Glauber Dynamics for Random Field Ising Models on Bounded Degree Graphs

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Abstract

We study the ferromagnetic random field Ising model (RFIM) on a graph $G = (V, E)$ having maximal degree Δ, where the external field at each vertex is an i.i.d. random variable. When the random field distribution is sufficiently anti-concentrated, we prove that with high probability over the quenched randomness of the external field, the Glauber dynamics of this RFIM mixes in polynomial time as a consequence of a Poincaré inequality. This model is relevant to the Griffiths phase where the correlations decay exponentially fast in expectation over the quenched random field, but contraction does not hold point-wise due to the existence of weak fields that lead to low-temperature behavior. Previously, fast mixing of Glauber dynamics under large disorder was only proven on the integer lattice, and for RFIM on general graphs, only a sampling algorithm based on self-avoiding walks was known.

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1 Introduction

Consider a finite graph $G = (V, E)$, the random field Ising model (RFIM) on G at inverse temperature $\beta > 0$ is defined as the following random probability measure on $\{-1, +1\}^V$:

$$\mu_G(\sigma) \propto \exp\{H(\sigma)\}, \quad \sigma \in \{-1, +1\}^V, \tag{1.1}$$

where H is the following Hamiltonian

$$H(\sigma) = \beta \sum_{(u,v)\in E} \sigma_u\sigma_v + \sum_{u\in V} h_u\sigma_u. \tag{1.2}$$

Here $h = (h_u)_{u\in V}$ are i.i.d. random variables. The vector h is called the external field, serving as a quenched source of disorder in this model.

There has been much recent attention on RFIM on (subgraphs of) the integer lattice $\mathbb{Z}^d$ in the statistical physics literature. In $d = 2$, the existence of an arbitrarily small random field is predicted by Imry and Ma [20] to suppress the low temperature phase of the zero field ($h = 0$) Ising model, so that uniqueness and decay of correlations hold for any $\beta > 0$. The prediction was confirmed by Aizenman and Wehr [1], who proved that the expected influence on the spin at the origin by different boundary conditions at distance r away should decay with r. For the Gaussian case $h_u \sim \mathcal{N}(0, \sigma^2)$, the breakthrough work [12] confirmed that the influence decays exponentially in r at all $\beta > 0$ and $\sigma > 0$. This no longer holds for RFIM with dimension $d \geq 3$ and small disorder, where there is a phase transition as β varies. For small disorder, there is no decay of influence when $\beta = \infty$ [19] and when β sufficiently large [3], and this is recently proven throughout the entire low temperature regime [11].

Beyond the integer lattice, the RFIM on a general graph G has also attracted interest from a computational complexity perspective. For a large $\beta > 0$ and general external field h, estimating the partition function of μ_G up to a multiplicative error is a #BIS-hard problem in the worst case [4], and polynomial-time approximation algorithms are not expected to exist. However, the random field may turn this hard computational problem into a feasible one: for bounded degree graphs G and large enough σ where $h_u \sim \mathcal{N}(0, \sigma^2)$, [18] constructed a polynomial time approximation algorithm for the partition function of μ_G over a $1 - o(1)$ probability realization of the external fields h.

For a general graph G, it becomes much less tractable to study the influence decay of a RFIM with small or moderate disorder h_u (that is, smaller σ if $h_u \sim \mathcal{N}(0, \sigma^2)$). This is because current RFIM papers on $\mathbb{Z}^d$ that capture the influence decay at $d = 2$ or the phase change at $d = 3$ use the geometry of $\mathbb{Z}^d$ in a fundamental way and such a delicate description is hard to expect on a general graph G. As a natural next step, in this paper we study effective sampling algorithms for the RFIM μ_G on a general graph G. For sufficiently large disorder (say, larger σ), it is straightforward to verify the decay of correlations for this RFIM via a percolation argument.

In contrast, even for relatively large disorder, the RFIM still possesses major difficulties from a sampling perspective. We consider the Glauber dynamics for RFIM, where we recall that Glauber dynamics (introduced in [16]; see also [22] for more background) is a discrete time random process $(X_n(v))_{v \in V, n \in \mathbb{N}_+}$ reversible w.r.t. μ_G, that at each step n selects a vertex $v \in V$ uniformly with probability $\frac{1}{|V|}$ and then resamples $X_n(v)$ according to the conditional distribution

$$X_n(v) \sim \mu_G(\sigma_v \in \cdot \mid (\sigma_w)_{w \neq v} = (X_{n-1}(w))_{w \neq v}), \tag{1.3}$$

and where we set $X_n(w) = X_{n-1}(w)$ for all $w \neq v$. If we assume that all the $|h_u|$ are uniformly larger than some fixed constant, then it is straightforward to show that the Glauber dynamics for RFIM mixes in time $O(n \log n)$, as in [18], Theorem 7. However, in the typical RFIM setting, although most external fields h_u are large, there are large regions where the absolute values of external fields $|h_u|$ are small, and the Glauber dynamics does not contract at these vertices (see [18], Remark 7). To avoid this problem, [18] constructed a polynomial sampling algorithm for RFIM μ_G on general bounded degree graphs G utilizing the Weitz self-avoiding walk tree [24], which is a global algorithm that differs in spirit from the Glauber update rule. As a future direction, [18] asked whether one can use Markov chain based algorithms (such as Glauber dynamics) to effectively sample from RFIM.

For Glauber dynamics of RFIM on $\mathbb{Z}^d, d \geq 2$, [14] conducted a detailed study for the mixing time of μ_G and designed a sampling algorithm at weak disorder. Under a notion of weak spatial mixing that measures averaged influence decay within a ball, they constructed a polynomial time sampling algorithm for RFIM based on a weak Poincaré inequality for Glauber dynamics. Under a stronger notion of influence decay and in particular when random fields have large disorder (h_u have good anti-concentration), they prove that Glauber dynamics for μ_G mixes in polynomial time as a corollary of a full Poincaré inequality. However, the proof in [14] is highly dependent on the lattice structure of $\mathbb{Z}^d$ due to a coarse graining argument in Section 5, and in Section 4 also somewhat relies on a volume growth exponent on G that rules out graphs of exponential volume growth. Therefore, the Glauber dynamics of μ_G on general graphs G had remained open.

The main result of this paper is a Poincaré inequality for μ_G which implies polynomial time mixing for μ_G.

Let $\mathcal{E}_G$ denote the Dirichlet form of the (discrete time) Glauber dynamics, so that for any test function $\varphi : \{-1,+1\}^V \to \mathbb{R}$,

$$\mathcal{E}_G(\varphi,\varphi) := \frac{1}{2|V|} \sum_{\sigma\sim\sigma'} \frac{\mu_G(\sigma)\mu_G(\sigma')}{\mu_G(\sigma)+\mu_G(\sigma')} (\varphi(\sigma)-\varphi(\sigma'))^2, \tag{1.4}$$

so that the summation is taken over pairs $\sigma, \sigma' \in \{-1,1\}^V$ differing only at one coordinate. Then we define the spectral gap of the chain via

$$\mathbf{gap}_G := \inf_{\varphi} \frac{\mathcal{E}_G(\varphi,\varphi)}{\mathrm{Var}_G(\varphi)} \tag{1.5}$$

with the infimum taken over all test functions $\varphi : \{-1,1\}^V \to \mathbb{R}$ with non-zero variance.

