Almost All Graphs Are Vertex-Minor Universal

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Abstract

Answering a question of Claudet, we prove that the uniformly random graph $G \sim \mathbb{G}(n, 1/2)$ is $\Omega(\sqrt{n})$-vertex-minor universal with high probability. That is, for some constant $\alpha \approx 0.911$, any graph on any $\alpha\sqrt{n}$ specified vertices of G can be obtained as a vertex-minor of G. This has direct implications for quantum communications networks: an n-vertex k-vertex-minor universal graph corresponds to an n-qubit k-stabilizer universal graph state, which has the property that one can induce any stabilizer state on any k qubits using only local operations and classical communications.

We further employ our methods in two other contexts. We obtain a bipartite pivot-minor version of our main result, and we use it to derive a universality statement for minors in random binary matroids. We also introduce the *vertex-minor Ramsey number* $R_{\mathrm{vm}}(k)$ to be the smallest value n such that every n-vertex graph contains an independent set of size k as a vertex-minor. Supported by our main result, we conjecture that $R_{\mathrm{vm}}(k)$ is polynomial in k. We prove $\Omega(k^2) \le R_{\mathrm{vm}}(k) \le 2^k - 1$.

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1 Background and main results

Vertex-minors have been the star of numerous fascinating problems in graph theory since the 1980s. They often serve as a dense analogue of the usual minor relation, and so similarly have many deep applications to width parameters, algorithms, and logic [30, 22, 15, 5]. For more background on vertex-minors and their applications, see the recent survey on vertex-minors by Kim and Oum [28]. We begin with their definition.

▶ **Definition 1.** *Given a graph G and a vertex $v \in V(G)$, a* local complementation *at v is an operation which replaces $G[N(v)]$, the subgraph of G induced on the neighborhood of v, by its complement. The resulting graph is denoted $G * v$ and has the same vertex set as G. A graph H with $V(H) \subseteq V(G)$ is a* vertex-minor *of G if it can be obtained from G through a sequence of local complementations and vertex deletions.*

Vertex-minors also have fascinating connections to quantum information theory, on which we elaborate in Section 1.1. The following definition, introduced by Claudet et al. [12] and further studied by Cautrès et al. [8], provides a bridge between the purely graph-theoretic concept of vertex-minors and the quantum-theoretic notion of k-stabilizer universal states.

▶ **Definition 2.** *A graph G is k-*vertex-minor universal *on $V(G)$ if for every vertex subset $U \subseteq V(G)$ with $|U| = k$, every graph H on U is a vertex-minor of G.*

We emphasize that we require each H to be obtained exactly on each specified vertex set U. In particular, there are $\binom{|V(G)|}{k}2^{\binom{k}{2}}$ total choices of pairs U and H.

Our main result, resolving a question of Claudet [11], is that for some constant C, *almost all* graphs on Ck^2 vertices are k-vertex-minor universal. That is, a uniformly random graph on Ck^2 vertices is k-vertex-minor universal with probability tending exponentially fast to 1 as k increases. The quadratic dependence is best possible up to the constant factor C [12]. We use the notation $\mathbb{G}(n, p)$ for the Erdős-Renyi distribution on graphs on n vertices where each edge is included independently with probability p; in particular, $\mathbb{G}(n, 1/2)$ is the uniform distribution on graphs on n vertices.

▶ **Theorem 3.** *For all $c > 0$, if $G \sim \mathbb{G}(n, 1/2)$ with $n \geq (1 + c)\frac{1}{2\log_2(4/3)}k^2$, then G is k-vertex-minor universal with probability at least $1 - 2^{-(1+o(1))ck^2/2}$.*

On the practical side, our explicit bound on the probability of success converges exponentially quickly beyond the threshold value. Thus we have a probabilistic algorithm to produce a k-vertex-minor universal graph on $O(k^2)$ vertices. Explicitly, letting $C = \frac{1}{2\log_2(4/3)} \approx 1.205$, we have that for any $\varepsilon > 0$ and $k \geq 3$, a uniformly random graph on $n = \lceil Ck^2 + 16k \log k + 4 \log(1/\varepsilon)\rceil$ vertices is k-vertex-minor universal with probability at least $1 - \varepsilon$. This follows from keeping track of the error terms in the union bound (see (3)).

The proof of Theorem 3 is non-constructive, and we do not obtain an algorithm for finding a desired vertex-minor in $\mathbb{G}(n, 1/2)$. We note that this problem is NP-hard in general [17, 16], but this does not preclude the possibility of an efficient algorithm which succeeds with high probability when the input is a random graph.

▶ **Question.** *Does there exist a polynomial-time algorithm which takes as input $G \sim \mathbb{G}(n, 1/2)$ and a graph H on $O(\sqrt{n})$ vertices, and with high probability outputs a sequence of local complementations verifying that H is a vertex-minor of G?*

For comparison to known results regarding k-vertex-minor universal graphs, Claudet et al. [12] showed that there exist k-vertex-minor universal graphs with $O(k^4 \log k)$ vertices, while any k-vertex-minor universal graph must have at least $(k-2)^2/(2\log_2 3)$ vertices. Cautrès et al. [8] improved the existence result to match the lower bound up to a constant factor, establishing the existence of k-vertex-minor universal graphs on $(2+o(1))k^2$ vertices. Their construction is based on a random unbalanced bipartite graph G where one side has $\Theta(k^2)$ vertices and the other has $\Theta(k \log k)$ vertices, with edges between the two sides included independently with probability $1/2$. This yields a graph with $\Theta(k^3 \log k)$ edges with high probability, which is less than a constant proportion of possible edges. They also give a polynomial time algorithm to find a k-vertex graph H as a vertex-minor in G with high probability.

Our Theorem 3 improves the constant factor 2 from Cautrès et al. [8] to $\frac{1}{2\log_2(4/3)} \approx 1.205$, progressing closer to the lower bound [12] of $1/(2\log_2 3) \approx 0.315$. More conceptually important, however, is that our result states that almost all graphs – including dense and non-bipartite graphs – also have the vertex-minor universality property. This is perhaps surprising in the context of a paper of Gross, Flammia, and Eisert [24], which shows that all but an exponentially small fraction of quantum states are too entangled to be computationally useful. Our theorem, in contrast, shows that all but an exponentially small fraction of $O(k^2)$-qubit *graph* states are in fact k-stabilizer universal. We elaborate more on the quantum perspective in Section 1.1.

Although the vertex-minor problem addressed by Theorem 3 is most relevant to quantum information theory, our methods extend to a stronger notion of universality. The following definition plays a key role.

▶ **Definition 4.** *Given a graph G and an edge $uv \in E(G)$, a pivot at the edge uv is the operation which replaces G by $G \times uv := G * u * v * u$. It is easy to verify that $G * u * v * u = G * v * u * v$, so the operation is well-defined. A graph H with $V(H) \subseteq V(G)$ is a* pivot-minor *of G if it can be obtained from G through a sequence of pivots and vertex deletions. A graph G is k*-pivot-minor universal *on $V(G)$ if for every vertex subset $U \subseteq V(G)$ with $|U| = k$, every graph H on U is a pivot-minor of G.*

Our techniques naturally extend to pivot-minors. Pivot-minors in bipartite graphs are essentially equivalent to minors in binary matroids, and thus we also obtain the following two results. For brevity, we omit proofs of these results and precise definitions for matroid minor universality, but we include them in the full version of this manuscript [1].

▶ **Theorem 5.** *For all $c > 0$, if $G \sim \mathbb{G}(n, 1/2)$ with $n \geq (1+c)\frac{2}{\log_2(16/13)}k^2 \approx (1+c)6.68k^2$, then G is k-pivot-minor universal with probability at least $1 - 2^{-(1+o(1))ck^2}$.*

▶ **Theorem 6.** *Let $C = \frac{1}{2\log_2(8/7)} + \frac{1}{4} \approx 2.85$. For all $c > 0$, if M is a uniformly random binary matroid on ground set $[n]$ and $n \geq (1+c)2Ck^2$, then M is k-minor universal with probability at least $1 - 2^{-(1+o(1))ck^2/4}$.*

For G to be k-vertex-minor universal requires that we can obtain every graph on every k-subset of vertices of G as a vertex-minor. We can also ask a related question: how big does n need to be to ensure that *every* graph on n vertices contains a *particular* graph H as a vertex-minor on some k vertices? The answer is only finite if the graph H has no edges, or is an *independent set*; indeed, otherwise, a graph of any size n with no edges does not contain H as a vertex-minor.

▶ **Definition 7.** *Let $R_{\mathrm{vm}}(k)$ be the minimum n such that any n-vertex graph contains the independent set of size k as a vertex-minor.*

Recall the classical Ramsey number $R(s,t)$ which is the smallest n such that any n-vertex graph contains either an independent set of size s or a clique of size t. As observed in [6], we have $R_{\mathrm{vm}}(k) \leq R(k,k+1)$, since if a graph contains a clique of size $k+1$, locally complementing at one vertex forms an independent set on the remaining k vertices. The current best upper bound on $R(k,k+1)$ is roughly 3.8^k, due to Gupta, Ndiaye, Norin, and Wei [25] following the breakthrough work of Campos, Griffiths, Morris, and Sahasrabudhe [7]. Here we provide a nontrivial, but still exponential upper bound for $R_{\mathrm{vm}}(k)$, as well as a quadratic lower bound obtained from the random graph.

