Bisimulations and Modal Logics for Higher Dimensional Automata

Safa Zouari
Norwegian University of Science and Technology Gjøvik, Norway

Rob van Glabbeek
School of Informatics, University of Edinburgh, Scotland
School of Computer Science and Engineering, University of New South Wales, Sydney, Australia

Krzysztof Ziemiański
Institute of Mathematics, University of Warsaw, Poland

Abstract

Higher-Dimensional Automata (HDAs) provide a geometric model of true concurrency. While hereditary history-preserving (hhp) bisimilarity is the finest behavioural equivalence in van Glabbeek's spectrum, no modal logic has previously characterised it on HDAs. We introduce several new intermediate equivalences that sit strictly between ST- and hhp-bisimilarity. We show how separating similarity and subsumption of paths leads to a clean formulation of these equivalences, and we present a modal logic that characterises hhp-bisimilarity. Natural fragments characterise ST-bisimilarity and the intermediate notions.

2012 ACM Subject Classification Theory of computation → Concurrency; Theory of computation → Modal and temporal logics

Keywords and phrases higher-dimensional automata, bisimilarity, history-preserving bisimulation, modal logic, concurrency theory

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2026.49

Funding *Safa Zouari*: Supported by a PhD fellowship at the Department of Information Security and Communication Technology (IIK), Norwegian University of Science and Technology (NTNU).
Rob van Glabbeek: Supported by Royal Society Wolfson Fellowship RSWF\R1\221008.

1 Introduction

Higher-dimensional automata (HDAs) [20, 11] provide a geometric model of true concurrency: n-dimensional cells represent the concurrent execution of n independent events, without reducing concurrency to interleaving. HDAs occupy a central position in the spectrum of concurrency models: Petri nets [24], event structures [25], configuration structures [14], and asynchronous systems [5] embed into HDAs up to hereditary history-preserving (hhp) bisimilarity [12]. Thus HDAs form a unifying semantic framework for non-interleaving concurrency.

Hereditary history-preserving bisimilarity (hhp-bisimilarity) is one of the finest behavioural equivalences, respecting both branching time and causality. For several true-concurrency models, modal logics have been shown to characterise hhp-bisimilarity (e.g. on event structures [4], stable configuration structures [18], prime event structures and history-dependent automata [3]). For HDAs, however, despite being introduced 35 years ago, no modal logic has provided an exact characterisation of hhp-bisimilarity. Prisacariu's Higher-Dimensional Modal Logic [21] did not characterise even ST-bisimilarity [15], reaching only split bisimulation; a follow-up [22] still fell short of hp-bisimilarity. More recently, Zouari, Ziemiański and Fahrenberg [26], inspired by the open maps technique, succeeded in characterising

37th International Conference on Concurrency Theory (CONCUR 2026).
Editors: Ana Sokolova and Patrick Totzke; Article No. 49; pp. 49:1–49:19
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

ST-bisimilarity, but only in the absence of autoconcurrency. Our work is the first to provide exact modal characterisations of the full spectrum from ST- to hhp-bisimilarity on HDAs, including in the presence of autoconcurrency.

Decomposing path adjacency. A key technical contribution of this paper is a decomposition of the classical *ℓ-adjacency* relation on paths [12] into two conceptually distinct components. *Similarity* at a position ℓ in a path (written $\leftrightarrows^{\ell}$) permutes two adjacent independent start phases of events, or two adjacent end phases, and is inherently symmetric. *Subsumption* (written $\sqsubseteq^{\ell}$) replaces a sequential traversal around a higher-dimensional cell by a path that passes through it directly, increasing the degree of observed concurrency; it is directional. In the original formulation [12] these two operations are bundled into a single symmetric adjacency step. This forces any adjacency-based notion of bisimulation to require either both the forward and backward directions of subsumption, or neither of them. Making the decomposition explicit allows one to require subsumption in only one direction, yielding a strictly finer control over what behavioural information is preserved.

A spectrum of new bisimilarities. The decomposition of path adjacency into similarity and subsumption naturally gives rise to a family of bisimulation equivalences that fills the gap between ST-bisimilarity and hhp-bisimilarity in the spectrum of [13, 10]. This family is parameterised by which subset of the operations subsumption, unsubsumption, similarity and path restriction is required of the bisimulation relation. These intermediate notions are of independent interest: they isolate distinct behavioural properties of concurrent systems, allowing one to distinguish systems that differ in their capacity for concurrency refinement from those that differ in the symmetry of their concurrent structure.

We establish strict inclusions between these notions using explicit separating examples. Between two of the equivalences we establish inclusion without a separating example; hereby we pose the open question whether these equivalences coincide.

A general modal characterisation framework. To obtain modal characterisations uniformly across the entire spectrum, we implicitly apply transformations of HDAs into a kind of labelled transition system called *(I, M)-system*. These have state-labels as well as transition labels. For ST-bisimilarity, hhp-bisimilarity, and each of the new intermediate equivalences on HDAs, we apply a transformation that translates this equivalence on HDAs to strong bisimilarity on the associated (I, M)-system. The states in our (I, M)-system will be the paths in the HDA; they are labelled by their associated ST-trace, which describes the observable content of a path through a sequence of action-phases (starts and ends of actions), with a matching of each end phase with it corresponding start. The labelled transitions in the (I, M)-system will be the relations in $I \subseteq \{$path extension, subsumption, similarity, unsubsumption, path restriction$\}$. The upshot of this is that a form of the classical Hennessy-Milner logic on labelled transition systems can be used to reason about these path-based semantic equivalences on HDAs. Each bisimilarity in our spectrum is characterised by an appropriate choice of the index set I, yielding the corresponding modal logic as a natural fragment of a common language. This framework also explains, in a uniform way, why the logics characterising coarser equivalences are syntactic fragments of those for finer ones.

As a consequence of the above method, our logics on HDA are *path-based*, whereas previous proposals [21, 22] could be classified as *cell-based*.

Contributions

- We decompose ℓ-adjacency into ℓ-similarity and ℓ-subsumption, identifying these as independent behavioural conditions of conceptually different character.
- Using this decomposition, we introduce a family of intermediate bisimulation equivalences between ST- and hhp-bisimilarity, including in particular *semi-history-preserving* (shp) and *quasi-history-preserving* (qhp) bisimilarity, and establish strict inclusions between them via explicit separating examples.
- We introduce (I, M)-systems and prove a general Hennessy-Milner theorem: bisimilarity and logical equivalence coincide for any choice of index set I and label set M.
- Instantiating this framework to HDAs, we obtain exact modal characterisations of each bisimilarity in our spectrum as a natural fragment of a single modal language, providing in particular the first modal characterisation of hhp-bisimilarity for HDAs.

Why a finer spectrum matters. Bisimulation-based techniques have a strong track record in the verification of security protocols (e.g. for information-flow properties [9]), and can even uncover real-world privacy vulnerabilities [7, 17, 16]. More directly relevant to our setting, Aubert, Horne and Johansen [1] show that moving to finer non-interleaving bisimilarity notions expands the class of detectable attacks on privacy protocols, with the underlying transition systems coinciding exactly with HDAs; in [2] this line of work explicitly takes advantage of history-preserving bisimilarity. Our intermediate equivalences, each accompanied by a characterising modal logic, provide a principled way to calibrate the attacker model between ST- and hhp-bisimilarity, potentially uncovering attacks beyond ST without requiring the full strength of hhp. Developing these connections into concrete verification tools is left as future work.

A *testing scenario* is a method to explain the non-equivalence of two systems in terms of an observation that could be made for one but not the other. One of the few testing scenarios that distinguishes ST-bisimilar system is “Teams can see pomsets” of Plotkin & Pratt [19]. It might be that a suitable branching time version of this testing scenario corresponds with one of the semantic equivalences explored in our paper.

Organisation of the paper. Section 2 recalls the ordered precubical presentation of HDAs. Section 3 introduces paths and decomposes ℓ-adjacency into ℓ-similarity and ℓ-subsumption. Section 4 recalls the ST-trace of a path. Section 5 introduces the bisimulation equivalences arising from the decomposition and establishes the inclusion hierarchy. Section 6 develops the (I, M)-system framework and proves the general Hennessy-Milner theorem. Section 7 instantiates that framework to HDAs and derives modal characterisation for each bisimilarity in our spectrum.

2 Higher-dimensional automata

Higher-dimensional automata (HDAs) model concurrent computation geometrically: an n-dimensional cell represents the simultaneous execution of n independent events. Formally, HDAs are defined as presheaves over a precubical base category encoding how events start and terminate. In this section we recall the classical ordered precubical presentation of HDAs, following [12, 8]. Fix a set of actions Σ.

▶ **Definition 2.1** (The ordered precube category $\square$). Objects are *concurrency lists* (*conclists*) $((n), \lambda)$, where $(n) = \{1 < \cdots < n\}$ is a finite totally ordered set and $\lambda\colon (n) \to \Sigma$ is a labelling. We write (n, λ) instead of $((n), \lambda)$ for simplicity.

A morphism $(f, \varepsilon)\colon (m, \lambda) \to (n, \mu)$ consists of:

- an injective order- and label-preserving map $f\colon (m) \hookrightarrow (n)$;
- a status map $\varepsilon\colon (n) \to \{0, ⅃, 1\}$ satisfying $\varepsilon^{-1}(⅃) = f((m))$.

Intuitively, an object $U = (n, \lambda)$ represents a configuration in which n concurrent events are being performed; they are linearly ordered solely in order to distinguish them when relating events from different configurations. Given $(f, \varepsilon)\colon U \to V$, the status map ε classifies each event of V as *not started* (0), *executing* (⅃), or *terminated* (1), while f embeds the active events of U into those of V. Since f is injective and order-preserving, given $U = (m, \lambda)$ and $V = (n, \mu)$, it is uniquely determined by the active set $\varepsilon^{-1}(⅃) \subseteq (n)$. Additionally, the event labelling λ is completely determined by μ and f.

Composition of $(f, \varepsilon)\colon S \to T$ and $(g, \zeta)\colon T \to U$ is defined by

$$(g, \zeta) \circ (f, \varepsilon) = (g \circ f, \eta),$$

where

$$\eta(u) = \begin{cases} \varepsilon(t) & \text{if } u = g(t) \text{ for some } t \in T, \\ \zeta(u) & \text{otherwise.} \end{cases}$$

For $U = (n-1, \lambda)$, $V = (n, \lambda')$ and $k \in \{0, 1\}$, the *coface map* $d^k_{i,n}\colon U \to V$ $(1 \le i \le n)$ is the unique morphism (ι_i, ε) where $\iota_i : (n-1) \to (n)$ omits position i from (n), $\varepsilon(i) = k$, and $\varepsilon(j) = ⅃$ for all $j \neq i$. When n is clear we write d^k_i. Intuitively, when V is seen as an n-dimensional cube, representing the concurrent execution of n events, U represents its $n-1$-dimensional face in which the i^{th} event of V has not yet started (if k=0) or is already finished (if k=1). The coface maps satisfy the cubical identities

$$d^\ell_{j,n} \circ d^k_{i,n-1} = d^k_{i,n} \circ d^\ell_{j-1,n-1} \qquad (i < j), \quad k, \ell \in \{0, 1\},$$

and generate the category $\square$, in the sense that any morphism can be obtained by composing coface maps. A *precubical set* is a *presheaf over* $\square$, that is, a functor

$$X\colon \square^{op} \to \mathbf{Set}.$$

If X is a presheaf over $\square$ and $U = (n, \lambda)$ is an object, $X[U]$ represents the set of those configurations in X in which exactly n events are active, with labels $(\lambda(i))_{i=1}^n$. Elements of $X[U]$, for all objects U of $\square$, form the set of cells Cell_X of X. Specifically, we have:

$$\mathrm{Cell}_X = \bigsqcup_{U \in \mathrm{obj}(\square)} X[U].$$

Let $\square\,(U, V)$ denote the set of morphisms from U to V. If the dimension of U is one lower than that of V, these are coface maps. For a coface map d^k_i we write $\delta^k_i := X[d^k_i]$; it is a function that for each n-dimensional cell yields its starting or end face (depending on k) in dimension i.

