sum_{y \in g^{-1}(o)} \mathcal{D}(h)(b)(y)$. It satisfies the condition (i) in Def. 15 as follows:

$$
\begin{aligned}
(\mathsf{flat}^O_f \circ \alpha^O_f)(b)(x) &= \sum_{b'} \alpha^O_f(b)(b') \cdot b'(x) \\
&= \Big(\sum_{x' \in X \text{ s.t. } f(x') = f(x)} b(x')\Big) \cdot \frac{b(x)}{\sum_{x' \in X \text{ s.t. } f(x') = f(x)} b(x')} = b(x).
\end{aligned}
$$

It is easy to see that the condition (ii) in Def. 15 holds for $(\alpha^O)_O$. ◀

Next, we see that the two modalities $\rho\colon [0,\infty] \times \mathbb{R}_{\geq 0} \to [0,\infty]$ and $\sigma\colon \mathcal{D}([0,\infty]) \to [0,\infty]$ satisfy the first assumption in $(\star)$.

▶ **Proposition 44.** *The morphisms ρ and σ satisfy the first assumption in $(\star)$.*

Proof. It is straightforward to see that ρ is a monotone algebra, and σ is a monotone Eilenberg-Moore algebra. We see that $\perp_{\mathcal{D}(X)} = \sigma \circ \mathcal{D}(\perp_X)$ in the following:

$$
\big(\sigma \circ \mathcal{D}(\perp_X)\big)(\nu) = \sum_{x \in X} \nu(x) \cdot \perp_X(x) = 0 = \perp_{\mathcal{D}(X)}(\nu).
$$

We next see that $\sigma \circ \mathcal{D}(\rho) = \rho \circ (\sigma \times \mathbb{R}_{\geq 0}) \circ \lambda_\Omega$ as follows:

$$
\begin{aligned}
\big(\sigma \circ \mathcal{D}(\rho)\big)(\nu) &= \sum_r \mathcal{D}(\rho)(\nu)(r) \cdot r = \sum_{r_1, r_2} \nu(r_1, r_2) \cdot (r_1 + r_2) \\
\big(\rho \circ (\sigma \times \mathbb{R}_{\geq 0}) \circ \lambda_\Omega\big)(\nu) &= \big(\pi_2 \circ \lambda_\Omega\big)(\nu) + \sum_r (\pi_1 \circ \lambda_\Omega)(\nu)(r) \cdot r \\
&= \Big(\sum_{r_1, r_2} \nu(r_1, r_2) \cdot r_2\Big) + \Big(\sum_r \Big(\sum_{r'} \nu(r, r')\Big) \cdot r\Big) \\
&= \sum_{r_1, r_2} \nu(r_1, r_2) \cdot (r_1 + r_2).
\end{aligned}
$$

◀

Lastly, we check the second assumption in $(\star)$.

▶ **Proposition 45.** *The second assumption in $(\star)$ holds for POMDPs.*

Proof. We have

$$\big(\mathcal{D}\langle \mathrm{id}, \mathrm{obs}\rangle \circ \iota_{\mathrm{obs}}\big)(\nu)(x,o) = \begin{cases} \nu(x) & \text{if } o = \mathrm{obs}(x), \\ 0 & \text{otherwise,} \end{cases}$$

$$\big(\mathrm{st}^{\mathcal{D}} \circ \langle \iota_{\mathrm{obs}}, \overline{\mathcal{D}}\mathrm{obs}\rangle\big)(\nu)(x,o) = \big(\mathrm{st}^{\mathcal{D}}(\nu, o')\big)(x,o) = \begin{cases} \nu(x) & \text{if } o = \mathrm{obs}(x), \\ 0 & \text{otherwise,} \end{cases}$$

where $o' = \mathrm{obs}(x')$ for some $x' \in \mathrm{supp}(\nu)$. ◀

B.2 Omitted Definitions in §8.2

Given a coalgebra $c\colon S \to \mathcal{R}(S) \times \mathbb{R}_{\geq 0}$, we define a weighted transition system $c'\colon S \to \mathcal{R}(S + \{\checkmark\})$ such that $c'(s)(s') := (\pi_1 \circ c)(s)(s')$ and $c'(s)(\checkmark) := (\pi_2 \circ c)(s)$. A *terminating path* p on c is the path p on c' such that p ends at $\checkmark$. The *weight* $w(p)$ over the terminating path $p = s_1 \cdots s_n \cdot \checkmark$ is given by

$$w(p) := \Big(\prod_{i \in [1, n-1]} c'(s_i, s_{i+1}) \Big) \cdot c'(s_n, \checkmark).$$

The characterization of the objective $V^*_{i,c}$ given in §8.2 is a corollary of the following characterization.