▶ **Theorem 8.** *For all $k \geq 0$,*

$$(1+o(1))\frac{1}{2\log_2 3}k^2 \leq R_{\mathrm{vm}}(k) \leq 2^k - 1.$$

The lower bound comes from showing that with high probability, the uniformly random graph $G \sim \mathbb{G}(n,1/2)$ does not contain an independent set of size k as a vertex-minor if $k \geq (1+\varepsilon)\sqrt{2\log_2(3)n}$, for any $\varepsilon > 0$. Due to Theorem 3, this is best possible up to the constant factor, and we believe the lower bound is closer to the true answer.

▶ **Conjecture 9.** $R_{\mathrm{vm}}(k) \leq \mathsf{poly}(k)$.

Paper organization. We conclude this section by discussing the connection between vertex-minor universality and quantum information theory. Then, in Section 2, we provide the technical graph-theoretic and probabilistic tools we need for the proof of Theorem 3, state some preliminary lemmas, and give a detailed overview of the proof. In Section 3, we carry out our proof strategy for Theorem 3. Finally, in Section 4, we prove Theorem 8 about the vertex-minor Ramsey number and conclude with some remaining open questions.

1.1 Motivation from quantum information theory

Although the study of k-vertex-minor universal graphs has only taken off in recent years, the relationship between quantum graph states and vertex-minors dates back to 2004 with the pioneering work of Van den Nest, Dehaene, and De Moor [31]. In the years since, many authors have studied various models of random graph states and their algorithmic and quantum entanglement properties; see e.g. [32, 16, 27, 23] and the references therein for non-exhaustive coverage of the subject.

Recent work in quantum information theory has emphasized the role of entangled multi-partite states as reusable resources for large-scale quantum communication networks. In such networks, separate parties are restricted to Local Operations and Classical Communication (LOCC), with no ability to perform joint quantum operations once the resource state has been distributed. A central question in this setting is to understand which global entangled states allow pairs of parties to generate entangled EPR pairs.

Motivated by this perspective, Bravyi, Sharma, Szegedy, and de Wolf [4] introduced the notion of k-pairable quantum states. An n-party state $|\psi\rangle$ is k-pairable if, for any choice of k disjoint pairs of parties, there exists a LOCC protocol that starts with $|\psi\rangle$ and ends in a state such that each of those pairs of parties share an EPR pair. This definition reflects a worst-case notion of network universality: a single fixed resource state must, under LOCC, generate EPR pairs for any choice of k disjoint pairs of parties. Bravyi et al. [4] established upper and lower bounds on the achievable value of k in terms of the number of parties and number of qubits per party.

Subsequently, Claudet, Mhalla, and Perdrix [12] and Cautrès, Claudet, Mhalla, Perdrix, Savin, and Thomassé [8] elaborated on the ideas of Bravyi et al. [4], focusing on the case where each party has just one qubit and studying a generalization of k-pairable states which Cautrés et al. [8] call *k-stabilizer universal* states. A state is k-stabilizer universal if any stabilizer state on any subset of k qubits can be obtained through LOCC protocols. Stabilizer universality is a stronger notion than k-pairability: EPR pairs are stabilizer states, so any $2k$-stabilizer universal state is also k-pairable. For more background on stabilizer states, see for example [26].

The quantum-theoretic contributions of [12] and [8], as well as those of this article, rely on the connection between quantum states and graph theory. Vertex-minors capture an important sequence of operations on quantum states: if H can be obtained as a vertex-minor of G, then the quantum graph state corresponding to H can be obtained from the graph state corresponding to G through just local Clifford operations, local Pauli measurements, and classical communications. (The converse also holds if H has no isolated vertices [18].) It follows that if G is a k-vertex-minor universal graph then the quantum graph state corresponding to G is k-stabilizer universal. This enables explicit constructions of universal resource states for quantum communication protocols [12, 8].

2 Preliminaries and proof overview

We denote the symmetric difference of sets by $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set respectively. For $U \subseteq V(G)$, we use $G[U]$ to denote the graph induced on U, and $G - U$ to denote the graph obtained by deleting U. The *adjacency matrix* of a graph G (over $\mathbb{F}_2$) is the symmetric matrix $A \in \mathbb{F}_2^{V(G)\times V(G)}$ such that $A_{u,v} = 1$ if $uv \in E(G)$ and $A_{u,v} = 0$ otherwise. Note that as we assume graphs to be simple, A has 0 along the diagonal. For disjoint sets X, Y, we define the set $[X, Y] \coloneqq \{\{x, y\} \mid x \in X, y \in Y\}$ and let $G[X, Y]$ be the induced bipartite subgraph with edge set $E(G) \cap [X, Y]$.

2.1 Vertex-minor preliminaries

Throughout the paper it will be convenient to have notation which represents both the local complementation and pivot operations. To that end we define the following. Let $x \in \{\varnothing, \{v\}, \{u, v\}\}$. Then

$$G \circ x = \begin{cases} G & x = \varnothing \\ G * v & x = \{v\} \\ G \times uv & x = \{u, v\}. \end{cases}$$

Implicitly we assume that if $x = \{u, v\}$, then $uv \in E(G)$, otherwise the operation is not defined. If $(x_i)_{i=1}^{\ell}$ is a sequence of such x, then we often use $G \circ (x_i)_{i=1}^{\ell}$ as shorthand for $G \circ x_1 \circ x_2 \circ \cdots \circ x_n$. We similarly write $G * (v_i)_{i=1}^{\ell}$ as shorthand for $G * v_1 * \cdots * v_\ell$.

Observe that an equivalent way to view the pivot operation is that for $vu \in E(G)$, $G \times vu$ is the graph obtained from G by taking the symmetric difference of G with the complete tripartite graph with parts $N(v) \cap N(u)$, $N(v) \setminus (N(u) \cup \{u\})$, $N(u) \setminus (N(v) \cup \{v\})$, and then swapping the labels of u and v.

We now discuss some well-known facts about vertex-minors and pivot-minors. We say two graphs are *locally equivalent* if one can be obtained from the other by a sequence of local complementations. Similarly we say that two graphs are *pivot equivalent* if one can be obtained from the other by a sequence of pivots. It is easy to see that if H is a vertex-(pivot-) minor of G, then $H \subseteq G'$ for some graph G' which is locally (pivot) equivalent to G. The following fact is well-known.

▶ **Lemma 10** ([30, Proposition 2.5]). *If $vu \in E(G)$, then $uw \in E(G \times vu)$ if and only if $vw \in E(G)$. Furthermore, if both edges exist, we have*

$$G \times vu \times uw = G \times vw.$$

The following result, proven independently by Bouchet [3] and Fon-Der-Flaass [21], will be crucial in reordering the operations when finding a vertex-minor.

▶ **Theorem 11** ([3, 21]). *Suppose H is a vertex-minor of G and $v \in V(G) \setminus V(H)$. Then H is a vertex-minor of*

1. *$G - v$,*
2. *$G * v - v$, or*
3. *$G \times vu - v$ for some $u \in N(v)$.*

We note that Lemma 10 implies that the choice of u does not matter up to pivot-equivalence. That is, if H is a vertex-minor of $G \times vu - v$ for some $u \in N(v)$, then H is a vertex-minor of $G \times vu' - v$ for every $u' \in N(v)$. This helps us prove our key structural lemma below. We call this our "reordering lemma," because it allows us to reorganize a sequence of local complementations in a convenient way.

▶ **Lemma 12.** *Fix a set $\hat{V}$. Let $v_1, \dots, v_\ell \in \hat{V}$ be a (possibly repeating) sequence and let $\hat{G}$ be a graph with $V(\hat{G}) = \hat{V}$. Then there is a sequence $x_1, \dots, x_r$ of singletons and pairs from $\hat{V}$ such that for any graph G with $\hat{V} \subseteq V(G)$ and $G[\hat{V}] = \hat{G}$, we have*

- **(I)** *$G * v_1 * v_2 \cdots * v_\ell - \hat{V} = G \circ x_1 \cdots \circ x_r - \hat{V}$.*
- **(II)** *The sets $x_1, \dots, x_r$ are pairwise disjoint.*
- **(III)** *If $v \in \hat{V}$ appears exactly once in $(v_i)_{i=1}^{\ell}$, then $v \in x_j$ for some $j \in [r]$.*

Moreover, any choice of $(x_i)_i$ satisfying (I) and (II) for all G also satisfies (III).

We emphasize that the sequence $(x_i)_i$ depends *only* on the subgraph $\hat{G} = G[\hat{V}]$, and is independent of the edges between $\hat{V}$ and $V(G) \setminus \hat{V}$. Without this careful tracking of the dependency, this lemma would be implied by known results, (see [14, Proposition 6] or [13, Proposition 7]). In Section 3.3, we give a short proof by induction using Theorem 11.