Let 0 denote the unique object (n, λ) in $\square$ with $n = 0$.

▶ **Definition 2.2** (Higher-dimensional automaton). An *HDA* is a pair $\mathcal{X} = (X, i_{\mathcal{X}})$ where X is a precubical set and $i_{\mathcal{X}} \in X(0)$ is a distinguished initial 0-cell.

3 Paths in HDAs

We recall the notion of paths in HDAs, fixing notation for later use.

▶ **Definition 3.1.** A *path* of length n in a precubical set X is a sequence $\alpha = (x_0, \varphi_1, x_1, \varphi_2, \ldots, \varphi_n, x_n)$, where $x_j \in X[U_j]$ are cells, and for all $j \in \{1, \ldots, n\}$, either

- $\varphi_j = d^0_{i_j} \in \square(U_{j-1}, U_j)$, a coface map, and $x_{j-1} = \delta^0_{i_j}(x_j)$ (up-step), or
- $\varphi_j = d^1_{i_j} \in \square(U_j, U_{j-1}), \delta^1_{i_j}(x_{j-1}) = x_j$ (down-step).

A *path in an HDA* $\mathcal{X} = (X, i_{\mathcal{X}})$ is a path in the underlying precubical set X whose first cell is the initial cell $i_{\mathcal{X}}$. We write Path_X for the sets of all paths in X, and $\mathrm{Path}_{\mathcal{X}} \subseteq \mathrm{Path}_X$ for the sets of paths starting at the initial cell.

If $\alpha = (x_0, \varphi_1, \ldots, \varphi_n, x_n)$ and $\beta = (y_0, \psi_1, \ldots, \psi_m, y_m)$ are paths in X with $x_n = y_0$, their *concatenation* is the path

$$\alpha * \beta = \big(x_0, \varphi_1, \ldots, \varphi_n, x_n, \psi_1, y_1, \ldots, \psi_m, y_m\big).$$

In this situation, we say that $\alpha * \beta$ is an *extension* of α and that α is a *restriction* or *prefix* of $\alpha * \beta$; we write $\alpha \hookrightarrow \alpha * \beta$.

▶ **Definition 3.2** (Similarity of paths)**.** We say that paths α and β are *ℓ-similar*, written $\alpha \leftrightarrows^\ell \beta$, if one is obtained from the other by replacing

$$(1) \quad (d^0_i,\ x_\ell,\ d^0_j) \ \text{ by } \ (d^0_{j-1},\ x'_\ell,\ d^0_i), \qquad \text{or} \qquad (2) \quad (d^1_j,\ x_\ell,\ d^1_i) \ \text{ by } \ (d^1_i,\ x'_\ell,\ d^1_{j-1})$$

for indices $i < j$. *Similarity* $\leftrightarrows$ is the equivalence relation generated by $\leftrightarrows^\ell$ for all $\ell \geq 1$.

▶ **Definition 3.3** (Subsumption of paths)**.** We say that path β *subsumes* α *at position* ℓ, written $\alpha \sqsubseteq^\ell \beta$, if α is obtained from β by replacing

$$(3) \quad (d^0_i,\ x_\ell,\ d^1_j) \ \text{ by } \ (d^1_{j-1},\ x'_\ell,\ d^0_i), \qquad \text{or} \qquad (4) \quad (d^0_j,\ x_\ell,\ d^1_i) \ \text{ by } \ (d^1_i,\ x'_\ell,\ d^0_{j-1})$$

for indices $i < j$. The *global subsumption* relation $\sqsubseteq$ is the reflexive and transitive closure of all ℓ-subsumptions:

$$\alpha \ \sqsubseteq\ \beta \quad \iff \quad \exists\, \ell_1, \ldots, \ell_k \text{ such that } \alpha = \alpha^{(0)} \sqsubseteq^{\ell_1} \alpha^{(1)} \sqsubseteq^{\ell_2} \cdots \sqsubseteq^{\ell_k} \alpha^{(k)} = \beta.$$

Geometrically, if $\alpha \sqsubseteq^\ell \beta$, then β passes through the n-dimensional cell x_ℓ, while α bypasses it by following the corresponding $(n-2)$-dimensional face x'_ℓ instead. In particular, β explores a region of the automaton with more concurrency; see Fig. 1 for the $n = 2$ case.

▶ **Definition 3.4** (Adjacency of paths [12])**.** Paths α and β are ℓ-adjacent, written $\alpha \overset{\ell}{\leftrightsquigarrow} \beta$, if they are either ℓ-similar or ℓ-subsumed. *Homotopy* is the equivalence relation generated by similarity and subsumption. This is, two paths α and β are *homotopic* if one can be obtained from the other by a sequence of ℓ-similarities and ℓ-subsumptions.

Let $\alpha = (x_0, \varphi_1, x_1, \varphi_2, \ldots, \varphi_m, x_m)$ be a path of length m and $1 \leq \ell < m$. If φ_ℓ and $\varphi_{\ell+1}$ are both up-steps or both down-steps, then there exists a unique path $\alpha^{(\ell)}$ that is ℓ-similar to α. If $\varphi_\ell = d^0_i$ is an up-step and $\varphi_{\ell+1} = d^1_j$ is a down-step, then there exists a unique $\alpha^{(\ell)}$ such that $\alpha^{(\ell)} \sqsubseteq^\ell \alpha$ if $i \neq j$, and no ℓ-adjacent paths otherwise. In all these cases we have $\alpha \overset{\ell}{\leftrightsquigarrow} \alpha^{(\ell)}$. Finally, if φ_ℓ is a down-step and $\varphi_{\ell+1}$ is an up-step, there may be any number (including none) of paths β such that $\alpha \sqsubseteq^\ell \beta$.

In the original work [12], ℓ-similarity and ℓ-subsumption are bundled into the single symmetric ℓ-adjacency relation. In this paper, we decompose ℓ-adjacency into two conceptually distinct components: *ℓ-similarity* and *ℓ-subsumption*. This separation is not merely cosmetic.

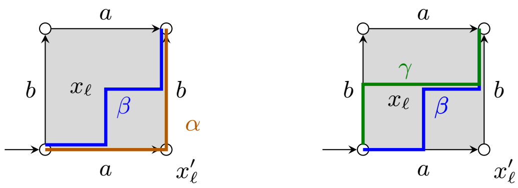

Figure 1 Subsumption step as a bypass of a higher-dimensional cell (illustrated for $n = 2$). Here β could be a path $(x_0, d^0_{1,1}, x_1, d^0_{2,2}, x_2, d^1_{1,2}, x_3, d^1_{1,1})$ in which x_0, x_1, x_2, x_3, x_4 are cells of dimensions $0, 1, 2, 1, 0$, respectively, $d^0_{i,n}$ indicates the start and $d^1_{i,n}$ the end of an action, numbered i out of n. Here $\ell = 2$. Now α becomes $(x_0, d^0_{1,1}, x_1, d^1_{1,1}, x'_2, d^0_{1,1}, x_3, d^1_{1,1})$, where x'_2 is a cell of dimension 0. In this case $\beta \sqsupseteq^2 \alpha$, so β subsumes α; in β the two actions overlap, whereas in α they don't. Moreover, γ could be a path $(x_0, d^0_{1,1}, x'_1, d^0_{1,2}, x_2, d^1_{1,2}, x_3, d^1_{1,1})$, where x'_1 is a cell of dimension 1. Here $\gamma \leftrightarrows^r \beta$ with $r = 1$, so that γ and β are similar paths.

While similarity is inherently symmetric, subsumption is directional and cannot be treated symmetrically without loss of structure. Making this distinction explicit is a key technical step of the present work and directly enables our novel formulation of bisimilarity. In particular, it allows us to define several variants of hereditary history-preserving bisimilarity (in Section 5) in which subsumption is required to hold only in a single direction.

4 ST-trace

Fix a down-step $(x_\ell, d^1_{i_{\ell+1}}, x_{\ell+1})$ occurring in a path

$$\alpha = (x_0, \varphi_1, x_1, \ldots, \varphi_\ell, x_\ell, d^1_{i_{\ell+1}}, x_{\ell+1}, \varphi_{\ell+2}, \ldots, x_m)$$

in an HDA. By the text above Definition 21 in [12], there exists a unique index $k \leq \ell$ such that

$$\alpha \overset{\ell}{\leftrightsquigarrow} \alpha^{(\ell)} \overset{\ell-1}{\leftrightsquigarrow} \alpha^{(\ell-1)} \overset{\ell-2}{\leftrightsquigarrow} \cdots \overset{k+1}{\leftrightsquigarrow} \alpha^{(k+1)} \overset{k}{\not\leftrightsquigarrow} \alpha^{(k)},$$

where each step is obtained by applying the unique adjacency replacement at the indicated position.

This means that the down-step can be moved successively towards the beginning of the path by adjacency replacements in the sense of Definition 3.4, until it sits immediately after its matching up-step and cannot be moved further.

More explicitly, the down-step $d^1_{i_{\ell+1}}$ can be commuted one position to the left precisely when the adjacent segment $(d^k_{i_\ell}, x_\ell, d^1_{i_{\ell+1}})$ satisfies $k = 1$ or matches one of the subsumption patterns of Definition 3.3, that is, when $i_\ell \neq i_{\ell+1}$.

The process stops exactly at position k because this condition fails: in $\alpha^{(k+1)}$ the adjacent segment has the form $(d^0_{i_k}, x_{k+1}, d^1_{i_k})$, where the indices coincide and no adjacency replacement is applicable. We then write $\mathsf{start}(\ell{+}1) = k$. It says that the down-step $d^1_{i_{\ell+1}}$ at position $\ell{+}1$ marks the end of an event that started with the up-step $d^0_{i_k}$ at position k.

▶ **Definition 4.1** (ST-trace [12])**.** Let $\alpha = (x_0, d^{k_1}_{i_1}, x_1, \ldots, d^{k_n}_{i_n}, x_n) \in \mathrm{Path}_{\mathcal{X}}$, and let $\lambda(i_j)$ denote the label of the event whose start or termination is represented by the face map $d^{k_j}_{i_j}$. For each step define

$$\sigma^{ST}_j := \begin{cases} + & \text{if } k_j = 0, \\ start(j) & \text{if } k_j = 1. \end{cases}$$

Then $\mathsf{ST}(\alpha) = (\lambda(i_1)^{\sigma^{ST}_1}, \ldots, \lambda(i_n)^{\sigma^{ST}_n})$.

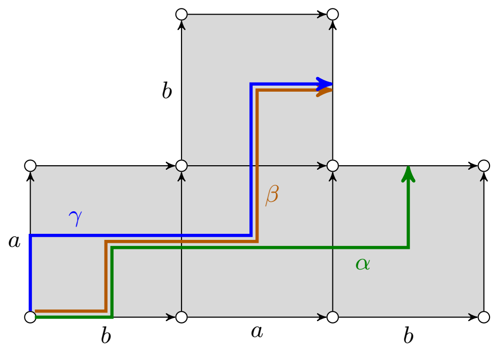

Figure 2 Examples of paths in an HDA.

The *ST-trace* $\mathsf{ST}(\alpha)$ records not only when each action starts (+) and terminates, but also links each termination to the position of its corresponding start via the index *start*(j). In this way, the ST-trace encodes the causal pairing between the beginning and ending of individual actions, capturing their overlap and nesting (see [12] for the original formulation).