1.1 Polynomial time mixing for Glauber dynamics of RFIM with large disorder

Before stating the main technical result, we outline the anti-concentration assumptions on the external field h in the following:

▶ **Assumption 1.1.** *Let $\Delta \geq 3$ and $p_0 \in (0,1)$ be a constant such that $p_0(\Delta-1) < 1$, let $K = K(p_0) > 0$ be a constant such that*

$$\rho = \rho(\beta,\Delta,K) := \frac{e^{\Delta\beta-K}}{e^{\Delta\beta-K}+e^{-\Delta\beta+K}}, \tag{1.6}$$

is less than $\frac{p_0}{4}$. Suppose that the RFIM μ_G with external field h is such that $(h_u)_{u\in V}$ are i.i.d. and such that

$$\mathbb{P}_h(|h_u| \leq K) < \frac{p_0}{2}.$$

We then denote by

$$\alpha_* = \frac{1}{2}\log\frac{(\Delta-2)^{\Delta-2}}{p_0(\Delta-1)^{\Delta-1}(1-p_0)^{\Delta-2}}.$$

Note that for fixed $\Delta > 0$, $\alpha_* \uparrow \infty$ as $p_0 \downarrow 0$.

For a graph $G = (V,E)$, we use $|V|$ for the number of vertices in G.

Our first main result on Glauber dynamics for RFIM for any bounded degree graph G is as follows:

▶ **Theorem 1.2.** *Let $G = (V,E)$ be a graph of degree at most Δ. Assume that Assumption 1.1 is satisfied for some $p_0 \in (0,1)$, then with $\mathbb{P}_h$- probability at least $1 - O_{\beta,\Delta,p_0}(\frac{1}{|V|})$ over the external field (here O_{β,Δ,p_0} denotes constants that may depend on β, Δ and p_0), the spectral gap of RFIM measure μ_G satisfies*

$$\mathbf{gap}_G \geq |V|^{-1}\exp\left(-\frac{16\beta\Delta\ln|V|}{\alpha_*}\right),$$

and the ϵ- mixing time of Glauber dynamics for μ_G is

$$t_{mix}(P,\epsilon) = O\left(|V|^{1+\frac{16\beta\Delta}{\alpha_*}}(\beta\Delta|V| + \|h\|_1 + \log(1/\epsilon))\right).$$

Here $\|h\|_1 := \sum_{x\in V}|h_x|$.

▶ Remark 1.3. One may also consider the following more general model with different edge weights β_{uv}. Suppose that for each edge $(u,v) \in E$ we assign a constant $\beta_{uv} \in \mathbb{R}$ such that $|\beta_{uv}| \leq \beta$. Consider the following Hamiltonian $\widetilde{H}$ and Ising model $\mu_{\widetilde{G}}$:

$$\widetilde{H}(\sigma) = \sum_{(u,v)\in E} \beta_{uv}\sigma_u\sigma_v + \sum_{u\in V} h_u\sigma_u, \quad \mu_{\widetilde{G}}(\sigma) \approx \exp\{\widetilde{H}(\sigma)\}, \quad \sigma \in \{-1,1\}^V. \tag{1.7}$$

In the ferromagnetic case $\beta_{uv} \geq 0$ with nonconstant edge weights, the same strategy should be adaptable by using edge-dependent tilting rates. We do not pursue this extension here. However, when some $\beta_{uv} < 0$, the edge-tilt localization scheme that we will use breaks down, so Theorem 1.2 does not readily extend to non-monotone systems.

▶ Remark 1.4. We note that on the integer lattice $\mathbb{Z}^d$, [14], Section 5 obtained a mixing time $O(|V|^{1+o(1)})$ for a high probability realization of h via a coarse graining argument (they considered a continuous time Glauber dynamics where vertices are updated at rate 1, and we rewrite their runtime in the discrete setting). We consider much more general graphs (such as expander graphs), and by setting $p_0 > 0$ sufficiently small (so that $\frac{16\Delta\beta}{\alpha_*} \leq \epsilon$) we can achieve $O(|V|^{2+\epsilon})$ mixing time for any $\epsilon > 0$. Note that the spectral gap scales like $|V|^{-1-\epsilon}$ when $p_0 > 0$ is small, which is almost optimal up to the $\epsilon > 0$.

We also note that the anti-concentration condition is designed for a worst case percolation argument to work through, and thus may not be the weakest possible condition. Previous papers concerning a large external field also work under these types of conditions, see [18] and [14], Section 5.

1.2 Overview of the proof

The RFIM measure μ_G is difficult to study by itself since it does not satisfy the Dobrushin condition or standard notions of spectral independence when β is large. Motivated by the recent application of localization schemes of Chen and Eldan [9], we will seek a continuous (and stochastic) deformation of μ_G to a target measure that is already known to satisfy a functional inequality such as Poincaré inequality, and that the deformation approximately preserves certain statistics, such as variance, of μ_G.

How can we choose a proper localization scheme for RFIM? The first candidate choice is the stochastic localization scheme in [9] where we use a Brownian motion on the localizing path. The exact path of localization still requires a delicate choice: one standard deformation is to use the localization path to weaken the quadratic interaction, so that μ_G is deformed to

$$\nu_t^1(\sigma) \propto \exp\{\beta(1-t) \sum_{(u,v)\in E} \sigma_u\sigma_v + \sum_{u\in V}(h_u + y_u^1(t))\sigma_u\}, \quad \forall\sigma \in \{-1,+1\}^V,$$

where $(y_u^1(t))_{u\in V}$ is a random external field defined by stochastic integrals of the adjacency matrix of G with respect to a Brownian motion, see [9], Fact 14 for the formula. This deformation, while useful for many important models (cf. [15], [9]), does not work well for RFIM μ_G because the external field h_u is changed to a random external field $h_u + y_u^1(t)$ that differs at each time t, and the quadratic interaction is also changed along the path. We find it hard to uniformly control ν_t^1 over all realizations of the stochastic localization path $y_u^1(t)$ with respect to **a single quenched realization** of the external field $(h_u)_{u\in V}$. In contrast, in the RFIM setting the authors of [14] chose another stochastic localization path that fixes the quadratic interaction and changes only the external field. Then μ_G deforms to

$$\nu_t^2(\sigma) \propto \exp\{\beta \sum_{(u,v)\in E} \sigma_u\sigma_v + \sum_{u\in V}(h_u + y_u^2(t))\sigma_u\}, \quad \forall\sigma \in \{-1,+1\}^V.$$

The distribution of this random external field $y_u^2(t)$ follows $t\sigma^* + B_t$ where B_t is a standard Brownian motion at time t and σ^* is independently sampled from $\sigma^* \sim \nu_0^2$. In this second localization, although the external field gains more randomness, the quadratic form is unchanged and the property WSM(C) can be passed along the localization path in a high probability sense (but not for every realization of the localization path). Thanks to this high probability preservation of WSM(C), the authors of [14] can transfer functional inequalities for ν_T^2 for a large T back to weak functional inequalities for μ_G. This idea is called virtually increasing the external field since the measure ν_T^2 is a RFIM with the same quadratic term but having a much stronger disorder in the random field. Via this strong disorder, they used a coarse graining idea on $\mathbb{Z}^d$ to verify that, for a sufficiently large T, the measure ν_T^2 satisfies a Poincaré inequality. This transfers back into weak Poincaré inequalities for μ_G on $\mathbb{Z}^d$, which shows rapid mixing of Glauber dynamics from a warm start. Then they constructed a polynomial time sampling algorithm from μ_G via Glauber dynamics on increasingly large subsets of the graph. However, the final step of coarse graining of ν_T^2 significantly restricts the geometry of G where the algorithm converges.