Because of the dependence only on $\hat{G}$, if $G \sim \mathbb{G}(n, 1/2)$, then by revealing only the edges of $\hat{G}$ we can rewrite any arbitrary sequence of local complementations $*(v_i)_i$ in the desired form $\circ(x_j)_j$, and then study the statistics of the resulting graph.

2.2 Probability preliminaries

We use $\mathbb{G}(n,p)$ to denote the Erdős-Rényi random graph distribution on vertex set $[n]$, where each edge is present independently with probability p. We let $\mathcal{G}_n$ denote the set of all labeled graphs on vertex set $[n]$. The following simple proposition is crucial for our proof.

▶ **Proposition 13.** *If $G \sim \mathbb{G}(n,1/2)$ and $v \in [n]$, then $G * v \sim \mathbb{G}(n,1/2)$.*

Proof. Since the map $G \mapsto G*v$ is an involution on $\mathcal{G}_n$, it preserves the uniform distribution. Indeed, for any graph $G' \in \mathcal{G}_n$, we have $\mathbb{P}[G*v = G'] = \mathbb{P}[G = G'*v] = \frac{1}{|\mathcal{G}_n|}$. ◀

The basic probabilistic tool we rely on is Chebyshev's inequality, which we use in the following form.

▶ **Lemma 14** (Chebyshev's inequality)**.** *For a random variable X and $\alpha > 0$, we have*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \alpha] \leq \frac{\mathrm{Var}[X]}{\alpha^2}.$$

In particular, setting $\alpha = \mathbb{E}[X]$ gives $\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2}$.

2.3 Proof overview

Before the complete proof in Section 3, we outline the argument for Theorem 3. Let $G \sim \mathbb{G}(n+k,1/2)$. Fix a vertex set $U \subseteq V(G)$ of size $|U| = k$, and fix a graph H on U. Denote the vertices of G by $V = \{v_1, \dots, v_n\} \sqcup U$. To form a copy of H on vertex set U, we locally complement on some subset of the vertices in $\{v_1, \dots, v_n\}$, given by index set J. Complementing on this subset J produces a random graph on U, which we denote by $G_J[U]$. If two choices for the subset J are sufficiently different, then the resulting graphs $G_J[U]$ will likely also be different. There are 2^n choices for J and $2^{\binom{k}{2}}$ choices for the graph H. Hence for $k = \Theta(\sqrt{n})$, we are likely to find a choice of J for which the graph $G_J[U]$ is equal to H. The core difficulty of the proof is analyzing the difference, or more precisely the correlation, between G_J and $G_{J'}$ for different subsets J, J'.

We now proceed with a more technical overview of our proof. For each subset $J \subseteq [n]$, where $J = \{i_1 < i_2 < \dots < i_{|J|}\}$ we define

$$G_J := G * v_{i_1} * v_{i_2} * \cdots * v_{i_{|J|}}, \tag{1}$$

so we can track 2^n distinct procedures for locally complementing on G. For example, $G_\emptyset = G$ and $G_{[n]} = G * v_1 * \cdots * v_n$. Let X be the number of "successful" procedures, i.e. $X = \sum_{J \subseteq [n]} X_J$ where X_J the indicator that $G_J[U] = H$. We want to show that $X > 0$, and hence H is a vertex-minor of G, with high probability. By Proposition 13, for each J we have $G_J \sim \mathbb{G}(n,1/2)$, so each X_J equals 1 with probability $2^{-\binom{k}{2}}$. Thus $\mathbb{E}[X] = 2^n 2^{-\binom{k}{2}} \to \infty$, and we upper bound $\mathbb{P}[X = 0]$ using Chebyshev's inequality. Finally, we apply a union bound over all choices of U and H to show that G is k-vertex-minor universal.

The central challenge for this argument is to control $\mathrm{Var}[X]$, and therefore to show that the various $G_J, G_{J'}$ are not too correlated. For example, comparing $G_\emptyset = G$ and $G_{[n]}$, we find that each time we locally complement at a vertex v_i, a random subset of the edges in U are flipped. Hence, we expect that after sufficiently many flips, the distributions of $G[U]$ and $G_{[n]}[U]$ will be almost entirely uncorrelated.

To make this precise, we define two random walks on the space of graphs with vertex set U (see Section 3.1). The first random walk chooses a uniformly random subset of U and complements the edges with both ends in that subset. This corresponds to locally complementing at a vertex outside of U that has a random neighborhood. The second random walk chooses two uniformly random subsets $S, T \subseteq U$ and toggles adjacency between pairs of vertices which lie in two different sets among $S \setminus T, T \setminus S$, and $S \cap T$. This corresponds to pivoting on an edge outside U where each end has a random neighborhood in U.

We view these random walks as linear transformations over the space of all graphs, and we compute their eigenvectors and eigenvalues exactly (see Lemma 16). We use this spectral calculation to conclude that both random walks quickly approach the uniform distribution (Lemma 20).

We then use these random walks to study $G_J[U] \triangle G_{J'}[U]$ when J and J' differ. Specifically, we argue that if $|J \triangle J'|$ is large, then $G_J[U] \triangle G_{J'}[U]$ is close to uniformly random, and thus the covariance between X_J and $X_{J'}$ is very small (see Lemma 24). Suppose $J = \{i_1 < i_2 < \cdots < i_{|J|}\}$ and $J' = \{j_1 < j_2 < \cdots < j_{|J'|}\}$. Because local complementation is an involution, we have

$$G_{J'} = G_J * v_{i_{|J|}} * \cdots * v_{i_1} * v_{j_1} * \cdots * v_{j_{|J'|}},$$

meaning that $G_{J'}[U]$ is formed from $G_J[U]$ by locally complementing on vertices outside of U. This sounds like our random walk, but as stated, the same vertex could be locally complemented on multiple times, and thus the steps in the corresponding random walk may be correlated. To combat this, we apply our reordering Lemma 12 to obtain a sequence $(x_i)_{i=1}^{\ell}$ where

$$G_{J'}[U] = G_J \circ (x_i)_{i=1}^{\ell}[U]$$

such that the x_i are pairwise disjoint. Then we may consider $G_J[U] \triangle (G_J \circ (x_i)_{i=1}^{t})[U]$ as a random walk starting at $t = 0$ with the empty graph and ending at $t = \ell$ with $G_J[U] \triangle G_{J'}[U]$. Property (III) of Lemma 12 implies that $|\bigcup_{i=1}^{\ell} x_i| \geq |J \triangle J'|$, so if $|J \triangle J'|$ is sufficiently large, then this random walk mixed for many steps, implying that $G_J[U] \triangle G_{J'}[U]$ is approximately uniformly random. Since typical sets J, J' differ in $n/2$ many indices, this gives the desired correlation bounds, implying that the variance of X is small and hence the probability that H is a vertex-minor of G is large (Lemma 26).

We note some subtlety in the sets of edges which must stay independent in this approach. In order to apply Lemma 12, we first reveal the edges of $G \setminus U$. Then we reveal the edges between x_i and U one at a time so that at each step, each vertex in x_i has a uniformly random neighborhood in U. We use Lemma 21 to argue that conditional on all previous steps, the unexposed edges remain uniformly random throughout the process. This guarantees that each step of the random walk is independent of the previous steps, and so each step brings us closer to the uniform distribution.

3 Proof of vertex-minor universality of the random graph

3.1 Random walks on the space of graphs

We first define and consider two random walks Com and Piv on the space of graphs with vertex set $U = [k]$. We begin with a graph Γ drawn from some starting distribution μ_0 on all graphs with vertex set U. Then at each time t, the Com walk simulates locally complementing (and deleting) at vertex v outside of U with a uniformly random neighborhood. Similarly, the Piv walk simulates pivoting on an edge outside U with uniformly random neighborhoods. Formally, we define the random walks as follows.

▶ **Definition 15.** *Suppose μ is a probability distribution on $\mathcal{G}_k$, the set of labeled graphs with vertex set $[k]$. Identify μ with a vector in $[0,1]^{\mathcal{G}_k}$, and we write $\mu(H)$ to be the probability of the graph $H \in \mathcal{G}_k$. First, let $\mathrm{Com}(\mu)$ be the distribution of the graph $\Gamma \triangle K_S$, where $\Gamma \sim \mu$, and $S \subseteq [k]$ is a uniformly random subset independent of Γ. Next, for $S, T \subseteq [k]$ we let $K^{\triangle}_{S,T}$ be the complete tripartite graph with tripartition $\{S \setminus T, T \setminus S, S \cap T\}$. Then let $\mathrm{Piv}(\mu)$ be the distribution of $\Gamma \triangle K^{\triangle}_{S,T}$ where $\Gamma \sim \mu$ and $S, T \subseteq [k]$ are uniformly random and independent of Γ. Finally, let μ_{unif} be the uniform distribution on $\mathcal{G}_k$.*

It is easy to see that the uniform distribution μ_{unif} is stationary under both the Com and Piv operators, and we will be interested in how fast the distribution μ_0 approaches μ_{unif} under these operators. It is helpful to think of these as steps in a random walk on a Cayley graph of the additive group $\mathbb{F}_2^{E(K_k)}$. For example, if we construct the Cayley graph with multiset of generators $\{\mathbb{1}_{E(K_S)} : S \subseteq [k]\}$, then one step in this walk gives the convolution $\mu \mapsto \mathrm{Com}(\mu)$. A nice survey of random walks on groups is given by Diaconis [20]. As symmetric random walks, both $\mathrm{Com}, \mathrm{Piv} : \mathbb{R}^{\mathcal{G}_n} \to \mathbb{R}^{\mathcal{G}_n}$ are symmetric linear transformations over $\mathbb{R}$. Because they are walks on a Cayley graph, we can explicitly compute their eigenvectors.