▶ **Proposition 46.** *Let $c\colon S \to \mathcal{R}(S) \times \mathbb{R}_{\geq 0}$, and $\Phi_c\colon \mathbb{C}(S, [0,\infty]) \to \mathbb{C}(S, [0,\infty])$ be the monotone map defined in §8.2. We have*

$$\bigvee_{n \in \mathbb{N}} \Phi_c^n(\bot)(s) = \sum_{p \in \mathrm{TPath}(s)} w(p),$$

where $\mathrm{TPath}(s)$ *is the set of terminating paths starting from s.*

C Omitted Proofs

C.1 Proof of Prop. 10

Proof. The functor $\overline{T}_O$ forms a monad, which comes from the sliced adjunction of the Eilenberg–Moore adjunction $F_T \dashv U_T\colon \mathbb{C}^T \to \mathbb{C}$ at O:

$$\begin{tikzcd} \mathbb{C}/O \arrow[r, bend left, "(F_T)_{/O}"] \arrow[r, phantom, "\perp"] & \mathbb{C}^T/F_T(O) \arrow[l, bend left, "(U_T)_{/O}"] \end{tikzcd}$$

Here $(F_T)_{/O}$ is the functor induced by applying F_T to the objects and morphisms of $\mathbb{C}/O$, and $(U_T)_{/O}$ is the functor defined by applying U_T and taking the pullback along the unit of the adjunction $F_T \dashv U_T$. Therefore, the induced monad $(U_T)_{/O}(F_T)_{/O}$ is $\overline{T}_O$.

Let $u\colon O \to O'$. Then we have the equality $(F_T)_{/O'} \circ \Sigma_u = \Sigma_{F_T(u)} \circ (F_T)_{/O}$. Hence by applying Lem. 8 to the functors Σ_u, $\Sigma_{F_T(u)}$ and the adjunctions $(F_T)_{/O'} \dashv (U_T)_{/O'}$ and $(F_T)_{/O} \dashv (U_T)_{/O}$, the statement about the monad morphism follows. The explicit description of the natural transformation θ_u follows from the construction of the monad morphism in Lem. 8. ◀

C.2 Proof of Prop. 12

Proof. Apply Lem. 8 to the forgetful functors $\mathbb{C}/O \to \mathbb{C}$ and $\mathbb{C}^T/F_T(O) \to \mathbb{C}^T$ and the adjunctions $F_T \dashv U_T$ and $(F_T)_{/O} \dashv (U_T)_{/O}$. ◀

C.3 Proof of Prop. 17

Proof. Let $h \coloneqq d\overline{T}(f) \circ d(\theta_g)_{\mathrm{obs}}$. We show that (h, g) is a morphism of I-pointed PO coalgebras. First, since (Σ_g, θ_g) is a monad morphism, its compatibility with units yields $h \circ \eta^{\overline{T}}_{\mathrm{obs}} = \eta^{\overline{T}}_{\mathrm{obs}'} \circ f$. Hence $h \circ \eta^{\overline{T}}_{\mathrm{obs}} \circ i = \eta^{\overline{T}}_{\mathrm{obs}'} \circ f \circ i = \eta^{\overline{T}}_{\mathrm{obs}'} \circ i'$.

Next, by construction, h is a morphism $\Sigma_g(\overline{T}(\mathrm{obs})) \to \overline{T}(\mathrm{obs}')$ in the slice category $\mathbb{C}/O'$. Therefore the observation part commutes: $g \circ \overline{T}(\mathrm{obs}) = \overline{T}(\mathrm{obs}') \circ h$.

The transition part also commutes by the following equations:

$$\begin{aligned}
FTh \circ c^{\mathsf{Bel}} &= FTh \circ F\alpha_{\mathrm{obs}} \circ \mathrm{Det}(\delta) \circ \iota_{\mathrm{obs}} \\
&= F\alpha_{\mathrm{obs}'} \circ FTf \circ \mathrm{Det}(\delta) \circ \iota_{\mathrm{obs}} \\
&= F\alpha_{\mathrm{obs}'} \circ \mathrm{Det}(\delta') \circ Tf \circ \iota_{\mathrm{obs}} \\
&= F\alpha_{\mathrm{obs}'} \circ \mathrm{Det}(\delta') \circ \iota_{\mathrm{obs}'} \circ h \\
&= \delta'^{\mathsf{Bel}} \circ h.
\end{aligned}$$