▶ **Example 4.2.** The paths depicted in Fig. 2 have all different ST-traces:

$$\mathsf{ST}(\alpha) = b^+a^+b^1a^+a^4b^+a^2, \quad \mathsf{ST}(\beta) = b^+a^+b^1a^+a^2b^+a^4, \quad \mathsf{ST}(\gamma) = a^+b^+b^2a^+a^1b^+a^4.$$

5 Bisimulations for Higher Dimensional Automata

In this section, we review the main notions of bisimilarity defined for HDAs. Additionally, we introduce several new bisimilarities that lie between history-preserving bisimilarity and ST-bisimilarity in the spectrum established in [13].

Hereditary history preserving bisimilarity (hhp-bisimilarity) and history preserving bisimilarity (hp-bisimilarity) are equivalence notions that were originally introduced for other concurrency models [13, 23, 6], and were later adapted to higher-dimensional automata (HDA) [12]. Within the settings of this work, (h)hp-bisimilarity is defined as follows.

▶ **Definition 5.1.** A *history-preserving bisimulation* (*hp-bisimulation*) between HDAs $\mathcal{Y}$ and $\mathcal{Z}$ is a symmetric relation $\mathfrak{R}$ between paths of $\mathcal{Y}$ and $\mathcal{Z}$ such that:

1. *Initial condition.* $(i_{\mathcal{Y}}, i_{\mathcal{Z}}) \in \mathfrak{R}$. Here $i_{\mathcal{Y}}$ is the unique path in $\mathcal{Y}$ of length 0.
2. *Label equality.* If $(\alpha, \beta) \in \mathfrak{R}$, then $\mathsf{ST}(\alpha) = \mathsf{ST}(\beta)$.
3. *Path extension.* If $(\alpha, \beta) \in \mathfrak{R}$ and $\alpha \hookrightarrow \alpha'$ (i.e. α is an initial subpath of α'), then there exists β' such that $\beta \hookrightarrow \beta'$ and $(\alpha', \beta') \in \mathfrak{R}$.
4. *Subsumption.* If $(\alpha, \beta) \in \mathfrak{R}$ and $\alpha \sqsubseteq^\ell \alpha'$, then there exists β' such that $\beta \sqsubseteq^\ell \beta'$ and $(\alpha', \beta') \in \mathfrak{R}$.
5. *Unsubsumption.* If $(\alpha, \beta) \in \mathfrak{R}$ and $\alpha' \sqsubseteq^\ell \alpha$, then there exists β' with $\beta' \sqsubseteq^\ell \beta$ and $(\alpha', \beta') \in \mathfrak{R}$.
6. *Similarity.* If $(\alpha, \beta) \in \mathfrak{R}$ and $\alpha' \leftrightarrows^\ell \alpha$, then there exists β' such that $\beta' \leftrightarrows^\ell \beta$ and $(\alpha', \beta') \in \mathfrak{R}$.

The relation $\mathfrak{R}$ is called a *hereditary history-preserving bisimulation* (*hhp-bisimulation*) if, in addition, it satisfies:

7. *Path restriction.* If $(\alpha, \beta) \in \mathfrak{R}$ and $\alpha' \hookrightarrow \alpha$, then there exists $\beta' \hookrightarrow \beta$ such that $(\alpha', \beta') \in \mathfrak{R}$.

We say that $\mathcal{X}$ and $\mathcal{Y}$ are *(hereditary) history preserving bisimilar* (*(h)hp-bisimilar*), denoted $\mathcal{X} \leftrightarroweq_{(h)hp} \mathcal{Y}$, if there exists a (hereditary) history-preserving bisimulation ((h)hp-bisimulation) between them. This clearly is an equivalence relation.

We will address the clauses 3–7 of Definition 5.1 by the symbols $\hookrightarrow$, $\sqsubseteq$, $\sqsupseteq$, $\leftrightarrows$ and $\hookleftarrow$, respectively. An *ST-bisimulation* between HDAs $\mathcal{Y}$ and $\mathcal{Z}$ is a symmetric relation $\mathfrak{R}$ between paths of $\mathcal{Y}$ and $\mathcal{Z}$ satisfying Clauses 1–3 of Definition 5.1. We say that $\mathcal{X}$ and $\mathcal{Y}$ are *ST-bisimilar*, denoted $\mathcal{X} \mathrel{\underline{\leftrightarrow}}_{ST} \mathcal{Y}$, if an ST-bisimulation between them exists. So by definition $\mathrel{\underline{\leftrightarrow}}_{hhp} \subseteq \mathrel{\underline{\leftrightarrow}}_{hp} \subseteq \mathrel{\underline{\leftrightarrow}}_{ST}$.

From Definition 5.1 we derive 2^4 bisimulation equivalences that are included between $\mathrel{\underline{\leftrightarrow}}_{ST}$ and $\mathrel{\underline{\leftrightarrow}}_{hhp}$. Namely for any subset $I \subseteq \{\sqsubseteq, \sqsupseteq, \leftrightarrows, \hookleftarrow\}$ we define an I-bisimulation as a relation between the paths of two HDAs that satisfies Clauses 1, 2 and 3 of Def. 5.1, as well as the clauses from I; two HDAs are I-bisimilar iff an I-bisimulation between them exists. It is trivial to check that all these notions are in fact equivalence relations.

Thus $\emptyset$-bisimilarity is ST-bisimilarity, $\{\sqsubseteq, \sqsupseteq, \leftrightarrows, \hookleftarrow\}$-bisimilarity is hhp-bisimilarity and $\{\sqsubseteq, \sqsupseteq, \leftrightarrows\}$ is hp-bisimilarity. We give names to six more of these equivalence notions that may yield new and different equivalences on HDAs: *semi–history-preserving (shp) bisimilarity* ($\mathrel{\underline{\leftrightarrow}}_{shp}$) has $I = \{\sqsubseteq\}$ and *quasi–history-preserving (qhp) bisimilarity* ($\mathrel{\underline{\leftrightarrow}}_{qhp}$) has $I = \{\sqsubseteq, \leftrightarrows\}$; in both cases we get *hereditary* versions ($\mathrel{\underline{\leftrightarrow}}_{hshp}$ and $\mathrel{\underline{\leftrightarrow}}_{hqhp}$) by adding clause $\hookleftarrow$. We denote $I = \{\leftrightarrows, \hookleftarrow\}$-bisimilarity as $\mathrel{\underline{\leftrightarrow}}_{reST}$ and $I = \{\leftrightarrows, \hookleftarrow\}$-bisimilarity as $\mathrel{\underline{\leftrightarrow}}_{ureST}$; here the letters r, e and u stand for "path *r*estriction", "similarity *e*quivalence" and "*u*nsubsumption".

The following table summarises which conditions are satisfied by each of these bisimilarities.

Condition	$\mathrel{\underline{\leftrightarrow}}_{ST}$	$\mathrel{\underline{\leftrightarrow}}_{reST}$	$\mathrel{\underline{\leftrightarrow}}_{ureST}$	$\mathrel{\underline{\leftrightarrow}}_{shp}$	$\mathrel{\underline{\leftrightarrow}}_{hshp}$	$\mathrel{\underline{\leftrightarrow}}_{qhp}$	$\mathrel{\underline{\leftrightarrow}}_{hqhp}$	$\mathrel{\underline{\leftrightarrow}}_{hp}$	$\mathrel{\underline{\leftrightarrow}}_{hhp}$
Initial paths are related	✓	✓	✓	✓	✓	✓	✓	✓	✓
Labels of related paths	ST	ST	ST	ST	ST	ST	ST	ST	ST
Extension	✓	✓	✓	✓	✓	✓	✓	✓	✓
Subsumption				✓	✓	✓	✓	✓	✓
Similarity		✓	✓			✓	✓	✓	✓
Unsubsumption			✓					✓	✓
Path restriction		✓	✓		✓		✓		✓

Now we establish the relationships between the various notions of equivalence that were defined above. Our findings are summarised in Fig.3.

▶ **Theorem 5.2.** If $\mathcal{X} \mathrel{\underline{\leftrightarrow}}_{ST} \mathcal{Y}$ then $\mathcal{X}$ and $\mathcal{Y}$ are related by a $\{\hookleftarrow\}$-bisimulation, that is, an ST-bisimulation that also satisfies path restriction.

Proof. Assume that there exists an ST-bisimulation $\mathfrak{K}$ between $\mathcal{X}$ and $\mathcal{Y}$. Let $\mathfrak{R}$ be the relation defined by

$$(\alpha, \beta) \in \mathfrak{R} \iff (\alpha', \beta') \in \mathfrak{K} \text{ for all } \alpha', \beta' \text{ such that } \alpha' \hookrightarrow \alpha,\ \beta' \hookrightarrow \beta \text{ and } |\alpha'| = |\beta'|.$$

Note that in Clause 3 (path extension) of Def. 5.1 one can without limitation of generality restrict to the case where $|\alpha'| = |\alpha| + 1$, that is, α' extends α by exactly one pair (φ, x) of a coface map and a cell. Using this, all Clauses 1–3 of Def. 5.1 hold for $\mathfrak{R}$ because they hold for $\mathfrak{K}$. Moreover, Clause 7 (path restriction) holds by construction. ◀

▶ **Theorem 5.3.** If $\mathcal{X} \mathrel{\underline{\leftrightarrow}}_{ST} \mathcal{Y}$ then $\mathcal{X}$ and $\mathcal{Y}$ are related by a $\{\sqsupseteq, \leftrightarrows\}$-bisimulation, that is, an ST-bisimulation that also satisfies unsubsumption and similarity.

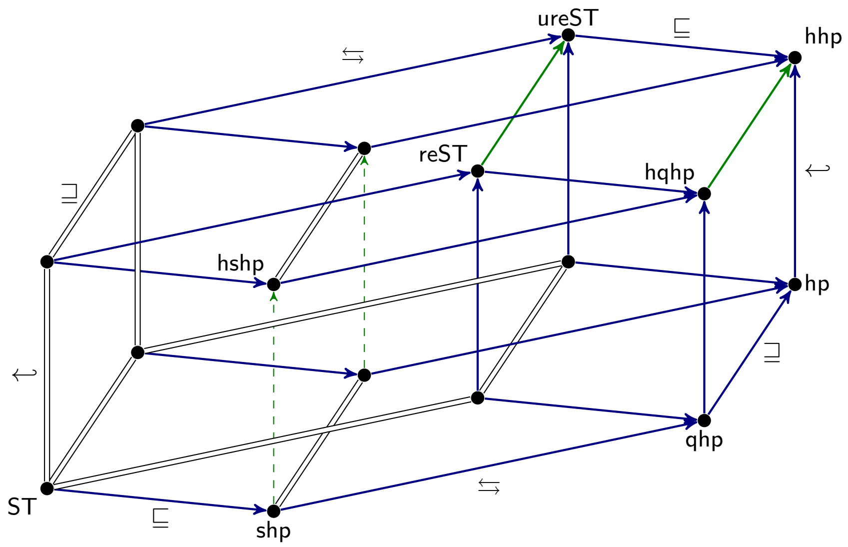

Figure 3 Hierarchy of bisimulations on HDAs. Blue arrows point to finer equivalences, double lines denote coinciding equivalences, dashed green lines are conjectured coincidences. Green arrows point to finer equivalences that are not proven in this paper.