▶ **Lemma 16.** *The eigenvectors of Com and Piv are given by $\{\chi_G \mid G \in \mathcal{G}_k\}$ where $\chi_G \in \mathbb{R}^{\mathcal{G}_k}$ is defined as $\chi_G(H) = (-1)^{|E(G) \cap E(H)|}$. For each $G \in \mathcal{G}_k$, let $\lambda_G^{\mathrm{Com}}, \lambda_G^{\mathrm{Piv}}$ denote the eigenvalues of χ_G under Com, Piv respectively. Then we have*

$$\lambda_G^{\mathrm{Com}} = \mathbb{E}_S\left[(-1)^{|E(G) \cap E(K_S)|}\right]$$
$$\lambda_G^{\mathrm{Piv}} = \mathbb{E}_{S,T}\left[(-1)^{|E(G) \cap E(K^{\triangle}_{S,T})|}\right]$$

where the expectation is over $S, T \subseteq [k]$ drawn uniformly at random.

Proof. The statement for Com follows from a simple calculation:

$$\begin{aligned}
\mathrm{Com}(\chi_G)(H) &= 2^{-k} \sum_{S \subseteq [k]} \chi_G(H \triangle K_S) \\
&= 2^{-k} \sum_{S \subseteq [k]} (-1)^{|E(G) \cap E(H)|} (-1)^{|E(G) \cap E(K_S)|} \\
&= \chi_G(H) \cdot 2^{-k} \sum_{S \subseteq [k]} (-1)^{|E(G) \cap E(K_S)|} \\
&= \chi_G(H) \cdot \mathbb{E}_S\left[(-1)^{|E(G) \cap E(K_S)|}\right].
\end{aligned}$$

The proof for Piv is similar. ◀

We can now bound the eigenvalues as follows.

▶ **Lemma 17.** *Let $G \in \mathcal{G}_k$ have adjacency matrix $A \in \mathbb{F}_2^{k \times k}$. Then*

$$(\lambda_G^{\mathrm{Com}})^2 \leq \lambda_G^{\mathrm{Piv}} = 2^{-\mathrm{rank}_{\mathbb{F}_2}(A)}.$$

Proof. We first compute λ_G^{Piv}. Recall that for $S, T \subseteq [k]$, if x_S and x_T are the indicator vectors of S and T, then $x_S^\top A x_T = |\{(u,v) \mid u \in S, v \in T, uv \in E(G)\}|$ over $\mathbb{R}$. For example, $x_S^\top A x_S = 2|E(G) \cap E(K_S)|$. We observe that $|E(G) \cap E(K_{S,T}^{\triangle})| \equiv x_S^\top A x_T \pmod 2$. This is because every edge from S to T is counted, with edges in $S \cap T$ counted twice. Using this and Lemma 16,

$$\lambda_G^{\mathrm{Piv}} = \mathbb{E}\left[(-1)^{|E(G \cap K_{S,T}^{\triangle})|}\right] = 2^{-2k} \sum_{S,T \subseteq [k]} (-1)^{x_T^\top A x_S} = 2^{-2k} \sum_{u \in \mathbb{F}_2^k} \left(\sum_{v \in \mathbb{F}_2^k} (-1)^{v^\top A u} \right).$$

Consider the inner sum. If $u \in \mathrm{Null}_{\mathbb{F}_2}(A)$, then (over $\mathbb{F}_2$) we have $Au = 0$, so the inner sum is 2^k. Otherwise, Au is a nonzero vector, so exactly half of vectors $v \in \mathbb{F}_2^k$ have $v^\top A u = 0$, and hence the inner sum is 0. Thus,

$$\lambda_G^{\mathrm{Piv}} = 2^{-2k}\Big(2^k |\mathrm{Null}(A)|\Big) = 2^{-k} \cdot 2^{k - \mathrm{rank}(A)} = 2^{-\mathrm{rank}(A)}.$$

Next, we use this to bound λ_G^{Com}. For subsets $S, T \subseteq [k]$, let $R = S \triangle T$. Observe the following two identities:

$$\begin{aligned} |E(G) \cap E(K_S)| + |E(G) \cap E(K_T)| &\equiv |E(G) \cap E(K_{S \triangle T})| + |E(G) \cap E(K_{S,T}^{\triangle})| \pmod 2 \\ &\equiv |E(G) \cap E(K_R)| + |E(G) \cap E(K_{S,R}^{\triangle})| \pmod 2. \end{aligned}$$

The first can be checked combinatorially, or using linear algebra (over $\mathbb{R}$):

$$\frac{x_S^\top A x_S}{2} + \frac{x_T^\top A x_T}{2} = \frac{(x_S + x_T)^\top A (x_S + x_T)}{2} - x_T^\top A x_S.$$

Evaluating each side mod 2 yields the first identity. For the second, note that by definition $K_{S,T}^{\triangle} = K_{S,R}^{\triangle}$, or using algebra $x_T^\top A x_S \equiv (x_T + x_S)^\top A x_S \pmod 2$. As a result, we have the following change of variables (where again we use Lemma 16):

$$\begin{aligned} (\lambda_G^{\mathrm{Com}})^2 = \mathbb{E}\big[(-1)^{|E(G) \cap E(K_S)|}\big]^2 &= 2^{-2k} \sum_{S \subseteq [k]} (-1)^{|E(G) \cap E(K_S)|} \sum_{T \subseteq [k]} (-1)^{|E(G) \cap E(K_T)|} \\ &= 2^{-2k} \sum_{S,T \subseteq [k]} (-1)^{|E(G) \cap E(K_{S \triangle T})|} (-1)^{|E(G) \cap E(K_{S,T}^{\triangle})|} \\ &= 2^{-2k} \sum_{R \subseteq [k]} (-1)^{|E(G) \cap E(K_R)|} \sum_{S \subseteq [k]} (-1)^{|E(G \cap K_{S,R}^{\triangle})|} \\ &\leq 2^{-2k} \sum_{R \subseteq [k]} \left| \sum_{S \subseteq [k]} (-1)^{x_S^\top A x_R} \right| = \lambda_G^{\mathrm{Piv}}. \end{aligned}$$

In the last line, we used the fact again that the inner sum is either 2^k or 0. Thus, we conclude that $(\lambda_G^{\mathrm{Com}})^2 \leq \lambda_G^{\mathrm{Piv}} = 2^{-\mathrm{rank}(A)}$, as desired. ◀

▶ Remark 18. We note that A is an alternating matrix over $\mathbb{F}_2$, and so the rank of A is even. Thus in particular, if $G = I_k$, $\lambda_G^{\mathrm{Com}} = \lambda_G^{\mathrm{Piv}} = 1$, and for every other $G \in \mathcal{G}_k$, $\lambda_G^{\mathrm{Com}} \leq 1/2$ and $\lambda_G^{\mathrm{Piv}} \leq 1/4$.

We note that μ_{unif} is a multiple of χ_{I_k}, and so the crude bound of 1/2 is enough to get exponential convergence to the uniform distribution. To get even sharper bounds, we count the number of graphs whose adjacency matrix has a given rank. This is explicitly enumerated by MacWilliams [29], but we only require the following bound.

▶ **Theorem 19** ([29, Theorem 3]). *Let $\mathcal{G}_{k,r}$ denote the set of labeled graphs whose adjacency matrix (a symmetric matrix in $\mathbb{F}_2^{k\times k}$ with 0 diagonal) has rank r. Then if $r \geq 1$ is even,*

$$|\mathcal{G}_{k,r}| \leq 2^{rk-2}$$

and if r is odd, $|\mathcal{G}_{k,r}| = 0$.

We are now ready to show that the two random walks (or any combination of the two) converge exponentially fast to the uniform distribution.

▶ **Lemma 20.** *Let $\mu_0 = \delta_{I_k}$ be the point mass distribution on the independent graph $I_k \in \mathcal{G}_k$. Then for $t = 1, \ldots, \ell$, define either $\mu_t = \mathrm{Com}(\mu_{t-1})$ or $\mu_t = \mathrm{Piv}(\mu_{t-1})$. Suppose that the* Com *transformation is applied m_1 times and the* Piv *transformation is applied m_2 times, and let $m = m_1 + 2m_2$. Then if $m > 2k$, we have*

$$|\mu_\ell(H) - \mu_{\mathrm{unif}}(H)| \leq 2^{2k-\binom{k}{2}-m}$$

for every graph $H \in \mathcal{G}_k$.