Here the second equality uses the compatibility of α with base change together with the naturality of $\alpha^{O'}$, the third equality uses the functoriality of ordinary coalgebraic determinization, and the fourth equality holds by definition of h. ◀

C.4 Proof for α in Example 18

By definition, we can see that $(\overline{\mathcal{P}_\emptyset})_O \colon \mathbf{Set}/O \to \mathbf{Set}/O$ is defined by for any $f \colon S \to O$, $(\overline{\mathcal{P}_\emptyset})_O(f) \colon B_f \to O$ and $(\overline{\mathcal{P}_\emptyset})_O(f)(U) = f(u)$ for some $u \in U$, where $B_f \coloneqq \{U \in \mathcal{P}_\emptyset(S) \mid \exists o \in O \text{ s.t. } U \subseteq f^{-1}(o)\}$. For any $h \colon f \to g$ in $\mathbf{Set}/O$, we have $(\overline{\mathcal{P}_\emptyset})_O(h)(U) = h(U)$.

We see that α^O is a natural transformation: given $h \colon f \to g$ in $\mathbf{Set}/O$ and $U \in (\mathcal{P}_\emptyset d)(f)$, we have

$$\begin{aligned}
\big(\alpha^O_g \circ (\mathcal{P}_\emptyset d)(h)\big)(U) &= \big\{\{v \in h(U) \mid g(v) = o\} \,\big|\, o \in O \text{ s.t. } g^{-1}(o) \cap h(U) \neq \emptyset\big\} \\
&= \big\{\{v \in h(U) \mid g(v) = o\} \,\big|\, o \in O \text{ s.t. } f^{-1}(o) \cap U \neq \emptyset\big\} \\
&= \big\{\{h(u) \mid u \in U \text{ and } f(u) = o\} \,\big|\, o \in O \text{ s.t. } f^{-1}(o) \cap U \neq \emptyset\big\} \\
&= \big((\mathcal{P}_\emptyset d\overline{\mathcal{P}_\emptyset})(h) \circ \alpha^O_f\big)(U).
\end{aligned}$$

It is straightforward to see that α^O is a section of flat^O, since α^O simply creates the partition based on observations.

Next, we show that α satisfies the condition (ii) in Def. 15. For each injective function $u \colon O \to O'$ and each function $f \colon X \to O$, the component $(\mathcal{P}_\emptyset d_{O'} \theta_u \circ \alpha^O)_f$ is the function mapping $U \in \mathcal{P}_\emptyset X$ to $\{f^{-1}(o) \cap U \mid o \in O \text{ s.t. } f^{-1}(o) \cap U \neq \emptyset\}$, and the component $(\alpha^{O'} \Sigma_u)_f$ is the function mapping $U \in \mathcal{P}_\emptyset X$ to $\{(u \circ f)^{-1}(o') \cap U \mid o' \in O' \text{ s.t. } (u \circ f)^{-1}(o') \cap U \neq \emptyset\}$. They are equal because u is injective.

C.5 Proof of Prop 20

Proof. Note that $f \coloneqq \overline{T}(\mathrm{id}) \colon d\overline{T}(\mathrm{id}_S) \to S$ is an isomorphism because it is obtained as the pullback of the isomorphism id_{TS}. From the pullback square we have $\iota_{\mathrm{id}_S} = \eta_S \circ f$.

By the assumption $\mu_S \circ T\iota_{\mathrm{id}_S} \circ \alpha_{\mathrm{id}_S} = id_{TS}$, the following equations hold:

$$\mathrm{id}_{TS} = \mu_S \circ T\iota_{\mathrm{id}_S} \circ \alpha_{\mathrm{id}_S} = \mu_S \circ T\eta_S \circ Tf \circ \alpha_{\mathrm{id}_S} = Tf \circ \alpha_{\mathrm{id}_S}.$$

By the following equations, $f\colon c^{\mathsf{Bel}} \to \delta$ is an FT-coalgebra morphism:

$$FTf \circ c^{\mathsf{Bel}} = FTf \circ F\alpha_{\mathrm{id}_S} \circ \mathrm{Det}(\delta) \circ \iota_{\mathrm{id}_S} = \mathrm{Det}(\delta) \circ \iota_{\mathrm{id}_S} = \mathrm{Det}(\delta) \circ \eta_S \circ f = \delta \circ f.$$