Proof. Let $\mathfrak{K} \subseteq \mathrm{Path}_{\mathcal{X}} \times \mathrm{Path}_{\mathcal{Y}}$ be an ST-bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Define a relation $\mathfrak{R} \subseteq \mathrm{Path}_{\mathcal{X}} \times \mathrm{Path}_{\mathcal{Y}}$ as the least relation with $\mathfrak{K} \subseteq \mathfrak{R}$ that satisfies the following closure conditions:

$$\begin{aligned} &\text{(Similarity)} && (\alpha,\beta) \in \mathfrak{R},\ \alpha \leftrightarrows^{\ell} \alpha',\ \beta \leftrightarrows^{\ell} \beta' \implies (\alpha',\beta') \in \mathfrak{R},\\ &\text{(Unsubsumption)} && (\alpha,\beta) \in \mathfrak{R},\ \alpha' \sqsubseteq^{\ell} \alpha,\ \beta' \sqsubseteq^{\ell} \beta \implies (\alpha',\beta') \in \mathfrak{R}. \end{aligned}$$

It suffices to show that $\mathfrak{R}$ is an ST-bisimulation. Clause 1. (the initial paths are related) is valid by construction since $\mathfrak{K} \subseteq \mathfrak{R}$. The remaining conditions need to be shown for each pair $(\alpha,\beta) \in \mathfrak{R}$; we will do so by induction on the derivation of $(\alpha,\beta) \in \mathfrak{R}$ from the closure conditions above. The induction base is that $(\alpha,\beta) \in \mathfrak{K}$, and in this case the conditions hold because they hold for $\mathfrak{K}$. For the induction step, we may assume that the required conditions have been shown already to hold for the pair $(\alpha,\beta) \in \mathfrak{R}$, and now consider a pair $(\alpha',\beta') \in \mathfrak{R}$ obtained by one of the above closure conditions.

For Clause 2. (preservation of ST-traces), note that if $\alpha \leftrightarrows^{\ell} \alpha'$, then $\mathsf{ST}(\alpha')$ is completely determined by ℓ and $\mathsf{ST}(\alpha)$, and the same applies when $\alpha' \sqsubseteq^{\ell} \alpha$. Hence from the induction hypothesis that $\mathsf{ST}(\alpha) = \mathsf{ST}(\beta)$ one immediately derives that $\mathsf{ST}(\alpha') = \mathsf{ST}(\beta')$.

For Clause 3 (path extension), let $\alpha' \hookrightarrow \alpha' * \gamma$. Then $\alpha \hookrightarrow \alpha * \gamma$, so by Clause 3 for the pair $(\alpha,\beta) \in \mathfrak{R}$, there exists a path δ such that $\beta \hookrightarrow \beta * \delta$ and $(\alpha * \gamma, \beta * \delta) \in \mathfrak{R}$. Now $\beta' * \delta$ is an extension of β' and $(\alpha' * \gamma, \beta' * \delta) \in \mathfrak{R}$ by the closure properties of $\mathfrak{R}$.

It remains to show that $\mathfrak{R}$ satisfies Clauses 5 and 6 (unsubsumption and similarity). So suppose that $(\alpha,\beta) \in \mathfrak{R}$ and $\alpha' \sqsupseteq^{\ell} \alpha$. This means that right before position ℓ in the path α one event started and right after ℓ another event ended, while in α' these two action phases are reversed. Now since $\mathsf{ST}(\beta) = \mathsf{ST}(\alpha)$, in β the situation is similar, so there must be a β' such that $\beta' \sqsupseteq^{\ell} \beta$. The closure property of $\mathfrak{R}$ for unsubsumption yields that $(\alpha',\beta') \in \mathfrak{R}$. The case for similarity goes likewise. ◀

The above two theorems imply that $\{\sqsupseteq\}$-, $\{\leftrightarrows\}$-, $\{\sqsupseteq,\leftrightarrows\}$- and $\{\hookleftarrow\}$-bisimilarity coincide with ST-bisimilarity. Interestingly, although adding to ST-bisimilarity one of the clauses $\leftrightarrows$ or $\hookleftarrow$ makes no difference, adding both of them yields a strictly finer equivalence ($\underline{\leftrightarrow}_{reST}$). This will be shown by Thm. 5.9.

▶ **Theorem 5.4.** If $\mathcal{X} \underline{\leftrightarrow}_{shp} \mathcal{Y}$ then $\mathcal{X}$ and $\mathcal{Y}$ are related by a $\{\sqsubseteq,\sqsupseteq\}$-bisimulation, that is, an shp-bisimulation that also satisfies unsubsumption.

Proof. Let $\mathfrak{K} \subseteq \text{Path}_\mathcal{X} \times \text{Path}_\mathcal{Y}$ be an shp-bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Define $\mathfrak{R}$ by closing $\mathfrak{K}$ under unsubsumption, just as in the proof of Thm. 5.3. That $\mathfrak{R}$ satisfies Clauses 1–3 and 5 (unsubsumption) of Def. 5.1 follows exactly as in the proof of Thm. 5.3.

For Clause 4 (subsumption), let $\alpha' \sqsubseteq^k \alpha''$. Recall that by induction the pair $(\alpha,\beta) \in \mathfrak{R}$ is supposed to satisfy subsumption, and we now want to show this for the pair $(\alpha',\beta') \in \mathfrak{R}$ obtained using that $\alpha' \sqsubseteq^\ell \alpha$ and $\beta' \sqsubseteq^\ell \beta$. The special case that $k = \ell$ yields $\alpha'' = \alpha$, so picking $\beta'' := \beta$ finishes the argument. Now consider the case $k \neq \ell$. Given that $\sqsupseteq$ moves a start-phase of an event past the termination of another event, while $\sqsubseteq$ does the opposite, the modifications $\alpha \sqsupseteq^\ell \alpha' \sqsubseteq^k \alpha''$ do not interfere with each other, i.e., $|k-\ell| \geq 2$, so there must be a path α''' such that $\alpha \sqsubseteq^k \alpha''' \sqsupseteq^\ell \alpha''$. Since $(\alpha,\beta) \in \mathfrak{R}$ satisfies subsumption, there exists a path β''' such that $\beta \sqsubseteq^k \beta'''$ and $(\alpha''',\beta''') \in \mathfrak{R}$. Since $\mathfrak{R}$ is closed under unsubsumption, there is a β'' such that $\beta''' \sqsupseteq^\ell \beta''$ and $(\alpha'',\beta'') \in \mathfrak{R}$. Using that $|k-\ell| \geq 2$, there must be a $\beta^\dagger$ such that $\beta \sqsupseteq^\ell \beta^\dagger \sqsubseteq^k \beta''$. When $\beta \sqsupseteq^\ell \beta^\dagger$, there is in fact no choice for $\beta^\dagger$; it is completely determined by β and ℓ. Hence $\beta^\dagger = \beta$, which finishes the proof. ◀

▶ **Theorem 5.5.** If $\mathcal{X} \underline{\leftrightarrow}_{\{\sqsubseteq,\hookleftarrow\}} \mathcal{Y}$ then $\mathcal{X} \underline{\leftrightarrow}_{\{\sqsubseteq,\sqsupseteq,\hookleftarrow\}} \mathcal{Y}$.

Proof. Let $\mathfrak{K} \subseteq \text{Path}_\mathcal{X} \times \text{Path}_\mathcal{Y}$ be an $\{\sqsubseteq,\hookleftarrow\}$-bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Define $\mathfrak{R}$ by closing $\mathfrak{K}$ under unsubsumption, just as in the proof of Thm. 5.4. That $\mathfrak{R}$ satisfies Clauses 1–5 of Def. 5.1 follows exactly as in the proof of Thm. 5.4.

For Clause 7 (path restriction), let $\alpha'' \hookrightarrow \alpha'$. Here $\alpha' \sqsubseteq^\ell \alpha$ and $\beta' \sqsubseteq^\ell \beta$, and by induction the pair $(\alpha,\beta) \in \mathfrak{R}$ is supposed to satisfy path restriction. Let β'' be the unique path such that $|\alpha''| = |\beta''|$ and $\beta'' \hookrightarrow \beta$. We need to show that $(\alpha'',\beta'') \in \mathfrak{R}$.

In case $|\alpha''| < \ell$, meaning that we restrict α' to below the place where it differs from α, we have $\alpha'' \hookrightarrow \alpha$. Hence there exists a $\beta^\dagger \hookrightarrow \beta$ such that $(\alpha'',\beta^\dagger) \in \mathfrak{R}$. Since $|\beta''| = |\beta^\dagger| < \ell$ we have $\beta\dagger = \beta''$, so we are done.

In case $|\alpha''| > \ell$, there exists an $\alpha^\dagger \hookrightarrow \alpha$ with $\alpha'' \sqsubseteq^\ell \alpha^\dagger$. By induction there is a $\beta^\dagger \hookrightarrow \beta$ with $(\alpha^\dagger,\beta^\dagger) \in \mathfrak{R}$, and since $\mathfrak{R}$ is closed under unsubsumption there is a $\beta^\ddagger$ such that $\beta^\ddagger \sqsubseteq^\ell \beta^\dagger$ and $(\alpha'',\beta^\ddagger) \in \mathfrak{R}$. It must be that $\beta^\dagger = \beta''$, so again we are done.

The remaining case that $|\alpha''| = \ell$ is the most interesting. Let $\alpha''' \hookrightarrow \alpha'$ and $\beta''' \hookrightarrow \beta'$ be such that $|\alpha'''| = |\beta'''| = \ell - 1$. By the above, $(\alpha''',\beta''') \in \mathfrak{R}$. Since $\alpha''' \hookrightarrow \alpha''$ there exists a $\beta''' \hookrightarrow \beta^\dagger$ such that $(\alpha'',\beta^\ddagger) \in \mathfrak{R}$, using Clause 4. We also have $\beta''' \hookrightarrow \beta''$ and $|\beta^\ddagger| = |\beta''|$. Since the extension of $\beta^\ddagger$ w.r.t. β''' is the end-phase of a well-defined event occurring in β''', it is uniquely determined by β'''. Hence $\beta^\dagger = \beta''$. ◀

▶ **Theorem 5.6.** If $\mathcal{X} \underline{\leftrightarrow}_{\{\hookleftarrow\}} \mathcal{Y}$ then $\mathcal{X} \underline{\leftrightarrow}_{\{\sqsupseteq,\hookleftarrow\}} \mathcal{Y}$.

Proof. This follows by a straightforward simplification of the proof above. ◀

▶ **Theorem 5.7.** $\underline{\leftrightarrow}_{shp} \subsetneq \underline{\leftrightarrow}_{ST}$. In fact, even $\underline{\leftrightarrow}_{shp} \not\supseteq \underline{\leftrightarrow}_{ureST}$.

Proof. The inclusion follows directly from the definitions: every shp-bisimulation satisfies all clauses required for ST-bisimilarity, since ST-bisimilarity can be obtained from shp-bisimilarity by dropping Clause 4 (subsumption).

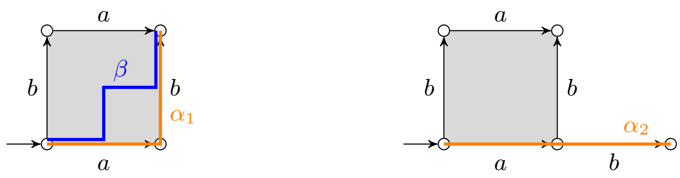

Figure 4 Two HDA $\mathcal{Y}_1$ on the left and $\mathcal{Y}_2$ on the right that are ST-bisimilar.

Its strictness is established by the example of the HDAs $\mathcal{Y}_1$ and $\mathcal{Y}_2$ illustrated in Figure 4. For ST-bisimilarity, the relation $\mathfrak{R}$ given by

$$(\alpha, \beta) \in \mathfrak{R} \quad \Longleftrightarrow \quad \mathsf{ST}(\alpha) = \mathsf{ST}(\beta).$$

is easily seen to be an ST-bisimulation, in that it satisfies the first three conditions of Def. 5.1. In fact, it is even a $\{\sqsupseteq, \leftrightarrows, \hookleftarrow\}$-bisimulation. Note that it relates the highlighted paths α_1 and α_2. Thus $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are ST-bisimilar.