Proof. Note that $\{\chi_G \mid G \in \mathcal{G}_k\}$ is an orthogonal basis of $\mathbb{R}^{\mathcal{G}_k}$. Because $\mu_0 = \delta_{I_k}$ has the same inner product with each eigenvector, we have exactly $\mu_0 = 2^{-\binom{k}{2}} \sum_{G\in\mathcal{G}_k} \chi_G$. By induction on t, we compute

$$\mu_\ell = 2^{-\binom{k}{2}} \sum_{G\in\mathcal{G}_k} (\lambda_G^{\mathrm{Com}})^{m_1} (\lambda_G^{\mathrm{Piv}})^{m_2} \chi_G.$$

We note that $2^{-\binom{k}{2}}\chi_{I_k} = \mu_{\mathrm{unif}}$ and $\lambda_{I_k}^{\mathrm{Com}} = \lambda_{I_k}^{\mathrm{Piv}} = 1$, so

$$\mu_\ell - \mu_{\mathrm{unif}} = 2^{-\binom{k}{2}} \sum_{G\neq I_k} (\lambda_G^{\mathrm{Com}})^{m_1} (\lambda_G^{\mathrm{Piv}})^{m_2} \chi_G.$$

We now decompose the sum into the families $\mathcal{G}_{k,2r}$ for all $r \geq 0$, noting that $\mathcal{G}_{k,0} = \{I_k\}$. Using the eigenvalue bounds in Lemma 17 and the counting bounds of Theorem 19, we have the following for any $H \in \mathcal{G}_k$:

$$\begin{aligned}
|\mu_\ell(H) - \mu_{\mathrm{unif}}(H)| &\leq 2^{-\binom{k}{2}} \sum_{G\neq I_k} |\lambda_G^{\mathrm{Com}}|^{m_1} \cdot |\lambda_G^{\mathrm{Piv}}|^{m_2} \cdot |\chi_G(H)| \\
&\leq 2^{-\binom{k}{2}} \sum_{r=1}^{\lfloor k/2 \rfloor} \sum_{G\in\mathcal{G}_{k,2r}} 2^{-rm_1} 2^{-2rm_2} \\
&\leq 2^{-\binom{k}{2}} \sum_{r=1}^{\infty} 2^{2rk-2} 2^{-rm} \\
&\leq 2^{-\binom{k}{2}+2k-m},
\end{aligned}$$

where we use that $m > 2k$ so the sum converges. ◀

We will use Lemma 20 to show that, in a graph $G \sim \mathbb{G}(n, 1/2)$ with $U \subseteq V(G)$, by repeatedly locally complementing/pivoting at distinct vertices outside U, the distribution of $G \circ (x_t)_t[U]$ becomes quickly uncorrelated from the initial state $G[U]$. However, to apply Lemma 20, we need that the local complementation/pivot steps are actually independent; this relies on the initial graph being uniformly random. In fact, we need that the steps of the random walk are independent even when we choose the sequence $(x_t)_t$ based on $G - U$. Towards that end, we prove the following technical lemma.

▶ **Lemma 21.** *Let $W \subseteq \hat{V} \subseteq [n]$ and $U = [n] \setminus \hat{V}$, and fix a graph $\hat{G}$ with vertex set $\hat{V}$. Let $v_1, \dots, v_t \in W$ be a (possibly repeating) sequence of vertices depending on $\hat{G}$. Let G be a random graph with vertex set $[n]$ drawn from $\mathbb{G}(n, 1/2)$ conditioned on $G[\hat{V}] = \hat{G}$. Partition $\binom{[n]}{2}$ into $E_1 = \binom{\hat{V}}{2} \cup [W, U]$ and $E_2 = \binom{U}{2} \cup [\hat{V} \setminus W, U]$. Let $G' = G * (v_i)_{i=1}^t$. Then $E(G') \cap E_2$ is uniformly random and independent of $E(G) \cap E_1$.*

We think of our random walk as first revealing the edges of $G[\hat{V}] = \hat{G}$ to choose the sequence $v_1, \dots, v_t$. As we perform the random walk, we reveal edges between the corresponding vertices $v_1, \dots, v_t$ and U. The set W will correspond to these revealed vertices. Lemma 21 shows that the next step in our random walk is independent of all revealed information.

Proof of Lemma 21. Fix a subset A of E_1. It suffices to show that conditioned the event that $E(G) \cap E_1 = A$, the distribution of $E(G') \cap E_2$ is uniformly random. We proceed by induction on t. If $t = 0$ then $G' = G$, so by definition of $\mathbb{G}(n, 1/2)$, $E(G') \cap E_2$ is uniformly random.

Suppose $t \geq 1$ and let $G'' = G * (v_i)_{i=1}^{t-1}$; by the induction hypothesis, $E(G'') \cap E_2$ is uniformly random conditioned on $E(G) \cap E_1$. Observe that $G' = G'' * v_t$ is obtained by the involutive map $H \mapsto H \triangle K_T$, where $T = N_{G''}(v_t)$ is determined exactly by $E(G) \cap E_1$. This induces an involutive map $E(G'') \cap E_2 \mapsto E(G') \cap E_2$, and therefore preserves the uniform measure. ◀

▶ Remark 22. We note that because the above proposition holds for every graph $\hat{G}$, it also holds if we let $\hat{G}$ be uniformly random, and thus it holds when $G \sim \mathbb{G}(n, 1/2)$ and $v_1, \dots, v_t$ is a random sequence depending on $G[\hat{V}]$.

Observe that Lemma 21 implies that conditioning on $G - U$ and some previous mixing steps $x_1, \dots, x_{t-1} \subseteq W$, as long as x_t is disjoint from W and U, we have that both $G[U]$ and the edges from x_t to U are uniformly random. This implies that each step of our random walk is independent of the previous steps. With this, we conclude the final random walk result.

▶ **Lemma 23.** *Let $G \sim \mathbb{G}(n + k, 1/2)$ with partition $V(G) = \hat{V} \sqcup U$, $|U| = k$. Let $\hat{G}$ be a graph on vertex set $\hat{V}$ and let $(x_i)_{i=1}^\ell$ be a sequence of disjoint singletons and pairs from $\hat{V}$ such that $\hat{G} \circ (x_i)_{i=1}^\ell$ is well-defined and $|\bigcup_i x_i| > 2k$. Then,*

$$\left| \mathbb{P}\left[G \circ (x_i)_{i=1}^\ell [U] = G[U] \;\middle|\; G[\hat{V}] = \hat{G} \right] - 2^{-\binom{k}{2}} \right| \leq 2^{2k - \binom{k}{2} - |\bigcup_i x_i|}.$$

Proof. For times $t = 0, 1, \dots, \ell$, define $G_t \coloneqq G \circ (x_i)_{i=1}^t$, and define the probability distribution $\mu_t \in [0,1]^{\mathcal{G}_k}$ as the distribution of $H_t \coloneqq (G_t \triangle G)[U]$, conditioned on $G[\hat{V}] = \hat{G}$. For example, μ_0 is the point mass distribution δ_{I_k}.

At time t, H_{t-1} is a function of $G[\hat{V}] = \hat{G}$ and $N_G(w)$ for $w \in \bigcup_{i=1}^{t-1} x_i$. If $x_t = \{v\}$, then $H_t = H_{t-1} \triangle K_S$ where $S = N_{G_{t-1}}(v) \cap U$. Similarly, if $x_t = \{v, v'\}$, then $H_t = H_{t-1} \triangle K_{S,T}^{\triangle}$, where $S = N_{G_{t-1}}(v) \cap U$ and $T = N_{G_{t-1}}(v') \cap U$. By Lemma 21 (applied with $W = \bigcup_{i<t} x_i$), we have that S (resp. T) is a uniformly random subset of U, independent of $G[\hat{V}]$ and $N_G(u)$ for $u \in \bigcup_{i<t} x_i$ and thus S (resp. T) is independent of H_{t-1}. Then by definition, if $|x_t| = 1$ then we have exactly $\mu_t = \mathrm{Com}(\mu_{t-1})$, and if $|x_t| = 2$ then $\mu_t = \mathrm{Piv}(\mu_{t-1})$. We then apply Lemma 20 (noting that the x_i are disjoint so $m = |\bigcup_i x_i|$) to conclude

$$\left| \mathbb{P}\left[(G_\ell \triangle G)[U] = I_k \;\middle|\; G[\hat{V}] = \hat{G} \right] - 2^{-\binom{k}{2}} \right| = |\mu_\ell(I_k) - \mu_{\mathrm{unif}}(I_k)| \;\leq\; 2^{2k - \binom{k}{2} - |\bigcup_i x_i|},$$

as desired. ◀

3.2 Proof of Theorem 3

We now use the reordering Lemma 12 to show that when J, J' are sufficiently different, the graphs G_J and $G_{J'}$ are sufficiently uncorrelated by applying the random walk described in Section 3.1.