We take a different view towards designing localizing schemes of RFIM measures on general graphs. We need a localization path that (1) maintains a good control of the relevant covariance/correlation matrix of ν_t along the localization path, and the estimate holds for any possible pinning with respect to a single quenched external field h; and (2) the terminal measure is easy to analyze. The very recent paper [8] introduced an edge field localization scheme which is a piecewise deterministic jump Markov process that does not use SDEs. Notably, this localization path does not change the external field of RFIM and allows a quenched probability control over h. The construction is as follows. We first re-parametrize RFIM as a probability measure on $\{0,1\}^V$ rather than $\{-1,+1\}^V$. Then for any target measure μ on $\{0,1\}^V$, proceed as follows. Pick a family of events on the configuration $\sigma \in \{0,1\}^V$: $A_{(u,v)} := \{\sigma_u = \sigma_v = 1\}$ for each edge $(u,v) \in E$. Let $\mathcal{A}$ denote the collection of these events $A_{(u,v)}$. For the configuration σ denote by

$$Z_{A_{(u,v)}}(\sigma) = \mathbf{1}_{\sigma \in A_{(u,v)}}, \quad Z(\sigma) = \{A_{(u,v)} \in \mathcal{A} : \sigma \in A_{(u,v)}\}.$$

We first sample $X \sim \mu$, then for each $A_{(u,v)} \in \mathcal{A}$ we independently sample a uniform random variable $U_{(u,v)} \sim \mathrm{Unif}[0,1]$. Then we define the set of events revealed at time t:

$$T_t := \{A_{(u,v)} \in \mathcal{A} : X \in A, U_{A_{(u,v)}} \leq t\}.$$

Then the localization path is the following posterior distribution, for each $S \subset \mathcal{A}$:

$$\mu_t^S(\sigma) = \mathbb{P}(X = \sigma \mid T_t = S) \propto \mu(\sigma) \cdot (1-t)^{|Z(\sigma)|} \cdot \mathbf{1}[S \subset Z(\sigma)].$$

Then observe that μ_t^S is still an Ising model on G with vertices along edges that are associated to an event in S pinned at 1, and the quadratic interactions reduce from 4β to $4\beta + \ln(1-t)$. It remains to upper bound the operator norm of a certain correlation matrix of μ_t^S uniformly over all pinnings on S and uniformly over all quadratic interaction coefficients $4\beta + \ln(1-t)$ whenever it is within $[0, 4\beta]$. As the random external fields h are unchanged along the path, we can apply a percolation argument and take a grand coupling of all these tilted RFIM models to deduce, with very high probability over h, a uniform upper bound for operator norm of the correlation matrix. Finally, we take the terminal time $\theta_* = 1 - \exp(-4\beta)$, where the conditional measure is a product measure for any pinning S and approximate tensorization of variance directly follows. This approach only relies on a percolation argument and completely bypasses fine local geometries of $\mathbb{Z}^d$. Unlike other terminal-time choices based on high-temperature tensorization, the current edge-field denoising flow is stopped when the posterior becomes a product measure; therefore, no separate high-temperature case results are needed.

1.3 Facts, notations and conventions

1.3.1 Graph distance and boundary

Let $G = (V, E)$ be a graph. For any two vertices $u, v \in V$ we use $d(u, v)$ to denote the graph distance between u and v in G.

For a subset $A \subset V$, denote by $\partial A := \{v \in G : v \notin A, \exists w \in A, (v, w) \in E\}$ the boundary of A. For any fixed configuration $\tau \in \{-1, 1\}^{\partial A}$ on ∂A, we denote by μ_A^τ the restriction of the Ising model μ_G to A with the boundary condition τ.

Let $(u, v) \in E$ be an edge of G, for any $l \in \mathbb{N}_+$ we denote by $B((u, v), l)$ the l-neighborhood of the edge (u, v):

$$B((u, v), l) := \{w \in G : d(w, u) \leq l \text{ or } d(w, v) \leq l\} = B(u, l) \cup B(v, l). \tag{1.8}$$

1.3.2 Markov chain mixing times

Let P be a Markov operator on Ω with invariant measure ν, P being reversible with respect to ν. Let μ be an initial distribution that is absolutely continuous with respect to ν.

The total-variation mixing time is defined by

$$t_{mix}(P, \epsilon; \mu) = \min\{t > 0; |P^t[\mu](A) - \nu(A)| \leq \epsilon, \forall A \subset \Omega\},$$

and we set

$$t_{mix}(P, \epsilon) = \max_{x \in \Omega} t_{mix}(P, \epsilon, \delta_x).$$

Then we have the following standard fact, see, for example, [21, Chapters 12–13]:

▶ **Fact 1.5.** *Assume that for all $x \in \Omega$, $\nu(\{x\}) \geq \eta$. Then*

$$t_{mix}(P, \epsilon) \leq C\mathbf{gap}(P)^{-1}(\log(1/\eta) + \log(1/\epsilon)),$$

and

$$t_{mix}(P, \epsilon) \leq C\rho_{LS}(P)^{-1}(\log\log(1/\eta) + \log(1/\epsilon)).$$

1.3.3 FKG and coupling

If we assume that all the RFIM interactions are positive, then μ_G satisfies useful correlation inequalities including the FKG inequality. That is, for a subset $A \subset V$ and a non-decreasing function $\varphi : \{-1, 1\}^A \to \mathbb{R}$, the mean of φ is monotone in the boundary condition: $\mu_A^\tau(\varphi) \leq \mu_A^{\tau'}(\varphi)$ whenever boundary conditions satisfy $\tau \leq \tau'$ pointwise.

Again assuming monotonicity, for the two boundary conditions $\tau \leq \tau'$, it is standard to use the monotone coupling to couple μ_A^τ and $\mu_A^{\tau'}$. Specifically, we fix an ordering $v_1, \cdots, v_k$ of vertices in A and we then draw uniform random variables $U_1, \cdots, U_k$ independently. Then the coupling $(\sigma^\tau, \sigma^{\tau'})$ can be constructed sequentially where for each $i \geq 1$ we shall sample $\sigma_{v_i}^\tau$, $\sigma_{v_i}^{\tau'}$ from the conditional distributions given $(\sigma_{v_j}^\tau)_{j<i}$, τ and $(\sigma_{v_j}^{\tau'})_{j<i}, \tau'$ where we use the common randomness in U_i. The coupling is clearly monotone in the boundary conditions: $\sigma^\tau \leq \sigma^{\tau'}$ whenever $\tau \leq \tau'$.

Without assuming monotonicity, let $A \subset V$ and consider a class of Ising measures $\mu_1, \mu_2, \cdots, \mu_t$ on A with boundary conditions $\tau_1, \cdots, \tau_t$ on ∂A, where each Ising measure may have different interactions and external fields, and each μ_i may be further pinned at some $A_i \subset A$ by some value $\tau_i' \in \{-1,1\}^{A_i}$. Then we can use the same uniform random variable to couple $\mu_1, \cdots, \mu_t$ as above. Specifically, we again fix an ordering $v_1, \cdots, v_k$ of A and draw uniform random variables $U_1, \cdots, U_k$ independently. Then we construct a coupling of $(\sigma^1, \cdots, \sigma^t)$ where $\sigma^1 \sim \mu_1, \cdots, \sigma^t \sim \mu_t$ sequentially so that for each $i \geq 1$ we sample $\sigma^1_{v_i}, \cdots, \sigma^t_{v_i}$ from their respective conditional distributions given $(\sigma^r_{v_j})_{j<i}, 1 \leq r \leq t$ and $\tau_1, \cdots, \tau_t$, using U_i as common randomness. If v_i is already pinned in μ_r then we output the pinned value for $\sigma^r_{v_i}$.

1.4 Change of coordinates for RFIM

The current RFIM measure is defined on $\{-1,+1\}^V$ but we will frequently change its coordinates to a measure on $\{0,1\}^V$. The following fact summarizes the change of coordinates:

▶ **Fact 1.6.** *Let μ_G be the RFIM measure* (1.1) *defined on $\{-1,+1\}^V$. Then it admits an equivalent expression*

$$\mu_G(\sigma') \propto \exp\{4\beta \sum_{(u,v)\in E} \sigma'_u \sigma'_v + \sum_u (2h_u - 2\beta d_u)\sigma'_u\}, \quad \forall \sigma' \in \{0,1\}^V,$$

where $\sigma'_u = \frac{\sigma_u + 1}{2}$ for each $u \in V$. Here d_u is the degree of u in G for each $u \in V$. Similarly, the RFIM measure $\mu_{\widetilde{G}}$ defined in (1.7) *admits an equivalent expression*

$$\mu_{\widetilde{G}}(\sigma') \propto \exp\{4 \sum_{(u,v)\in E} \beta_{uv} \sigma'_u \sigma'_v + \sum_u (2h_u - 2 \sum_{v:(u,v)\in E} \beta_{uv})\sigma'_u\}, \quad \forall \sigma' \in \{0,1\}^V.$$

Namely, after the change of coordinates, the external fields are shifted by a deterministic, vertex dependent constant but remain independent.