By definition, if an shp-bisimulation $\mathfrak{R}$ exists between $\mathcal{Y}_1$ and $\mathcal{Y}_2$, then $(\alpha_1, \alpha_2) \in \mathfrak{R}$. This is because the initial paths of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ must be related, and if we extend the initial path of $\mathcal{Y}_2$ to α_2, the only path in $\mathcal{Y}_1$ with the same ST-trace is α_1. However, there exists a path $\beta \in \mathrm{Path}_{Y_1}$, indicated in blue, such that $\alpha_1 \sqsubseteq \beta$, while no path $\beta' \in \mathrm{Path}_{Y_2}$ satisfies $\alpha_2 \sqsubseteq \beta'$, which violates condition 4 of an shp-bisimulation. ◀

▶ **Theorem 5.8.** $\mathrel{\underline{\leftrightarrow}}_{qhp} \subsetneq \mathrel{\underline{\leftrightarrow}}_{shp}$. In fact, even $\mathrel{\underline{\leftrightarrow}}_{qhp} \not\supseteq \mathrel{\underline{\leftrightarrow}}_{\{\sqsubseteq, \sqsupseteq, \hookleftarrow\}}$.

Proof. The inclusions follow directly from the semantics of the relations in question.

To prove strictness, consider the HDAs $\mathcal{X}$ and $\mathcal{Y}$ in Fig. 5. Suppose, towards a contradiction, that there exists an qhp-bisimulation $\mathfrak{R}$ between $\mathcal{X}$ and $\mathcal{Y}$. By Clause 1, $(i_{\mathcal{X}}, i_{\mathcal{Y}}) \in \mathfrak{R}$. Now $i_{\mathcal{Y}}$ can be extended to the path β, indicated in orange in Fig. 5. So by Clauses 3 and 2, there must be a path α in $\mathcal{X}$ such that $(\alpha, \beta) \in \mathfrak{R}$ and $\mathsf{ST}(\alpha) = a^+c^+a^1c^2b^+d^+b^5d^6$. The only two candidates, α_1 and α_2, are indicated in orange and purple respectively.

In $\mathcal{Y}$ we have a subsumption $\beta \sqsubseteq^4 \beta'$, indicated in blue. Clause 4 therefore forces the existence of a path α' in $\mathcal{X}$ such that $\alpha \sqsubseteq^4 \alpha'$ and $(\alpha', \beta') \in \mathfrak{R}$. However, in $\mathcal{X}$ the path α_2 has no subsumption successor of this form (again by inspection of Fig. 5), and hence is disqualified as candidate for α.

In $\mathcal{Y}$ we have a similarity $\beta \leftrightarrows^3 \beta'' \leftrightarrows^5 \beta'''$, obtained by swapping a^- and c^-, as well as b^+ and d^+, and a subsumption $\beta''' \sqsubseteq^4 \beta''''$, where β'''' is indicated in green in Fig. 5. Clauses 6 and 4 therefore force the existence of a paths $\alpha'', \alpha''', \alpha''''$ in $\mathcal{X}$ such that $\alpha_1 \leftrightarrows^3 \alpha'' \leftrightarrows^5 \alpha''' \sqsubseteq^4 \alpha''''$ and $(\alpha'''', \beta'''') \in \mathfrak{R}$. However, whereas α'' and α''' can be found, there is no such α''''. This contradiction shows that $\mathcal{X} \not\mathrel{\underline{\leftrightarrow}}_{qhp} \mathcal{Y}$.

Why $\mathcal{X} \mathrel{\underline{\leftrightarrow}}_{shp} \mathcal{Y}$. We now exhibit an shp-bisimulation $\mathfrak{S} \subseteq \mathrm{Path}_X \times \mathrm{Path}_Y$. Each path in $\mathcal{Y}$ that does not pass through the middle state is related to the unique path in $\mathcal{X}$ that has the same ST-trace. For paths that do go through the middle state, if from there they enter a wing of $\mathcal{Y}$, the matching path of $\mathcal{X}$ enters the same wing of $\mathcal{X}$; and if the first move after the middle state doesn't enter a wing, neither does the corresponding path of $\mathcal{X}$. The continuations of these paths after that move are completely determined by their ST-traces. It is trivial to check that $\mathfrak{S}$ satisfies all clauses of a shp-bisimulation. In fact, it is even an $\{\sqsubseteq, \sqsupseteq, \hookleftarrow\}$-bisimulation.

Note that $\mathfrak{S}$ relates α_1 with the path β considered above. Yet, the counterexample cannot be completed, because an shp-bisimulation lacks the similarity clause. ◀

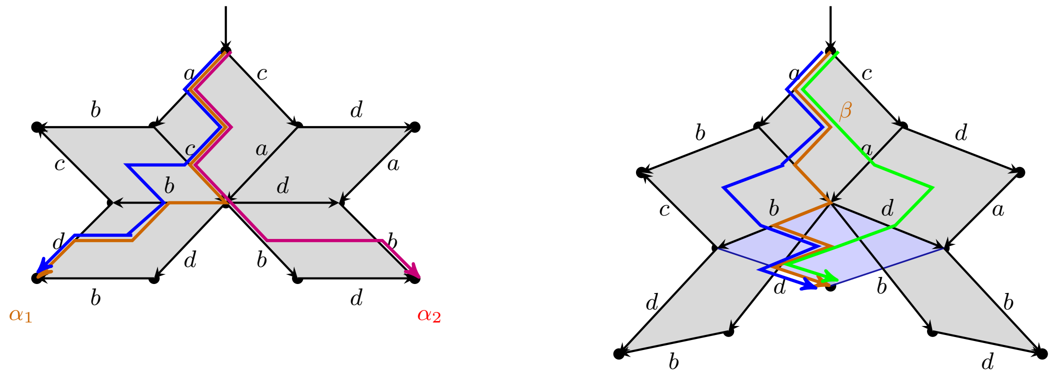

Figure 5 Two dimensional HDAs $\mathcal{X}$ on the left and $\mathcal{Y}$ on the right that are shp-bisimilar but not qhp-bisimilar (start vertex marked by an incoming arrow).

▶ **Theorem 5.9.** $\underline{\leftrightarrow}_{reST} \subsetneq \underline{\leftrightarrow}_{ST}$ and $\underline{\leftrightarrow}_{hhp} \subsetneq \underline{\leftrightarrow}_{hp}$. In fact, even $\underline{\leftrightarrow}_{hp} \not\subseteq \underline{\leftrightarrow}_{\{\leftrightarrows, \hookleftarrow\}} = \underline{\leftrightarrow}_{reST}$ and $\underline{\leftrightarrow}_{hshp} = \underline{\leftrightarrow}_{\{\sqsubseteq, \sqsupseteq, \hookleftarrow\}} \not\subseteq \underline{\leftrightarrow}_{reST}$.

Proof. The inclusions are immediate from the definitions. Strictness follows from the absorption-law example in Fig. 6. This example is used in [13, 4] to separate the hereditary and non-hereditary variants, and the same argument applies verbatim here. To distinguish these HDAs one needs both $\leftrightarrows$ and $\hookleftarrow$. For completeness, we will give an independent separation argument later in Section 6, exhibiting a distinguishing $\mathcal{L}_{reST}$-formula for the two systems. ◀

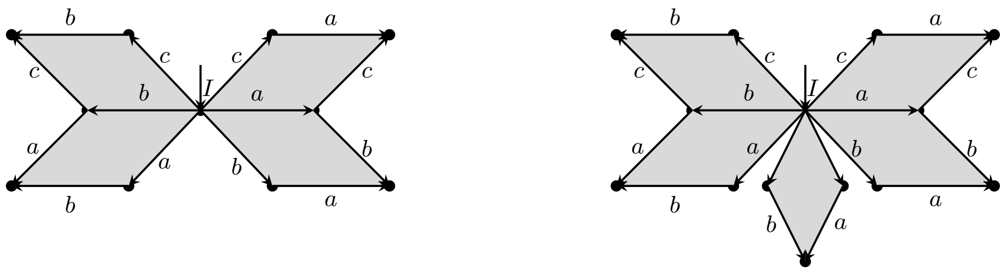

Figure 6 The absorption law from [4, 13] depicting two systems $E_1 = \big(a \mid (b+c)\big) + \big(b \mid (a+c)\big)$ on the left and $E_2 = \big(a \mid (b+c)\big) + a \mid b + \big(b \mid (a+c)\big)$ on the right. These HDA are hp-bisimilar as well as $\{\sqsubseteq, \sqsupseteq, \hookleftarrow\}$-bisimilar, but not reST-bisimilar.

In order to show the difference between $\underline{\leftrightarrow}_{qhp}$- and $\underline{\leftrightarrow}_{hp}$-bisimilarity, let $\mathcal{X}_n$ for $n \geq 1$ be the HDA that is obtained by gluing HDAs $\mathcal{A}$ and $\mathcal{B}_1, \dots, \mathcal{B}_n$ displayed in Fig. 7. For the following observations, Figure 8 is helpful.

▶ **Observation 5.10.** $\mathrm{Path}_{\mathcal{X}_n} = \mathrm{Path}_{\mathcal{X}_1} \cup \bigcup_{r \geq 2} \mathrm{Path}_{\mathcal{B}_r}$.

▶ **Observation 5.11.** Let $\alpha \in \mathrm{Path}_{\mathcal{X}_n}$ be a path that can be obtained from (i, a_r) by applying the closure operations

1. $\hookrightarrow, \sqsubseteq^\ell, \leftrightarrows$,
2. $\hookrightarrow, \sqsupseteq^\ell, \leftrightarrows$, or
3. $\hookrightarrow, \sqsubseteq^\ell, \sqsupseteq^\ell, \hookleftarrow$.

Then $\alpha \in \mathrm{Path}_{\mathcal{X}_r}$ for all r, and $\alpha \in \mathrm{Path}_{\mathcal{B}_{r-1}} \cup \mathrm{Path}_{\mathcal{B}_r}$ for $r \geq 3$.

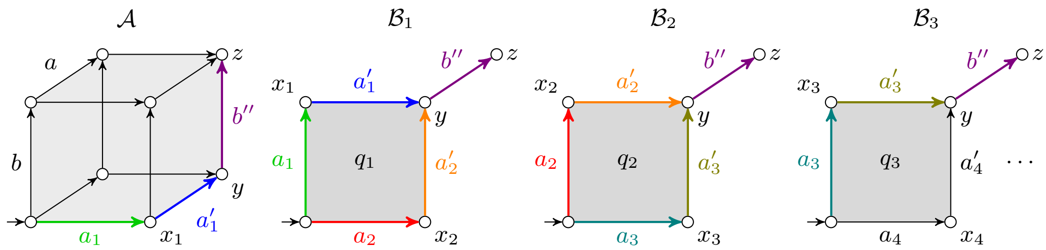

Figure 7 The HDA $\mathcal{X}_n$ is obtained from the disjoint union of HDAs $\mathcal{A}$, $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathfrak{B}_n$ by identification of edges and vertices having the same names. All edges a_i and a'_i are labelled with a, and b'' is labelled by b.

▶ **Proposition 5.12.** The HDAs $\mathcal{X}_m$ and $\mathcal{X}_n$ for $3 \le m < n$ in Fig. 7 are:
1. not $\{\sqsubseteq, \sqsupseteq, \leftrightarrows\}$-bisimilar (hp).
2. not $\{\hookleftarrow, \leftrightarrows\}$-bisimilar (reST).
3. $\{\sqsubseteq, \leftrightarrows\}$-bisimilar (qhp).
4. $\{\sqsupseteq, \leftrightarrows\}$-bisimilar (= ST).
5. $\{\sqsubseteq, \sqsupseteq, \hookleftarrow\}$-bisimilar (= hshp).