▶ **Lemma 24.** *Let $G \sim \mathbb{G}(n+k, 1/2)$, $U \subseteq V(G)$ with $|U| = k$, and let $J, J' \subseteq [n]$ with $|J \triangle J'| > 2k$. Defining $G_J, G_{J'}$ as in Equation (1), we have*

$$\left|\mathbb{P}[G_J[U] = G_{J'}[U]] - 2^{-\binom{k}{2}}\right| \leq 2^{2k-\binom{k}{2}-|J\triangle J'|}$$

Proof. Let $J = \{i_1 < \ldots < i_\ell\}$ and $J' = \{j_1 < \ldots < j_{\ell'}\}$. Observe that

$$G_{J'} = G_J * v_{i_\ell} * \cdots * v_{i_2} * v_{i_1} * v_{j_1} * v_{j_2} * \cdots * v_{j_{\ell'}}. \tag{2}$$

Fix $\hat{G} \in \mathcal{G}_n$, and condition on the event $G[\hat{V}] = \hat{G}$, which depends only on the edges inside $\hat{V}$. By the reordering Lemma 12, there is a sequence $(x_i)_{i=1}^r$ (depending on $\hat{G}$) with the three properties from that lemma such that $G_{J'}[U] = G_J \circ (x_i)_{i=1}^r[U]$. By property (III), we note that $|\bigcup_i x_i| \geq |J \triangle J'|$, since a vertex v_i appears exactly once in Equation (2) if $i \in J \triangle J'$. Let $\hat{G}_J = \hat{G} * (v_{i_t})_{t=1}^{\ell}$, so $G[\hat{V}] = \hat{G}$ if and only if $G_J[\hat{V}] = \hat{G}_J$. Then by Lemma 23,

$$\begin{aligned}\left|\mathbb{P}\left[G_{J'}[U] = G_J[U] \;\middle|\; G[\hat{V}] = \hat{G}\right] - 2^{-\binom{k}{2}}\right| &= \left|\mathbb{P}\left[G_J \circ (x_i)_{i=1}^r[U] = G_J[U] \;\middle|\; G_J[\hat{V}] = \hat{G}_J\right] - 2^{-\binom{k}{2}}\right| \\ &\leq 2^{2k-\binom{k}{2}-|\bigcup_{i\leq r} x_i|} \\ &\leq 2^{2k-\binom{k}{2}-|J\triangle J'|},\end{aligned}$$

as desired. We apply the law of total probability over all $\hat{G} \in \mathcal{G}_n$ to obtain the unconditioned bound. ◀

Before we complete the second moment method calculation, we need one more fact about the independence implicit in our construction.

▶ **Proposition 25.** *Let $G \sim \mathbb{G}(n+k, 1/2)$, $U \subseteq V(G)$ with $|U| = k$, and let $J, J' \subseteq [n]$. Defining $G_J, G_{J'}$ as in Equation (1), the graphs $G_J[U]$ and $(G_J \triangle G_{J'})[U]$ are independent.*

Proof. We use a different but equivalent sampling procedure to obtain G_J and $G_{J'}$. First, sample $G_J \sim \mathbb{G}(n, 1/2)$ uniformly at random, which is the correct distribution by Proposition 13. In particular, the sets $E(G_J[U])$ and $E(G_J) \setminus E(G_J[U])$ are independent. Next, compute $G_{J'}$ directly using Equation (2). Observe that the set $E(G_J \triangle G_{J'})$ is a function only of $E(G_J) \setminus E(G_J[U])$ since this procedure for obtaining $G_{J'}$ depends on the edges incident to the v_i's. Thus the graphs $G_J[U]$ and $(G_J \triangle G_{J'})[U]$ are independent, as desired. ◀

We now perform the second moment calculation.

▶ **Lemma 26.** *Let $G \sim \mathbb{G}(n+k, 1/2)$. Fix a vertex set $U \subseteq V(G)$ with $|U| = k$ and a graph H on U. Then H is a vertex-minor of G with probability at least $1 - 2^{2k}(3/4)^n - 2^{-n+\binom{k}{2}+2k\log_2 n}$.*

Proof. For all $J \subseteq [n]$, construct G_J as in (1). Let X_J be the indicator that $G_J[U] = H$, and let $X = \sum_{J\subseteq[n]} X_J$. Recall that $\mathbb{E}[X] = 2^{n-\binom{k}{2}}$ by linearity of expectation and Proposition 13. We compute

$$\begin{aligned}
\mathrm{Var}[X] &= \sum_{J,J'\subseteq[n]} \mathbb{E}[X_J X_{J'}] - \mathbb{E}[X_J]\mathbb{E}[X_{J'}] \\
&= \sum_{J,J'\subseteq[n]} \mathbb{P}[G_J[U] = H \wedge G_J[U] = G_{J'}[U]] - \mathbb{P}[G_J[U] = H]\mathbb{P}[G_{J'}[U] = H] \\
&= \sum_{J,J'\subseteq[n]} \mathbb{P}[G_J[U] = H]\mathbb{P}[G_J[U] = G_{J'}[U]] - \mathbb{P}[G_J[U] = H]\mathbb{P}[G_{J'}[U] = H],
\end{aligned}$$

where we use the independence from Proposition 25. Recall that $G_J[U], G_{J'}[U]$ are both uniformly random, so by Lemma 24 we have

$$\begin{aligned}
\mathrm{Var}[X] &= \sum_{J\subseteq[n]} 2^{-\binom{k}{2}} \sum_{J'\subseteq[n]} \left(\mathbb{P}[G_J[U] = G_{J'}[U]] - 2^{-\binom{k}{2}}\right) \\
&= \sum_{J\subseteq[n]} 2^{-\binom{k}{2}} \sum_{J'\triangle J\subseteq[n]} \left(\mathbb{P}[G_J[U] = G_{J'}[U]] - 2^{-\binom{k}{2}}\right) \\
&\leq \sum_{J\subseteq[n]} 2^{-\binom{k}{2}} \left(\sum_{\substack{J'\triangle J\subseteq[n] \\ |J'\triangle J|>2k}} 2^{2k-\binom{k}{2}-|J'\triangle J|} + \sum_{\substack{J'\triangle J\subseteq[n] \\ |J'\triangle J|\leq 2k}} 1 \right) \\
&= \sum_{J\subseteq[n]} 2^{-\binom{k}{2}} \left(\sum_{i=2k+1}^{n} \binom{n}{i} 2^{2k-\binom{k}{2}-i} + \sum_{i=0}^{2k} \binom{n}{i} \right) \\
&\leq \sum_{J\subseteq[n]} 2^{-\binom{k}{2}} \left(2^{2k-\binom{k}{2}} \sum_{i=0}^{n} \binom{n}{i} 2^{-i} + n^{2k} \right) \\
&\leq \sum_{J\subseteq[n]} 2^{-\binom{k}{2}} \left(2^{2k-\binom{k}{2}} (3/2)^n + n^{2k} \right) \\
&= 2^{n-\binom{k}{2}} \left(2^{2k-\binom{k}{2}} (3/2)^n + n^{2k} \right).
\end{aligned}$$

(To obtain the bound $\sum_{i=0}^{2k} \binom{n}{i} \leq n^{2k}$, consider for example the surjection $f : [n]^{2k} \to \binom{n}{\leq 2k}$ given by $f(x_1, \ldots, x_{2k}) = \{x_1, \ldots, x_{2k}\}$, except $f(2k, 2k-1, \ldots, 1) = \emptyset$.) In conclusion, by Chebyshev's inequality (Lemma 14) we have

$$\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \leq \frac{2^{n-\binom{k}{2}} \left(2^{2k-\binom{k}{2}} (3/2)^n + n^{2k} \right)}{2^{2n-2\binom{k}{2}}} = 2^{2k}(3/4)^n + 2^{-n+\binom{k}{2}} n^{2k}$$

as desired. ◀

We are now ready to prove our main result via the union bound over all choices of $U \subseteq V(G)$ and graphs H with vertex set U. We first restate the theorem for convenience.

▶ **Theorem 3.** *For all $c > 0$, if $G \sim \mathbb{G}(n, 1/2)$ with $n \geq (1+c)\frac{1}{2\log_2(4/3)}k^2$, then G is k-vertex-minor universal with probability at least $1 - 2^{-(1+o(1))ck^2/2}$.*

Proof. Let $G \sim \mathbb{G}(n, 1/2)$, where $n = (1+c)\frac{1}{2\log_2(4/3)}k^2$. Here we allow $c = c(k)$ to be increasing with k, but it is lower bounded by some positive constant. By Lemma 26, the probability that a given graph H on k given vertices of $V(G)$ is not a vertex-minor of G is at most $2^{2k}(3/4)^{n-k} + 2^{-n+k+\binom{k}{2}}(n-k)^{2k} = 2^{2k}(3/4)^{(1+o(1))n}$.