2 Edge-field localization schemes and spectral independence

In this section we introduce the general framework of edge tilted localization process of [8]. Then we introduce an edge field localization process for the RFIM μ_G and provide criteria for its spectral stability.

2.1 Entropy conservation and localization schemes

Consider a domain $D \subseteq \mathbb{R}$ and a convex function $\phi : D \to \mathbb{R}$. Consider μ a distribution on a finite set Ω and sample $X \sim \mu$. For any real-valued function $f : \Omega \to \mathbb{R}$, define the ϕ-entropy of f with respect to μ by

$$\mathrm{Ent}^\phi_\mu[f] := \mathrm{Ent}^\phi[f(X)] := \mathbb{E}[\phi(f(X))] - \phi(\mathbb{E}[f(X)]).$$

If ν is a probability measure on Ω absolutely continuous with respect to μ, we define the ϕ-divergence between ν and μ via

$$D_\phi(\nu \,||\, \mu) := \mathrm{Ent}^\phi_\mu\left[\frac{\nu}{\mu}\right].$$

In this paper we focus on the following two specific cases

- $\phi(x) = \chi^2(x) := x^2$, so that χ^2-entropy of f is equivalent to the variance of f, and
- $\phi(x) = \mathbf{KL}(x) := x \log x$, which induces the KL-divergence

$$D_{KL}(\nu \,||\, \mu) = \sum_{\sigma\in\Omega} \nu(\sigma) \log \frac{\nu(\sigma)}{\mu(\sigma)}.$$

We now introduce some notations from [8] on the conservation and decay of entropy for a Markov operator. To be consistent with the notations in [8], we use the state space $\{0,1\}^n$ rather than $\{-1,1\}^n$. The formula for the RFIM measure μ_G on the new coordinate $\{0,1\}^n$ is explicitly written in Fact 1.6.

▶ **Definition 2.1** (Approximate tensorization of ϕ-entropy). *We say that a distribution μ on $\{0,1\}^n$ satisfies K-approximate tensorization of ϕ-entropy if for any function $f:\Omega\to\mathbb{R}$ and any $X\sim\mu$,*

$$\mathrm{Ent}^{\phi}[f(X)] \le K\cdot\sum_{i=1}^{n}\mathbb{E}[\mathrm{Ent}^{\phi}[f(X)\mid X_{-i}]],$$

where we denote by X_{-i} the configuration X restricted to $[n]\setminus\{i\}$.

▶ **Definition 2.2** (Decay of entropy). *Consider a Markov chain $(X_t)_{t\ge 0}$ on Ω having transition matrix P and stationary distribution μ. Then we say that it satisfies decay of ϕ-entropy with rate κ if for all functions $f:\Omega\to\mathbb{R}$,*

$$\mathrm{Ent}^{\phi}_{\mu}[Pf] \le (1-\kappa)\,\mathrm{Ent}^{\phi}_{\mu}[f].$$

Then we introduce the general notion of a localization process in [8]. We will later specify the exact localization scheme that we shall use.

▶ **Definition 2.3.** *Consider μ a probability distribution on Ω. Then a localization scheme having μ as target distribution is made up of a pair of continuous time processes:*
- *The noising process $(X_t)_{t\in[0,1]}$. This is a Markov process such that $X_0\sim\mu$ and X_1 follows a Dirac measure.*
- *A denoising process $(Y_t)_{t\in[0,1]}$. This process is defined as the reversal of $(X_t)_{t\in[0,1]}$: $Y_\theta = X_{1-\theta}$. Thus Y_0 is a Dirac measure and $Y_1\sim\mu$.*

Since X_t is a time reversal of Y_t, we use the denoising process $(Y_t)_{t\in[0,1]}$ to represent the localization scheme and call it the localization process.

We would like the localization path Y_t to have the following nice property:

▶ **Definition 2.4** (Approximate conservation of ϕ-entropy). *For a given denoising process $(Y_t)_{t\in[0,1]}$ and $\theta\in[0,1]$, the process $(Y_t)_{t\in[0,1]}$ satisfies R-conservation of ϕ-entropy up to time θ if for all functions $f:\Omega\to\mathbb{R}_+$,*

$$\mathrm{Ent}^{\phi}[f(Y_1)] \le R\cdot\mathbb{E}[\mathrm{Ent}^{\phi}[f(Y_1)\mid Y_\theta]].$$

This general notion of localization schemes gives rise to a boosting of mixing properties from a parameter regime where mixing is known, to a new parameter regime where we expect to prove mixing.

▶ **Lemma 2.5** ([6], Lemma 3.13). *Consider $(Y_t)_{t\in[0,1]}$ a denoising process for a distribution μ. Assume we can find some $\theta\in[0,1]$ such that*
1. *The conditional distribution $\mathrm{Law}(Y_1\mid Y_\theta)$ satisfies K-approximate tensorization of ϕ-entropy.*
2. *The denoising process $(Y_t)_{t\in[0,1]}$ satisfies R-approximate conservation of ϕ-entropy up to time θ.*

Then μ satisfies (KR)-approximate tensorization of ϕ-entropy.

The following property is very useful in verifying approximation conservation of ϕ-entropy:

▶ **Definition 2.6.** *Consider the denoising process $(Y_t)_{t\in[0,1]}$ for a measure μ on Ω. Then the process $(Y_t)_{t\in[0,1]}$ is said to be ϕ-entropically stable with rate C at $\theta \in [0,1]$ if for any given function $f : \Omega \to \mathbb{R}$, denoting by $F = f(Y_1)$, then for any configuration T with $\mathbb{P}(Y_\theta = T) > 0$, we have*

$$\lim_{h\to 0^+} \frac{1}{h} \operatorname{Ent}^\phi[\mathbb{E}[F \mid Y_{\theta+h}] \mid Y_\theta = T] \leq \frac{C}{1-\theta} \operatorname{Ent}^\phi[F \mid Y_\theta = T].$$

The property of ϕ-entropic stability implies approximate conservation of ϕ-entropy, as outlined in the following lemma:

▶ **Lemma 2.7** ([6], Theorem 3.16)**.** *Given $C : [0,1] \to \mathbb{R}_{\geq 0}$. Assume that a denoising process $(Y_t)_{t\in[0,1]}$ is ϕ-entropically stable with rate $C(t)$ throughout $t \in [0,1]$. Then for each given $\theta \in [0,1]$, the process $(Y_t)_{t\in[0,1]}$ will satisfy R-approximate conservation of ϕ-entropy up to time θ where we take*

$$R = \exp\left(\int_0^\theta \frac{C(t)}{1-t} dt\right).$$

Therefore, in order to deduce approximate tensorization of ϕ-entropy of μ, we only need to construct a denoising process Y_t which is ϕ-entropically stable.

From approximate tensorization to functional inequalities for Glauber dynamics

We recall here the fact that approximate tensorization of variance (χ^2 entropy) implies a Poincaré inequality for Glauber dynamics:

▶ **Fact 2.8.** *A distribution μ on $\{0,1\}^n$ satisfies C_1- approximate tensorization of variance if and only if the spectral gap for the Glauber dynamics for μ satisfies* $\mathbf{gap}_{GL} \geq \frac{1}{C_1 n}$.

A similar statement holds for modified log-Sobolev inequalities:

▶ **Fact 2.9** ([10], Fact 3.5)**.** *If a distribution μ on $\{0,1\}^n$ satisfies C_1- approximate tensorization of entropy, then the modified log-Sobolev inequality holds for the Glauber dynamics with constant $\rho_0 \geq \frac{1}{C_1 n}$.*

2.2 A special case: vertex field denoising

An important case of localization schemes is the vertex-tilting field dynamics introduced in [7]. Our edge-tilting localization scheme will also be built on this vertex field version, so we give a quick review here. We consider distributions on $\{0,1\}^n$ and identify a configuration $\sigma \in \{0,1\}^n$ with the subset $\{v \in [n] : \sigma_v = 1\}$.