Proof. Note that $\mathcal{X}_m \subseteq \mathcal{X}_n$.

1. Let $\mathfrak{R}$ be a bisimulation between $\mathcal{X}_m$ and $\mathcal{X}_n$. Let ω_r be the path $(i, a_r, x_r, a'_r, y, b'', z)$. Note that ω_1 cannot be related to ω_r for $r \ge 2$: ω_1 can be subsumed by a path having trace $a^+a^1a^+b^+b^4a^3$ while ω_r cannot. Consider a sequence of paths:

$$\omega_r \sqsubseteq^2 \alpha \leftrightarrows^{1,3} \alpha' \sqsupseteq^2 \alpha''.$$

Then $\alpha'' = \omega_{r-1}$ or α_{r+1}. As a consequence, ω_2 cannot be related to any path ω_r for $r \ge 3$. Following this argument we obtain that $\mathcal{X}_m$ and $\mathcal{X}_n$ can be bisimilar only for $m = n$.

2. The argument is similar. Let $\eta_r = (i, a_r)$. Again, η_1 cannot be bisimilar to η_r for $r > 1$: η_1 can be extended to a path with ST-trace a^+b^+ while η_r cannot. Furthermore, the sequence

$$\eta_r \hookrightarrow \beta \leftrightarrows^1 \beta' \hookleftarrow \beta''$$

with $\mathsf{ST}(\beta) = \mathsf{ST}(\beta') = a^+a^+$ and $\mathsf{ST}(\beta'') = a^+$ can only produce $\beta'' = \eta^s$ for $s \in \{r-1, r+1\}$. Thus, we show by induction that $\eta_r \mathfrak{R} \eta_s \implies r = s$.

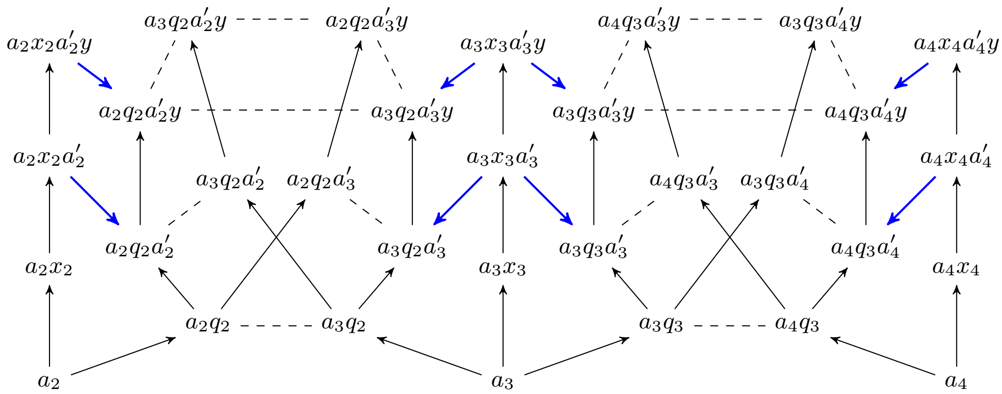

Figure 8 A part of the diagram of paths in $\mathcal{X}_n$. Regular arrows denote path extensions, dashed lines denote similarity and blue arrows denote subsumption. Various $\mathcal{X}_n$'s are bisimilar if and only if the diagram can be traversed sideways using arrows having the given set of types.

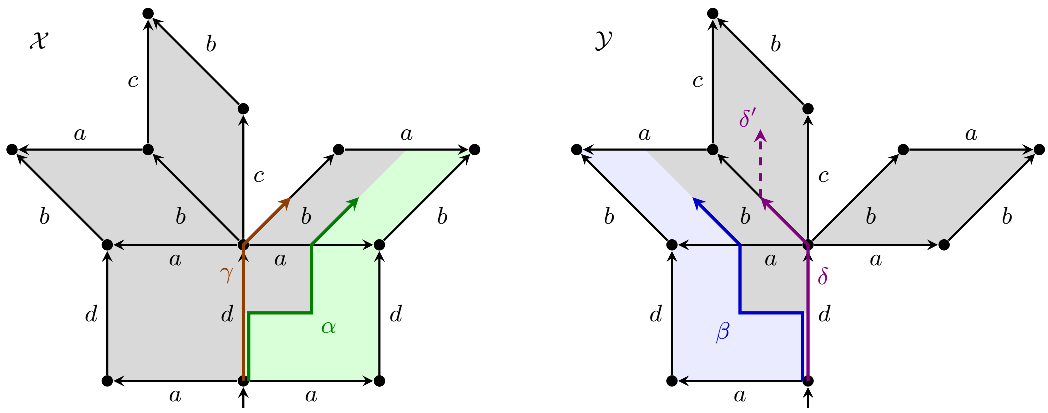

Figure 9 An example of HDAs that are reST-bisimilar but not ureST-bisimilar.

3. Let $\mathcal{P}^n_{\geq 3} \subseteq \mathrm{Path}_{\mathcal{X}_n}$ be the set of paths that can be obtained from (i, a_r), $r \geq 3$, by applying the closure operations $\hookrightarrow$, $\sqsubseteq$ and $\leftrightarrows$. Namely, $\mathcal{P}^n_{\geq 3}$ consists of paths (i, a_r) for $r \geq 3$ and paths of length ≥ 2 that cross x_r for $r \geq 3$ or q_r for $r \geq 2$. Observation 5.11 implies that $\mathcal{P}^n_{\geq 3} \subseteq \bigcup_{r \geq 2} \mathrm{Path}_{\mathcal{B}_r}$. Define a relation $\mathfrak{R} \subseteq \mathrm{Path}_{\mathcal{X}_m} \times \mathrm{Path}_{\mathcal{X}_n}$ by

$$\alpha \mathfrak{R} \beta \iff \alpha = \beta \ \vee\ (\alpha \in \mathcal{P}^m_{\geq 3} \ \wedge\ \beta \in \mathcal{P}^n_{\geq 3} \ \wedge\ \mathsf{ST}(\alpha) = \mathsf{ST}(\beta)).$$

Another consequence of Observation 5.11 is that $\mathrm{Path}_{\mathcal{X}_n} \setminus \mathcal{P}^n_{\geq 3} = \mathrm{Path}_{\mathcal{X}_m} \setminus \mathcal{P}^m_{\geq 3}$; denote it $\mathfrak{A}$. Paths in $\mathfrak{A}$ are never related to paths outside $\mathfrak{A}$.

Assume that $\alpha \mathfrak{R} \beta$ and $\beta \ltimes \beta'$ for $\ltimes \in \{\hookrightarrow, \sqsubseteq, \leftrightarrows\}$. If $\alpha \in \mathfrak{A}$, then $\beta = \alpha$ and we have $\alpha \ltimes \beta'$ (and $\beta' \mathfrak{R} \beta'$).

If $\alpha \in \mathcal{P}^m_{\geq 3}$, then $\beta' \in \mathcal{P}^n_{\geq 3}$. This implies that $\beta, \beta' \in \mathrm{Path}_{\mathcal{B}_s}$ for some $s \geq 2$. Similarly, $\alpha \in \mathrm{Path}_{\mathcal{B}_r}$ for $r \geq 2$. Moreover, there exists a map $f : \mathcal{B}_s \to \mathcal{B}_r$, which is either

$$(q_s, a_s, a_{s+1}, a'_s, a'_{s+1}, x_s, x_{s+1}) \mapsto (q_r, a_r, a_{r+1}, a'_r, a'_{r+1}, x_r, x_{r+1})$$
$$(q_s, a_s, a_{s+1}, a'_s, a'_{s+1}, x_s, x_{s+1}) \mapsto (q_r, a_{r+1}, a_r, a'_{r+1}, a'_r, x_{r+1}, x_r)$$

(an isomorphism with a possible swap), and the identity on $i_{\mathcal{X}_n}$, b'' and z, such that $f(\beta) = \alpha$. We put $\alpha' = f(\beta')$. Clearly $\alpha \ltimes \alpha'$, $\mathsf{ST}(\alpha') = \mathsf{ST}(\beta')$, and $\alpha', \beta' \notin \mathfrak{A}$ so $\alpha' \mathfrak{R} \beta'$.

4., 5. A similar argument applies. ◀

▶ **Corollary 5.13.** $\leftrightarroweq_{hp} \subsetneq \leftrightarroweq_{qhp}$.

▶ **Theorem 5.14.** $\leftrightarroweq_{ureST} \subsetneq \leftrightarroweq_{reST}$.

Proof. Let $\mathcal{X}$, $\mathcal{Y}$ be the HDAs from Figure 9. Note that $\mathcal{Y} \subseteq \mathcal{X}$. Define a relation $\mathfrak{R} \subseteq \mathrm{Path}_{\mathcal{X}} \times \mathrm{Path}_{\mathcal{Y}}$ that contains all pairs (α, α) for $\alpha \in \mathrm{Path}_{\mathcal{Y}}$ and all pairs (α, β), where $\alpha \in \mathrm{Path}_{\mathcal{X}} \setminus \mathrm{Path}_{\mathcal{Y}}$ (green area) and β is the reflection of α along the vertical axis (blue area). Restrictions, extensions and similarities of paths in the green area stay in the green area (or in $\mathrm{Path}_{\mathcal{Y}}$) and then can be matched with similar operations performed in the blue area. Likewise, restrictions, extensions and similarities of paths of $\mathcal{X}$ that lay in $\mathcal{Y}$ stay within $\mathcal{Y}$ (or in the green area) and can be matched by themselves. Thus $\mathfrak{R}$ is a reST-bisimulation.

Assume that $\mathcal{X}$ and $\mathcal{Y}$ are ureST-bisimilar. Then the path $\alpha \in \mathrm{Path}_{\mathcal{X}}$ must be matched by $\beta \in \mathrm{Path}_{\mathcal{Y}}$, which is the only path in $\mathcal{Y}$ that has the same ST-trace. There is a sequence of operations $\beta \sqsupseteq^2 \beta' \leftrightarrows^3 \beta'' \hookleftarrow \delta$ which can be answered only by $\alpha \sqsupseteq^2 \alpha' \leftrightarrows^3 \alpha'' \hookleftarrow \gamma$. But now the extension of δ to δ' cannot be matched by γ. ◀

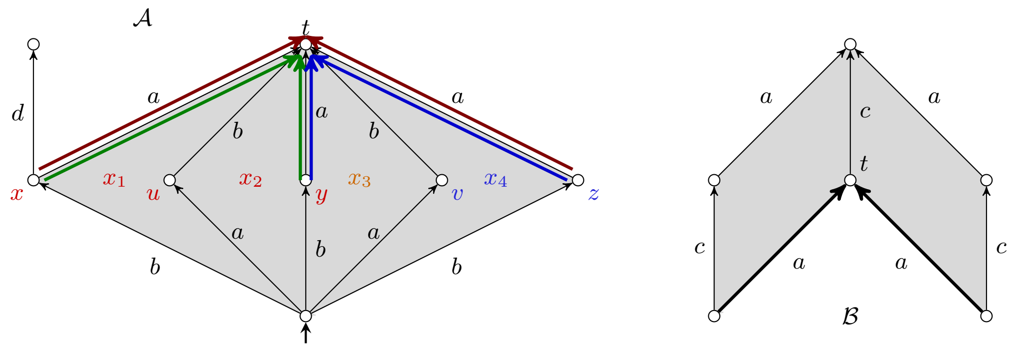

Figure 10 The HDA $\mathcal{X}$ is obtained be gluing $\mathcal{A}$ with two copies of $\mathcal{B}$: $\mathcal{B}_1$ along the edges $x \to t \leftarrow y$ (green) and $\mathcal{B}_2$ along $y \to t \leftarrow z$ (blue). The HDA $\mathcal{Y}$ is obtained by, additionally, gluing an extra copy $\mathcal{B}_3$ along $x \to t \leftarrow z$ (brown).