Thus $(f, \mathrm{id}_S)\colon \mathsf{Bel}(\mathrm{id}_S, \langle\delta, id_S\rangle) \to (\mathrm{id}_S, \langle\delta, \mathrm{id}_S\rangle)$ is an isomorphism in $\mathrm{Coalg}^{\mathbb{O}}_S(FT)$. ◀

C.6 Proof of Prop 32

Proof. By Prop. 26, it suffices to show that there is a morphism from $(\mathsf{nil} \circ i, \mathsf{Hist}(c)_u)$ to $(\langle \mathrm{id}, (u \circ \mathsf{cons}_O)^\dagger \circ \mathrm{obs}\rangle \circ i, \mathsf{Sch}(c))$ for each $u\colon O^+ \to A$. For each $u\colon O^+ \to A$, define a morphism $\mathrm{cmp}_u \coloneqq \langle \mathrm{last}, \chi_u^\dagger\rangle\colon S^+ \to S \times A^{O^*}$ where $\chi_u\colon S^+ \times O^* \to A$ is the composite

$$S^+ \times O^* \xrightarrow{\mathrm{obs}^+ \times \mathrm{id}} O^+ \times O^* \xrightarrow{\mathsf{ext}_O} O^+ \xrightarrow{u} A.$$

We show that $\mathrm{cmp}_u\colon (\mathsf{nil} \circ i, \mathsf{Hist}(c)_u) \to (\langle \mathrm{id}, (u \circ \mathsf{cons}_O)^\dagger \circ \mathrm{obs}\rangle \circ i, \mathsf{Sch}(c))$ in $\mathrm{Coalg}_I(FT)$.

First, cmp_u is an FT-coalgebra morphism $\mathrm{cmp}_u\colon \mathsf{Hist}(c)_u \to \mathsf{Sch}(c)$ by the following commutative diagram:

$$\begin{tikzcd}
S^+ \arrow[r, "{\langle \delta\circ\mathrm{last}_{\mathrm{id}}, u\circ\mathrm{obs}^+, \mathrm{id}\rangle}"] \arrow[d, "\mathrm{cmp}_u"'] & FTS^A \times A \times S^+ \arrow[r, "\mathrm{st}\circ\mathrm{ev}\times\mathrm{id}_{S^+}"] \arrow[d, dashed, "FTS^A \times A \times f"'] & FT(S \times S^+) \arrow[r, "\cong"] \arrow[d, "FT(S\times f)"] & FT(S^+ \times S) \arrow[r, "FT\mathrm{ext}"] & FT(S^+) \arrow[d, "FT(\mathrm{cmp}_u)"] \\
S \times A^{O^*} \arrow[r, "{\langle \delta, \cong\rangle}"] & FTS^A \times A \times A^{O^+} \arrow[r, "\mathrm{st}\circ\mathrm{ev}\times\mathrm{id}"] & FT(S \times A^{O^+}) \arrow[rr, "{FT\langle \pi_1, \mathrm{ev}\circ\mathrm{obs}\times\mathrm{id}\rangle}"] & & FT(S \times A^{O^*})
\end{tikzcd}$$

where $f\colon S^+ \to A^{O^+}$ is the transpose of $S^+ \times O^+ \xrightarrow{\mathrm{obs}^+ \times \mathrm{id}} O^+ \times O^+ \xrightarrow{\mathsf{ext}} O^+ \xrightarrow{u} A$. The leftmost square commutes because, under the canonical isomorphism $A^{O^*} \cong A \times A^{O^+}$, the morphism $\chi_u^\dagger$ corresponds to the pair $\langle u \circ \mathrm{obs}^+, f\rangle\colon S^+ \to A \times A^{O^+}$.

Next, we show that $\mathrm{cmp}_u \circ \mathsf{nil}_S = \langle \mathrm{id}_S, (u \circ \mathsf{cons})^\dagger \circ \mathrm{obs}\rangle$, which induces that cmp_u preserves the point. For the first component, we use $\mathrm{last} \circ \mathsf{nil}_S = \mathrm{id}_S$. For the second component, by adjoint transposition, it suffices to show $\chi_u \circ (\mathsf{nil}_S \times \mathrm{id}_{O^*}) = u \circ \mathsf{cons} \circ (\mathrm{obs} \times \mathrm{id}_{O^*})$.

This equality holds because of the naturality of nil and $\mathsf{cons} = \mathsf{ext} \circ (\mathsf{nil} \times \mathrm{id})$. ◀