▶ **Definition 2.10.** *Consider a distribution μ on $\{0,1\}^n$ and a sample $X \sim \mu$. Assign, for each $v \in [n]$, an independent random variable $U_v \sim \mathrm{Uniform}[0,1]$. Then the vertex field denoising process with respect to μ is defined by the following process $(Y_t)_{t\in[0,1]}$:*

$$Y_t := \{v \in [n] : X_v = 1 \wedge U_v \leq t\}.$$

For a $\theta \in [0,1]$, the tilted distribution $\theta * \mu$ is defined as

$$(\theta * \mu)(\sigma) \propto \mu(\sigma) \cdot \theta^{|\sigma|}, \quad \forall \sigma \in \{0,1\}^n,$$

where $|\sigma| := |\{v \in [n] : \sigma_v = 1\}|$.

▶ **Lemma 2.11** ([8], Proposition 2.14). *For a distribution μ, let $(Y_t)_{t\in[0,1]}$ be the corresponding vertex field denoising process. Then we have*

$$\mathrm{Law}(Y_1 \mid Y_t) = (1-t) * \mu^{Y_t},$$

such that μ^{Y_t} is the conditional law obtained by setting all variables in Y_t pinned to be 1.

Spectral stability of vertex-field denoising process has the following characterization:

▶ **Lemma 2.12** ([5], Proposition 3.3). *Let $(Y_t)_{t\in[0,1]}$ be a vertex-field denoising process with respect to μ on $\{0,1\}^n$. Then for any fixed $\theta \in [0,1]$, the following two claims are equivalent:*
- *$(Y_t)_{t\in[0,1]}$ is spectrally stable with rate C at time θ;*
- *Given any subset S with $\mathbb{P}[Y_\theta = S] > 0$,*

  $$\mathrm{Cov}((1-\theta) * \mu^S) \lesssim C \cdot \mathrm{diag}\{\mathbf{m}((1-\theta) * \mu^S)\},$$

  where for a distribution ν, $\mathbf{m}(\nu) := \mathbb{E}_{X\sim\nu}[X]$ is its mean vector.

2.3 Edge-field tilting and its localization schemes

The vertex-field denoising process introduced in the last section changes the strength of the external field of an Ising model but keeps quadratic interaction unchanged. This is not sufficient for the RFIM where we wish to weaken the quadratic interaction but to keep the external field fixed. The recent paper [8] pointed out that our goal can be fulfilled by considering a tilting of the edge-field instead of vertex-field, so we remove edges rather than vertices. Before presenting our concrete constructions of edge-field denoising, we outline the abstract framework of [8].

For a given distribution μ on $\{0,1\}^V$ where V is a ground set, let $\mathcal{A}$ be a collection of events, so that each $A \in \mathcal{A}$ is a subset of $\{0,1\}^V$. We take $X \sim \mu$ and for any event $A \in \mathcal{A}$ define $Z_A = Z_A(X) := \mathbf{1}[X \in A]$. Let $Z = Z(X) := (Z_A)_{A\in\mathcal{A}}$.

We let π denote the joint distribution of (X, Z), so that

$$\pi := \mathrm{Law}(X, Z(X)),$$

so that π is a distribution over $\{0,1\}^{V\cup\mathcal{A}}$. Then by construction, $\mu = \pi_V$ and $\mathrm{Law}(Z) = \pi_{\mathcal{A}}$.

▶ **Definition 2.13** (Event-field dynamics, [8]). *Let $\theta \in [0,1]$. The event-field dynamics for μ with respect to $\mathcal{A}$ and tilt θ is the following defined Markov chain on the state space $\Omega(\mu)$:*

From the current state $X_t \in \Omega(\mu)$, the next state X_{t+1} is generated via
1. *($V \to \mathcal{A}$). Set $Z_t := Z(X_t) = \{A \in \mathcal{A} \mid X_t \in A\}$.*
2. *(Down-$\mathcal{A}$) Then generate a random subset $T \subset Z_t$ where we independently remove each $A \in Z_t$ with probability θ;*
3. *(Up-$\mathcal{A}$) Sample $Z_{t+1} \sim (\theta * \pi_{\mathcal{A}})(\cdot \mid Z_{t+1} \supseteq T)$, where we define*

   $$\theta * \pi_{\mathcal{A}}(Z) \propto \pi_A(Z) \cdot \prod_{A\in\mathcal{A}} \theta^{\mathbf{1}(Z_A=1)}, \quad \forall Z \in \{0,1\}^{\mathcal{A}},$$

   the distribution obtained from $\pi_{\mathcal{A}}$ by tilting the occurrence of each event A by θ.
4. *Sample $X_{t+1} \sim \pi_V(\cdot \mid Z_{t+1})$.*

The only example that we will use in this paper is the following edge-field dynamics:

▶ **Example 2.14** (Edge-field dynamics). Let $G = (V, E)$ be a graph and μ a distribution on $\{0,1\}^V$. We set

$$\mathcal{A} := \{A_{uv} \mid (u,v) \in E\}, \quad \text{where } A_{uv} := \{\sigma : \sigma_u = \sigma_v = 1\}, \quad \forall (u,v) \in E.$$

Given $T \subseteq \mathcal{A}$ generated in the (Down-$\mathcal{A}$) step, we can compute that

$$\mathbb{P}(X_{t+1} = \sigma \mid T) \propto \mu(\sigma) \cdot \theta^{|\mathcal{A}(\sigma)|} \cdot \mathbf{1}[\mathcal{A}(\sigma) \supseteq T],$$

where $\mathcal{A}(\sigma) := \{A_{uv} \in \mathcal{A} : \sigma \in A_{uv}\}$.

Next we recall the definition of an event-field denoising process from [8]. This is mostly a projection of the vertex-field denoising process on the collection of events $\mathcal{A}$.

▶ **Definition 2.15** (Event-field denoising). *Let μ be a distribution on $\{0,1\}^V$ and consider a collection of events $\mathcal{A}$. Denote by π the joint distribution on $\{0,1\}^{V \cup \mathcal{A}}$ and let $(Y'_t)_{t\in[0,1]}$ be the vertex-field denoising process for the marginal $\pi_{\mathcal{A}}$ with $\pi_{\mathcal{A}} = \mathrm{Law}(Y'_1)$. Then event-field denoising process for μ through $\mathcal{A}$, which we denote by $(Y_t)_{t\in[0,1]}$ is defined via*

1. *Sample $Y_1 \sim \pi_V(\cdot \mid Y'_1)$,*
2. *For $t \in [0,1)$, we set $Y_t = Y'_t$.*

In other words, the process follows the vertex field denoising process applied to the event indicators, and finally at $t = 1$ projects onto the vertices in V via the conditional law $\pi_V(\cdot \mid Y'_1)$.

Since $Y_t = Y'_t$ for all $t \in [0,1)$, the spectral stability of $(Y'_t)_{t\in[0,1]}$ implies the spectral stability of $(Y_t)_{t\in[0,1]}$:

▶ **Lemma 2.16** (ϕ-Entropic stability upgrades, [8], Proposition 4.12). *Let $(Y_t)_{t\in[0,1]}$ and $(Y'_t)_{t\in[0,1]}$ be the two denoising processes in Definition 2.15. Then for any $\theta \in [0,1)$, suppose that $(Y'_t)_{t\in[0,1]}$ is ϕ-entropically stable with rate C at time θ, then $(Y_t)_{t\in[0,1]}$ is also ϕ-entropically stable with rate C at time θ.*

Then we specialize to the edge-field dynamics in Example 2.14 and define its own denoising process:

▶ **Definition 2.17** (Edge-field denoising process). *Let $G = (V, E)$ be a graph and μ a distribution on $\{0,1\}^V$. Recall that we define*

$$\mathcal{A} := \{A_{uv} \mid (u,v) \in E\}, \quad \textit{where } A_{uv} := \{\sigma : \sigma_u = \sigma_v = 1\}, \quad \forall (u,v) \in E.$$

Let $(Y_t)_{t\in[0,1]}$ be the event-field denoising process for μ with respect to this family $\mathcal{A}$. Then the resulting process $(Y_t)_{t\in[0,1]}$ is called the edge-field denoising process for μ on G.