▶ **Theorem 5.15.** $\leftrightarroweq_{hhp} \subsetneq \leftrightarroweq_{hqhp}$.

Proof. Let $\mathcal{X}$, $\mathcal{Y}$ be HDAs illustrated in Figure 10. First we show that they are not hhp-bisimilar. In $\mathcal{Y}$, every path α with ST-trace b^+b^1 (there are three such paths, terminating in x or y or z) can be extended to a path β with ST-trace $b^+b^1c^+c^3a^+a^5$ (there are two possible extension for each α, we choose the one crossing $\mathcal{B}_1$ or $\mathcal{B}_3$), which in turn can be transformed using similarities, subsumptions and unsubsumptions into a path γ with ST-trace $b^+b^1c^+c^3a^+a^5$ that finally can be restricted to a path δ with ST-trace b^+b^1 and then extended to a (unique) path ζ with ST-trace $b^+b^1d^+$.

This cannot be done for the path α leading to z in $\mathcal{X}$: the only possible extension β terminates at the top vertex of $\mathcal{B}_2$ and then also γ terminates at the same vertex. But the only possible δ terminates at y or z and thus cannot be extended to ζ.

Now we prove that $\mathcal{X}$ and $\mathcal{Y}$ are hqhp-bisimilar. Denote $P_0 = \mathrm{Path}_{\mathcal{A}}$ and $P_i = \mathrm{Path}_{\mathcal{A}\cup\mathcal{B}_i} \setminus \mathrm{Path}_{\mathcal{A}}$. Thus we have $\mathrm{Path}_{\mathcal{X}} = P_0 \sqcup P_1 \sqcup P_2$ and $\mathrm{Path}_{\mathcal{Y}} = P_0 \sqcup P_1 \sqcup P_2 \sqcup P_3$. Further, we call a path red if it passes through x, x_1, u, x_2 or y, orange if it passes through x_3, and blue if it passes through v, x_4 or z. Let $\mathfrak{R} \subseteq \mathrm{Path}_{\mathcal{X}} \times \mathrm{Path}_{\mathcal{Y}}$ be the set of pairs (α, β) such that $\mathsf{ST}(\alpha) = \mathsf{ST}(\beta)$ and

1. If $\beta \in \mathrm{Path}(\mathcal{X})$, then $\alpha = \beta$.
2. If $\beta \in P_3$, then the initial segments of α and β that lie in $\mathcal{A}$ are equal.
3. If $\beta \in P_3$ is red, then $\alpha \in P_1$.
4. If $\beta \in P_3$ is blue, then $\alpha \in P_2$.

These conditions imply that a path β of $\mathcal{Y}$ is related to exactly two paths of $\mathcal{X}$ if $\beta \in P_3$ and β is orange, and to exactly one path of $\mathcal{X}$ otherwise.

We have to check that the symmetric closure of $\mathfrak{R}$ is an hqhp-bisimulation. Initial paths are trivially related, and related paths have the same ST-traces. The path restriction condition holds by construction. To see that the extension property holds it suffices to note that every red path in P_0 can be extended into a path in P_1 and every orange and every blue path in P_0 can be extended into a path in P_2.

The subsumption condition holds because (a) no further subsumption of an orange path is possible (b) applying subsumption to a blue path yields either a blue or an orange path (c) applying subsumption to a red path yields either a red or an orange path. The similarity condition holds because applying similarity to a path does not change its colour.

Note that the unsubsumption condition would fail, as this can turn an orange path into a red or blue one. ◀

6 Generalised bisimulation logic

▶ **Definition 6.1.** Let I and M be sets, and let $m_0 \in M$ be a distinguished element. An (I, M)-*system* is a tuple $(A, \alpha_0, \lambda, \{\ltimes_i\}_{i\in I})$, where

- A is a set (of *paths*) with a distinguished element $\alpha_0 \in A$,
- $\lambda : A \to M$ is a function (a *labelling*) such that $\lambda(\alpha_0) = m_0$,
- $\ltimes_i \subseteq A \times A$ is a relation for every $i \in I$.

▶ **Example 6.2.** Let $\mathcal{X}$ be an HDA. Then $(\mathrm{Path}_{\mathcal{X}}, i_{\mathcal{X}}, \mathsf{ST}, \{\hookrightarrow, \sqsubseteq\})$ is a $(\{1, 2\}, \mathsf{ST})$-system.

▶ **Definition 6.3.** A *bisimulation* between (I, M)-systems $A = (A, \alpha_0, \lambda^A, \{\ltimes_i^A\}_{i\in I})$ and $B = (B, \beta_0, \lambda^B, \{\ltimes_i^B\}_{i\in I})$ is a relation $\mathfrak{R} \subseteq A \times B$ such that

1. $\alpha_0 \mathfrak{R} \beta_0$,
2. $\alpha\mathfrak{R}\beta \implies \lambda^A(\alpha) = \lambda^B(\beta)$,
3. $\alpha\mathfrak{R}\beta \;\wedge\; \alpha \ltimes_i^A \alpha' \implies \exists\, \beta' :\; \alpha'\mathfrak{R}\beta' \;\wedge\; \beta \ltimes_i^B \beta'$,
4. $\alpha\mathfrak{R}\beta \;\wedge\; \beta \ltimes_i^B \beta' \implies \exists\, \alpha' :\; \alpha'\mathfrak{R}\beta' \;\wedge\; \alpha \ltimes_i^A \alpha'$,

for all $\alpha, \alpha' \in A$, $\beta, \beta' \in B$, $i \in I$.

▶ **Definition 6.4.** The set $\mathcal{L}_{(I,M)}$ of (I, M)-*formulas* is generated by

$$F ::= \top \mid \neg F \mid \bigwedge_{j\in J} F_j \mid \Lambda m \mid \langle i\rangle F,$$

where $m \in M$ and $i \in I$. We write $[i]F := \neg\langle i\rangle\neg F$.

▶ **Definition 6.5.** Let P be an (I, M)-system. The *satisfaction relation* $\models\, \subseteq P \times \mathcal{L}_{(I,M)}$ is defined by structural induction:

- $\alpha \models \top$;
- $\alpha \models \neg F \iff \neg(\alpha \models F)$;
- $\alpha \models \bigwedge_{j\in J} F_j \iff \forall\, j \in J :\; \alpha \models F_j$;
- $\alpha \models \Lambda m \iff \lambda(\alpha) = m$;
- $\alpha \models \langle i\rangle F \iff \exists \beta \in P :\; \alpha \ltimes_i \beta \;\wedge\; \beta \models F$.

A formula holds for an (I, M)-system $A = (A, \alpha_0, \lambda^A, \{\ltimes_i^A\}_{i\in I})$ iff it holds for α_0.

▶ **Definition 6.6.** (I, M)-systems P and Q are *logically equivalent* if $\alpha_0^P \models F \iff \alpha_0^Q \models F$ for every $F \in \mathcal{L}_{(I,M)}$.

▶ **Theorem 6.7.** (I, M)-systems P and Q are bisimilar iff they are logically equivalent.

Proof. ($\Rightarrow$) Assume that $\mathfrak{R}$ is a bisimulation between P and Q. By induction on F, we show that for all $F \in \mathcal{L}_{(I,M)}$ the formula

$$E(F) ::= \forall\alpha \in P,\ \forall\beta \in Q :\; \alpha\mathfrak{R}\beta \implies (\alpha \models F \iff \beta \models F).$$

is satisfied. It is clear that $E(\top)$, $E(F) \implies E(\neg F)$ and $(\forall_{j\in J} :\; E(F_j)) \implies E(\bigwedge_{j\in J} F_j)$. The formula $E(\Lambda P)$ follows from condition 6.3.2. Assume that $\alpha \in P$, $\beta \in Q$ and $\alpha\mathfrak{R}\beta$. Then

$$\begin{aligned}
\alpha \models \langle i\rangle F &\iff \exists_{\alpha'\in A}\, \alpha \ltimes_i^A \alpha' \;\wedge\; \alpha' \models F \\
&\implies \exists_{\alpha'\in A}\, \exists_{\beta'\in B}\, \alpha'\mathfrak{R}\beta' \;\wedge\; \beta \ltimes_i^B \beta' \;\wedge\; \alpha' \models F && \text{(by 6.3.3)} \\
&\implies \exists_{\beta'\in B}\, \beta \ltimes_i \beta' \;\wedge\; \beta' \models F && \text{(by inductive assumption)} \\
&\iff \beta \models \langle i\rangle F.
\end{aligned}$$

Similarly we show that $\beta \models \langle i\rangle F$ implies $\alpha \models \langle i\rangle F$. Consequently, $E(F) \implies E(\langle i\rangle F)$.

($\Leftarrow$) Assume that A and B and logically equivalent. Define a relation $\mathfrak{R} \subseteq A \times B$ by

$$\alpha \mathfrak{R} \beta \iff \forall_{F \in \mathcal{L}_{(I,M)}} (\alpha \models F \iff \beta \models F).$$

By assumption, $\alpha_0 \mathfrak{R} \beta_0$. Moreover, if $\alpha \mathfrak{R} \beta$ then for all $m \in M$ one has $\alpha \models \Lambda m$ iff $\beta \models \Lambda m$, and thus $\lambda(\alpha) = m$ iff $\lambda(\beta) = m$. Hence also the second clause of Def. 6.3 is satisfied. Assume that $\alpha \ltimes_i^A \alpha'$ and $\alpha \mathfrak{R} \beta$ for $\alpha, \alpha' \in A$ and $\beta \in B$. We need to show that there exists $\beta' \in B$ such that $\beta \ltimes_i \beta'$ and $\alpha' \mathfrak{R} \beta'$. Denote

$$C = \{\beta'_j\}_{j \in J} = \{\beta' \in B \mid \beta \ltimes_i \beta'\}.$$

If there exists $\beta'_j \in B$ such that $\alpha' \models F \iff \beta'_j \models F$ for all F, then we have $\alpha' \mathfrak{R} \beta'_j$. Otherwise, for every $j \in J$ there exists $F_j \in \mathcal{L}_{(I,M)}$ such that $\alpha' \models F_j$ and $\beta'_j \models \neg F_j$. Finally,

$$\alpha \models \langle i \rangle \bigwedge_{j \in J} F_j \qquad \text{and} \qquad \beta \models \neg \langle i \rangle \bigwedge_{j \in J} F_j,$$

a contradiction. ◀

7 Modal characterisations

Depending on the semantic equivalence we want to focus on, every HDA $\mathcal{X}$ defines an (I, M)-system $(A, \alpha_0, \lambda, \{\ltimes_i\}_{i \in I})$ with $M :=$ the set of ST traces, $A := \mathrm{Path}_\mathcal{X}$, $\alpha_0 := i_\mathcal{X}$, $\lambda := \mathsf{ST} : \mathrm{Path}_\mathcal{X} \to M$, and

- (ST) $I := \{\hookrightarrow\}$,
- (hp) $I := \{\hookrightarrow, \sqsubseteq^\ell, \sqsupseteq^\ell, \leftrightarrows^\ell \mid \ell \in \mathbb{N}\}$,
- (hhp) $I := \{\hookrightarrow, \sqsubseteq^\ell, \sqsupseteq^\ell, \leftrightarrows^\ell, \hookleftarrow \mid \ell \in \mathbb{N}\}$,
- (qhp) $I := \{\hookrightarrow, \sqsubseteq^\ell, \leftrightarrows^\ell \mid \ell \in \mathbb{N}\}$,
- (shp) $I := \{\hookrightarrow, \sqsubseteq^\ell \mid \ell \in \mathbb{N}\}$,

etc., with $\ltimes_\hookrightarrow := \hookrightarrow \subseteq \mathrm{Path}_\mathcal{X} \times \mathrm{Path}_\mathcal{X}$, and likewise for the other elements of I.