By the union bound, we have

$$\begin{aligned}\mathbb{P}[G \text{ is not } k\text{-vertex-minor universal}] &\leq \binom{n}{k} 2^{\binom{k}{2}} \cdot \left(2^{2k}(3/4)^{(1+o(1))n}\right) \qquad (3)\\ &\leq \exp_2\left(k\log_2 n + \frac{k^2}{2} + 2k - (1+o(1))\log_2(4/3)n\right)\\ &= \exp_2\left(O(k\log(ck)) + \frac{k^2}{2} - (1+o(1))\frac{1+c}{2}k^2\right)\\ &= \exp_2\left(O(k\log(ck)) - (1+o(1))\,ck^2/2\right).\end{aligned}$$

Thus, whenever $n = (1+c)\frac{1}{2\log_2(4/3)}k^2$, we have $\mathbb{P}[G \text{ is } k\text{-vm-universal}] \geq 1 - 2^{-(1+o(1))ck^2/2}$, as desired. ◀

3.3 Proof of reordering Lemma 12

First, we restate the vertex-minor reordering lemma:

▶ **Lemma 12.** *Fix a set $\hat{V}$. Let $v_1, \ldots, v_\ell \in \hat{V}$ be a (possibly repeating) sequence and let $\hat{G}$ be a graph with $V(\hat{G}) = \hat{V}$. Then there is a sequence $x_1, \ldots, x_r$ of singletons and pairs from $\hat{V}$ such that for any graph G with $\hat{V} \subseteq V(G)$ and $G[\hat{V}] = \hat{G}$, we have*

(I) *$G * v_1 * v_2 \cdots * v_\ell - \hat{V} = G \circ x_1 \cdots \circ x_r - \hat{V}$.*

(II) *The sets $x_1, \ldots, x_r$ are pairwise disjoint.*

(III) *If $v \in \hat{V}$ appears exactly once in $(v_i)_{i=1}^{\ell}$, then $v \in x_j$ for some $j \in [r]$.*

Moreover, any choice of $(x_i)_i$ satisfying (I) and (II) for all G also satisfies (III).

We note that if $\hat{V}$ is an independent set in G, the complementations $*v_i$ are commutative and self-inverse, so we can choose $(x_i)_i$ to be any ordering of those vertices v_i which appear in the sequence an odd number of times. If not, then the complementations interact in some fixed way determined only by $G[\hat{V}]$. The proof idea is to pull out one *special* vertex $v \in \hat{V}$ at a time, and use Theorem 11 to force that v appears at most once in the resulting sequence $(x_i)_i$. We do this via the following inductive lemma in which the ordering may depend on the host graph G. We will then show that the ordering does in fact work for all graphs G with $G[\hat{V}] = \hat{G}$.

▶ **Lemma 27.** *For any graph G with $V(G) = \hat{V} \sqcup U$ and $v_1, \ldots, v_\ell \in \hat{V}$, and for any special vertex $v \in \hat{V}$, there exists a sequence $(y_i)_{i=1}^r$ satisfying (I) and (II) for G such that one of the following holds:*

1. *we have $v \notin \bigcup_i y_i$,*
2. *we have $v \in y_1$, or*
3. *we have $y_2 = \{v\}$ and $y_1 = \{w\}$ for some $w \in \hat{V}$ with $vw \in E(G)$.*

Proof. Let $H = G * v_1 * \cdots * v_\ell[U]$, so H is a vertex-minor of G. We proceed by induction on $n := |V(G)|$; the base case $n = 1$ is trivial. Let $v \in \hat{V}$ be arbitrary.

By Theorem 11, H is a vertex-minor of $G - v$, $G * v - v$, or $G \times vw - v$ for some w adjacent to v. In the first case, we apply the induction hypothesis to $G - v$, and clearly the resulting sequence $(y_i)_i$ satisfies item 1. In the second case, we apply the induction hypothesis to $G * v - v$ to obtain the sequence $(y_i)_{i \geq 2}$. Then clearly setting $y_1 = \{v\}$ satisfies item 2.

In the final case, we apply the induction hypothesis to $G \times vw - v$ for some w adjacent to v such that w is the special vertex to obtain the sequence $(y_i)_{i \geq 2}$. We then split into cases based on the location of w. If $w \notin \bigcup_{i \geq 2} y_i$, then setting $y_1 = \{v, w\}$ gives the desired sequence

$(y_i)_{i\geq 1}$ satisfying item 2. If $y_2 = \{w\}$, then note $G \times vw * w = G * w * v$, and so replacing y_2 with $\{v\}$ and setting $y_1 = \{w\}$ gives the desired sequence $(y_i)_{i\geq 1}$ satisfying item 3. If $y_2 = \{w, u\}$, then by Lemma 10, $G \times vw \times wu = G \times vu$, and so replacing y_2 with $\{v, u\}$ yields the desired sequence $(y_i)_{i\geq 2}$ satisfying item 2. Finally, if $y_2 = \{u\}, y_3 = \{w\}$ with $uw \in E(G \times vw)$, then we use the fact that $G \times vw * u * w = G \times vw \times uw * u = G \times vu * u = G * u * v$, and so by replacing y_2, y_3 with $\{u\}, \{v\}$ respectively, we get the desired sequence $(y_i)_{i\geq 2}$ satisfying item 3. ◀

Finally, to construct a sequence $(x_i)_i$ which satisfies Lemma 12 for all graphs G with $G[\hat{V}]$, we apply 27 to a special gadget which encodes all possible structures that can appear in G.

Proof of Lemma 12. Let $\hat{V}$ be a set, and fix a sequence $v_1, \ldots, v_\ell \in \hat{V}$ and a graph $\hat{G}$ on vertex set $\hat{V}$. To show that we can pick $(x_i)_i$ to depend only on $\hat{G}$, we apply Lemma 27 to a universal "gadget" graph defined as follows. Let $\mathbf{G}$ have vertex set $\hat{V} \cup [3 \cdot 2^{3|\hat{V}|}]$ by setting $\mathbf{G}[\hat{V}] = \hat{G}$, placing $2^{3|\hat{V}|}$ disjoint copies of the 3 vertex path P_3, and adding edges from $\hat{V}$ to the copies of P_3 such that all of the $2^{3|\hat{V}|}$ possible sets of edges occur between $\hat{V}$ and a copy of P_3. This has the property that for any graph G on vertex set $\hat{V} \cup \{u_1, u_2\}$ with $G[\hat{V}] = \hat{G}$, there exist j_1, j_2 such that $\mathbf{G}[\hat{V} \cup \{j_1, j_2\}]$ is isomorphic to G with the isomorphism $\hat{V} \mapsto \hat{V}$ and $u_i \mapsto j_i$. Such an isomorphism can be found by taking the copy of P_3 with the correct adjacency to $\hat{V}$ and letting j_1, j_2 be either adjacent or nonadjacent vertices in the P_3.

▷ Claim 12.1. For all $k \geq 2$, if the sequence $(x_i)_{i=1}^r$ satisfies properties (I) and (II) for the gadget $\mathbf{G}$, then it satisfies property (III).

Proof. Suppose $v \in \hat{V}$ appears exactly once in $\{v_i\}_i$ in position i^*, and suppose $(x_i)_i$ does not contain v. Expanding out $\circ(x_i)_i = *(w_i)_{i=1}^m$ as a sequence of local complementations for $w_i \in \hat{V} \setminus \{v\}$, for $\mathbf{G}' = \mathbf{G} * v_1 * \cdots * v_{i^*-1}$, we have

$$\mathbf{G}' * v = \mathbf{G}' * v_{i^*-1} * \cdots * v_1 * w_1 * \cdots * w_m * v_\ell * \cdots * v_{i^*+1}. \tag{4}$$

We show this is impossible. Let RHS denote the right-hand side of (4). Choose vertices $j_1, j_2 > n$ such that $j_1 j_2 \notin E(\mathbf{G})$ and $N_{\mathbf{G}}(j_1) \cap \hat{V} = N_{\mathbf{G}}(j_2) \cap \hat{V} = \{v\}$; these vertices exist by definition of $\mathbf{G}$. Then we note $j_1 j_2 \notin E(\mathbf{G}')$ and $N_{\mathbf{G}'}(j_1) = N_{\mathbf{G}}(j_1)$, $N_{\mathbf{G}'}(j_2) = N_{\mathbf{G}}(j_2)$. Then $j_1 j_2 \in E(\mathbf{G}' * v)$ but $j_1 j_2 \notin E(\text{RHS})$, because no local complementation step can add an edge incident to j_1, j_2. This contradicts (4), so $(x_i)_i$ contains v. ◁

We now show that if a sequence satisfies Lemma 27 for $\mathbf{G}$, then it satisfies Lemma 12. Let $(x_i)_{i=1}^r$ be the sequence given by Lemma 27 for the gadget $\mathbf{G}$. Let G be an arbitrary graph with $\hat{V} \subseteq V(G)$ and $G[\hat{V}] = \hat{G}$. We need to show that the sequence satisfies condition (I) for G. For this it suffices to show that for all $u_1, u_2 \in V(G) \setminus \hat{V}$, we have $u_1 u_2 \in E(G * (v_i)_{i=1}^\ell)$ if and only if $u_1 u_2 \in E(G \circ (x_i)_{i=1}^j)$. Indeed, letting $G' = G[\hat{V} \cup \{u_1, u_2\}]$, there exist $j_1, j_2 \in V(\mathbf{G})$ such that $\mathbf{G}[\hat{V} \cup \{j_1, j_2\}]$ is isomorphic to G with isomorphism $\hat{V} \mapsto \hat{V}$ and $u_i \mapsto j_i$. In particular, this means $G' * (v_i)_i - \hat{V} = G' \circ (x_i)_i - \hat{V}$, so we conclude that $u_1 u_2 \in E(G' * (v_i)_i)$ if and only if $u_1 u_2 \in E(G' * (v_i)_i)$, as desired. Thus we conclude $G * (v_i)_i - \hat{V} = G \circ (x_i)_i - \hat{V}$, so all three properties hold. ◀

4 Vertex-minor Ramsey and concluding remarks

We now prove our bounds on the vertex-minor Ramsey problem stated in the introduction.