In the following, we give an explicit description of the process Y_t. We shall use a notation $\theta \otimes \mu$ for the distribution obtained by tilting the interactions by a factor $\theta > 0$:

$$\theta \otimes \mu(\sigma) \propto \mu(\sigma) \cdot \theta^{m(\sigma)}, \quad \forall \sigma \in \{0,1\}^V,$$

where we denote by $m(\sigma) := |\{(u,v) \in E : \sigma_u = \sigma_v = 1\}|$ the number of edges whose two endpoints are both equal to 1.

The edge field denoising process $(Y_t)_{t\in[0,1]}$ satisfies that its posterior distribution is the target distribution μ with tilted interaction:

▶ **Proposition 2.18.** *Let $(Y_t)_{t\in[0,1]}$ denote the edge-field denoising process. Then its posterior distribution satisfies*

$$\mathrm{Law}(Y_1 \mid Y_t) = (1-t) \otimes \mu^{Y_t},$$

where we recall that μ^{Y_t} denotes the conditional law of μ conditioning on the event that for every edge $(u,v) \in E$ with $Y_t(u,v) = 1$, then A_{uv} occurs, so that $\sigma_u = \sigma_v = 1$.

Proof. For any $\sigma \in \{0,1\}^V$, denote by $Z(\sigma) \in \{0,1\}^{\mathcal{A}}$ the vector that indicates whether each $A_{uv} \in \mathcal{A}$ occurs in the configuration σ. Take $X \sim \mu$, recall that $\pi := \mathrm{Law}(X, Z(X))$ on $\{0,1\}^{V\cup\mathcal{A}}$.

Let (Y'_t) be the vertex-field denoising process for $\pi_{\mathcal{A}}$. Then for any configuration T,

$$\begin{aligned}
&\mathbb{P}[Y_1 = \sigma \mid Y_t = T] \\
&= \mathbb{P}[Y_1 = \sigma \mid Y'_1 = Z(\sigma)] \cdot \mathbb{P}[Y'_1 = Z(\sigma) \mid Y'_t = T] \\
&\propto \mathbb{P}[Y_1 = \sigma \mid Y'_1 = Z(\sigma)] \cdot (1-t)^{|Z(\sigma)|} \cdot \pi_{\mathcal{A}}(Z(\sigma)) \cdot \mathbf{1}[T \subseteq Z(\sigma)] \\
&\propto (1-t)^{m(\sigma)} \cdot \pi(\sigma, Z(\sigma)) \cdot \mathbf{1}[T \subseteq Z(\sigma)].
\end{aligned}$$

This implies $\mathrm{Law}(Y_1 \mid Y_t) = (1-t) \otimes \mu^{Y_t}$ since σ determines $Z(\sigma)$. ◀

2.4 Spectral stability of edge-field dynamics

Thanks to Lemma 2.16, we only need to prove the spectral stability of the denoising process Y'_t. Motivated by the criterion in Lemma 2.12, we define the following correlation matrix for Y'_t:

▶ **Definition 2.19.** *(Second-order correlation matrix) Let μ be a distribution on $\{0,1\}^V$ and $G = (V,E)$ be a graph. Then for $u, v \in V$, denote by uv the event that*

$$uv := \{\sigma \in \{0,1\}^V \mid \sigma_u = \sigma_v = 1\}.$$

Then we define the second order correlation matrix $\mathrm{Cor}^{(2)}_\mu \in \mathbb{R}^{E\times E}$ via

$$\forall (u,v),(w,z) \in E, \quad \mathrm{Cor}^{(2)}_\mu(uv, wz) := \begin{cases} \mu(wz \mid uv) - \mu(wz), & \text{if } \mu(uv) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

▶ Remark 2.20. We define the correlation matrix $\mathrm{Cor}^{(2)}_\mu$ in this form due to the following fact: let μ be a distribution on $\{0,1\}^V$, then $(\mathrm{diag}(\mathbb{E}(\mu)))^{-1} \cdot \mathrm{Cov}_\mu = \mathrm{Cor}_\mu$.

Although $\mathrm{Cor}^{(2)}_\mu$ is not a symmetric matrix, its eigenvalues are all real.

Then we can translate spectral stability into the following criterion:

▶ **Corollary 2.21.** *(Spectral stability via second-order correlations) Let $(Y_t)_{t\in[0,1]}$ denote the edge-field denoising process for μ on the graph $G = (V,E)$. Then for fixed $\theta \in (0,1)$, assume that for every $\Lambda \subset V$ and any feasible pinning $\tau \in \Omega(\mu_\Lambda)$ we have that*

$$\lambda_{max}(\mathrm{Cor}^{(2)}_{(1-\theta)\otimes\mu^\tau}) \le C,$$

then the process $(Y_t)_{t\in[0,1]}$ is spectrally stable with rate C at time θ.

Proof. This follows from combining Definition 2.15, Lemma 2.16 and Lemma 2.12. Note that a conditioning on the event field events A_{uv} is equivalent to conditioning on the vertex events $\sigma_u = \sigma_v = 1$. ◀

2.5 Approximate tensorization at the end of the flow

We first record how the parameter of RFIM model changes under the edge-field denoising process:

▶ **Fact 2.22.** *Let μ be the RFIM measure μ_G. Then for any $\theta \in (0, 1)$ and any pinning τ, the Ising measure $(1-\theta) \otimes \mu^\tau$ has quadratic coefficient $4\beta + \ln(1-\theta)$ (on the coordinate $\{0,1\}^V$ in its Hamiltonian form) on edges of G whose both endpoints are not pinned by τ. In particular, when we take $\theta_* = 1 - \exp(-4\beta)$, then $(1-\theta_*) \otimes \mu^\tau$ is a product measure.*

Proof. By Fact 1.6, in the coordinate $\{0,1\}^V$ the quadratic coefficient of $\mu = \mu_G$ in the Hamiltonian is 4β. For each unpinned edge, the denoising process Y_t shrinks the probability by multiplying by $1-\theta$ before normalizing, which means changing 4β to $4\beta + \ln(1-\theta)$ in the Hamiltonian coefficient. ◀

For sake of completeness, we note here that when β is sufficiently small, this model is within the realm of spectral independence where approximate tensorization of variance directly follows. We, however, do not use this estimate as we stop the denoising process at θ_* where the conditional law is the product measure.

▶ **Lemma 2.23.** *Let $G = (V, E)$ be a graph of maximal degree at most Δ, and for each $(u, v) \in E$ we are given constants β_{uv} where $|\beta_{uv}| \le \beta$ for all edges. Fix an arbitrary external field $h = (h_u)_{u \in V}$, an arbitrary subset $A \subset V$ and configuration $\tau \in \{0,1\}^A$. Then whenever $\beta \le \frac{1}{\Delta}$, the measure $\mu_{\beta,h}$ defined on $\{0,1\}^{V \setminus A}$ via*

$$\mu_{\beta,h}(\sigma) \propto \exp\Big(\sum_{(u,v)\in E} \beta_{uv}\sigma_u\sigma_v + \sum_{u \in V\setminus A} h_u\sigma_u\Big), \quad \forall \sigma \in \{0,1\}^V : \sigma|_A = \tau$$

satisfies approximate tensorization of variance and approximate tensorization of entropy with constant 2.