By Thm. 6.7, this immediately yields a modal characterisation for each of the eight semantic equivalences above. In the case of hhp-bisimilarity, for instance, the set of formulae is generated by

$$F ::= \top \mid \neg F \mid \bigwedge_{j \in J} F_j \mid \Lambda m \mid \langle \hookrightarrow \rangle F \mid \langle \sqsubseteq^\ell \rangle F \mid \langle \sqsupseteq^\ell \rangle F \mid \langle \leftrightarrows^\ell \rangle F \mid \langle \hookleftarrow \rangle F,$$

with m an ST trace . We address this logic as $\mathcal{L}_{hhp}$, and similarly for its fragments.

▶ **Example 7.1.** We can show that the HDAs of Fig 4 are not shp-bisimilar by the distinguishing $\mathcal{L}_{shp}$ formula $F := \langle \hookrightarrow \rangle \big(\Lambda(a^+ a^1 b^+ b^3) \wedge \neg \langle \sqsubseteq^2 \rangle \top\big)$. Here $\langle \hookrightarrow \rangle \varphi$ is a formula that holds for an HDA iff it has a path that satisfies φ, namely an extension of the initial path. The formula F says that there exists a path with ST-trace $a^+ a^1 b^+ b^3$ (thus executing a and b in succession) that has no subsumption at position 2. It holds for the right-hand HDA, but not for the left one.

▶ **Example 7.2.** The HDAs of Fig. 5 are not qhp-bisimilar, as witnessed by the $\mathcal{L}_{qhp}$ formula $F := \langle \hookrightarrow \rangle \big(\Lambda(a^+ c^+ a^1 c^2 b^+ d^+ b^5 d^6) \wedge \langle \sqsubseteq^4 \rangle \top \wedge \langle \leftrightarrows^3 \rangle \langle \leftrightarrows^5 \rangle \langle \sqsubseteq^4 \rangle \top\big)$. This formula says that there exists a path with the indicated ST-trace, that allows a subsumption at position 4, and also allows similarities at positions 3 and 5, following by a subsumption at position 4. It holds for the right-hand HDA of Fig. 5, but not for the left one.

▶ **Example 7.3.** The two HDAs of Fig. 6 are distinguished by the following $\mathcal{L}_{hhp}$-formula:

$$F := \langle\hookrightarrow\rangle\big(\Lambda(a^+b^+)\wedge\langle\hookleftarrow\rangle(\Lambda(a^+)\wedge\neg\langle\hookrightarrow\rangle\Lambda(a^+c^+))\wedge\langle\leftrightarrows^1\rangle\langle\hookleftarrow\rangle(\Lambda(b^+)\wedge\neg\langle\hookrightarrow\rangle\Lambda(b^+c^+))\big)\ .$$

It says that there exists a path in which first a and then b start, such that (1) the restriction of this path to the start of a cannot be extended with the start of c, and (2) after swapping the starts of a and b, the restriction of this path to the start of b cannot be extended with the start of c. This formula holds only for the right-hand side of Fig. 6. The formulas sits even in $\mathcal{L}_{reST}$.

▶ **Example 7.4.** The HDAs $\mathcal{X}_1$ and $\mathcal{X}_2$ of Fig. 7 can be distinguished with the formula $[\hookrightarrow]\big(\Lambda(a^+)\to\langle\hookrightarrow\rangle\big(\Lambda(a^+a+)\wedge\langle\leftrightarrows\rangle\langle\hookleftarrow\rangle\big(\Lambda(a^+)\wedge\langle\hookrightarrow\rangle\Lambda(a^+b+)\big)\big)\big)$, which holds for $\mathcal{X}_1$ but not for $\mathcal{X}_2$. It says that any path with label a^+ can be extended to have label a^+a^+, then undergo a similarity swap, following by a restriction to the label a^+, so that the result of that can be extended to have label a^+b^+. This is an $\mathcal{L}_{reST}$-formula, showing that $\mathcal{X}_1 \not\leftrightarrow_{reST} \mathcal{X}_2$.

They can also be distinguished by

$$F := [\hookrightarrow]\big(\Lambda(a^+a^1a^+a^3b^+)\to\langle\sqsubseteq^2\rangle\langle\leftrightarrows^1\rangle\langle\leftrightarrows^3\rangle\langle\sqsupseteq^2\rangle\langle\sqsubseteq^4\rangle\Lambda(a^+a^1a^+b^+a^3)\big)\ .$$

It says that on any path with the indicated ST-trace one can apply the operators $\sqsubseteq^2$, $\leftrightarrows^1$, $\leftrightarrows^3$, $\sqsupseteq^2$ and $\sqsubseteq^4$, in that order, to obtain a path with label $a^+a^1a^+b^+a^3$. Again it holds for $\mathcal{X}_1$ but not for $\mathcal{X}_2$. This is an $\mathcal{L}_{hp}$-formula, showing that $\mathcal{X}_1 \not\leftrightarrow_{hp} \mathcal{X}_2$.

References

1. Clément Aubert, Ross Horne, and Christian Johansen. Diamonds for security: A non-interleaving operational semantics for the applied π-calculus. In *Proceedings of CONCUR 2022*, volume 243 of *LIPIcs*. Schloss Dagstuhl, 2022. doi:10.4230/LIPIcs.CONCUR.2022.30.
2. Clément Aubert, Ross Horne, Christian Johansen, and Sjouke Mauw. Unlinkability and history preserving bisimilarity. *Computer Security*, 165, 2026. doi:10.1016/j.cose.2025.104819.
3. Paolo Baldan and Silvia Crafa. Hereditary history-preserving bisimilarity: Logics and automata. In Jacques Garrigue, editor, Proc. *Programming Languages and Systems* (APLAS'14), volume 8858 of *lncs*, pages 469–488, 2014. doi:10.1007/978-3-319-12736-1_25.
4. Paolo Baldan and Silvia Crafa. A logic for true concurrency. *J. ACM*, 61(4):24:1–24:36, 2014. doi:10.1145/2629638.
5. Marek A. Bednarczyk. *Categories of Asynchronous Systems.* PhD thesis, University of Sussex, 1987.
6. Marek A. Bednarczyk. Hereditary history preserving bisimulations, or what is the power of the future perfect in program logics. Technical report, Polish Academy of Sciences, 1991.
7. Sergiu Bursuc, Ross Horne, Sjouke Mauw, and Semen Yurkov. Provably unlinkable smart card-based payments. In *Proceedings of the 2023 ACM SIGSAC Conference on Computer and Communications Security (CCS '23)*, pages 1392–1406, 2023. doi:10.1145/3576915.3623109.
8. Uli Fahrenberg, Christian Johansen, Georg Struth, and Krzysztof Ziemiański. Languages of higher-dimensional automata. *Math. Struct. Comput. Sci.*, 31(5):575–613, 2021. doi:10.1017/S0960129521000293.
9. Riccardo Focardi and Roberto Gorrieri. Classification of security properties. In *Foundations of Security Analysis and Design (FOSAD 2000)*, volume 2171 of *lncs*. springer, 2001. doi:10.1007/3-540-45608-2_6.
10. Rob J. van Glabbeek. The refinement theorem for ST-bisimulation semantics. In M. Broy and C.B. Jones, editors, Proceedings IFIP TC2 Working Conference on *Programming Concepts and Methods,* Sea of Gallilee, Israel, April 1990, pages 27–52. North-Holland, 1990. URL: http://kilby.stanford.edu/~rvg/pub/STbisimulation.pdf.

11 Rob J. van Glabbeek. Bisimulations for higher dimensional automata. Email message, July 7, 1991, 1991. URL: `http://theory.stanford.edu/~rvg/hda`.
12 Rob J. van Glabbeek. On the expressiveness of higher dimensional automata. *Theoretical Computer Science*, 356(3):265–290, 2006. Expressiveness in Concurrency. `doi:10.1016/j.tcs.2006.02.012`.
13 Rob J. van Glabbeek and Ursula Goltz. Refinement of actions and equivalence notions for concurrent systems. *Acta Informatica*, 37(4):229–327, 2001. `doi:10.1007/S002360000041`.
14 Rob J. van Glabbeek and Gordon D. Plotkin. Configuration structures. In *Proceedings of Tenth Annual IEEE Symposium on Logic in Computer Science*, pages 199–209. IEEE, 1995. `doi:10.1109/LICS.1995.523257`.
15 Rob J. van Glabbeek and Frits W. Vaandrager. Petri net models for algebraic theories of concurrency (extended abstract). In J.W. de Bakker, A.J. Nijman, and P.C. Treleaven, editors, Proceedings *PARLE, Parallel Architectures and Languages Europe,* Eindhoven, The Netherlands, June 1987, Vol. II: Parallel Languages, volume 259 of *lncs*, pages 224–242. springer, 1987. `doi:10.1007/3-540-17945-3_13`.
16 Ross Horne and Sjouke Mauw. Discovering epassport vulnerabilities using bisimilarity. *Logical Methods in Computer Science*, 17, 2021. `doi:10.23638/LMCS-17(2:24)2021`.
17 Ross Horne, Sjouke Mauw, and Semen Yurkov. When privacy fails, a formula describes an attack. *Theoretical Computer Science*, 959, 2023. `doi:10.1016/j.tcs.2023.113842`.
18 Iain Phillips and Irek Ulidowski. Event identifier logic. *Mathematical Structures in Computer Science*, 24(2), 2014. `doi:10.1017/S0960129513000510`.
19 Gordon D. Plotkin and Vaughan R. Pratt. Teams can see pomsets. In Doron A. Peled, Vaughan R. Pratt, and Gerard J. Holzmann, editors, *Partial Order Methods in Verification, Proceedings of a DIMACS Workshop, Princeton, New Jersey, USA, July 24-26, 1996*, volume 29 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 117–128. DIMACS/AMS, 1996. `doi:10.1090/DIMACS/029/07`.
20 Vaughan R. Pratt. Modeling concurrency with geometry. In Proc. *Principles of Programming Languages* (POPL'91), pages 311–322. ACM, 1991. `doi:10.1145/99583.99625`.
21 Cristian Prisacariu. Modal logic over higher dimensional automata. In *CONCUR*, volume 6269, pages 494–508, 2010. `doi:10.1007/978-3-642-15375-4_34`.
22 Cristian Prisacariu. The glory of the past and geometrical concurrency, 2013. Manuscript.
23 Alexander Rabinovich and Boris A Trakhtenbrot. Behavior structures and nets. *Fundamenta Informaticae*, 11(4):357–403, 1988. `doi:10.3233/FI-1988-11404`.
24 Wolfgang Reisig. *Petri Nets: An Introduction*, volume 4 of *EATCS Monographs on Theoretical Computer Science*. springer, 1985. `doi:10.1007/978-3-642-69968-9`.
25 Glynn Winskel. Event structures. In W. Brauer, W. Reisig, and G. Rozenberg, editors, *Petri Nets: Applications and Relationships to Other Models of Concurrency*, lncs, pages 325–392. springer, 1987. `doi:10.1007/3-540-17906-2_31`.
26 Safa Zouari, Krzysztof Ziemiański, and Uli Fahrenberg. Bisimulations and logics for higher-dimensional automata. In Chutiporn Anutariya and Marcello M. Bonsangue, editors, *Theoretical Aspects of Computing* (ICTAC'24), pages 132–150. springer, 2025. `doi:10.1007/978-3-031-77019-7_8`.