▶ **Theorem 8.** *For all $k \geq 0$,*

$$(1 + o(1))\frac{1}{2\log_2 3}k^2 \leq R_{\mathrm{vm}}(k) \leq 2^k - 1.$$

Proof. We first prove the upper bound. When $k = 0, 1$ this is satisfied with equality. If $k \geq 2$, let $N = R_{\mathrm{vm}}(k) - 1$, so there is a graph G on N vertices with $U \subseteq V(G)$ an independent set in G such that $|U| \leq k-1$ and G does not contain a larger independent set as a vertex-minor. For all $S \subseteq U$, let $V_S = \{v \in V(G) \setminus U : N(v) \cap U = S\}$. Then we have the disjoint union $V(G) = U \cup \bigcup_{S \subseteq U} V_S$. We now claim that $|V_S| \leq \min\{2, |S|\}$ for all $S \subseteq U$. Assuming this, we conclude

$$R_{\mathrm{vm}}(k) - 1 = N \leq (k-1) + 1 \cdot \binom{k-1}{1} + 2 \cdot (2^{k-1} - k) = 2^k - 2.$$

Note that if $x \in V_\emptyset$, then $U \cup \{x\}$ is an independent set, contradicting the maximality of U. Suppose towards a contradiction that $x, y \in V_{\{u\}}$ for some $u \in U$. By locally complementing at u, we may assume that $xy \notin E(G)$. Then $U \cup \{x, y\} \setminus \{u\}$ is an independent set, contradicting the maximality of U.

Suppose towards a contradiction that $|S| \geq 2$ and $x, y, z \in V_S$. Let $u \in S$. By locally complementing at u, we may assume that $G[\{x, y, z\}]$ contains at most 1 edge. If $\{x, y, z\}$ is an independent set in G, then $U \cup \{y, z\} \setminus \{u\}$ is an independent set in $G \times ux$. Otherwise without loss of generality $yz \in E(G)$. Then $U \cup \{y, z\} \setminus \{u\}$ is an independent set in $G * x * u$. In every case we obtain an independent set of size larger than U as a vertex-minor, a contradiction.

For the lower bound, let $G \sim \mathbb{G}(n, 1/2)$. Note that G contains I_k as a vertex-minor if and only if G is locally equivalent (l.e.) to some graph H with $\alpha(H) \geq k$. A result of Bahramgiri and Beigi [2] states that for any graph H on n vertices, there are at most 3^n graphs locally equivalent to H. Then, by a union bound,

$$\begin{aligned}\mathbb{P}[G \text{ l.e. } H,\ \alpha(H) \geq k] &\leq 3^n \mathbb{P}[\alpha(G) \geq k] \\ &\leq 3^n \binom{n}{k} 2^{-\binom{k}{2}} \leq \exp_2\left(n \log_2 3 + k \log_2 n - k^2/2 + k/2\right) < 1,\end{aligned}$$

as long as $n \leq (1 + o(1))\frac{1}{2\log_2 3}k^2$. ◀

We note that our upper bound is the correct answer for $k \leq 3$. (For $k = 3$, it is not hard to verify that the wheel on 6 vertices does not contain an independent of size 3 as a vertex-minor.) However, since the lower bound in Theorem 8 comes from the random graph, it is natural to believe that the polynomial lower bound is closer to the truth for sufficiently large k than our modest upper bound. This belief leads us to Conjecture 9, restated below.

▶ **Conjecture 9.** $R_{\mathrm{vm}}(k) \leq \mathsf{poly}(k)$.

By Theorem 3, the random graph $\mathbb{G}(n, 1/2)$ cannot be a counterexample, and indeed it is possible that $R_{\mathrm{vm}}(k) = O(k^2)$. We note that proper vertex-minor closed classes have the Erdős-Hajnal property [10]. This implies that if we restrict our graphs to any proper vertex-minor closed class, they contain an independent set of size $\Omega(n^c)$ as a vertex-minor for some $c > 0$ possibly depending on the class. We also make the connection between pivot-minor universality and its corresponding Ramsey-type question.

▶ **Definition 28.** *Let $R_{\mathrm{piv}}(k)$ be the minimum n such that any n-vertex graph contains an independent set or clique of size k as a pivot-minor.*

Note that unlike the definition of $R_{\mathrm{vm}}(k)$, we need to allow either a clique or an independent set in this definition since K_n does not contain any non-trivial independent set as a pivot-minor.

We note that trivially $R_{\mathrm{vm}}(k-1) \leq R_{\mathrm{piv}}(k) \leq R(k)$, the diagonal Ramsey number. Theorem 5 supports the following conjecture, which would imply Conjecture 9.

▶ **Conjecture 29.** $R_{\mathrm{piv}}(k) \leq \mathsf{poly}(k)$.

We note that proper pivot-minor closed classes also have the Erdős-Hajnal property [19], so this conjecture holds if we restrict to any proper pivot-minor closed class. We also improve the trivial upper bound of $R(k)$ for $R_{\mathrm{piv}}(k)$.

▶ **Theorem 30.** *For all $k \geq 0$, $R_{\mathrm{piv}}(k) \leq (k-1)(2^k - 1) + 1$.*

Proof. Let G be a graph with no independent set or clique of size k as a pivot-minor on $R_{\mathrm{piv}}(k) - 1$ vertices. By possibly replacing G with a pivot-equivalent graph, we let U be an independent set in G such that G does not contain a larger independent set as a pivot-minor. For all $S \subseteq U$, let $V_S = \{v \in V(G) \setminus U : N(v) \cap U = S\}$. Clearly $V_\emptyset$ is empty by the maximality of U. For S containing a vertex u, we claim that V_S can not contain an independent set of size 3. Indeed otherwise if $\{x, y, z\} \in V_S$ is independent, then $U \cup \{y, x\} \setminus \{u\}$ is an independent set in $G \times ux$. We also note that V_S can not contain an induced path on 3 vertices. Indeed if x, y, z are the vertices of an induced path in V_S in that order, then $U \cup \{y, x\} \setminus \{u\}$ is an independent set in $G \times ux$. Clearly V_S can not contain a clique of size k by definition of G. It follows that $G[V_S]$ is a disjoint union of cliques and thus $|V_S| \leq 2(k-1)$. Thus

$$R_{\mathrm{piv}}(k) - 1 \leq (k-1) + (2^{k-1} - 1)(2k-2) = (k-1)(2^k - 1). \qquad ◀$$

We make some final conjectures about vertex-minor universality. Theorem 3 and our proof of the lower bound in Theorem 8 together show that for any $\varepsilon > 0$, the random graph $G \sim \mathbb{G}(n, 1/2)$ is with high probability $(\alpha - \varepsilon)\sqrt{n}$-vertex-minor universal but not $(\beta + \varepsilon)\sqrt{n}$-vertex-minor universal, where $\alpha = \sqrt{2\log_2(4/3)} \approx 0.911$ and $\beta = \sqrt{2\log_2(3)n} \approx 1.78$. It is natural to conjecture that there is a sharp threshold for universality, as follows.

▶ **Conjecture 31.** *There exists $C \in [\sqrt{2\log_2(4/3)}, \sqrt{2\log_2(3)}]$ such that for any $\varepsilon > 0$, with high probability $G \sim \mathbb{G}(n, 1/2)$ is $(C - \varepsilon)\sqrt{n}$-vertex-minor universal and not $(C + \varepsilon)\sqrt{n}$-vertex-minor universal.*

It is also natural to ask for what values of p the random graph $\mathbb{G}(n, p)$ is k-vertex-minor universal for n polynomial in k. The main difficulty in extending our approach to $p \neq 1/2$ is that the random walk defined in Section 3.1 now has steps which are correlated to each other: after flipping an edge which was not yet revealed, its probability of existing is now $1 - p$

instead of p. If one could get around this issue, the resulting random walk would then have the second largest eigenvalue bounded in absolute value by the max of $1-2p^2, 1-2(1-p)^2$, and $1-2p(1-p)$. Assuming without loss of generality that $p \leq 1/2$, this means it would suffice to run the random walk for $O(k^2/p^2)$ steps. Thus we conjecture the following.

▶ **Conjecture 32.** *If $p = \omega(1/\sqrt{n})$ is at most $1/2$ and $G \sim \mathbb{G}(n,p)$ or $G \sim \mathbb{G}(n, 1-p)$, then G is k-vertex-minor universal with high probability for some $k = \Omega(p\sqrt{n})$.*

If true, this would qualitatively generalize Theorem 3, although unlike Theorem 3 it is not necessarily tight up to a constant for all p. Conjecture 32 would also imply that $\mathbb{G}(n,p)$ is not a counterexample to Conjecture 9 for any p. This follows for instance by noting that if p is outside the range $[n^{-1/4}, 1-n^{-1/4}]$, then $\mathbb{G}(n,p)$ already contains a clique or independent set of size $\Omega(n^{1/4}\log n)$.

After the appearance of an earlier draft of this paper, Chao and Xu [9] verified Conjecture 32 up to a logarithmic factor.

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