Proof. Let Q denote the quadratic interaction matrix in $\mu_{\beta,h}$ where $Q_{uv} = \beta_{uv}$ if $(u, v) \in E$ and $u \notin A$ and $v \notin A$, and set $Q_{uv} = 0$ otherwise. Then $\lambda_{max}(Q) \le \beta\Delta$ and $\lambda_{min}(Q) \ge -\beta\Delta$, where $\lambda_{min}(Q), \lambda_{max}(Q)$ are the smallest and largest eigenvalues of Q. Then via Fact 1.6, we can rewrite $\mu_{\beta,h}$ as an Ising model on $\{-1,1\}^V \setminus A$ where the quadratic interaction matrix is $\frac{1}{4}Q$. Then by [2], Theorem 12, $\mu_{\beta,h}$ satisfies approximate tensorization of variance and entropy with constant $\frac{1}{1-(\lambda_{max}(\frac{1}{4}Q)-\lambda_{min}(\frac{1}{4}Q))} \le 2$. ◀

Now we summarize what we have gotten so far, and what remains to be proven in the remaining sections. We will apply the edge field denoising process Y_t to the RFIM μ_G, where we take

$$\theta_* = 1 - \exp(-4\beta) \tag{2.1}$$

as a threshold value. This value of θ_* is fixed throughout the paper. We need to upper bound the integral in R in Lemma 2.7 up to time θ_*, and then tensorization of variance/entropy is free for $\mathrm{Law}(Y_1 \mid Y_{\theta_*})$ since the latter is a product measure. For the integral in R, by Proposition 2.18, it suffices to control the second correlation matrix for $(1-\theta) \otimes \mu^\tau$ for all $0 \le \theta \le \theta_*$ and all pinning τ. In the next section, we will focus on the operator norm control of that matrix.

3 The spectral norm of correlation matrix with a random field

In this section, we make the following assumption on the external field of RFIM, which is more general than Assumption 1.1 as we only require $|h_v|$ to be independent, but the signs of h_v are not necessarily independent.

▶ **Assumption 3.1.** *Let $p_0 \in (0,1)$ be given such that $p_0(\Delta-1)<1$, let $K=K(p_0)>0$ be a constant such that*

$$\rho = \rho(\beta,\Delta,K) := \frac{e^{\Delta\beta-K}}{e^{\Delta\beta-K}+e^{-\Delta\beta+K}}, \tag{3.1}$$

is less than $\frac{p_0}{4}$. Suppose that the RFIM μ_G with external field h is such that $(|h_u|)_{u\in V}$ are i.i.d. and such that

$$\mathbb{P}_h(|h_u|\le K) < \frac{p_0}{2}.$$

▶ **Proposition 3.2.** *Let $(u,v)\in E$ be any edge of the graph $G=(V,E)$ of maximal degree Δ. Assume that Assumption 3.1 holds for the given $p_0\in(0,1)$, then we have, for any $m\ge 1$,*

$$\begin{aligned}&\mathbb{P}\left(\sup_{\theta\in[0,\theta_*],\Lambda,\tau_0\in\Omega(\Lambda)}\sum_{(wz)\in E}|\operatorname{Cor}^{(2)}_{(1-\theta)\otimes\mu^{\tau_0}}(uv,wz)|\ge\Delta m\right)\\&\quad\le\frac{2}{(1-e^{-\alpha_*})(1-e^{-2\alpha_*})}e^{2\gamma_*}e^{-\alpha_* m},\end{aligned} \tag{3.2}$$

where $\gamma_* := \ln\frac{1-p_0}{p_0(\Delta-2)}$ and $\alpha_* = \frac{1}{2}\log\frac{(\Delta-2)^{\Delta-2}}{p_0(\Delta-1)^{\Delta-1}(1-p_0)^{\Delta-2}}$.

Proof. For fixed edge (u,v) and the Ising measure $\nu := (1-\theta)\otimes\mu^{\tau_0}$, we construct a coupling of two spins $\sigma^1\sim\nu$ and $\sigma^2\sim\nu(\cdot\mid uv)$, where in σ^2 the vertices u, v are both fixed to be 1 and then we reveal σ^1 at u,v and then reveal both σ^1,σ^2 at vertices in $V\setminus\{u,v\}$. Assign uniform random variables $U_x, x\in V$ at each vertex of V, and generate $(\sigma^1)_u$ by the given probability law of ν, then generate $(\sigma^1)_v$ by the given probability law of ν conditioning on σ^1_u. Then we iteratively reveal the vertices of distance 1 to (u,v) in both configurations σ^1,σ^2 in any given sequence, and update the value of σ^1 and σ^2 with respect to the probability laws ν and $\nu(\cdot\mid uv)$, conditioning on the vertices already revealed. Use the same uniform random variable to update each vertex. Note that when we update both σ^1 and σ^2 at $w\neq u,v$, then when $|h_w|>K$, then the probability that the updated spins in σ^1,σ^2 have the sign of h_w is at least $1-\frac{p_0}{4}$, by Assumption 3.1. (the probability that σ^i_w has the same sign as h_w can be computed in the original coordinate $\{-1,1\}^V$ where the expression $\rho(\beta,\Delta,K)$ in Assumption 3.1 is more apparent) Therefore, the subset of V where σ^1 and σ^2 disagree is controlled by the connected component of (u,v) in the percolation process

$$\{|h_x|\le K\}\cup\{\min(U_x,1-U_x)\le\frac{p_0}{4}\} \tag{3.3}$$

at each $x\in V$, since outside this connected component, σ^1 and σ^2 are updated via the identity coupling. By definition of total variation distance, we have

$$|\operatorname{Cor}^{(2)}_\nu(uv,wz)|\le\mathbb{P}(\sigma^1_{wz}\neq\sigma^2_{wz})\le\mathbb{P}(\sigma^1_w\neq\sigma^2_w)+\mathbb{P}(\sigma^1_z\neq\sigma^2_z)$$

for our specific coupling (here $\sigma^1_{wz}\neq\sigma^2_{wz}$ means $\sigma^1_w\neq\sigma^2_w$ or $\sigma^1_z\neq\sigma^2_z$), and thus since each vertex is adjacent to at most Δ edges, we multiply by Δ and write

$$\sum_{(wz)\in E}|\operatorname{Cor}^{(2)}_\nu(uv,wz)|\le\Delta\cdot\mathbb{E}_U|\text{connected component of }(u,v)\text{ in }\mathrm{Ber}(p_0)\text{ percolation}|,$$

where the left hand side is a function of h only and the right hand side takes expectation with respect to the uniform measure $(U_x, x \in V)$ in defining the coupling, and the percolation is defined in (3.3). Let $\mathcal{C}_{h,U}(u,v)$ denote the component of (u,v) in this percolation, and denote by $S_{uv}(h) := \frac{1}{\Delta}\sum_{(wz)\in E} |\operatorname{Cor}^{(2)}_{\nu}(uv,wz)|$ (This notation has implicit dependence on θ, Λ, τ_0.) Then for any $s > 0$,

$$\mathbb{P}_h(S_{uv}(h) \geq m) \leq e^{-sm}\mathbb{E}_h \exp(s\mathbb{E}_U|\mathcal{C}_{h,U}|) \leq e^{-sm}\mathbb{E}_{h,U}\exp(s|\mathcal{C}_{h,U}|),$$

applying Jensen in the last step.

Moreover, for any $\theta \in [0,\theta_*]$, any Λ and any boundary condition τ_0, we use the same coupling for ν and $\nu(\cdot \mid uv)$ via the same uniform distribution, and thus the discrepancy region is simultaneously controlled by the same Bernoulli percolation (3.3). That is,

$$\mathbb{P}_h(\sup_{\theta\in[0,\theta_*],\Lambda,\tau_0} S_{uv}(h) \geq m) \leq e^{-sm}\mathbb{E}_{h,U}\exp(s|\mathcal{C}_{h,U}|).$$

In a subcritical Bernoulli percolation, the size of the connected component has exponential tails, see equation (3.5). Taking $s = \alpha_*$ defined below and using equation (3.5), we deduce that (3.2) holds and thus the Proposition is proven.

The rest of the proof is the derivation of (3.5) for the exponential moments for the size of the connected component. We consider a Bernoulli (p_0)-percolation on G whose law is denoted by $\mathbb{P}$, and we use $\mathcal{C}(u,v)$ denote the connected cluster in this percolation where (u,v) are forced open. Then we explore the open cluster of the edge (u,v). Since the maximal degree is at most Δ and (u,v) are forced open, the connected cluster is stochastically bounded by a Galton-Watson forest with two initial particles and offspring distribution $\operatorname{Bin}(\Delta-1,p_0)$, see for example [17]. By the Otter-Dwass formula [13] (and see [23] for the $r=2$ version), if we let $T^{(2)}$ denote the total progeny of the forest, then

$$\mathbb{P}(T^{(2)} = x) = \frac{2}{x}\mathbb{P}(\operatorname{Bin}((\Delta-1)x,p_0) = x-2), \quad \forall x \in \mathbb{N}_+.$$

By Chernoff, whenever $p_0(\Delta-1) < 1$, we have

$$\mathbb{P}(T^{(2)} = x) \leq \frac{2}{x}(\frac{1-p_0}{p_0(\Delta-2)})^2 e^{-\xi_* x},$$

where we take $\xi_* = \log\frac{(\Delta-2)^{\Delta-2}}{p_0(\Delta-1)^{\Delta-1}(1-p_0)^{\Delta-2}}$. Then we sum up over integers $x \geq m$ and get

$$\mathbb{P}(|\mathcal{C}(u,v)| \geq m \mid (u,v) \text{ open }) \leq \frac{2}{1-e^{-\xi_*}}(\frac{1-p_0}{p_0(\Delta-2)})^2 e^{-\xi_* m}. \tag{3.4}$$

Then we can compute the exponential moment of $\mathcal{C}(u,v)$: take $\alpha_* = \frac{1}{2}\xi_*$, then

$$\mathbb{E}[\exp(\alpha_*|\mathcal{C}(u,v)|) \mid (u,v) \text{ open }] \leq \frac{2}{(1-e^{-2\alpha_*})(1-e^{-\alpha_*})}(\frac{1-p_0}{p_0(\Delta-2)})^2. \tag{3.5}$$

◀

Via exactly the same argument, we have the following estimate:

▶ **Proposition 3.3.** *Let $(w,z) \in E$ be any edge of the graph $G = (V,E)$ of maximal degree Δ. Assume that Assumption 3.1 holds for the given $p_0 \in (0,1)$, then we have, for any $m \geq 1$,*

$$\begin{aligned}&\mathbb{P}\left(\sup_{\theta\in[0,\theta_*],\Lambda,\tau_0\in\Omega(\Lambda)} \sum_{(uv)\in E} |\operatorname{Cor}^{(2)}_{(1-\theta)\otimes\mu^{\tau_0}}(uv,wz)| \geq 2\Delta m\right)\\ &\leq \frac{2}{(1-e^{-\alpha_*})(1-e^{-2\alpha_*})}e^{2\gamma_*}e^{-\alpha_* m}.\end{aligned} \tag{3.6}$$

Proof. Consider the same Bernoulli percolation process as in (3.3), and for each $\theta \in (0, \theta_*)$ and pinning τ_0, use the same uniform coupling to generate $\sigma^1 \sim \nu := (1-\theta) \otimes \mu^{\tau_0}$ and $\sigma^2 \sim \nu(\cdot \mid wz)$. By the same argument in the proof of Proposition 3.2, we have

$$\begin{aligned}
&\sum_{(uv)\in E} |\operatorname{Cor}^{(2)}_\nu(uv, wz)| \\
&\le 2\mathbb{E}_U \sum_{(uv)\in E} \mathbf{1}((wz) \text{ is in the connected component of } (uv) \text{ in this percolation}) \\
&= 2\mathbb{E}_U \sum_{(uv)\in E} \mathbf{1}((uv) \text{ is in the connected component of } (wz) \text{ in this percolation}) \\
&\le 2\Delta \cdot \mathbb{E}_U |\text{connected component of } (wz) \text{ in } \mathrm{Ber}(p_0) \text{ percolation}|.
\end{aligned}$$

Denote by $S^{\mathrm{col}}_{wz}(h) := \frac{1}{2\Delta} \sum_{(uv)\in E} |\operatorname{Cor}^{(2)}_\nu(uv, wz)|$. Then

$$S^{\mathrm{col}}_{wz}(h) \le \mathbb{E}_U |\mathcal{C}_{h,U}(w, z)|.$$

Then the same Chernoff bound and exponential moment applies, concluding the proof. ◀

For a square matrix $A \in \mathbb{R}^{n\times n}$, its operator norm satisfies the following interpolation inequality

$$\|A\| \le \sqrt{\left(\sup_{i\in[n]} \sum_{j=1}^n |a_{ij}|\right)\left(\sup_{j\in[n]} \sum_{i=1}^n |a_{ij}|\right)}. \tag{3.7}$$

Taking a union bound in Proposition 3.2, we obtain an operator norm upper bound for all Ising models $(1-\theta) \otimes \mu^{\tau_0}$:

▶ **Proposition 3.4.** *Under the same assumption in Proposition 3.2, with probability at least $1 - O_{\beta,\Delta,p_0}(\frac{1}{|V|})$ over the randomness in the external field h,*

$$\sup_{\theta\in[0,\theta_*],\Lambda,\tau_0\in\Omega(\Lambda)} \| \operatorname{Cor}^{(2)}_{(1-\theta)\otimes\mu^{\tau_0}} \|_{op} \le \frac{4\Delta \ln |V|}{\alpha_*}.$$

Proof. We take $m = \frac{2\ln|V|}{\alpha_*}$ in the estimate (3.2) for each edge (u, v) and take the same value of m in the estimate (3.6) for each edge (w, z). Then with probability at least $1 - 2|V|\Delta \cdot \frac{2}{(1-e^{-\alpha_*})(1-e^{-2\alpha_*})} e^{2\gamma_*} \frac{1}{|V|^2}$, we have

$$\sup_{(uv)\in E} \sum_{(wz)\in E} |\operatorname{Cor}^{(2)}_{(1-\theta)\otimes\mu^{\tau_0}}(uv, wz)|, \sup_{(wz)\in E} \sum_{(uv)\in E} |\operatorname{Cor}^{(2)}_{(1-\theta)\otimes\mu^{\tau_0}}(uv, wz)| \le 2\Delta m.$$

Taking these two estimates into (3.7) for $A = \operatorname{Cor}^{(2)}_{(1-\theta)\otimes\mu^{\tau_0}}$ completes the proof. ◀

▶ **Corollary 3.5.** *Under the same assumption in Proposition 3.2, with $\mathbb{P}_h$-probability at least $1 - O_{\beta,\Delta,p_0}(\frac{1}{|V|})$, the edge field denoising process $(Y_t)_{t\in[0,1]}$ satisfies R- approximate conservation of variance up to time θ_* with constant*

$$R \le \exp\left(\frac{16\beta\Delta \ln |V|}{\alpha_*}\right).$$

Moreover, the RFIM measure μ_G satisfies approximate tensorization of variance with constant R.

Proof. Note that $\int_0^{\theta_*} \frac{dt}{1-t} = 4\beta$. The claim on (Y_t) follows directly from Proposition 3.4, Lemma 2.7 and Corollary 2.21. The R-approximate tensorization of variance of μ_G then follows from Lemma 2.5, since the posterior $\mathrm{Law}(Y_1 \mid Y_{\theta_*})$ is a product measure. ◀

Now we can conclude the proof of Theorem 1.2:

Proof of Theorem 1.2. The claim on Poincaré constant follows from Fact 2.8 combined with Corollary 3.5. The claim on mixing times follows from Fact 1.5. ◀

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